Homogenization of aligned "fuzzy fiber" composites

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Abstract

The aim of this work is to study composites in which carbon fibers coated with radially aligned carbon nanotubes are embedded in a matrix. The effective properties of these composites are identified using the asymptotic expansion homogenization method in two steps. Homogenization is performed in different coordinate system, the cylindrical and the Cartesian, and numerical examples are presented.

Key words: Carbon fibers, carbon nanotubes, asymptotic expansion homogenization method

1. Introduction

Despite their recent discovery by Iijima (1991), the carbon nanotubes (CNTs) have attracted considerable research attention. Nowadays a large variety of composites containing CNTs have been manufactured (Milo et al., 1999; Peigney et al., 2000; Potschke et al., 2004,Wagner et al., 1998; Lourie and Wagner, 1998; Star et al., 2001; McCarthy et al., 2002; Zhu et al., 2003). This scientific interest is derived from the CNTs exceptional properties. Carbon nanotubes are reported to have an axial Young's modulus in the range of 300 - 1000 GPa, up to five times the stiffness and with half the density of SiC fibers, while their theoretical elongation to break reaches 30-40% (Yakobson and Smalley, 1997; Yakobson et al., 1997; Yu et al., 2000; Wang et al., 2001; Salvetat-Delmotte and Rubio, 2002; Fisher et al., 2002; Popov, 2004).

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Modeling of composites containing CNTs has also grown significantly in recent years. The mechanical response in tension of a single CNT embedded in polymer via finite element analysis was studied by Liu and Chen (2003), while Odegard et al. (2003) have modeled aligned and misaligned CNT composites using the equivalent continuum method in conjunction with the Mori-Tanaka micromechanics method. Fisher et al. (2002, 2003) studied the effects of nanotube waviness on the effective composite properties using finite element analysis and the micromechanics Mori-Tanaka method. Hadjiev et al. (2006) considered buckling of CNTs within an epoxy matrix. Several efforts in CNT composite modeling have focused on the inclusion of less than ideal CNT adhesion to the matrix (Wagner, 2002; Frankland et al., 2003; Griebel and Hamaekers, 2004). The clustering of CNTs in the polymer matrix was studied in Seidel and Lagoudas (2006). In Spanos and Kontsos (2008) nanocomposite properties were computed using Monte Carlo finite element method. Molecular Dynamics (MD) simulations have used to obtain the stress-strain behavior of CNTs embedded in a polymer matrix (Frankland et al., 2002), or the properties of the interphase between CNTs and polymer (Awasthi et al., 2009). In all these modeling efforts, the carbon nanotubes are embedded directly in a polymer matrix.

In this work we focus on composites containing carbon fibers, which are coated with radially aligned carbon nanotubes ("fuzzy fibers"). In these "fuzzy fiber" composites, the interphase layer between the fiber and the matrix can be seen as a separate composite material consisting of CNTs in radial arrangement inside a matrix. In this perspective, the "fuzzy fiber" can be studied as a two concentric cylinders material, the fiber and the interphase layer. The elastic response of homogeneous and non-homogeneous cylinders under different boundary conditions was studied by Chatterjee (1970); Horgan and Chan (1999a,b); Chen et al. (2000); Tarn and Wang (2001); Tarn (2002); Ruhi et al. (2005); Hosseini Kordkheili and Naghdabadi (2007); Chatzigeorgiou et al. (2008); Tsukrov and Drach (2010); Nie and Batra (2010a,b,c).

Our goal is to obtain the effective mechanical properties of unidirectional "fuzzy fiber" composites. In order to achieve it we use a multiscale approach based on the asymptotic expansion homogenization method (AEH). The AEH method is well documented in the literature for periodic composites (Sanchez-Palencia, 1978; Bensoussan et al., 1978; Kalamkarov and Kolpakov, 1997; Chung et al., 2001), whose periodicity can be represented in the Cartesian coordinate system. In this paper, due to the structure of the interphase layer (angular periodicity), we present a modified version of the AEH method, in which the homogenized properties of the composite are obtained in two steps. Initially, the properties of the interphase layer are computed through homogenization with respect to a cylindrical coordinate system. Then, the homogenized layer is introduced in the actual "fuzzy fiber" composite, which is homogenized with respect to a Cartesian coordinate system.

The structure of this paper is the following: in Section 2 we describe the characteristics of the "fuzzy fiber" composites and the mathematical assumptions. The first step of homogenization, which refers to the nanocomposite layer of a "fuzzy fiber" is presented in Section 3. Section 4 discusses the second step of homogenization in the mesoscale. A numerical example is presented in Section 5, and the final section includes the major conclusions.

2. "Fuzzy fiber" composites

The "fuzzy fiber" composite we want to study is a fiber composite material system, in which a carbon fiber (CF) is coated with radially aligned carbon nanotubes (CNTs) (Figure 1). The "fuzzy fiber" is embedded in a matrix, which can be an epoxy. The intermediate layer between the CF and the matrix, consisting of CNTs and matrix, will be denoted in the sequel as nanocomposite (NCP). The CF is represented here as a long cylinder, while the CNTs are represented as hollow tubes of very large length, compared to their diameter.

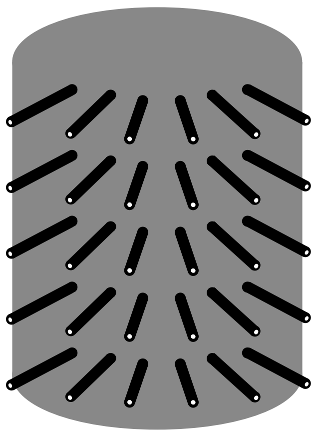
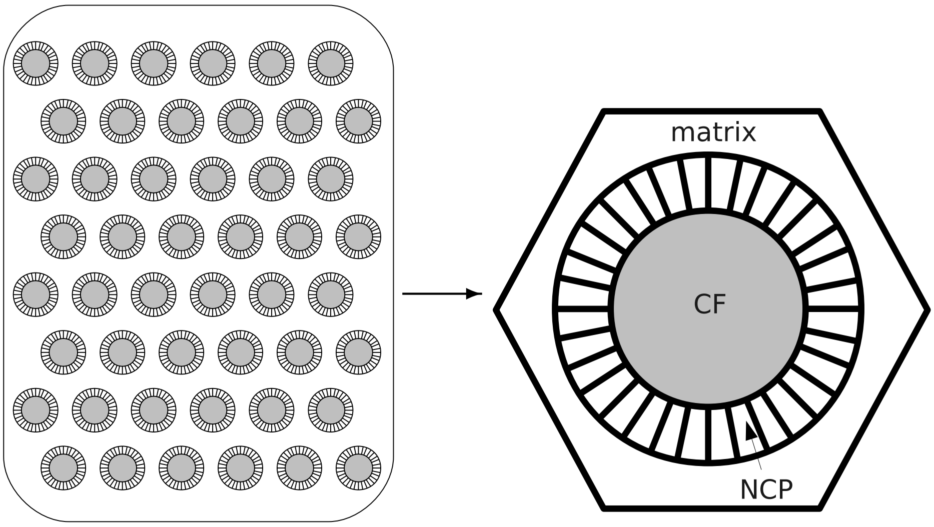


Figure 1: "Fuzzy fiber": Carbon fiber coated with radially aligned carbon nanotubes.

The composite we investigate consists of unidirectional "fuzzy fibers", distributed in a hexagonal form inside the matrix (Figure 2). The hexagonal distribution represents efficiently a random distribution of the fibers in the matrix (Hashin and Rosen, 1964).



(a) (b)

Figure 2: a) "Fuzzy fiber" composite and b) unit cell of the composite.

In order to obtain the effective properties of the composite, we are using the asymptotic expansion homogenization (AEH) method, which is a two scale homogenization method. The AEH is a well established method (Sanchez-Palencia, 1978; Bensoussan et al., 1978; Kalamkarov and Kolpakov, 1997), in which 2 scales are taken into account, the macroscale and the microscale. In a Cartesian coordinate system framework, the macroscale is described by the coordinates (x_1,x_2,x_3), while the microscale by the coordinates $(\frac{x_1}{\epsilon},\frac{x_2}{\epsilon},\frac{x_3}{\epsilon})$, where ϵ is the characteristic length of the periodic cell. The idea of the method is that the displacements are written in an asymptotic series form with respect to ϵ and the expanded forms of the equilibrium equations lead to ϵ^{-1} terms (microequations) and ϵ^0 terms (macroequations). From the microequations we obtain the necessary quantities, whose averages give as the homogenized properties used in the macroequations.

In our composite system, the unit cell of Figure 2_b consists of 3 different material systems, the carbon fiber, the nanocomposite and the matrix. The nanocomposite though by itself is a composite (Figure 3_a), whose periodic cell is represented in cylindrical coordinate system. Its structure can include CNTs in tetragonal (Figure 3_b) or hexagonal (Figure 3_c) array. In this case, in addition to the characteristic length ϵ, we need to introduce the characteristic length of the nanocomposite δ. This leads to three series of coordinates, the macroscale (x_1,x_2,x_3), the mesoscale ($\frac{x_1}{\epsilon}$, $\frac{x_2}{\epsilon}$,$\frac{x_3}{\epsilon}$) and the microscale ($\frac{x_1}{\delta}$,$\frac{x_2}{\delta}$,$\frac{x_3}{\delta}$). The characteristic lengths δ and ϵ can be related with one as the square power of the other. In this work though we prefer to use two independent characteristic lengths, in order to allow the independency of the two scales. Since the AEH method is based on the idea of ϵ or δ tending to zero, the microscale characteristic length δ and the mesoscale characteristic length ϵ can be seen as independently tending to zero. If one of the two characteristic lengths is not close to zero (for instance, if we have very few CNTs in the nanocomposite, δ is not close to zero) then the homogenization in this scale is not necessarily accurate.

In order to solve efficiently this 3 scale problem, we split it into two 2 scale problems. The first problem is describing the relation between the microscale and the mesoscale, and focuses in the computation of the effective properties of the NCP. The NCP effective properties are used in the second problem which deals with the connection between the mesoscale with the macroscale.

The homogenization of a “fuzzy fiber” composite can be put into a general framework of homogenization with multiple metrics. As in homogenization of a “fuzzy fiber” composite, there are many applications where small-scale features can have periodic or regular forms that can be handled with existing homogenization techniques. For example, one may need to select a certain coordinate system at a given scale to have periodicity or sufficient regularity. We will discuss some examples after presenting the homogenization. Consider a material whose properties are defined on N different scales

$$C(\frac{x}{\epsilon_1}, \frac{x}{\epsilon_2}, ..., \frac{x}{\epsilon_N}),$$

where we do not assume a periodicity at any scales. Assume that the periodicity can be achieved at any of the scales if one can consider an appropriate coordinate system. For example, the cylindrical coordinate system provides periodic microstructure at the scale N, while for the scale $N-1$, we may need to choose the spherical coordinate system to achieve periodicity at that scale. Any coordinate system is defined via a metric tensor $G = (g_{ij})$. We

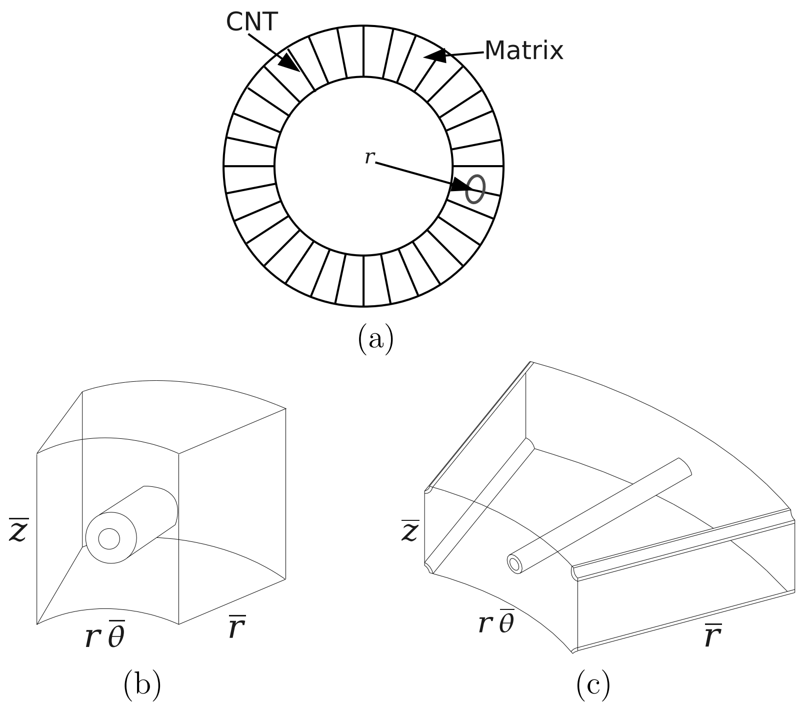


Figure 3: a) Cross section of the nanocomposite, b) tetragonal unit cell of the nanocomposite and c) hexagonal unit cell of the nanocomposite.

denote the metric that gives homogenization-amenable coordinate system for ith scale by G_i. Then, we re-write our material properties as

$$C(\frac{x^{G_1}}{\epsilon_1}, \frac{x^{G_2}}{\epsilon_2}, ..., \frac{x^{G_N}}{\epsilon_N}), \tag{1}$$

where $\epsilon_N \ll ... \ll \epsilon_1$ and G_i's are the metrics for ith coordinate system where homogenization is performed.

As we mentioned above that the metric elements $G = (g_{ij})$ can correspond to some well-known coordinate transformations (e.g., spherical or cylindrical) or can be non-trivial transformations as in the case of problems without scale separation (Owhadi and Zhang, 2007; Efendiev and Hou, 2009). In Owhadi and Zhang (2007) it was shown that in the case of no-scale separation, by choosing the metric correctly, one can smooth the solution and thus, the solution can be approximated by piecewise linear functions on new coordinate system. The metric here is defined via a solution of cell problems in a large domain. Consequently, in general, one needs to consider appropriate metrics at different scales to take advantage of the regularity of the solution or special features.

To carry out the homogenization in a media where different coordinate metrics are used at different scales, we start the homogenization from the smallest scale by freezing coordinates in all larger scales. Once the homogenization is performed at the corresponding metric, the material properties are transformed to the metric of the next scale. One requirement is that the metric needs not to vary rapidly at the scale of the homogenized variable so that when transforming to the next coordinate system, the smallest scales are removed. More precisely, assume that the scale ϵ_N is being homogenized in the metric G_N. Then,

$$C^*(\frac{x^{G_1}}{\epsilon_1}, \frac{x^{G_2}}{\epsilon_2}, ..., \frac{x^{G_{N-1}}}{\epsilon_N}) = H_{G_N}\left(C(\frac{x^{G_1}}{\epsilon_1}, \frac{x^{G_2}}{\epsilon_2}, ..., \frac{x^{G_N}}{\epsilon_N})\right),$$

where H_{G_N} is the homogenization operator (generally, nonlinear operator) in coordinate system with metric G_N. To do this homogenization, coefficients are frozen at the scales ϵ_1, ..., ϵ_{N-1}, and then transformed to the coordinate system with the metric G_N. At this scale, the coefficients are homogenized and then transformed to G_{N-1}. One needs to ensure that this transformation does not bring back ϵ_N scale. For this reason we need to assume that G_N does not depend on ϵ_N, i.e., the transformation metric does not vary rapidly at the scale ϵ_N. The mathematical error analysis will require some additional conditions on metrics such as boundedness of the transformation tensor. In this paper, our new framework is applied to an example problem.

3. Nanocomposite layer of a "fuzzy fiber" composite

In this section we are going to investigate the effective properties of the NCP. This intermediate layer of the "fuzzy fibers" composites consists of radially aligned CNTs and matrix (Figure 3_a). The unit cell of the NCP is shown in Figure 3_b. The effective mechanical properties of the nanocomposite will be obtained using the asymptotic expansion homogenization method. In this approach two scales are considered, the mesoscale and the microscale with characteristic length δ. In cylindrical coordinates we have the meso coordinates (r, θ, z)[1] and the micro coordinates $(\frac{r}{\delta}, \frac{\theta}{\delta}, \frac{z}{\delta}) \rightarrow (\bar{r}, \bar{\theta}, \bar{z})$. The choice of the cylindrical coordinate system has two main advantages: a) due to the NCP structure, there is no fast variation in the radial direction, reducing the microscale equations to 2-D, b) allows us to represent in a rigorous way the homogenization procedure and c) the periodicity of the microstructure is represented easier with respect to $\bar{\theta}$ and $\bar{z}$. For clarity and simplification, we denote the axes (r, θ, z) as $(1, 2, 3)$ and we use the Einstein summation rule for double indices. Additionally, we introduce the operators $\mathcal{L}_i$ for the mesoscale and $\bar{\mathcal{L}}_i$ for the microscale, where

$$\mathcal{L}_1 = \frac{\partial}{\partial r}, \quad \mathcal{L}_2 = \frac{1}{r}\frac{\partial}{\partial \theta}, \quad \mathcal{L}_3 = \frac{\partial}{\partial z}, \tag{2}$$

$$\bar{\mathcal{L}}_1 = \frac{\partial}{\partial \bar{r}}, \quad \bar{\mathcal{L}}_2 = \frac{1}{r}\frac{\partial}{\partial \bar{\theta}}, \quad \bar{\mathcal{L}}_3 = \frac{\partial}{\partial \bar{z}}. \tag{3}$$

The aim of the asymptotic expansion homogenization (AEH) method is to identify the behavior of the composite material, when the size of the microstructure becomes infinitesimally small, i.e. $\delta \rightarrow 0$. In all the quantities (displacements, strains, stresses, stiffness components) we will use the superscript δ, denoting that we refer to a material point, which can be in the matrix, in the CNT or in the void. The strain-displacement relation of the

[1]For simplicity in the expressions, we omit the mesoscale characteristic length ϵ

material system in cylindrical coordinates read[2]

$$\hat{\varepsilon}^{\delta}_{11} = \mathcal{L}_1 \hat{u}^{\delta}_1, \quad \hat{\varepsilon}^{\delta}_{22} = \mathcal{L}_2 \hat{u}^{\delta}_2 + \frac{\hat{u}^{\delta}_1}{r}, \quad \hat{\varepsilon}^{\delta}_{33} = \mathcal{L}_3 \hat{u}^{\delta}_3,$$
$$\hat{\varepsilon}^{\delta}_{23} = \frac{1}{2}\Big(\mathcal{L}_2 \hat{u}^{\delta}_3 + \mathcal{L}_3 \hat{u}^{\delta}_2\Big), \quad \hat{\varepsilon}^{\delta}_{13} = \frac{1}{2}\Big(\mathcal{L}_1 \hat{u}^{\delta}_3 + \mathcal{L}_3 \hat{u}^{\delta}_1\Big), \tag{4}$$
$$\hat{\varepsilon}^{\delta}_{12} = \frac{1}{2}\Big(\mathcal{L}_1 \hat{u}^{\delta}_2 + \mathcal{L}_2 \hat{u}^{\delta}_1 - \frac{\hat{u}^{\delta}_2}{r}\Big).$$

Ignoring inertia and body forces, the equilibrium equations are written as

$$\mathcal{L}_j \hat{\sigma}^{\delta}_{1j} + \frac{\hat{\sigma}^{\delta}_{11} - \hat{\sigma}^{\delta}_{22}}{r} = 0, \quad \mathcal{L}_j \hat{\sigma}^{\delta}_{2j} + 2\frac{\hat{\sigma}^{\delta}_{12}}{r} = 0, \quad \mathcal{L}_j \hat{\sigma}^{\delta}_{3j} + \frac{\hat{\sigma}^{\delta}_{13}}{r} = 0. \tag{5}$$

Finally, the Hooke's law is written

$$\hat{\sigma}^{\delta}_{ij} = \hat{C}^{\delta}_{ijkl} \hat{\varepsilon}^{\delta}_{kl}. \tag{6}$$

The stiffness components $\hat{C}^{\delta}_{ijkl}$ are generally spatially dependent. At the microscale level it depends on the microcoordinates $\bar{\theta}$ and $\bar{z}$. The material parameters vary very slowly in the radial direction and depend on the meso-coordinate r. Due to the geometry of the CNT (only its center is independent on r), the stiffness components present local discontinuity with respect to r. The discontinuity appears only when we move from the void to the CNT and from the CNT to the matrix. So we can write

$$\hat{C}^{\delta}_{ijkl} = \hat{C}^{\delta}_{ijkl}(r, \bar{\theta}, \bar{z}), \quad \text{slow variation with respect to } r. \tag{7}$$

In the AEH method, the displacements are represented in a series expansion form

$$\hat{u}^{\delta}_i = \hat{u}^{(0)}_i(r, \theta, z) + \delta \hat{u}^{(1)}_i(r, \theta, z, \bar{\theta}, \bar{z}) + \delta^2 \hat{u}^{(2)}_i(r, \theta, z, \bar{\theta}, \bar{z}) + ..., \tag{8}$$

where $\hat{u}^{(0)}_i$ denotes the mesodisplacement and $\hat{u}^{(1)}_i$, $\hat{u}^{(2)}_i$ e.t.c. are periodic functions and represent the oscillating terms. The derivatives can be written in the form

$$\mathcal{L}_i = \mathcal{L}_i + \frac{1}{\delta}\bar{\mathcal{L}}_i. \tag{9}$$

[2]In the sequel, we will use ˆ to denote that the specific quantity refers to cylindrical coordinate system.

Using (8) and (9), the strains in (4) can be written in the form

$$\hat{\varepsilon}_{ij}^{\delta} = \hat{\varepsilon}_{ij}^{(0)} + \delta \hat{\varepsilon}_{ij}^{(1)} + ..., \tag{10}$$

where

$$\hat{\varepsilon}_{ij}^{(m)} = \hat{\varepsilon}_{ij}^{(m*)} + \frac{1}{2}\Big(\bar{\mathcal{L}}_i \hat{u}_j^{(m+1)} + \bar{\mathcal{L}}_j \hat{u}_i^{(m+1)}\Big), \quad m = 0, 1, 2, ... \tag{11}$$

and

$$\begin{gathered}
\hat{\varepsilon}_{11}^{(m*)} = \mathcal{L}_1 \hat{u}_1^{(m)}, \quad \hat{\varepsilon}_{22}^{(m*)} = \mathcal{L}_2 \hat{u}_2^{(m)} + \frac{\hat{u}_1^{(m)}}{r}, \quad \hat{\varepsilon}_{33}^{(m*)} = \mathcal{L}_3 \hat{u}_3^{(m)}, \\
\hat{\varepsilon}_{23}^{(m*)} = \frac{1}{2}\Big(\mathcal{L}_3 \hat{u}_2^{(m)} + \mathcal{L}_2 \hat{u}_3^{(m)}\Big), \quad \hat{\varepsilon}_{13}^{(m*)} = \frac{1}{2}\Big(\mathcal{L}_1 \hat{u}_3^{(m)} + \mathcal{L}_3 \hat{u}_1^{(m)}\Big), \\
\hat{\varepsilon}_{12}^{(m*)} = \frac{1}{2}\Big(\mathcal{L}_2 \hat{u}_1^{(m)} + \mathcal{L}_1 \hat{u}_2^{(m)} - \frac{\hat{u}_2^{(m)}}{r}\Big).
\end{gathered} \tag{12}$$

From the Hooke's law (6) and equation (10) we can write the expanded form of the stresses

$$\hat{\sigma}_{ij}^{\delta} = \hat{\sigma}_{ij}^{(0)} + \delta \hat{\sigma}_{ij}^{(1)} + ..., \tag{13}$$

where

$$\hat{\sigma}_{ij}^{(m)} = \hat{C}_{ijkl} \hat{\varepsilon}_{kl}^{(m*)} + \hat{C}_{ijkl} \bar{\mathcal{L}}_k \hat{u}_l^{(m+1)}. \tag{14}$$

Using the expanded form of the stresses (13) and the equations (9) the equilibrium equations take the form

$$\frac{1}{\delta}\Big(\bar{\mathcal{L}}_j \hat{\sigma}_{1j}^{(0)}\Big) + \mathcal{L}_j \hat{\sigma}_{1j}^{(0)} + \frac{\hat{\sigma}_{11}^{(0)} - \hat{\sigma}_{22}^{(0)}}{r} + \bar{\mathcal{L}}_j \hat{\sigma}_{1j}^{(1)} + \delta ... = 0, \tag{15}$$

$$\frac{1}{\delta}\Big(\bar{\mathcal{L}}_j \hat{\sigma}_{2j}^{(0)}\Big) + \mathcal{L}_j \hat{\sigma}_{2j}^{(0)} + 2\frac{\hat{\sigma}_{12}^{(0)}}{r} + \bar{\mathcal{L}}_j \hat{\sigma}_{2j}^{(1)} + \delta ... = 0, \tag{16}$$

$$\frac{1}{\delta}\Big(\bar{\mathcal{L}}_j \hat{\sigma}_{3j}^{(0)}\Big) + \mathcal{L}_j \hat{\sigma}_{3j}^{(0)} + \frac{\hat{\sigma}_{13}^{(0)}}{r} + \bar{\mathcal{L}}_j \hat{\sigma}_{3j}^{(1)} + \delta ... = 0. \tag{17}$$

According to the classical procedure of the AEH method, the micro-equations are defined from the δ^{-1} terms

$$\bar{\mathcal{L}}_j \hat{\sigma}_{ij}^{(0)} = 0, \quad i = 1, 2, 3, \tag{18}$$

which, using equation (14) for $m = 0$, can be written as

$$\bar{\mathcal{L}}_j\left(\hat{C}_{ijkl}\right)\hat{\varepsilon}_{kl}^{(0*)} + \bar{\mathcal{L}}_j\left(\hat{C}_{ijkl}\bar{\mathcal{L}}_k\hat{u}_l^{(1)}\right) = 0. \tag{19}$$

In equation (19) $\hat{\varepsilon}_{ij}^{(0*)}$ depends only on the meso-displacements $\hat{u}_i^{(0)}$. By assuming that

$$\hat{u}_i^{(1)} = \hat{N}_i^{mn}\hat{\varepsilon}_{mn}^{(0*)}, \tag{20}$$

the micro-equations (19) are written

$$\bar{\mathcal{L}}_j\left(\hat{C}_{ijmn} + \hat{C}_{ijkl}\bar{\mathcal{L}}_k\hat{N}_l^{mn}\right) = 0. \tag{21}$$

The final form of the micro-equations are solved for the unknown functions $\hat{N}_i^{mn}$, which are periodic in the $(\bar{\theta}, \bar{z})$ space. Also, we need to impose the necessary continuity conditions

$$\left[\!\left[\hat{N}_i^{mn}\right]\!\right] = 0, \quad \left[\!\left[\left(\hat{C}_{ijmn} + \hat{C}_{ijkl}\bar{\mathcal{L}}_k\hat{N}_l^{mn}\right)n_j\right]\!\right] = 0, \tag{22}$$

where n_i is the unit normal vector to the surface of discontinuity.

The meso-equations can be obtained from the δ^0 terms of the equilibrium equations. When δ approaches zero, periodic functions attain their weak limit, which is equal to the area integral of the functions in the periodic unit cell. We introduce the area integral symbol on the area A of the 2-D unit cell in $(\bar{\theta},\, \bar{z})$,

$$\langle\phi\rangle = \frac{1}{A}\int_{-\bar{z}'/2}^{\bar{z}'/2}\int_{-\bar{\theta}'/2}^{\bar{\theta}'/2} r\phi(r, \bar{\theta}, \bar{z})\mathrm{d}\bar{\theta}\mathrm{d}\bar{z}. \tag{23}$$

By setting ω_i as the outer unit normal vector to the boundary and ∂A the boundary surface of the unit cell, we can use the Gauss theorem and the periodicity of $\hat{\sigma}_{ij}^{(1)}$ to show that

$$\langle\bar{\mathcal{L}}_j\hat{\sigma}_{ij}^{(1)}\rangle = \frac{r}{A}\int_{\partial A}\hat{\sigma}_{ij}^{(1)}\omega_j\mathrm{d}S = 0. \tag{24}$$

The meso-equations then are obtained from the weak limit of the δ^0 terms of the equilibrium equations

$$\begin{gathered}\mathcal{L}_j\langle\hat{\sigma}_{1j}^{(0)}\rangle + \frac{\langle\hat{\sigma}_{11}^{(0)}\rangle - \langle\hat{\sigma}_{22}^{(0)}\rangle}{r} = 0, \quad \mathcal{L}_j\langle\hat{\sigma}_{2j}^{(0)}\rangle + 2\frac{\langle\hat{\sigma}_{12}^{(0)}\rangle}{r} = 0,\\ \mathcal{L}_j\langle\hat{\sigma}_{3j}^{(0)}\rangle + \frac{\langle\hat{\sigma}_{13}^{(0)}\rangle}{r} = 0,\end{gathered} \tag{25}$$

where

$$\langle \hat{\sigma}_{ij}^{(0)} \rangle = \langle \hat{C}_{ijmn} + \hat{C}_{ijkl} \bar{\mathcal{L}}_k \hat{N}_l^{mn} \rangle \hat{\varepsilon}_{mn}^{(0*)}. \tag{26}$$

From the last equation it becomes obvious that the effective, or homogenized, properties $\hat{C}_{ijkl}^{\mathrm{NCP}}$ are given by

$$\hat{C}_{ijmn}^{\mathrm{NCP}} = \langle \hat{C}_{ijmn} + \hat{C}_{ijkl} \bar{\mathcal{L}}_k \hat{N}_l^{mn} \rangle, \tag{27}$$

where the functions $\hat{N}_i^{mn}$ are determined by solving the equations (21). It is important to note that one can possibly derive the homogenized equations using curvilinear periodicity cells. However, our approach provides a rigorous foundation of performing homogenization in curvilinear system.

In the microlevel and at a specific radius r, equations (21) represent 2 anti-plane strain problems and 4 plane strain problems. Due to the large difference in $\bar{\theta}$ and $\bar{z}$ scales, it is more preferable to solve the microequations in the $r\bar{\theta} - \bar{z}$. In the sequel, we will use $\hat{y}_2^*$ for the $r\bar{\theta}$ coordinate and $\hat{y}_3$ for the $\bar{z}$ coordinate and we will adopt the Voigt notation[3]. In the 2-D form, the anti-plane problems are given for $i = 1$, $\alpha = 5, 6$

$$\begin{aligned} -\frac{\partial}{\partial \hat{y}_2^*}\Big(\hat{C}_{66}\frac{\partial \hat{N}_1^{\alpha}}{\partial \hat{y}_2^*} + \hat{C}_{56}\frac{\partial \hat{N}_1^{\alpha}}{\partial \hat{y}_3}\Big) - \frac{\partial}{\partial \hat{y}_3}\Big(\hat{C}_{56}\frac{\partial \hat{N}_1^{\alpha}}{\partial \hat{y}_2^*} + \hat{C}_{55}\frac{\partial \hat{N}_1^{\alpha}}{\partial \hat{y}_3}\Big) \\ = \frac{\partial \hat{C}_{6\alpha}}{\partial \hat{y}_2^*} + \frac{\partial \hat{C}_{5\alpha}}{\partial \hat{y}_3}, \end{aligned} \tag{28}$$

The plane strain problems are given for $i = 2$ and $i = 3$, $\alpha = 1, 2, 3, 4$ from the system of equations

$$\begin{aligned} -\frac{\partial}{\partial \hat{y}_2^*}\Big(\hat{C}_{22}\frac{\partial \hat{N}_2^{\alpha}}{\partial \hat{y}_2^*} + \hat{C}_{24}\frac{\partial \hat{N}_2^{\alpha}}{\partial y_3} + \hat{C}_{24}\frac{\partial \hat{N}_3^{\alpha}}{\partial y_2^*} + \hat{C}_{23}\frac{\partial \hat{N}_3^{\alpha}}{\partial \hat{y}_3}\Big) \\ -\frac{\partial}{\partial \hat{y}_3}\Big(\hat{C}_{24}\frac{\partial \hat{N}_2^{\alpha}}{\partial \hat{y}_2^*} + C_{44}\frac{\partial \hat{N}_2^{\alpha}}{\partial \hat{y}_3} + \hat{C}_{44}\frac{\partial \hat{N}_3^{\alpha}}{\partial \hat{y}_2^*} + \hat{C}_{34}\frac{\partial \hat{N}_3^{\alpha}}{\partial \hat{y}_3}\Big) \\ = \frac{\partial \hat{C}_{2\alpha}}{\partial \hat{y}_2^*} + \frac{\partial \hat{C}_{4\alpha}}{\partial \hat{y}_3}. \end{aligned} \tag{29}$$

[3]We note that the Voigt notation is a way to rewrite a fourth order symmetric tensor A_{ijkl} in a 6×6 matrix form $A_{\alpha\beta}$, by applying the substitutions: 11→1, 22→2, 33→3, 23→4, 13→5, 12→6.

and

$$
\begin{aligned}
&-\frac{\partial}{\partial \hat{y}_2^*}\Big(\hat{C}_{24}\frac{\partial \hat{N}_2^\alpha}{\partial \hat{y}_2^*}+\hat{C}_{44}\frac{\partial \hat{N}_2^\alpha}{\partial \hat{y}_3}+\hat{C}_{44}\frac{\partial \hat{N}_3^\alpha}{\partial \hat{y}_2^*}+\hat{C}_{34}\frac{\partial \hat{N}_3^\alpha}{\partial \hat{y}_3}\Big)\\
&-\frac{\partial}{\partial \hat{y}_3}\Big(\hat{C}_{23}\frac{\partial \hat{N}_2^\alpha}{\partial \hat{y}_2^*}+\hat{C}_{34}\frac{\partial \hat{N}_2^\alpha}{\partial \hat{y}_3}+\hat{C}_{34}\frac{\partial \hat{N}_3^\alpha}{\partial \hat{y}_2^*}+\hat{C}_{33}\frac{\partial \hat{N}_3^\alpha}{\partial \hat{y}_3}\Big)\\
&\qquad=\frac{\partial \hat{C}_{4\alpha}}{\partial \hat{y}_2^*}+\frac{\partial \hat{C}_{3\alpha}}{\partial \hat{y}_3}.
\end{aligned}
\tag{30}
$$

The effective properties of the nanocomposite are obtained by the relations

$$
\begin{aligned}
\hat{C}_{1\alpha}^{\mathrm{NCP}} &= \Big\langle \hat{C}_{1\alpha}+\hat{C}_{12}\frac{\partial \hat{N}_2^\alpha}{\partial \hat{y}_2^*}+\hat{C}_{14}\frac{\partial \hat{N}_2^\alpha}{\partial \hat{y}_3}+\hat{C}_{14}\frac{\partial \hat{N}_3^\alpha}{\partial \hat{y}_2^*}+\hat{C}_{13}\frac{\partial \hat{N}_3^\alpha}{\partial \hat{y}_3}\Big\rangle,\\
\hat{C}_{2\alpha}^{\mathrm{NCP}} &= \Big\langle \hat{C}_{2\alpha}+\hat{C}_{22}\frac{\partial \hat{N}_2^\alpha}{\partial \hat{y}_2^*}+\hat{C}_{24}\frac{\partial \hat{N}_2^\alpha}{\partial \hat{y}_3}+\hat{C}_{24}\frac{\partial \hat{N}_3^\alpha}{\partial \hat{y}_2^*}+\hat{C}_{23}\frac{\partial \hat{N}_3^\alpha}{\partial \hat{y}_3}\Big\rangle,\\
\hat{C}_{3\alpha}^{\mathrm{NCP}} &= \Big\langle \hat{C}_{3\alpha}+\hat{C}_{23}\frac{\partial \hat{N}_2^\alpha}{\partial \hat{y}_2^*}+\hat{C}_{34}\frac{\partial \hat{N}_2^\alpha}{\partial \hat{y}_3}+\hat{C}_{34}\frac{\partial \hat{N}_3^\alpha}{\partial \hat{y}_2^*}+\hat{C}_{33}\frac{\partial \hat{N}_3^\alpha}{\partial \hat{y}_3}\Big\rangle,\\
\hat{C}_{4\alpha}^{\mathrm{NCP}} &= \Big\langle \hat{C}_{4\alpha}+\hat{C}_{24}\frac{\partial \hat{N}_2^\alpha}{\partial \hat{y}_2^*}+\hat{C}_{44}\frac{\partial \hat{N}_2^\alpha}{\partial \hat{y}_3}+\hat{C}_{44}\frac{\partial \hat{N}_3^\alpha}{\partial \hat{y}_2^*}+\hat{C}_{34}\frac{\partial \hat{N}_3^\alpha}{\partial \hat{y}_3}\Big\rangle,\\
\hat{C}_{5\alpha}^{\mathrm{NCP}} &= \Big\langle \hat{C}_{5\alpha}+\hat{C}_{56}\frac{\partial \hat{N}_1^\alpha}{\partial \hat{y}_2^*}+\hat{C}_{55}\frac{\partial \hat{N}_1^\alpha}{\partial \hat{y}_3}\Big\rangle,\\
\hat{C}_{6\alpha}^{\mathrm{NCP}} &= \Big\langle \hat{C}_{6\alpha}+\hat{C}_{66}\frac{\partial \hat{N}_1^\alpha}{\partial \hat{y}_2^*}+\hat{C}_{56}\frac{\partial \hat{N}_1^\alpha}{\partial \hat{y}_3}\Big\rangle.
\end{aligned}
\tag{31}
$$

4. Effective properties of the "fuzzy fiber" composite

Having defined the effective properties of the NCP, the homogenization of the actual composite can be determined using again the AEH method. The actual composite can be described easier in Cartesian coordinates, which necessitates to transfer the effective properties of the NCP from cylindrical (r,θ,z) to Cartesian (x_1,x_2,x_3) coordinates. The obtained NCP effective properties can be transformed from a cylindrical coordinates form $\hat{C}^{\mathrm{NCP}}$ to a Cartesian coordinates form C^{NCP} according to the rotation formula for fourth order tensors

$$
C_{ijkl}^{\mathrm{NCP}} = R_{im}R_{jn}R_{ko}R_{lp}\hat{C}_{mnop}^{\mathrm{NCP}}, \tag{32}
$$

with

$$\boldsymbol{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{33}$$

Here we need to mention that, according to the analysis of the previous section, the effective coefficients $\hat{C}_{ijkl}^{\mathrm{NCP}}$ of the NCP are functions of the radius r. In Cartesian coordinates, we have $r^2 = x_1^2 + x_2^2$. Using the Voigt notation, the stiffness tensor of the NCP is given in Cartesian coordinates by

$$\boldsymbol{C}^{\mathrm{NCP}} = \begin{pmatrix} C_{11}^{\mathrm{NCP}} & C_{12}^{\mathrm{NCP}} & C_{13}^{\mathrm{NCP}} & 0 & 0 & C_{16}^{\mathrm{NCP}} \\ C_{12}^{\mathrm{NCP}} & C_{22}^{\mathrm{NCP}} & C_{23}^{\mathrm{NCP}} & 0 & 0 & C_{26}^{\mathrm{NCP}} \\ C_{13}^{\mathrm{NCP}} & C_{23}^{\mathrm{NCP}} & C_{33}^{\mathrm{NCP}} & 0 & 0 & C_{36}^{\mathrm{NCP}} \\ 0 & 0 & 0 & C_{44}^{\mathrm{NCP}} & C_{45}^{\mathrm{NCP}} & 0 \\ 0 & 0 & 0 & C_{45}^{\mathrm{NCP}} & C_{55}^{\mathrm{NCP}} & 0 \\ C_{16}^{\mathrm{NCP}} & C_{26}^{\mathrm{NCP}} & C_{36}^{\mathrm{NCP}} & 0 & 0 & C_{66}^{\mathrm{NCP}} \end{pmatrix}, \tag{34}$$

where

$$C_{11}^{\mathrm{NCP}} = \frac{\hat{C}_{11}^{\mathrm{NCP}} x_1^4 + 2(\hat{C}_{12}^{\mathrm{NCP}} + 2\hat{C}_{66}^{\mathrm{NCP}}) x_1^2 x_2^2 + \hat{C}_{22}^{\mathrm{NCP}} x_2^4}{(x_1^2 + x_2^2)^2},$$

$$C_{12}^{\mathrm{NCP}} = \frac{\hat{C}_{12}^{\mathrm{NCP}} (x_1^4 + x_2^4) + (\hat{C}_{11}^{\mathrm{NCP}} + \hat{C}_{22}^{\mathrm{NCP}} - 4\hat{C}_{66}^{\mathrm{NCP}}) x_1^2 x_2^2}{(x_1^2 + x_2^2)^2},$$

$$C_{13}^{\mathrm{NCP}} = \frac{\hat{C}_{13}^{\mathrm{NCP}} x_1^2 + \hat{C}_{23}^{\mathrm{NCP}} x_2^2}{x_1^2 + x_2^2},$$

$$C_{16}^{\mathrm{NCP}} = \frac{(\hat{C}_{11}^{\mathrm{NCP}} - \hat{C}_{12}^{\mathrm{NCP}} - 2\hat{C}_{66}^{\mathrm{NCP}}) x_1^3 x_2 + (\hat{C}_{12}^{\mathrm{NCP}} - \hat{C}_{22}^{\mathrm{NCP}} + 2\hat{C}_{66}^{\mathrm{NCP}}) x_1 x_2^3}{(x_1^2 + x_2^2)^2},$$

$$C_{22}^{\mathrm{NCP}} = \frac{\hat{C}_{11}^{\mathrm{NCP}} x_2^4 + 2(\hat{C}_{12}^{\mathrm{NCP}} + 2\hat{C}_{66}^{\mathrm{NCP}}) x_1^2 x_2^2 + \hat{C}_{22}^{\mathrm{NCP}} x_1^4}{(x_1^2 + x_2^2)^2},$$

$$C_{23}^{\mathrm{NCP}} = \frac{\hat{C}_{13}^{\mathrm{NCP}} x_2^2 + \hat{C}_{23}^{\mathrm{NCP}} x_1^2}{x_1^2 + x_2^2},$$

$$C_{26}^{\mathrm{NCP}} = \frac{(\hat{C}_{11}^{\mathrm{NCP}} - \hat{C}_{12}^{\mathrm{NCP}} - 2\hat{C}_{66}^{\mathrm{NCP}}) x_1 x_2^3 + (\hat{C}_{12}^{\mathrm{NCP}} - \hat{C}_{22}^{\mathrm{NCP}} + 2\hat{C}_{66}^{\mathrm{NCP}}) x_1^3 x_2}{(x_1^2 + x_2^2)^2},$$

$$C_{33}^{\mathrm{NCP}} = \hat{C}_{33}^{\mathrm{NCP}},$$

$$C_{36}^{\mathrm{NCP}} = \frac{(\hat{C}_{13}^{\mathrm{NCP}} - \hat{C}_{23}^{\mathrm{NCP}})x_1x_2}{x_1^2 + x_2^2},$$

$$C_{44}^{\mathrm{NCP}} = \frac{\hat{C}_{44}^{\mathrm{NCP}}x_1^2 + \hat{C}_{55}^{\mathrm{NCP}}x_2^2}{x_1^2 + x_2^2},$$

$$C_{45}^{\mathrm{NCP}} = \frac{(\hat{C}_{55}^{\mathrm{NCP}} - \hat{C}_{44}^{\mathrm{NCP}})x_1x_2}{x_1^2 + x_2^2},$$

$$C_{55}^{\mathrm{NCP}} = \frac{\hat{C}_{44}^{\mathrm{NCP}}x_2^2 + \hat{C}_{55}^{\mathrm{NCP}}x_1^2}{x_1^2 + x_2^2},$$

$$C_{66}^{\mathrm{NCP}} = \frac{(\hat{C}_{11}^{\mathrm{NCP}} - 2\hat{C}_{12}^{\mathrm{NCP}} + \hat{C}_{22}^{\mathrm{NCP}})x_1^2x_2^2 + \hat{C}_{66}^{\mathrm{NCP}}(x_1^2 - x_2^2)^2}{(x_1^2 + x_2^2)^2}.$$

The other two material components of the composite, the matrix and the carbon fiber, are generally assumed as homogeneous isotropic or transversely isotropic materials, with the axis of symmetry parallel to the axis of the fiber. Under these conditions, the application of the AEH method for the second periodic problem with characteristic length ϵ (Figure 2_b) follows the standard approach. The equations that describe the behavior of the constituents are the equilibrium equations (neglecting body forces)

$$\frac{\partial \sigma_{ij}^{\epsilon}}{\partial x_j^{\epsilon}} = 0, \tag{35}$$

the constitutive law

$$\sigma_{ij}^{\epsilon} = C_{ijkl}^{\epsilon}\frac{\partial u_k^{\epsilon}}{\partial x_l^{\epsilon}}, \tag{36}$$

and appropriate boundary conditions. The stiffness tensor in the mesoscale level is a periodic function of $\frac{x_1}{\epsilon}$ and $\frac{x_2}{\epsilon}$,

$$C_{ijkl}^{\epsilon} = C_{ijkl}^{\epsilon}(\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}) = C_{ijkl}(y_1, y_2). \tag{37}$$

Expanding the displacements in terms of the mesoscale characteristic length ϵ,

$$u_i^{\epsilon} = u_i^{(0)}(x_1, x_2, x_3) + \epsilon u_i^{(1)}(x_1, x_2, x_3, y_1, y_2) + \epsilon^2 u_i^{(2)}(x_1, x_2, x_3, y_1, y_2) + ..., \tag{38}$$

and substituting in equations (36) and (35) we get an expanded form of the equilibrium equations, in which the ϵ^{-1} terms (meso-equations) are written

$$-\frac{\partial}{\partial y_j}\Big(C_{ijkl}\frac{\partial u_k^{(1)}}{\partial y_l}\Big) = \frac{\partial u_k^{(0)}}{\partial x_l}\frac{\partial C_{ijkl}}{\partial y_j}. \tag{39}$$

Considering $u_k^{(0)}$ as known, we assume that $u_i^{(1)}$ is given, up to an additive function on x_1, x_2, x_3, from

$$u_i^{(1)} = N_i^{mn}\frac{\partial u_m^{(0)}}{\partial x_n}, \tag{40}$$

where N_i^{mn} is given from the auxiliary system

$$\frac{\partial}{\partial y_j}\left(C_{ijkl}\frac{\partial N_k^{mn}}{\partial y_l} + C_{ijmn}\right) = 0. \tag{41}$$

Returning to the equilibrium equations and the ϵ^0 terms, we can easily show that the effective properties of the actual composite are given by

$$C_{ijkl}^{\text{eff}} = \Big\langle\Big\langle C_{ijkl} + C_{ijmn}\frac{\partial N_m^{kl}}{\partial y_n}\Big\rangle\Big\rangle, \tag{42}$$

where

$$\langle\langle\phi\rangle\rangle = \frac{1}{V^{mes}}\int_{-{y_1}'/2}^{{y_1}'/2}\int_{-{y_2}'/2}^{{y_2}'/2}\int_{-{y_3}'/2}^{{y_3}'/2}\phi(y_1, y_2, y_3)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3, \tag{43}$$

and V^{mes} is the volume in the mesoscale level.

In the case of our composite, the meso-macroscale structure does not vary with y_3 (monoclinic materials) and the derivatives with respect to y_3 vanish. This leads, with the help of the Voigt notation, to the meso-equations

$$\begin{aligned}
&-\frac{\partial}{\partial y_1}\Big(C_{11}\frac{\partial N_1^\alpha}{\partial y_1} + C_{16}\frac{\partial N_1^\alpha}{\partial y_2} + C_{16}\frac{\partial N_2^\alpha}{\partial y_1} + C_{12}\frac{\partial N_2^\alpha}{\partial y_2}\Big)\\
&-\frac{\partial}{\partial y_2}\Big(C_{16}\frac{\partial N_1^\alpha}{\partial y_1} + C_{66}\frac{\partial N_1^\alpha}{\partial y_2} + C_{66}\frac{\partial N_2^\alpha}{\partial y_1} + C_{26}\frac{\partial N_2^\alpha}{\partial y_2}\Big)\\
&\qquad = \frac{\partial C_{1\alpha}}{\partial y_1} + \frac{\partial C_{6\alpha}}{\partial y_2},
\end{aligned} \tag{44}$$

$$
\begin{aligned}
&-\frac{\partial}{\partial y_1}\Big(C_{16}\frac{\partial N_1^\alpha}{\partial y_1}+C_{66}\frac{\partial N_1^\alpha}{\partial y_2}+C_{66}\frac{\partial N_2^\alpha}{\partial y_1}+C_{26}\frac{\partial N_2^\alpha}{\partial y_2}\Big)\\
&-\frac{\partial}{\partial y_2}\Big(C_{12}\frac{\partial N_1^\alpha}{\partial y_1}+C_{26}\frac{\partial N_1^\alpha}{\partial y_2}+C_{26}\frac{\partial N_2^\alpha}{\partial y_1}+C_{22}\frac{\partial N_2^\alpha}{\partial y_2}\Big)\\
&\qquad=\frac{\partial C_{6\alpha}}{\partial y_1}+\frac{\partial C_{2\alpha}}{\partial y_2},
\end{aligned}
\tag{45}
$$

for α=1,2,3,6 (plane strain problems) and

$$
\begin{aligned}
&-\frac{\partial}{\partial y_1}\Big(C_{55}\frac{\partial N_3^\alpha}{\partial y_1}+C_{45}\frac{\partial N_3^\alpha}{\partial y_2}\Big)-\frac{\partial}{\partial y_2}\Big(C_{45}\frac{\partial N_3^\alpha}{\partial y_1}+C_{44}\frac{\partial N_3^\alpha}{\partial y_2}\Big)\\
&\qquad=\frac{\partial C_{5\alpha}}{\partial y_1}+\frac{\partial C_{4\alpha}}{\partial y_2},
\end{aligned}
\tag{46}
$$

for α=4,5 (anti-plane strain problems). The effective properties are given by

$$
\begin{aligned}
C_{1\alpha}^{\text{eff}}&=\Big\langle\Big\langle C_{1\alpha}+C_{11}\frac{\partial N_1^\alpha}{\partial y_1}+C_{16}\frac{\partial N_1^\alpha}{\partial y_2}+C_{16}\frac{\partial N_2^\alpha}{\partial y_1}+C_{12}\frac{\partial N_2^\alpha}{\partial y_2}\Big\rangle\Big\rangle,\\
C_{2\alpha}^{\text{eff}}&=\Big\langle\Big\langle C_{2\alpha}+C_{12}\frac{\partial N_1^\alpha}{\partial y_1}+C_{26}\frac{\partial N_1^\alpha}{\partial y_2}+C_{26}\frac{\partial N_2^\alpha}{\partial y_1}+C_{22}\frac{\partial N_2^\alpha}{\partial y_2}\Big\rangle\Big\rangle,\\
C_{3\alpha}^{\text{eff}}&=\Big\langle\Big\langle C_{3\alpha}+C_{13}\frac{\partial N_1^\alpha}{\partial y_1}+C_{36}\frac{\partial N_1^\alpha}{\partial y_2}+C_{36}\frac{\partial N_2^\alpha}{\partial y_1}+C_{23}\frac{\partial N_2^\alpha}{\partial y_2}\Big\rangle\Big\rangle,\\
C_{6\alpha}^{\text{eff}}&=\Big\langle\Big\langle C_{6\alpha}+C_{16}\frac{\partial N_1^\alpha}{\partial y_1}+C_{66}\frac{\partial N_1^\alpha}{\partial y_2}+C_{66}\frac{\partial N_2^\alpha}{\partial y_1}+C_{26}\frac{\partial N_2^\alpha}{\partial y_2}\Big\rangle\Big\rangle,
\end{aligned}
$$

for α=1,2,3,6 and

$$
\begin{aligned}
C_{4\alpha}^{\text{eff}}&=\Big\langle\Big\langle C_{4\alpha}+C_{45}\frac{\partial N_3^\alpha}{\partial y_1}+C_{44}\frac{\partial N_3^\alpha}{\partial y_2}\Big\rangle\Big\rangle,\\
C_{5\alpha}^{\text{eff}}&=\Big\langle\Big\langle C_{5\alpha}+C_{55}\frac{\partial N_3^\alpha}{\partial y_1}+C_{45}\frac{\partial N_3^\alpha}{\partial y_2}\Big\rangle\Big\rangle,
\end{aligned}
$$

for α=4,5.

In the above methodology for obtaining effective properties of "fuzzy fiber" composites we utilize a two step homogenization method, because in the first step (nanocomposite layer) we express our equations in cylindrical coordinates and in the second step (actual composite) we use Cartesian coordinates. A one step homogenization whould require everything to be expressed in one coordinate system, e.g. Cartesian, leading to solve a curvilinear unit shell in the microscale level (nanocomposite layer).

Table 1: Mechanical properties of layers

T650 carbon fiber	
Young's Modulus	276 GPa
Poisson's Ratio	0.3

EPIKOTE 862 resin	
Young's Modulus	3 GPa
Poisson's Ratio	0.3

CNT	
Young's Modulus	1100 GPa
Poisson's Ratio	0.14

5. Examples

The numerical examples presented in this section are motivated by the experiments presented in Sager et al. (2009). T650 carbon fibers with diameter 5 μm are coated with radially aligned hollow carbon nanotubes of 2 μm length. The CNTs have internal radius 0.51 nm, external radius 0.85 nm. The "fuzzy fibers" are embedded in EPIKOTE 862 resin. The intermediate layer contains CNTs with average volume fraction 42.17%. The properties of the CNTs are assumed the same as the properties of the graphene (Seidel and Lagoudas, 2006). The mechanical properties of the carbon fibers, the resin and the CNTs are shown in Table 1.

For the computations we used the finite element program COMSOL Multiphysics. The effective properties for the nanocomposite were obtained for both the tetragonal and the hexagonal arrangement of the CNTs. Since the periodic structure of the nanocomposite depends on the radius, we needed to solve numerically several unit cells. Each unit cell represents a different profile of the nanocomposite with respect to radius and the volume fraction of the CNTs decreases as the radius increases. Figures 4 and 5 show several unit cells that were solved for tetragonal and hexagonal arrangement of the CNTs respectively. Here the arrangement of CNTs is exactly tetragonal or hexagonal only at the interphase between the carbon fiber and the nanocomposite. As we move closer to the matrix, the length of the unit cell at the $r\bar{\theta}$ direction elongates, disturbing the tetragonal or hexagonal symmetry.

The obtained effective properties are shown in Figures 6 and 7. As it can

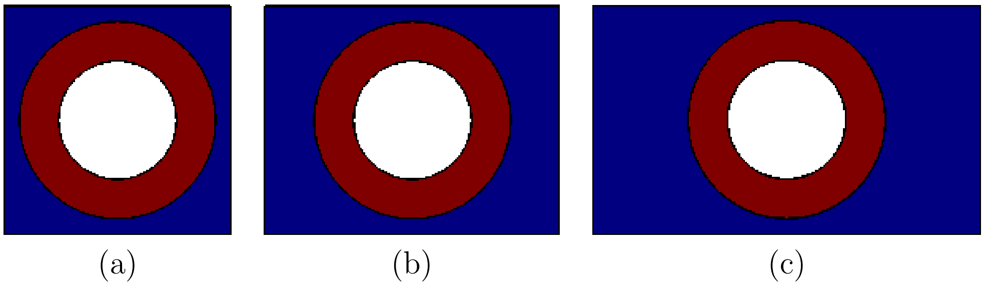


Figure 4: Unit cell of the NCP for r equal to a) 2.5 μm, b) 3.25 μm and c) 4.25 μm. Tetragonal arrangement.

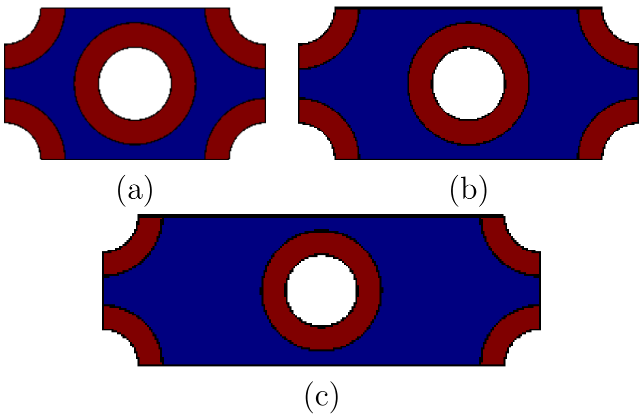
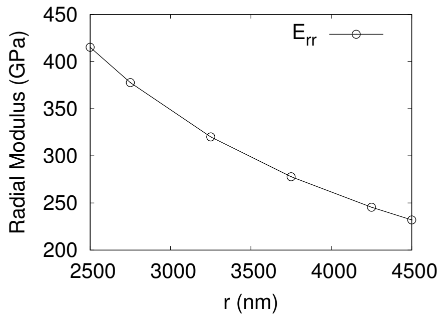
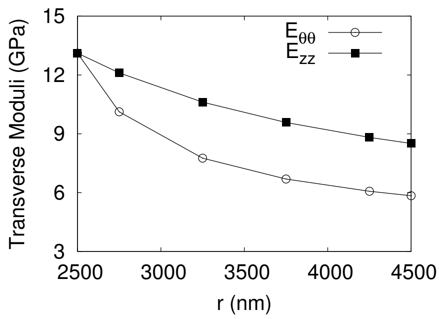


Figure 5: Unit cell of the NCP for r equal to a) 2.5 μm, b) 3.25 μm and c) 4.25 μm. Hexagonal arrangement.

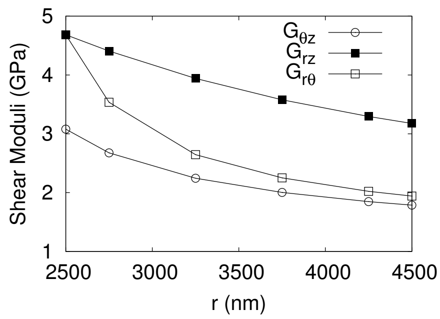
be seen from these figures, the NCP becomes more cylindrically orthotropic with the increase of radius. For the hexagonal arrangement the NCP in cylindrically transversely isotropic only at the interphase between NCP and carbon fiber (r=2.5 μm). Moreover the tetragonal arrangement favors the orthotropy more than the hexagonal arrangement. The radial Young's modulus has the same decrease in both cases with the increase of the NCP radius. From the results it is clear that the effective properties of the NCP are strongly affected by the radius. This radial dependency can be simulated by assuming that all the mechanical properties can be described with fifth order polynomials with respect to radius.



(a)



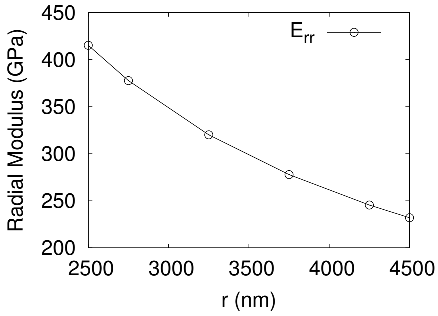
(b)



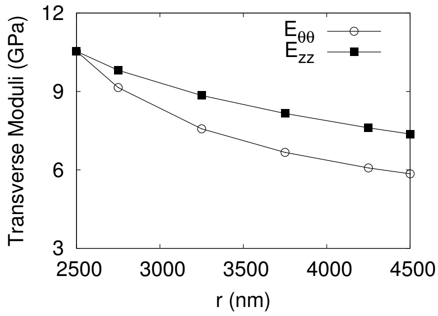
(c)

Figure 6: Effective properties of NCP for tetragonal arrangement of CNTs.

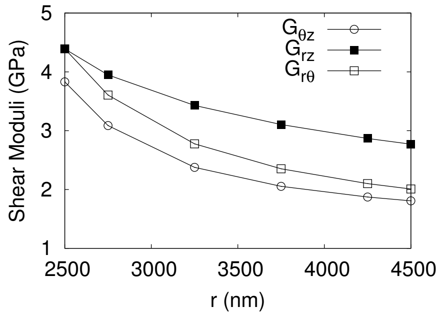
In the second step of the homogenization, the NCP is substituted by the effective medium, which is introduced in the mesoscale unit cell. As it can be seen in Figure 8, transferring the effective properties from the cylindrical to Cartesian coordinates, produces a fully anisotropic behavior for the NCP. Terms like C_{16} are no longer zero, only the average C_{16} over



(a)



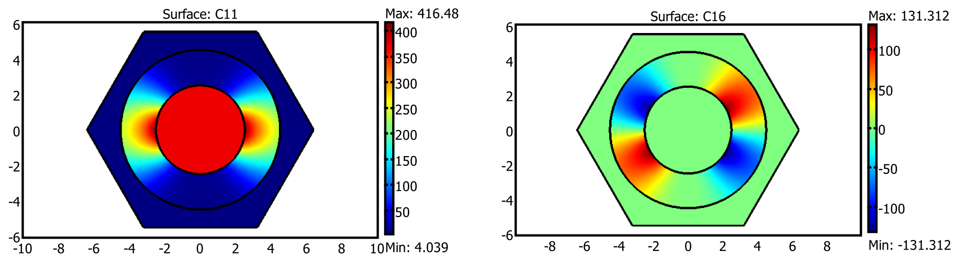
(b)



(c)

Figure 7: Effective properties of NCP for hexagonal arrangement of CNTs.

the whole NCP becomes zero. For the numerical example we use a volume fraction of 60% for the "fuzzy fiber" (carbon fiber plus NCP) inside the resin. The numerically obtained effective properties (Table 2) show that the arrangement of the CNTs in the NCP do not alter significantly the behavior of the actual composite. In both arrangements (tetragonal and hexagonal) the overall behavior of the "fuzzy fiber" composite is transversely isotropic with axis of symmetry the axis of the carbon fiber. This transverse isotropy is observed in typical fiber composites, indicating that the presence of the NCP changes the mechanical performance but not the level of anisotropy of the actual composite. In Table 2 we also present the results from an analytical approach, based on the Composite Cylinders Method and in the assumption of transversely isotropic nanocomposite layer with the axis of symmetry parallel to the axis of CNTs (for more details see Chatzigeorgiou et al., 2011). As it can be seen, the results from the analytical micromechanics approach are very close to the effective properties of "fuzzy fiber" composites with hexagonal CNTs arrangement.



(a) (b)

Figure 8: Distribution of the stiffness coefficients a) C_{11} and b) C_{16} in the mesoscale unit cell. The units are GPa.

6. Conclusions

In this paper we presented a two step homogenization approach, based on the asymptotic expansion homogenization method, in order to evaluate the effective properties of carbon fiber composites, in which the carbon fibers are coated with radially aligned carbon nanotubes (CNTs). From the theoretical analysis it is shown that the homogenization process can be split in two parts,

Table 2: Effective properties of “fuzzy fiber” composites

CNTs arrangement in NCP	Axial Young’s modulus (GPa)	Transverse Young’s modulus (GPa)	Axial shear modulus (GPa)	Transverse shear modulus (GPa)	Transverse bulk modulus (GPa)
tetragonal	56.57	9.03	2.82	3.10	8.53
hexagonal	55.92	9.07	2.70	3.12	8.53
analytical	55.34	9.50	2.55	3.33	8.50

in each one of which different coordinate system can be used. In the first part of homogenization, the nanocomposite layer which includes CNTs and matrix show cylindrically orthotropic behavior and the material properties depend on the radius. The results from the second step of homogenization, which includes the mesoscale unit cell, indicate that the nanocomposite does not influence the overall anisotropy of the composite. Additionally the arrangement of CNTs in the nanocomposite (tetragonal and hexagonal) does not change significantly the overall behavior of the “fuzzy fiber” composite.

7. Acknowledgements

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