SOME ERROR ESTIMATES FOR THE LUMPED MASS FINITE ELEMENT METHOD FOR A PARABOLIC PROBLEM

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Abstract. We study the spatially semidiscrete lumped mass method for the model homogeneous heat equation with homogeneous Dirichlet boundary conditions. Improving earlier results we show that known optimal order smooth initial data error estimates for the standard Galerkin method carry over to the lumped mass method whereas nonsmooth initial data estimates require special assumptions on the triangulation. We also discuss the application to time discretization by the backward Euler and Crank-Nicolson methods.

1. Introduction

We consider the model initial–boundary value problem

$$(1.1)\qquad \begin{aligned} u_t - \Delta u &= 0, \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega, \quad \text{for } t \geq 0,\\ u(0) &= v, \quad \text{in } \Omega, \end{aligned}$$

where Ω is a bounded convex polygonal domain in $\mathbb{R}^2$. For simplicity we restrict ourselves to the homogeneous heat equation, thus without a forcing term, so that the initial values v are the only data of the problem. This problem has a unique solution $u(t)$, under appropriate assumptions on v, and this solution is smooth for $t > 0$, even if v is not. More precisely, for $q \geq 0$ we denote by $\dot H^q \subset L_2(\Omega)$ the Hilbert space defined by the norm

$$|v|_q = \Big(\sum_{j=1}^{\infty} \lambda_j^q (v, \phi_j)^2\Big)^{1/2}, \quad \text{where } (v, w) = \int_\Omega vw\,dx,$$

and where $\{\lambda_j\}_{j=1}^\infty$, $\{\phi_j\}_{j=1}^\infty$ are the eigenvalues and eigenfunctions of $-\Delta$, with homogeneous Dirichlet boundary conditions on $\partial\Omega$. Thus $|v|_0 = \|v\| = (v, v)^{1/2}$ is the norm in $L_2 = L_2(\Omega)$, $|v|_1$ the norm in $H_0^1 = H_0^1(\Omega)$ and $|v|_2 = \|\Delta v\|$ is equivalent to the norm in $H^2(\Omega)$ when $v = 0$ on $\partial\Omega$. For the solution of (1.1) we then have the stability and smoothing estimate

$$|E(t)v|_p \leq Ct^{-(p-q)/2}|v|_q, \quad \text{for } 0 \leq q \leq p, \quad t > 0, \quad \text{where } u(t) = E(t)v.$$

We first recall some facts about the spatially semidiscrete standard Galerkin method for (1.1) in the piecewise linear finite element space

$$S_h = \{\chi \in \mathcal{C}(\Omega) : \ \chi|_\tau \ \text{ linear}, \ \forall\, \tau \in \mathcal{T}_h; \ \chi|_{\partial\Omega} = 0\},$$

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where $\{\mathcal{T}_h\}_{0<h<1}$ is a family of regular triangulations $\mathcal{T}_h = \{\tau\}$ of Ω, with h denoting the maximum diameter of the triangles $\tau \in \mathcal{T}_h$. We then seek an approximation $u_h(t) \in S_h$ of $u(t)$ for $t \geq 0$ from

$$(1.2) \qquad (u_{h,t}, \chi) + (\nabla u_h, \nabla \chi) = 0, \quad \forall \chi \in S_h, \quad \text{for } t \geq 0, \quad \text{with } u_h(0) = v_h,$$

where $v_h \in S_h$ is an approximation of v. It is well-known that we have the smooth data error estimate, valid uniformly down to $t = 0$, cf. [7, Theorem 3.1],

$$(1.3) \qquad \|u_h(t) - u(t)\| \leq Ch^2|v|_2, \quad \text{for } t \geq 0, \quad \text{if } \|v_h - v\| \leq Ch^2|v|_2.$$

We also have a nonsmooth data error estimate, for v only assumed to be in L_2, but which deteriorates for t tending to zero, cf. [7, Theorem 3.2], namely

$$(1.4) \qquad \|u_h(t) - u(t)\| \leq Ch^2t^{-1}\|v\|, \quad \text{if } v_h = P_h v, \quad \text{for } t > 0,$$

where P_h denotes the L_2-projection onto S_h. Note that the discrete initial data are not as general in this case as in (1.3).

We remark that the nonsmooth data error estimate (1.4) is of optimal order $O(h^2)$ for t bounded away from zero, but deteriorates as $t \to 0$. We emphasize that the triangulations $\mathcal{T}_h$ are assumed independent of t, and thus that the use of finer $\mathcal{T}_h$ for t small is not considered here.

We note for later use that a possible choice in (1.3) is $v_h = P_h v$, and that hence, by interpolation, we have the intermediate result between (1.3) and (1.4),

$$(1.5) \qquad \|u_h(t) - u(t)\| \leq Ch^2t^{-1/2}|v|_1, \quad \text{if } v_h = P_h v, \quad \text{for } t > 0.$$

As is easily seen, this error bound also holds for $v_h = R_h v$, the Ritz projection of v onto S_h defined in (2.4) below. In the sequel we shall not insist on generality in the choice of v_h in our various error estimates, and an estimate such as (1.3) would be expressed with $v_h = R_h v$. The above more general choice of v_h is then justified by the stability of (1.2) in v_h.

The object of study in this paper is the lumped mass modification of (1.2) obtained by replacing the first term on the left by a quadrature expression, or

$$(1.6) \qquad (\bar{u}_{h,t}, \chi)_h + (\nabla \bar{u}_h, \nabla \chi) = 0, \quad \forall \chi \in S_h, \quad \text{for } t \geq 0, \quad \text{with } \bar{u}_h(0) = v_h,$$

where, denoting by $\{z_j^\tau\}_{j=1}^3$ the vertices of τ and by $\pi_h : \mathcal{C}(\bar{\Omega}) \to S_h$ the finite element interpolation operator,

$$(1.7) \qquad (v, w)_h = \sum_{\tau \in \mathcal{T}_h} Q_{\tau,h}(vw), \quad \text{with } Q_{\tau,h}(f) = \frac{|\tau|}{3}\sum_{j=1}^{3} f(z_j^\tau) = \int_\tau \pi_h f \, dx.$$

This method has the advantage over the standard Galerkin method that, under the assumption that the triangulation is of Delaunay type, the solution satisfies the maximum-principle, cf., e.g. [7, Theorem 15.5]. Our aim here is to show analogues of (1.3), (1.4), and (1.5) for the solution of (1.6), namely, with the appropriate choices of v_h,

$$(1.8) \qquad \|\bar{u}_h(t) - u(t)\| \leq Ch^2t^{-(1-q/2)}|v|_q, \quad \text{for } t > 0, \quad q = 0, 1, 2.$$

We will prove this in Section 3 for $q = 1, 2$. However, for $q = 0$, we are only able to show this under an additional hypothesis, expressed in terms of the quadrature error operator $Q_h : S_h \to S_h$, defined by

$$(1.9) \qquad (\nabla Q_h \chi, \nabla \psi) = \epsilon_h(\chi, \psi) := (\chi, \psi)_h - (\chi, \psi), \quad \forall \psi \in S_h,$$

and requiring

$$\|Q_h\psi\| \le Ch^2\|\psi\|, \quad \forall \psi \in S_h. \tag{1.10}$$

It will be shown that this assumption is satisfied for symmetric triangulations. We will then give examples, in one space dimension, of nonsymmetric partitions such that (1.8) does not hold for $q = 0$. For finite difference methods, which are close in character to the lumped mass method with symmetric triangulations, it was shown in [5], that nonsmooth data estimates similar to (1.8) with $q = 0$ hold. Without condition (1.10) we are only able to show the nonoptimal order error estimate

$$\|\bar{u}_h(t) - u(t)\| \le Cht^{-1/2}\|v\|, \quad \text{for } t > 0.$$

Symmetry of the triangulations is a serious restriction which can only hold for special shapes of Ω.

We also discuss optimal order $O(h)$ error estimates for the gradient of $\bar{u}_h(t)-u(t)$, with a dependence of t depending on the smoothness of v.

Our analysis provides improvements of earlier results in [3], cf. also [7, Lemma 15.3 and p. 267], where, by mimicking the proof for the standard Galerkin method, it was shown that, e.g.,

$$\|\bar{u}_h(t) - u(t)\| \le \begin{cases} Ch^2|v|_3, \\ Ch^2t^{-1/2}|v|_2, \end{cases} \qquad \text{for } t > 0, \quad \text{if} \ \ v_h = R_h v,$$

thus requiring more regularity of the initial data than (1.8). Our approach here is to combine the error estimates (1.3), (1.4) and (1.5) for the standard Galerkin method with new bounds for the difference $\delta(t) = \bar{u}_h(t) - u_h(t)$, which satisfies

$$(\delta_t, \chi)_h + (\nabla\delta, \nabla\chi) = -\epsilon_h(u_{h,t}, \chi), \quad \text{for } \chi \in S_h. \tag{1.11}$$

After we had finished our research, we become aware of the paper [6], where the smooth data error estimate (1.8), with $q = 2$, is shown for a slightly more general parabolic equation and by a somewhat more lengthy argument than here. The nonsmooth data error estimate, with $q = 0$, is also stated but with an incomplete proof.

The following is an outline of the paper. In Section 2, we introduce notation and give some preliminary material needed for the analysis of the lumped mass method. Further, we derive smooth and nonsmooth initial data estimates for the gradient of the error in the standard Galerkin method, which will be used in the sequel. In Section 3 we derive error estimates for the lumped mass method for initial data with basic smoothness, or $v \in \dot{H}^q$ with $q = 1, 2$. In Section 4, we show the optimal order error bound for $v \in L_2$ under the assumption that (1.10) holds. In Section 5 we show that this assumption is valid for symmetric meshes, and in Section 6, in one space dimension, that the symmetry requirement can be somewhat relaxed. In Section 7 we give two nonsymmetric partitions in one space dimension for which optimal L_2-convergence for nonsmooth data does not hold. Finally, in Section 8 we consider briefly the application of our results for the spatially semidiscrete problem to the fully discrete backward Euler and Crank–Nicolson lumped mass methods.

2. Preliminaries

In this section we recall some basic known facts for the spatially semidiscrete standard Galerkin and lumped mass methods. We introduce notation and show smoothing properties of the solution operators for these two semidiscrete methods and some other preliminary results needed in the sequel.

Introducing the discrete Laplacian $\Delta_h : S_h \to S_h$ by

$$-(\Delta_h \psi, \chi) = (\nabla \psi, \nabla \chi), \quad \forall \psi,\, \chi \in S_h,$$

we may write the spatially discrete problem (1.2) as

$$u_{h,t} - \Delta_h u_h = 0, \quad \text{for } t \ge 0, \quad \text{with } u_h(0) = v_h. \tag{2.1}$$

With $u_h(t)$ its solution we define the solution operator $E_h(t) = e^{\Delta_h t}$ of (2.1) by $u_h(t) = E_h(t) v_h$. Letting $\{\lambda_j^h\}_{j=1}^N$, $\{\phi_j^h\}_{j=1}^N$ denote the eigenvalues and eigenfunctions of $-\Delta_h$, we have, by eigenfunction expansion,

$$u_h(t) = E_h(t) v_h = \sum_{j=1}^N e^{-\lambda_j^h t} (v_h, \phi_j^h) \phi_j^h, \quad \text{for } t \ge 0. \tag{2.2}$$

We shall need various smoothing properties of $E_h(t)$. First, we recall the following smoothing bounds for the exact solution u of (1.1), cf., e.g., [7],

$$|D_t^\ell E(t) v|_p \le C t^{-\ell-(p-q)/2} |v|_q, \quad \text{for } t > 0,\ p, q, \ell \ge 0,\ 2\ell + p \ge q,\ v \in \dot H^q. \tag{2.3}$$

In the following lemma, we show some discrete analogues of these bounds.

Lemma 2.1. *For $E_h(t)$ defined by (2.2) we have, for $v_h \in S_h$,*

$$\|\nabla^p D_t^\ell E_h(t) v_h\| \le C t^{-\ell-(p-q)/2} \|\nabla^q v_h\|, \quad \text{for } t > 0,\ \ell \ge 0,\ p, q = 0, 1,\ 2\ell + p \ge q.$$

Proof. By Parseval's relation, since $\lambda^s e^{-\lambda t} \le C_s t^{-s}$, for $\lambda > 0$, $s > 0$, we get

$$\|\nabla^p D_t^\ell E_h(t) v_h\|^2 = \sum_{j=1}^N (\lambda_j^h)^{2\ell+p} e^{-2\lambda_j^h t} (v_h, \phi_j^h)^2 \le C t^{-2\ell-p+q} \|\nabla^q v_h\|^2. \qquad \square$$

In addition to the L_2 projection $P_h : L_2 \to S_h$, satisfying

$$(P_h v, \chi) = (v, \chi), \quad \forall \chi \in S_h,$$

our error analysis will use the Ritz projection, $R_h : H_0^1 \to S_h$, defined by

$$(\nabla R_h v, \nabla \chi) = (\nabla v, \nabla \chi), \quad \text{for } \chi \in S_h. \tag{2.4}$$

It is well-known, cf. e.g. [7, Lemma 1.1], that R_h satisfies

$$\|R_h v - v\| + h \|\nabla (R_h v - v)\| \le C h^q |v|_q, \quad \text{for } v \in \dot H^q,\ q = 1, 2. \tag{2.5}$$

Next, we turn to the lumped mass method. As is well known, the norms $\|\cdot\|_h$ and $\|\cdot\|$ are equivalent on S_h, or, more precisely,

$$\tfrac{1}{2}\, \|\chi\|_h \le \|\chi\| \le \|\chi\|_h, \quad \forall\, \chi \in S_h. \tag{2.6}$$

We now introduce the discrete Laplacian $-\bar\Delta_h : S_h \to S_h$, corresponding to the inner product $(\cdot,\cdot)_h$, by

$$-(\bar\Delta_h \psi, \chi)_h = (\nabla \psi, \nabla \chi), \quad \forall \psi,\, \chi \in S_h. \tag{2.7}$$

The lumped mass method (1.6) can then be written in operator form as

$$\bar u_{h,t} - \bar\Delta_h \bar u_h = 0, \quad \text{for } t \ge 0, \quad \text{with } \bar u_h(0) = v_h. \tag{2.8}$$

With $\bar{E}_h(t) = e^{\bar{\Delta}_h t}$ the solution operator of (2.8), we have

$$\bar{u}_h(t) = \bar{E}_h(t)v_h = \sum_{j=1}^{N} e^{-\bar{\lambda}_j^h t}(v_h, \bar{\phi}_j^h)_h \bar{\phi}_j^h, \tag{2.9}$$

where $\{\bar{\lambda}_j^h\}_{j=1}^N$ and $\{\bar{\phi}_j^h\}_{j=1}^N$ are the eigenvalues and the corresponding orthonormal eigenfunctions, with respect to $(\cdot,\cdot)_h$, of the positive definite operator $-\bar{\Delta}_h$. We show the following analogue of Lemma 2.1 for $\bar{E}_h(t)$.

Lemma 2.2. *For $\bar{E}_h(t)$ defined by (2.9) we have, for $v_h \in S_h$,*

$$\|\nabla^p D_t^\ell \bar{E}_h(t)v_h\| \le Ct^{-\ell-(p-q)/2}\|\nabla^q v_h\|, \quad \text{for } t>0,\ \ell \ge 0,\ p,q=0,1,\ 2\ell+p \ge q.$$

Proof. Introducing the square root $\bar{G}_h = (-\bar{\Delta}_h)^{1/2} : S_h \to S_h$, of $-\bar{\Delta}_h$, we have

$$\|\nabla v_h\|^2 = ((-\bar{\Delta}_h)v_h, v_h)_h = \|\bar{G}_h v_h\|_h^2 = \sum_{j=1}^N \bar{\lambda}_j^h (v_h, \bar{\phi}_j^h)_h^2.$$

Since the norms $\|\cdot\|_h$ and $\|\cdot\|$ are equivalent on S_h we find

$$\begin{aligned}\|\nabla^p D_t^\ell \bar{E}_h(t)v_h\|^2 &\le C\|\bar{G}_h^p D_t^\ell \bar{E}_h(t)v_h\|_h^2 = C\sum_{j=1}^N (\bar{\lambda}_j^h)^{2\ell+p-q} e^{-2\bar{\lambda}_j^h t}(\bar{\lambda}_j^h)^q (v_h, \bar{\phi}_j^h)_h^2 \\ &\le C\,t^{-(2\ell+p-q)}\|\bar{G}_h^q v_h\|_h^2 \le C\,t^{-(2\ell+p-q)}\|\nabla^q v_h\|^2. \qquad \square\end{aligned}$$

We recall the following estimate for the error in the quadrature expression in (1.7).

Lemma 2.3. *Let $\epsilon_h(\chi,\psi) = (\chi,\psi)_h - (\chi,\psi)$. Then*

$$|\epsilon_h(\chi,\psi)| \le Ch^{p+q}\|\nabla^p\chi\|\,\|\nabla^q\psi\|, \quad \forall \chi,\psi \in S_h, \quad \textit{with } p,q=0,1.$$

Proof. For completeness we sketch the proof, cf. Lemma 15.1 in [7]. Since the quadrature formula is exact for linear functions over any triangle $\tau \in \mathcal{T}_h$, employing the Bramble–Hilbert lemma and a Sobolev inequality, we conclude that

$$\Big|Q_{\tau,h}(\chi\psi) - \int_\tau \chi\,\psi\,dx\Big| \le Ch_\tau^2 \sum_{|\alpha|=2}\|D^\alpha(\chi\psi)\|_{L_1(\tau)} \le Ch_\tau^2\|\nabla\chi\|_{L_2(\tau)}\|\nabla\psi\|_{L_2(\tau)},$$

with h_τ the maximal side length of τ. Now using an inverse inequality locally and summing over $\tau \in \mathcal{T}_h$, we obtain the desired result. $\square$

The following estimate holds for the quadrature error operator Q_h.

Lemma 2.4. *Let $\bar{\Delta}_h$ and Q_h be the operators defined by (2.7) and (1.9), respectively. Then*

$$\|\nabla Q_h\chi\| + h\|\bar{\Delta}_h Q_h\chi\|_h \le Ch^{p+1}\|\nabla^p\chi\|, \quad \forall \chi \in S_h,\ p=0,1. \tag{2.10}$$

Proof. By (1.9) and Lemma 2.3, with $\psi = Q_h\chi$ and $q=1$, it follows easily that

$$\|\nabla Q_h\chi\|^2 = \epsilon_h(\chi, Q_h\chi) \le Ch^{p+1}\|\nabla^p\chi\|\,\|\nabla Q_h\chi\|, \quad \text{for } p=0,1,$$

which shows the first estimate of (2.10). Also, by the definition of $\bar{\Delta}_h$, Lemma 2.3 with $q=0$ shows

$$\|\bar{\Delta}_h Q_h\chi\|_h^2 = -(\nabla Q_h\chi, \nabla\bar{\Delta}_h Q_h\chi) = -\epsilon_h(\chi, \bar{\Delta}_h Q_h\chi) \le Ch^p\|\nabla^p\chi\|\,\|\bar{\Delta}_h Q_h\chi\|_h,$$

for $p=0,1$, which gives the second bound in (2.10). $\square$

In our analysis of the lumped mass method (1.6) we shall be interested not only in the error estimates (1.8) but also in the corresponding estimates for the gradient of the error. Our approach will then require the following estimates for the standard Galerkin method.

Theorem 2.1. *Let u and u_h be the solutions of* (1.1) *and* (1.2)*, respectively. Then*

$$\|\nabla(u_h(t)-u(t))\| \le \begin{cases} Cht^{-(1-q/2)}|v|_q, & \text{for } q=1,2, \quad \text{if } v_h = R_h v,\\ Cht^{-1}\|v\|, & \text{if } v_h = P_h v.\end{cases}$$

Proof. In a customary way we split the error into two terms as

$$u_h - u = (u_h - R_h u) + (R_h u - u) = \vartheta + \varrho.$$

By (2.5) and (2.3) we have

$$\|\nabla\varrho(t)\| \le Ch|u(t)|_2 \le Cht^{-1+q/2}|v|_q, \quad \text{for } t>0, \quad v\in \dot H^q,\ q=0,1,2.$$

It remains to bound $\nabla\vartheta$ analogously. By our definitions we have

$$\vartheta_t - \Delta_h\vartheta = -P_h\varrho_t, \quad \text{for } t>0. \tag{2.11}$$

In the cases $q=1,2$ the Ritz projection $R_h v$ is well defined so that $\vartheta(0)=0$ and hence, by Duhamel's principle,

$$\vartheta(t) = -\int_0^t E_h(t-s)P_h\varrho_t(s)\,ds. \tag{2.12}$$

Using Lemma 2.1, the stability of P_h, (2.5) and (2.3), we find, for $2p+1\ge q$,

$$\begin{aligned}\|\nabla D_t^\ell E_h(t-s)P_hD_t^p\varrho(s)\| &\le C(t-s)^{-\ell-1/2}\,\|D_t^p\varrho(s)\|\\ &\le Ch(t-s)^{-\ell-1/2}\,|D_t^pu(s)|_1 \le Ch(t-s)^{-\ell-1/2}s^{-p-1/2+q/2}|v|_q.\end{aligned} \tag{2.13}$$

When $q=2$ we use this in (2.12) to obtain

$$\|\nabla\vartheta(t)\| \le Ch\int_0^t (t-s)^{-1/2}s^{-1/2}\,ds\,|v|_2 = Ch|v|_2,$$

which shows the desired estimate for $\nabla\vartheta$ in this case.

To treat the case $q=1$ we use (2.12) to write

$$\nabla\vartheta(t) = -\Big\{\int_0^{t/2} + \int_{t/2}^t\Big\}\nabla E_h(t-s)P_h\varrho_t(s)\,ds = T_1 + T_2.$$

Using (2.13), we find

$$\|T_2\| \le Ch\int_{t/2}^t (t-s)^{-1/2}s^{-1}\,ds\,|v|_1 \le Cht^{-1/2}|v|_1.$$

For T_1 we obtain by integration by parts

$$T_1 = -\Big[\nabla E_h(t-s)P_h\varrho(s)\Big]_0^{t/2} + \int_0^{t/2}\nabla D_sE_h(t-s)P_h\varrho(s)\,ds,$$

and hence

$$\|T_1\| \le Ch\,t^{-1/2}\,|v|_1 + Ch\int_0^{t/2}(t-s)^{-3/2}\,ds\,|v|_1 \le Cht^{-1/2}|v|_1.$$

Together these estimates show the desired bound for $\nabla\vartheta$ for $q=1$.

Finally, for $q = 0$, we multiply (2.11) by t to obtain

$$(t\vartheta)_t - \Delta_h(t\vartheta) = -tP_h\varrho_t + \vartheta, \quad \text{for } t > 0.$$

Although $\vartheta(0) = v_h - R_h v$ is not defined when $v \notin H_0^1$, we have $t\vartheta(t) \to 0$ as $t \to 0$. Indeed, using the estimate, cf. [7, formula (3.12)],

$$\|u_h(t) - u(t)\| \le Cht^{-1/2}\|v\|, \tag{2.14}$$

the error bound (2.5), and the regularity estimate $|u(t)|_1 \le Ct^{-1/2}\|v\|$ we get

$$\|\vartheta(t)\| \le \|u_h(t) - u(t)\| + \|R_h u(t) - u(t)\| \le Cht^{-1/2}\|v\|, \tag{2.15}$$

which shows that $t\vartheta(t) \to 0$ as $t \to 0$.

Hence we may integrate the above equation over $(0, t)$ to find

$$t\,\vartheta(t) = -\int_0^t sE_h(t-s)P_h\varrho_t(s)\,ds + \int_0^t E_h(t-s)\vartheta(s)\,ds,$$

so that

$$t\nabla\vartheta(t) = -\int_0^t s\nabla E_h(t-s)P_h\varrho_t(s)\,ds + \int_0^t \nabla E_h(t-s)\vartheta(s)\,ds = T_3 + T_4.$$

Using (2.13) with $l = 0$, $p = 1$, $q = 0$ we obtain

$$\|T_3\| \le Ch\int_0^t (t-s)^{-1/2}s^{-1/2}\,ds\,\|v\| = Ch\|v\|.$$

For T_4, we note that in view of (2.15) we have

$$\|T_4\| \le Ch\int_0^t (t-s)^{-1/2}s^{-1/2}\,ds\|v\| = Ch\|v\|.$$

Together these estimates complete the proof for $q = 0$ and thus of the theorem. Note that the choice $v_h = P_h v$ enters in the estimate for $u_h(t) - u(t)$ in (2.15). □

3. The lumped mass method with smooth initial data

In this section we derive optimal order error estimates for the lumped mass method (1.6), with initial data v in $\dot H^2$ and $\dot H^1$.

Theorem 3.1. *Let u be the solution of* (1.1) *and $\bar u_h$ that of* (1.6)*. Then*

$$\|\bar u_h(t) - u(t)\| + h\|\nabla(\bar u_h(t) - u(t))\| \le Ch^2t^{-(1-q/2)}|v|_q, \quad \textit{for } q = 1, 2, \ \textit{if } v_h = R_h v.$$

Proof. Since the corresponding error bounds hold for the solution u_h of the standard Galerkin method, by (1.3), (1.5) and Theorem 2.1, it suffices to show that

$$\|\delta(t)\| + h\|\nabla\delta(t)\| \le Ch^2t^{-(1-q/2)}|v|_q, \quad \text{for } t > 0, \ q = 1, 2, \quad \text{where } \delta = \bar u_h - u_h.$$

By (1.6), (1.2) and the definition (1.9) of the quadrature error operator Q_h, $\delta(t)$ satisfies (1.11) and hence

$$\delta_t - \bar\Delta_h\delta = \bar\Delta_h Q_h u_{h,t}, \quad \text{for } t \ge 0, \quad \text{with } \delta(0) = 0. \tag{3.1}$$

By Duhamel's principle this shows

$$\delta(t) = \int_0^t \bar E_h(t-s)\bar\Delta_h Q_h u_{h,t}(s)\,ds. \tag{3.2}$$

Using the fact that $\bar{E}_h(t)\bar{\Delta}_h = D_t\bar{E}_h(t)$, and Lemmas 2.2 and 2.4, we easily get

$$\begin{aligned}\|\bar{E}_h(t)\bar{\Delta}_h Q_h\chi\| + h\|\nabla\bar{E}_h(t)\bar{\Delta}_h Q_h\chi\| \\ \le Ct^{-1/2}\big(\|\nabla Q_h\chi\| + h\|\bar{\Delta}_h Q_h\chi\|\big) \le Ch^2t^{-1/2}\|\nabla\chi\|,\end{aligned} \tag{3.3}$$

and hence

$$\|\delta(t)\| + h\|\nabla\delta(t)\| \le Ch^2\int_0^t (t-s)^{-1/2}\|\nabla u_{h,t}(s)\|\,ds.$$

Here, since $\Delta_h R_h = P_h\Delta$, we obtain, by first applying Lemma 2.1

$$\|\nabla u_{h,t}(s)\| \le Cs^{-1/2}\|u_{h,t}(0)\| = Cs^{-1/2}\|\Delta_h R_h v\| = Cs^{-1/2}\|P_h\Delta v\| \le Cs^{-1/2}|v|_2,$$

and hence

$$\|\delta(t)\| + h\|\nabla\delta(t)\| \le Ch^2\int_0^t (t-s)^{-1/2}s^{-1/2}\,ds\,|v|_2 = C\,h^2|v|_2,$$

which completes the proof for $q = 2$.

To treat the case $q = 1$, we use (3.2) to write

$$\delta(t) = \Big\{\int_0^{t/2} + \int_{t/2}^t\Big\}\bar{E}_h(t-s)\bar{\Delta}_h Q_h u_{h,t}(s)\,ds = \delta_1(t) + \delta_2(t). \tag{3.4}$$

Here we have, in the same way as above,

$$\begin{aligned}\|\delta_2(t)\| + h\|\nabla\delta_2(t)\| &\le Ch^2\int_{t/2}^t (t-s)^{-1/2}\,\|\nabla u_{h,t}(s)\|\,ds \\ &\le Ch^2\int_{t/2}^t (t-s)^{-1/2}s^{-1}\,ds\,\|\nabla R_h v\| \le Ch^2t^{-1/2}\,|v|_1.\end{aligned}$$

Integrating by parts we obtain

$$\delta_1(t) = \Big[\bar{E}_h(t-s)\bar{\Delta}_h Q_h u_h(s)\Big]_0^{t/2} - \int_0^{t/2} D_s\bar{E}_h(t-s)\bar{\Delta}_h Q_h u_h(s)\,ds. \tag{3.5}$$

Employing (3.3) we now find, similarly to the above,

$$\begin{aligned}\|\delta_1(t)\| + h\|\nabla\delta_1(t)\| &\le Ch^2t^{-1/2}(\|\nabla u_h(t/2)\| + \|\nabla R_h v\|) \\ &+ Ch^2\int_0^{t/2}(t-s)^{-3/2}\,\|\nabla u_h(s)\|\,ds \le Ch^2t^{-1/2}|v|_1.\end{aligned}$$

Together these estimates complete the proof. □

4. The lumped mass method with nonsmooth initial data

In this section we discuss error estimates for the lumped mass method with $v \in L_2$, with discrete initial data $v_h = P_h v$. To derive an optimal order error bound analogous to (1.4) for the standard Galerkin method, we now need to impose a condition on the triangulations $\mathcal{T}_h$, expressed as a boundedness condition for the quadrature error operator Q_h. Without this condition we are only able to show a nonoptimal order $O(h)$ error bound in L_2, whereas for the gradient of the error an optimal order $O(h)$ still holds. We begin with the following theorem.

Theorem 4.1. *Let u be the solution of* (1.1) *with $v \in L_2$ and let $\bar{u}_h$ be the solution of* (1.6), *with $v_h = P_h v$. Then*

$$\|\bar{u}_h(t) - u(t) + \bar{E}_h(t)\bar{\Delta}_h Q_h v_h\| \le Ch^2 t^{-1}\|v\|, \quad \text{for } t > 0. \tag{4.1}$$

Hence, a necessary and sufficient condition for the nonsmooth data error bound

$$\|\bar{u}_h(t) - u(t)\| \le Ch^2 t^{-1}\|v\|, \quad \text{for } t > 0, \tag{4.2}$$

is that

$$\|\bar{E}_h(t)\bar{\Delta}_h Q_h P_h v\| \le Ch^2 t^{-1}\|v\|, \quad \text{for } t > 0. \tag{4.3}$$

Proof. Using again the notation $\delta = \bar{u}_h - u_h$, we write

$$\bar{u}_h(t) - u(t) + \bar{E}_h(t)\bar{\Delta}_h Q_h P_h v = (\delta(t) + \bar{E}_h(t)\bar{\Delta}_h Q_h P_h v) + (u_h(t) - u(t)).$$

Thus, in view of (1.4), it suffices to bound $\delta(t) + \bar{E}_h(t)\bar{\Delta}_h Q_h P_h v$. Using the representation (3.4) and (3.5) of δ, we obtain

$$\delta(t) + \bar{E}_h(t)\bar{\Delta}_h Q_h P_h v = \bar{E}_h(t/2)\bar{\Delta}_h Q_h u_h(t/2) + \delta_2(t) + \delta_3(t),$$

$$\text{where } \delta_3(t) = -\int_0^{t/2} D_s \bar{E}_h(t-s)\bar{\Delta}_h Q_h u_h(s)\, ds.$$

Here, similarly to the proof of Theorem 3.1, using the stability of P_h,

$$\|\bar{E}_h(t/2)\bar{\Delta}_h Q_h u_h(t/2)\| \le Ch^2 t^{-1/2}\|\nabla u_h(t/2)\| \le Ch^2 t^{-1}\|P_h v\| \le Ch^2 t^{-1}\|v\|.$$

Further,

$$\|\delta_2(t)\| \le Ch^2 \int_{t/2}^{t} (t-s)^{-1/2} s^{-3/2}\, ds\, \|P_h v\| \le Ch^2 t^{-1}\|v\|,$$

and, since $\|\nabla u_h(s)\| \le Cs^{-1/2}\|P_h v\|$,

$$\|\delta_3(t)\| \le Ch^2 \int_0^{t/2} (t-s)^{-3/2}\, \|\nabla u_h(s)\|\, ds \le Ch^2 t^{-1}\|v\|.$$

Together these estimates show the desired bound (4.1). □

We will now use this result to show that the $O(h^2)$ error bound (1.10) for the quadrature error operator $Q_h : S_h \to S_h$ defined in (1.9) is sufficient for the nonsmooth data error estimate (4.2) to hold.

Theorem 4.2. *Let u be the solution of* (1.1) *with $v \in L_2$ and let $\bar{u}_h$ be the solution of* (1.6), *with $v_h = P_h v$. Assume that Q_h satisfies* (1.10). *Then* (4.2) *holds.*

Proof. The result follows by Theorem 4.1 since, by Lemma 2.2 and (1.10), we have

$$\|\bar{E}_h(t)\bar{\Delta}_h Q_h v_h\| \le Ct^{-1}\|Q_h v_h\| \le Ch^2 t^{-1}\|v\|. \quad \square$$

The condition (1.10) will be discussed in more detail in Section 5 below. Note that, by Lemma 2.4, without additional assumptions on the mesh, we always have

$$\|Q_h \chi\| \le C\|\nabla Q_h \chi\| \le Ch\|\chi\|, \quad \forall \psi \in S_h,$$

and that the following lower order error estimate always holds.

Theorem 4.3. *Let u be the solution of* (1.1) *with $v \in L_2$ and $\bar{u}_h$ the solution of* (1.6), *with $v_h = P_h v$. Then*

$$\|\bar{u}_h(t) - u(t)\| \le Cht^{-1/2}\|v\|, \quad \text{for } t > 0.$$

Proof. Using the stability of $\bar{E}_h(t)$ and $E(t)$, and Lemma 2.4, we find

$$\|\bar{u}_h(t) - u(t) + \bar{E}_h(t)\bar{\Delta}_h Q_h P_h v\| \le C\|v\| + C\|\bar{\Delta}_h Q_h P_h v\| \le C\|v\|.$$

Combining this estimate with (4.1), we obtain

$$\|\bar{u}_h(t) - u(t) + \bar{E}_h(t)\bar{\Delta}_h Q_h v_h\| \le Cht^{-1/2}\|v\|.$$

But by Lemmas 2.2 and 2.4, we have

$$\|\bar{E}_h(t)\bar{\Delta}_h Q_h v_h\| \le Ct^{-1/2}\|\nabla Q_h P_h v\| \le Cht^{-1/2}\|v\|,$$

which shows the desired bound. □

We end this section by showing an optimal order H^1-norm nonsmooth data error estimate, which does not require the additional assumption (1.10).

Theorem 4.4. *Let u the solution of* (1.1) *with $v \in L_2$ and $\bar{u}_h$ the solution of* (1.6), *with $v_h = P_h v$. Then*

$$\|\nabla(\bar{u}_h(t) - u(t))\| \le Cht^{-1}\|v\|, \quad \text{for } t > 0.$$

Proof. In view of Theorem 2.1 it suffices to show

$$\|\nabla\delta(t)\| \le Cht^{-1}\|v\|, \quad \text{for } t > 0. \tag{4.4}$$

Multiplying (3.1) by t, we get

$$(t\delta)_t - \bar{\Delta}_h(t\delta) = t\bar{\Delta}_h Q_h u_{h,t} + \delta, \quad \text{for } t \ge 0.$$

Hence, by Duhamel's principle we get

$$t\nabla\delta(t) = \int_0^t s\nabla\bar{E}_h(t-s)\bar{\Delta}_h Q_h u_{h,t}(s)\,ds + \int_0^t \nabla\bar{E}_h(t-s)\delta(s)\,ds = I + II.$$

By (3.3), Lemma 2.1, and the stability of P_h, we find

$$\|I\| \le Ch\int_0^t (t-s)^{-1/2}s^{-1/2}\,ds\,\|v_h\| \le Ch\|v\|.$$

To bound II, we use (2.14) and Theorem 4.3 to obtain

$$\|\delta(t)\| \le \|\bar{u}_h(t) - u(t)\| + \|u_h(t) - u(t)\| \le Cht^{-1/2}\|v\|.$$

Therefore, Lemma 2.2 gives

$$\|II\| \le Ch\int_0^t (t-s)^{-1/2}s^{-1/2}\,ds\,\|v\| = Ch\|v\|.$$

Together these estimates show (4.4), which completes the proof. □

5. Symmetric triangulations

In this section we show that for triangulations that are symmetric, in a sense to be defined, assumption (1.10) is satisfied and therefore, by Theorem 4.2, the optimal order nonsmooth data error estimate (4.2) holds.

For z_i a vertex of the triangulation $\mathcal{T}_h$ we define the patch $\Pi_i = \{\cup\tau : \tau \in \mathcal{T}_h,\ z_i \in \partial\tau\}$, see Figure 1, and say that $\mathcal{T}_h$ is *symmetric at z_i* if the patch Π_i is symmetric around z_i, in the sense that if $x \in \Pi_i$, then $z_i - (x - z_i) = 2z_i - x \in \Pi_i$. Denoting by Z_h^0 the interior vertices of $\mathcal{T}_h$, we say that $\mathcal{T}_h$ is *symmetric* if it is symmetric at each $z_i \in Z_h^0$. The patch in Figure 1 is nonsymmetric with respect to z_i, whereas triangulations which are periodic repetition of the patches shown on

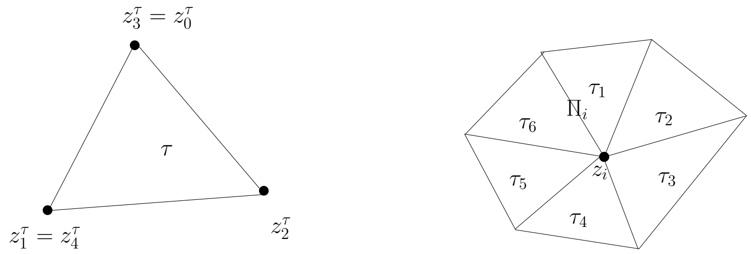

FIGURE 1. A triangle τ and a patch Π_i around a vertex z_i

Figure 2 are symmetric. Symmetric triangulations exist only for special domains, such as rectangles, but not for general polygonal domains.

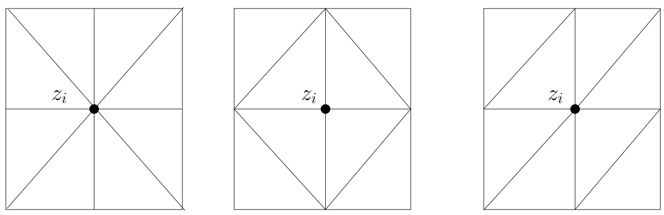

FIGURE 2. Patches which are symmetric with respect to the vertex z_i

We now show the sufficiency of symmetry for condition (1.10).

Theorem 5.1. *If the triangulation $\mathcal{T}_h$ is symmetric, then* (1.10) *holds.*

Proof. The proof is based on duality. For given $\chi \in S_h$ we define $\phi = \phi_\chi \in \dot{H}^1$ as the solution of the Dirichlet problem $(\nabla\phi, \nabla\eta) = (\chi, \eta)$ for all $\eta \in \dot{H}^1$. Since Ω is convex we have $\phi \in \dot{H}^2$ and $|\phi|_2 \le C\|\chi\|$. Letting π_h be the finite element interpolation operator into S_h, we then have, for any $\psi \in S_h$,

$$\begin{aligned}(5.1)\quad \|Q_h\psi\| &= \sup_{\chi\in S_h} \frac{(Q_h\psi, \chi)}{\|\chi\|} = \sup_{\chi\in S_h} \frac{(\nabla Q_h\psi, \nabla\phi)}{\|\chi\|} \\ &\le \sup_{\chi\in S_h} \frac{|(\nabla Q_h\psi, \nabla(\phi - \pi_h\phi))|}{\|\chi\|} + \sup_{\chi\in S_h} \frac{|(\nabla Q_h\psi, \nabla\pi_h\phi)|}{\|\chi\|} = I + II.\end{aligned}$$

By the obvious error estimate for π_h and Lemma 2.4 we have

$$|I| \le Ch \sup_{\chi\in S_h} \frac{\|\nabla Q_h\psi\|\,|\phi|_2}{\|\chi\|} \le Ch^2\|\psi\|. \tag{5.2}$$

To estimate II, we first rewrite the numerator in the form, cf. (1.7),

$$(\nabla Q_h\psi, \nabla\pi_h\phi) = \epsilon_h(\psi, \pi_h\phi) = (\psi, \pi_h\phi)_h - (\psi, \pi_h\phi) = \sum_{\tau\in\mathcal{T}_h} \int_\tau (\pi_h(\psi\phi) - \psi\,\pi_h\phi)dx.$$

Denoting the vertices of τ by $z_1^\tau, z_2^\tau, z_3^\tau$ and setting $z_4^\tau = z_1^\tau$, $z_0^\tau = z_3^\tau$, and $u(z_j^\tau) = u_j$ (cf. Figure 1), we obtain after a simple calculation

$$\int_\tau (\pi_h(\psi\phi) - \psi\,\pi_h\phi)dx = -\frac{|\tau|}{12}\sum_{j=1}^{3} \psi_j(\Delta^\tau\phi)_j,$$

where

$$(\Delta^\tau \phi)_j = \phi_{j-1} - 2\phi_j + \phi_{j+1},\ j = 1, 2, 3.$$

Hence, if $\tau \in \Pi_i$, with $(\Delta^\tau\phi)(z_i) = (\Delta^\tau\phi)_j$ if $z_j^\tau = z_i$,

$$\epsilon_h(\psi, \pi_h\phi) = -\sum_{z_i \in Z_h^0} \psi(z_i)(\Delta_h^*\phi)(z_i), \quad \text{where } (\Delta_h^*\phi)(z_i) = \frac{1}{12}\sum_{\tau\subset\Pi_i} |\tau|(\Delta^\tau\phi)(z_i).$$

We can look upon $(\Delta_h^*\phi)(z_i)$ as a finite difference approximation of $\Delta\phi$ at z_i, using the values of ϕ at the vertices of Π_i. Since $(\Delta_h^*\phi)(z_i)$ does not use information about the location of these vertices, it does not generally approximate the Laplacian $\Delta\phi$ at z_i. Such an example is the patch shown in Figure 1.

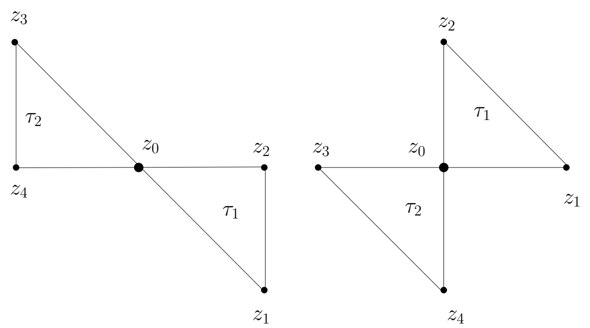

FIGURE 3. A pair of elements symmetric with respect to vertex z_0

Now let $\mathcal{T}_h$ be symmetric at the vertex z_0, so that the patch Π_0 is symmetric around z_0. Then for the triangle $\tau_1 \subset \Pi_0$ there is a triangle $\tau_2 \subset \Pi_0$, symmetric to τ_1 with respect to z_0, see Figure 3, so that $|\tau_1| = |\tau_2| = |\tau|$. Also, for ϕ a linear function in Π_0 we have

$$|\tau_1|(\Delta^{\tau_1}\phi)_0 + |\tau_2|(\Delta^{\tau_2}\phi)_0 = |\tau|(\phi_1 + \phi_2 - 2\phi_0 + \phi_3 + \phi_4 - 2\phi_0) = 0. \tag{5.3}$$

Thus, for a patch Π_i which is symmetric with respect z_i and ϕ linear in Π_i we have $(\Delta_h^*\phi)_i = 0$, since this expression will be a sum of symmetric pairs satisfying the relations (5.3). Applying the Bramble-Hilbert lemma we then obtain

$$|(\Delta_h^*\phi)(z_i)| \le Ch^2 |\Pi_i|^{1/2} \|\phi\|_{H^2(\Pi_i)}.$$

Employing this estimate, we get for any $\psi \in S_h$,

$$|(\nabla Q_h\psi, \nabla\pi_h\phi)| = |\epsilon_h(\psi, \pi_h\phi)| \le |\sum_{z_i\in Z_h^0} \psi(z_i)\Delta_h^*\phi(z_i)|$$
$$\le Ch^2 \sum_{z_i\in Z_h^0} |\psi(z_i)|\, |\Pi_i|^{\frac{1}{2}}\, \|\phi\|_{H^2(\Pi_i)} \le Ch^2\|\psi\|\, |\phi|_2 \le Ch^2\|\psi\|\, \|\chi\|,$$

and hence $|II| \le Ch^2\|\psi\|$. Together with (5.2) this completes the proof. □

6. "Almost" symmetric partitions in one dimension

In this section we shall consider the spatially one-dimensional analogue of the lumped mass method, and show that a nonsmooth data error estimate of type (4.2) holds for partitions which are somewhat more general than symmetric ones.

Let $\Omega = (0,1)$ and let $\mathcal{T}_h = \{I_i\}_{i=1}^N$, with $I_i = (x_{i-1}, x_i)$, be defined by the not necessarily uniform partition $0 = x_0 < x_1 < \cdots < x_N = 1$, and let S_h be the set of the continuous piecewise linear functions over $\mathcal{T}_h$. We set $h_i = x_i - x_{i-1}$ and $h = \max_i h_i$. Using the quadrature formula,

$$Q_{I_i,h_i}(f) = \frac{h_i}{2}(f(x_{i-1}) + f(x_i)) \approx \int_{x_{i-1}}^{x_i} f\,dx,$$

we now define the approximation of the inner product (v,w) in S_h by

$$(\psi,\chi)_h = \sum_{i=1}^{N} Q_{I_i,h_i}(\psi\chi) = \sum_{i=1}^{N-1} \psi_i\chi_i\hbar_i, \quad \text{with } \hbar_i = \frac{1}{2}(h_i + h_{i+1}).$$

The lumped mass finite element method is then defined by

$$(\bar{u}_{h,t},\chi)_h + (\bar{u}_h',\chi') = 0, \quad \forall \chi \in S_h, \quad \text{with } \bar{u}_h(0) = v_h. \tag{6.1}$$

It is easy to demonstrate that the analogues of our results in Sections 3–5 remain valid also for (6.1). Here we will show that assumption (1.10) for the operator Q_h holds for partitions which are "almost" symmetric in a sense to be defined below.

A direct computation shows that

$$\begin{aligned}((Q_h\chi)',\psi') &= \epsilon_h(\chi,\psi) = (\chi,\psi)_h - (\chi,\psi)\\ &= -\frac{1}{6}\sum_{i=1}^{N-1}(h_{i+1}(\chi_{i+1}-\chi_i) - h_i(\chi_i - \chi_{i-1}))\,\psi_i = (M_h\chi,\psi)_h,\end{aligned}$$

where, taking into account that $\chi_0 = \chi_N = 0$, we have

$$(M_h\chi)_i = -\frac{1}{6\hbar_i}\big(h_{i+1}(\chi_{i+1}-\chi_i) - h_i(\chi_i-\chi_{i-1})\big), \quad i = 1,\dots,N-1. \tag{6.2}$$

Similarly, direct computation of $(-\bar{\Delta}_h\chi,\psi)_h = (\chi',\psi')$ gives

$$-(\bar{\Delta}_h\chi)_i = -\frac{1}{\hbar_i}\left(\frac{\chi_{i+1}-\chi_i}{h_{i+1}} - \frac{\chi_i - \chi_{i-1}}{h_i}\right), \quad i = 1,\dots,N-1, \tag{6.3}$$

and we note that by the definition of the operator $-\bar{\Delta}_h$ we have

$$((Q_h\chi)',\psi') = -(\bar{\Delta}_h Q_h\chi,\psi)_h = (M_h\chi,\psi)_h,$$

so that

$$-\bar{\Delta}_h Q_h = M_h, \quad \text{or} \quad Q_h = (-\bar{\Delta}_h)^{-1}M_h. \tag{6.4}$$

Obviously, a partition that is symmetric with respect to each of its nodes is uniform, so that $h_i = h$ for all i. In this case (6.2) and (6.3) imply $6h^{-2}M_h = -\bar{\Delta}_h$ and $Q_h = \frac{1}{6}h^2 I$, where I is the identity operator, and hence assumption (1.10) is satisfied. More generally we have the following lemma which easily follows from (6.2) and (6.3) by checking the coefficients. Here, for $\widetilde{\omega} = (\omega_1,\dots,\omega_{N-1})$ and $\chi \in S_h$, we define $\widetilde{\omega}\,\chi \in S_h$ by $(\widetilde{\omega}\chi)_i = \omega_i\chi_i$. Further we set $(\widetilde{h}^2)_i = h_i^2$, $i = 1,\dots,N-1$.

Lemma 6.1. *Let the operator* $\bar\partial_h : S_h \to S_h$ *be defined by* $(\bar\partial_h \chi)_i = (\chi_i - \chi_{i-1})/\hbar_i$, $i = 1, \dots, N-1$. *Then*

$$M_h\chi = -\tfrac{1}{6}\bar\Delta_h(\widetilde{h}^2\chi) + \tfrac{1}{6}\bar\partial_h(\widetilde{\omega}\chi), \quad \textit{where } \omega_i = h_{i+1}\big(1 - (h_i/h_{i+1})^2\big).$$

In the following theorem we shall consider families of partitions that are almost uniform in the sense that, uniformly in h,

$$\Big|\frac{h_i}{h_{i+1}} - 1\Big| \le Ch, \quad i = 1, \dots, N-1. \tag{6.5}$$

Theorem 6.1. *If* (6.5) *holds, we have for* $Q_h = (-\bar\Delta_h)^{-1}M_h$, *uniformly in* h,

$$\|Q_h\chi\| \le C\,h^2\|\chi\|, \quad \textit{for } \chi \in S_h.$$

Proof. By (6.4) and Lemma 6.1 we have

$$(Q_h\chi)_i = ((-\bar\Delta_h)^{-1}M_h\chi)_i = \tfrac{1}{6}h_i^2\chi_i + \tfrac{1}{6}((-\bar\Delta_h)^{-1}\bar\partial_h(\widetilde{\omega}\chi))_i = I_i + II_i.$$

Clearly $\|I\|_h \le \frac{1}{6}h^2\|\chi\|_h$. To deal with II, we note that $(-\bar\Delta_h)^{-1}$, and hence also $(-\bar\Delta_h)^{-1/2}$, is bounded in $\|\cdot\|_h$, uniformly in h, and we shall show the following:

Lemma 6.2. *With our above definitions we have, uniformly in* h,

$$\|(-\bar\Delta_h)^{-1/2}\bar\partial_h\chi\|_h \le C\|\chi\|_h, \quad \textit{for } \chi \in S_h.$$

Using this lemma we find, since $|\omega_i| \le Ch^2$, that $\|II\|_h \le C\|\widetilde{\omega}\chi\|_h \le Ch^2\|\chi\|_h$, which completes the proof of the Theorem 6.1. □

Proof of Lemma 6.2. . We have

$$\|(-\bar\Delta_h)^{-1/2}\bar\partial_h\chi\|_h = \sup_{\psi\in S_h}\frac{((-\bar\Delta_h)^{-1/2}\bar\partial_h\chi,\psi)_h}{\|\psi\|_h} = \sup_{\psi\in S_h}\frac{(\bar\partial_h\chi,(-\bar\Delta_h)^{-1/2}\psi)_h}{\|\psi\|_h}.$$

Consider for $\phi \in S_h$, with $\partial_h\chi_i = (\chi_{i+1} - \chi_i)/\hbar_i$,

$$(\bar\partial_h\chi,\phi)_h = \sum_{i=1}^{N}(\chi_i - \chi_{i-1})\phi_i = -\sum_{i=0}^{N-1}\chi_i(\phi_{i+1} - \phi_i) = -(\chi,\partial_h\phi)_h \le \|\chi\|_h\|\partial_h\phi\|_h.$$

Note

$$\|\partial_h\phi\|_h^2 \le C\|\phi'\|^2 = C(-\bar\Delta_h\phi,\phi)_h, \quad \text{with } C > 0.$$

Now choose $\phi = (-\bar\Delta_h)^{-1/2}\psi$ to find

$$\|\partial_h(-\bar\Delta_h)^{1/2}\psi\|_h^2 \le C((-\bar\Delta_h)(-\bar\Delta_h)^{-1/2}\psi,(-\bar\Delta_h)^{-1/2}\psi)_h = C\|\psi\|_h^2.$$

Hence

$$(\bar\partial_h\chi,(-\bar\Delta_h)^{-1/2}\psi)_h \le C\|\chi\|_h\|\psi\|_h,$$

which completes the proof. □

As in Theorem 4.2 the result of Theorem 6.1 implies a nonsmooth data error estimate of the form (4.2) for $v_h = P_hv$.

7. Counterexamples

In this section we continue the discussion of the lumped mass method (6.1) in one space dimension and present two examples, where the necessary and sufficient condition for optimal convergence of Theorem 4.1 is not satisfied and hence the $O(h^2)$ nonsmooth data error estimate does not hold.

First, we consider a special nonuniform mesh by choosing $h = 4/(3N)$, where N a positive integer divisible by 4, and take

$$h_i = \tfrac{1}{2}h, \quad \text{for } i \text{ odd} \quad \text{and} \quad h_i = h, \quad \text{for } i \text{ even}, \quad i = 1, \dots, N. \tag{7.1}$$

Obviously $\hbar_i = \frac{3}{4}h$. This mesh consists of $J = N/2$ copies of the patch $(0, \frac{1}{2}h) \cup (\frac{1}{2}h, \frac{3}{2}h)$ and is not symmetric with respect to any mesh-point, see Figure 4.

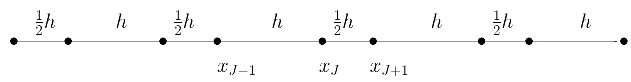

Figure 4. A nonsymmetric partition in one space dimension

By Lemma 6.1 since $\omega_i = \frac{3}{4}h$, for i odd, $\omega_i = -\frac{3}{2}h$, for i even, we have

$$M_h\chi = -\tfrac{1}{6}\bar{\Delta}_h(\widetilde{h}^2\chi) + \widetilde{M}_h\chi, \tag{7.2}$$

where

$$(\widetilde{M}_h\chi)_i = \frac{1}{6}\begin{cases} \chi_i + 2\chi_{i-1}, & \text{for } i \text{ odd}, \\ -2\chi_i - \chi_{i-1}, & \text{for } i \text{ even}. \end{cases} \tag{7.3}$$

In view of Theorem 4.1, the following proposition will show that the $O(h)$ error estimate for $t > 0$ in Theorem 4.3 is best possible.

Proposition 7.1. *Let $\mathcal{T}_h$ be defined by* (7.1). *For the lumped mass initial value problem* (6.1) *with $v_h = \sum_{j=0}^{\frac{1}{2}J-1}(\Phi_{J+2j} - 2\Phi_{J+2j+1})$, where Φ_j is the nodal basis function at x_j, we have, for each $t > 0$ and h small,*

$$\|\bar{E}_h(t)\bar{\Delta}_h Q_h v_h\|_h \geq ch\|v_h\|_h, \quad \textit{with } c = c(t) > 0.$$

Proof. In view of (7.2) and (6.4) we have

$$Q_h v_h = (-\bar{\Delta}_h)^{-1}M_h v_h = -\tfrac{1}{6}\widetilde{h}^2 v_h + (-\bar{\Delta}_h)^{-1}\widetilde{M}_h v_h. \tag{7.4}$$

Since, by $\bar{E}_h(t)\bar{\Delta}_h = D_t\bar{E}_h(t)$ and Lemma 2.2,

$$t\|\bar{E}_h(t)\bar{\Delta}_h(\widetilde{h}^2 v_h)\| \leq C\|\widetilde{h}^2 v_h\| \leq Ch^2\|v_h\|, \quad \text{for } t > 0,$$

it will suffice to consider the last term in the right of (7.4). We find at once from (7.3) that $\widetilde{M}_h(\Phi_{2j} - 2\Phi_{2j+1}) = (\Phi_{2j+2} - \Phi_{2j})/3$, where we have set $\Phi_N \equiv 0$, so that

$$\widetilde{M}_h v_h = \sum_{j=0}^{\frac{1}{2}J-1}\widetilde{M}_h(\Phi_{J+2j} - 2\Phi_{J+2j+1}) = \frac{1}{3}\sum_{j=0}^{\frac{1}{2}J-1}(\Phi_{J+2j+2} - \Phi_{J+2j}) = -\frac{1}{3}\Phi_J.$$

Hence, with $\bar{\lambda}_j^h$ and $\bar{\phi}_j^h$ be the eigenfunction and eigenvalues of $-\bar{\Delta}_h$, since $x_J = \frac{1}{2}$ and hence $(\widetilde{M}_h v_h, \bar{\phi}_1^h)_h = \frac{1}{3}\,(\Phi_J, \bar{\phi}_1^h)_h = \frac{1}{4}\,h\,\bar{\phi}_1^h(\frac{1}{2})$,

$$\|\bar{E}_h(t)\bar{\Delta}_h((-\bar{\Delta}_h)^{-1}\widetilde{M}_h v_h)\|_h^2 = \sum_{j=1}^{N-1} e^{-2\bar{\lambda}_j^h t}(\widetilde{M}_h v_h, \bar{\phi}_j^h)_h^2 \geq \tfrac{1}{16}\,h^2 e^{-2\bar{\lambda}_1^h t}\bar{\phi}_1^h(\tfrac{1}{2})^2.$$

Since, as is easily seen, $\|v_h\|_h = \frac{\sqrt{5}}{2}$, the proof is completed by showing that the last expression is bounded below by $c(t)h^2$.

Let $\phi_1(x) = \sqrt{2}\sin \pi x$ be the eigenfunction of $-u'' = \lambda u$, corresponding to the first eigenvalue $\lambda_1 = \pi^2$. We shall need the fact that

$$\|\bar{\phi}_1^h - \phi_1\|_{H^1} = O(h) \quad \text{and} \quad \bar{\lambda}_1^h \to \lambda_1, \text{ as } h \to 0, \tag{7.5}$$

see e.g. [4, pp. 87–92]. Using this, we have

$$\bar{\phi}_1^h(\tfrac{1}{2}) \geq \phi_1(\tfrac{1}{2}) - \|\bar{\phi}_1^h - \phi_1\|_{L_\infty} \geq \sqrt{2} - |\bar{\phi}_h^1 - \phi_1|_{H^1} \geq \sqrt{2} - Ch,$$

which shows our claim. The proof is now complete. □

Next we give a second example of a partition $\mathcal{T}_h$ for which the optimal order error estimate (4.2) does not hold, although $\mathcal{T}_h$ is symmetric with respect to all nodes of $\mathcal{T}_h$ but one. Let $J/N = 3/5$ and $h = 3/(4J)$, and let $\mathcal{T}_h$ be defined by, see Figure 5,

$$h_j = x_j - x_{j-1} = h, \quad \text{for } j \leq J \text{ and } h_j = \tfrac{1}{2}h, \quad \text{for } J < j \leq N. \tag{7.6}$$

By Lemma 6.1 we may write, since $\omega_J = -\frac{3}{2}\,h$ and $\omega_j = 0$ for $j \neq J$,

$$M_h\chi = -\tfrac{1}{6}\bar{\Delta}_h(\widetilde{h}^2\chi) + \widetilde{M}_h\chi, \quad \text{where } (\widetilde{M}_h\chi)_j = \frac{h}{4\hbar_j}\begin{cases} \chi_J, & \text{for } j = J+1, \\ -\chi_J, & \text{for } j = J, \\ 0, & \text{for } j \neq J, J+1, \end{cases}$$

and it follows that

$$(\widetilde{M}_h\chi, \psi)_h = \tfrac{1}{4}\,h\,\chi_J(\psi_{J+1} - \psi_J), \quad \text{for } \chi, \psi \in S_h. \tag{7.7}$$

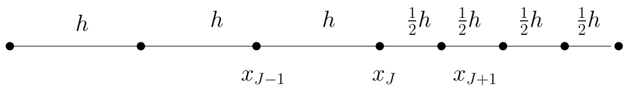

FIGURE 5. A nonsymmetric partition with respect to the point x_J

Proposition 7.2. *Let $\mathcal{T}_h$ be defined by (7.6). For the lumped mass initial value problem (6.1), with $\bar{u}_h(0) = \Phi_J$, the nodal basis function at $x_J = 3/4$, we have*

$$\|\bar{E}_h(t)\bar{\Delta}_h Q_h \Phi_J\|_h \geq c(t)h^{3/2}\|\Phi_J\|_h, \quad \textit{with } c(t) > 0, \quad \textit{for } t > 0, \ h \textit{ small.}$$

Proof. The proof is similar with Theorem 7.1. Using (7.7) we get

$$\|\bar{E}_h(t)\widetilde{M}_h\Phi_J\|_h^2 \geq e^{-2t\bar{\lambda}_1^h}(\widetilde{M}_h\Phi_J, \bar{\phi}_1^h)_h^2 \geq \tfrac{1}{16}\,e^{-2\bar{\lambda}_1^h t}h^2(\bar{\phi}_{1,J+1}^h - \bar{\phi}_{1,J}^h)^2.$$

Since (7.5) implies $|(\bar{\phi}_{1,J+1}^h - \bar{\phi}_{1,J}^h) - (\phi_{1,J+1} - \phi_{1,J})| \leq Ch^{3/2}$ and for $\phi_1 = \sqrt{2}\sin(\pi x)$ we have $|\phi_{1,J+1} - \phi_{1,J}| \geq \frac{\sqrt{2}}{2}\,h\pi|\cos(\pi x_J)| = \frac{1}{2}\,h\pi$, it follows that

$$\|\bar{E}_h(t)\widetilde{M}_h\Phi_J\|_h \geq c(t)h^2 = c(t)h^{3/2}\|\Phi_J\|_h,$$

since $\|\Phi_J\|_h = (3h/4)^{1/2}$. The proof is now complete. □

8. Some fully discrete schemes

In this final section we discuss briefly the generalization of our above results for the spatially semidiscrete lumped mass method to some basic fully discrete schemes, namely the backward Euler and Crank-Nicolson methods.

With $k > 0,\ t_n = n\,k,\ n = 0, 1, \ldots$, the backward Euler lumped mass method approximates $u(t_n)$ by $U^n \in S_h$ for $n \geq 0$ such that, with $\bar\partial U^n = (U^n - U^{n-1})/k$,

$$(\bar\partial U^n, \chi)_h + (\nabla U^n, \nabla\chi) = 0, \quad \forall \chi \in S_h, \quad \text{for } n \geq 1, \quad \text{with } U^0 = v_h,$$

or, with $A_h = -\bar\Delta_h$,

$$\bar\partial U^n + A_h U^n = 0, \quad \text{for } n \geq 1, \quad \text{with } U^0 = v_h. \tag{8.1}$$

Note that, for simplicity of notation, we write U^n instead of the perhaps more natural $\bar U^n$, and similarly below E_{kh} instead of $\bar E_{kh}$.

We shall have use for the following abstract lemma, in the case $\mathcal{H} = S_h$, normed by $\|\cdot\|_h$, and $A = A_h$.

Lemma 8.1. *Let A be a linear, selfadjoint, positive definite operator in a Hilbert space $\mathcal{H}$, with compact inverse, let $\mathbf{u} = \mathbf{u}(t)$ be the solution of*

$$\mathbf{u}' + A\mathbf{u} = 0, \quad \textit{for } t > 0, \quad \textit{with } \mathbf{u}(0) = \mathbf{v}, \tag{8.2}$$

and let $\mathbf{U} = \{\mathbf{U}^n\}_{n=0}^{\infty}$ be defined by

$$\bar\partial \mathbf{U}^n + A\mathbf{U}^n = 0, \quad \textit{for } n \geq 1, \quad \textit{with } \mathbf{U}^0 = \mathbf{v}. \tag{8.3}$$

Then, for $p = 0, 1,\ -1 \leq q \leq 3$, with $p + q \geq 0$, we have

$$\|A^{p/2}(\mathbf{U}^n - \mathbf{u}(t_n))\| \leq Ckt_n^{-(1-q/2)}\|A^{(p+q)/2}\mathbf{v}\|, \quad \textit{for } n \geq 1. \tag{8.4}$$

Proof. Solving for $\mathbf{U}^n$ we may write (8.3) as $\mathbf{U}^n = (I + kA)^{-1}\mathbf{U}^{n-1}$ and hence

$$\mathbf{U}^n = E_k^n \mathbf{v}, \quad \text{where } E_k = r(kA), \quad \text{with } r(\lambda) = 1/(1+\lambda).$$

Thus, since $\mathbf{u}(t_n) = e^{-t_n A}\mathbf{v} = e^{-nkA}\mathbf{v}$, we have

$$A^{p/2}(\mathbf{U}^n - \mathbf{u}(t_n)) = A^{p/2}F_n(kA)\mathbf{v}, \quad \text{with } F_n(\lambda) = r^n(\lambda) - e^{-n\lambda},$$

and therefore, by eigenfunction expansion and Parseval's relation,

$$\begin{aligned}\|A^{p/2}(\mathbf{U}^n - \mathbf{u}(t_n))\| &\leq \sup_{\lambda\in\sigma(A)} |\lambda^{-q/2}F_n(k\lambda)|\,\|A^{(p+q)/2}\mathbf{v}\| \\ &= k^{q/2} \sup_{\lambda\in\sigma(kA)} |\lambda^{-q/2}F_n(\lambda)|\,\|A^{(p+q)/2}\mathbf{v}\|.\end{aligned}$$

Hence, since $k^{q/2}n^{-(1-q/2)} = kt_n^{-(1-q/2)}$, it suffices for the proof of (8.4) to show

$$\lambda^{-q/2}|F_n(\lambda)| \leq C\,n^{-(1-q/2)}, \quad \text{for } \lambda > 0,\ n \geq 1. \tag{8.5}$$

For $0 < \lambda \leq 1$ we have $|r(\lambda)| \leq e^{-c\lambda}$, with $c > 0$, and $|r(\lambda) - e^{-\lambda}| \leq C\lambda^2$. Hence

$$\begin{aligned}\lambda^{-q/2}|F_n(\lambda)| &\leq \lambda^{-q/2}|r(\lambda) - e^{-\lambda}|\,\Big|\sum_{j=0}^{n-1} r^{n-1-j}(\lambda)e^{-j\lambda}\Big| \\ &\leq C\lambda^{2-q/2}n\,e^{-cn\lambda} \leq Cn^{-(1-q/2)}, \quad \text{for } n \geq 1.\end{aligned}$$

For $\lambda \geq 1$ we have $|r(\lambda)| \leq e^{-c}$, with $c > 0$, and since $\lambda^{-q/2}|r(\lambda)| \leq C$, we find

$$\lambda^{-q/2}|F_n(\lambda)| \leq \lambda^{-q/2}|r(\lambda)|\,|r(\lambda)|^{n-1} + \lambda^{-q/2}\,e^{-n\lambda} \leq Cn^{-(1-q/2)}, \quad \text{for } n \geq 1,$$

which shows (8.5) and thus completes the proof. □

We now show some optimal order error estimates for (8.1) with initial data in $\dot H^2$ and $\dot H^1$, and for initial data only in L_2, if (1.10) holds for Q_h.

Theorem 8.1. *Let U be the solution of* (8.1)*, and u that of* (1.1)*. Then*

$$\|U^n - u(t_n)\| \le C(h^2+k)t_n^{-1+q/2}|v|_q, \ \textit{for } n>0,\ q=1,2, \quad \textit{if}\ \ v_h = R_h v. \tag{8.6}$$

Further, if (1.10) *holds for Q_h,*

$$\|U^n - u(t_n)\| \le C(h^2+k)t_n^{-1}\|v\|, \ \textit{for } n>0, \quad \textit{if}\ \ v_h = P_h v. \tag{8.7}$$

Proof. We start with the estimates (8.6) and split the error as

$$U^n - u(t_n) = (U^n - \bar u_h(t_n)) + (\bar u_h(t_n) - u(t_n)) = \beta_n + \eta_n. \tag{8.8}$$

In view of Theorem 3.1, η_n is bounded as required. We obtain, by Lemma 8.1,

$$\|\beta_n\|_h = \|U^n - \bar u_n(t_n)\|_h \le Ckt_n^{-(1-q/2)}\|A_h^{q/2}R_h v\|_h \le Ckt^{-(1-q/2)}|v|_q,\ q=1,2,$$

where the last inequality follows from $\|A_h^{1/2}R_h v\|_h = \|\nabla R_h v\| \le |v|_1$ and

$$\|A_h R_h v\|_h^2 = (\nabla R_h v, \nabla A_h R_h v) = (\nabla v, \nabla A_h R_h v) = -(\Delta v, A_h R_h v),$$

for $q=1,2$, respectively. This completes the proof of (8.6).

We turn now to (8.7). Estimating η_n by Theorem 4.2, it remains to bound β_n as stated. Employing Lemma 8.1, we have

$$\|\beta_n\|_h = \|U^n - \bar u_h(t_n)\|_h \le Ckt_n^{-1}\|P_h v\|_h \le Ckt_n^{-1}\|v\|. \qquad \square$$

For the gradient of the error we have the following smooth and nonsmooth data error estimates, without additional assumptions on the triangulations.

Theorem 8.2. *Let U be the solution of* (8.1)*, and u that of* (1.1)*. Then*

$$\|\nabla(U^n - u(t_n))\| \le \begin{cases} C(h+k)|v|_3, & \textit{if}\ \ v_h = R_h v,\\ C(h\,t_n^{-1} + k\,t_n^{-3/2})\|v\|, & \textit{if}\ \ v_h = P_h v. \end{cases}$$

Proof. The estimates needed for η_n are contained in Theorems 3.1 and 4.4. To bound β_n in the smooth data case, we first show the error bound for $v_h = \widetilde v_h = -A_h^{-1}R_h\Delta v$. Since $\|\nabla\chi\| = \|A_h^{1/2}\chi\|_h$ for $\chi\in S_h$, we then have, by Lemma 8.1,

$$\|\nabla\beta_n\| \le Ck\|A_h^{3/2}\widetilde v_h\|_h \le Ck\|A_h^{1/2}R_h\Delta v\|_h = Ck\|\nabla(R_h\Delta v)\| \le Ck|v|_3. \tag{8.9}$$

In order to complete the proof it suffices to show

$$\|\nabla(R_h v - \widetilde v_h)\| \le Ch|v|_3. \tag{8.10}$$

In fact, setting $E_{kh} = (I+kA_h)^{-1}$ we have

$$\|\nabla E_{kh}^n\chi\| = \|A_h^{1/2}E_{kh}^n\chi\|_h \le \|A_h^{1/2}\chi\|_h = \|\nabla\chi\|,$$

and hence

$$\|\nabla E_{kh}^n(v_h - \widetilde v_h)\| \le \|\nabla(v_h - v)\| + \|\nabla(v - R_h v)\| + \|\nabla(R_h v - \widetilde v_h)\| \le Ch|v|_3.$$

The estimate (8.10) follows from

$$\begin{aligned}(\nabla(\widetilde v_h - R_h v), \nabla\chi) &= -(R_h\Delta v,\chi)_h + (\Delta v,\chi) = -\epsilon_h(R_h\Delta v,\chi) - ((R_h - I)\Delta v,\chi)\\ &\le Ch\|R_h\Delta v\|\,\|\nabla\chi\| + Ch\|\nabla\Delta v\|\,\|\chi\| \le Ch|v|_3\,\|\nabla\chi\|, \quad \text{for } \chi\in S_h.\end{aligned}$$

To bound β_n for nonsmooth data, we use Lemma 8.1 with $p=1,\ q=-1$ to find

$$\|\nabla\beta_n\| \le Ckt_n^{-3/2}\|P_h v\|_h \le Ckt_n^{-3/2}\|v\|.$$

This completes the proof of the theorem. $\square$

We now turn to the Crank-Nicolson method, defined by

$$\bar\partial U^n + A_h U^{n-\frac12} = 0, \text{ for } n \ge 1, \text{ with } U^0 = v_h, \quad U^{n-\frac12} = \tfrac12 (U^n + U^{n-1}). \tag{8.11}$$

This method does not have as advantageous smoothing properties as the backward Euler method, which is reflected in the following counterpart of Lemma 8.1.

Lemma 8.2. *Let A and $\mathbf{u}$ be as in Lemma 8.1 and let $\mathbf{U} = \{\mathbf{U}^n\}_{n=0}^\infty$ satisfy*

$$\bar\partial \mathbf{U}^n + A\mathbf{U}^{n-\frac12} = 0, \quad \textit{for } n \ge 1, \quad \textit{with } \mathbf{U}^0 = \mathbf{v}.$$

Then, for $p = 0, 1,\ q = 1, 2$, we have

$$\|A^{p/2}(\mathbf{U}^n - \mathbf{u}(t_n))\| \le Ck^2 t_n^{-(2-q)} \|A^{p/2+q}\mathbf{v}\|, \quad \textit{for } n \ge 1. \tag{8.12}$$

Proof. Here, as in the proof of Lemma 8.1, employing eigenvalue expansions, it suffices to show, for $F_n(\lambda) = r^n(\lambda) - e^{-n\lambda}$, with $r(\lambda) = (1 - \frac12\lambda)/(1 + \frac12\lambda)$, that

$$\lambda^{-q}|F_n(\lambda)| \le C\, n^{-(2-q)}, \quad \text{for } \lambda > 0,\ n \ge 1,\ q = 1, 2.$$

For $0 \le \lambda \le 1$ we have $|r(\lambda)| \le e^{-c\lambda}$, with $c > 0$, and $|r(\lambda) - e^{-\lambda}| \le C\lambda^3$, so that

$$\lambda^{-q}|F_n(\lambda)| \le C\lambda^{3-q} n\, e^{-cn\lambda} \le Cn^{-(2-q)}, \quad \text{for } 0 < \lambda \le 1,\ n \ge 1,\ q = 1, 2.$$

For $\lambda \ge 1$ we have $|r(\lambda)| \le e^{-c/\lambda}$, with $c > 0$, and hence

$$\lambda^{-q}|F_n(\lambda)| \le \lambda^{-q} e^{-cn/\lambda} + \lambda^{-q} e^{-n\lambda} \le C\, n^{-(2-q)}, \quad \text{for } n \ge 1,\ q = 1, 2. \qquad \square$$

We now show optimal order error estimates, where this time we need to require $v \in \dot H^4$ for the error bound to hold uniformly down to $t = 0$. Because of the limited smoothing in the Crank-Nicolson method, no error bound is given for v only in L_2.

Theorem 8.3. *Let U be the solution of* (8.11)*, and u that of* (1.1)*. Then, with $q = 1, 2$, we have*

$$\|U^n - u(t_n)\| \le C(h^2 + k^2 t_n^{-(2-q)})|v|_{2q}, \quad \textit{for } n \ge 1, \quad \textit{if } v_h = R_h v.$$

Proof. With our new U^n we may again split the error as in (8.8), and by Theorem 3.1 η_n is bounded as desired. To bound β_n, it suffices, using the stability of $E_{kh}^n = r^n(k\, A_h)$, with $r(k\, A_h) = (1 - \frac12 k\, A_h)(1 + \frac12 k\, A_h)^{-1}$, and Lemma 8.2, to find $\widetilde v_h$ such that

$$\|\widetilde v_h - R_h v\| \le Ch^2|v|_{2q} \quad \text{and} \quad \|A_h^q \widetilde v_h\|_h \le C\,|v|_{2q}. \tag{8.13}$$

For $q = 2$ we may choose $\widetilde v_h = -A_h^{-1} R_h \Delta v$, because $\|A_h^2 \widetilde v_h\|_h \le C|v|_4$ and

$$\|\widetilde v_h - R_h v\| \le C\|\nabla(\widetilde v_h - R_h v)\| \le Ch^2|v|_4,$$

which latter inequality follows, using our definitions and Lemma 2.3, from

$$\begin{aligned}(\nabla(\widetilde v_h - R_h v), \nabla\chi) &= -(R_h\Delta v, \chi)_h + (\Delta v, \chi) = -\epsilon_h(R_h\Delta v, \chi) - ((R_h - I)\Delta v, \chi)\\ &\le Ch^2\|\nabla R_h \Delta v\|\,\|\nabla\chi\| + Ch^2|\Delta v|_2\,\|\chi\| \le Ch^2|v|_4\,\|\nabla\chi\|, \quad \text{for } \chi \in S_h.\end{aligned}$$

For $q = 1$, (8.13) is obviously satisfied with $\widetilde v_h = R_h v$, completing the proof. $\square$

We now show corresponding error bounds for the gradient of the error.

Theorem 8.4. *Let U be the solution of* (8.11)*, and u that of* (1.1)*. Then, for $q = 1, 2$, we have*

$$\|\nabla(U^n - u(t_n))\| \le C(h + k^2 t_n^{-(2-q)})|v|_{2q+1}, \quad \textit{for } n \ge 1, \quad \textit{if } v_h = R_h v.$$

Proof. Again, by Theorem 3.1, η_n is bounded as desired. To estimate β_n, we now want to find $\widetilde{v}_h$ such that

$$\|\nabla(\widetilde{v}_h - R_h v)\| \le Ch|v|_{2q+1} \quad \text{and} \quad \|A_h^{1/2+q}\widetilde{v}_h\|_h \le C|v|_{2q+1}. \tag{8.14}$$

For $q = 2$ we choose $\widetilde{v}_h = A_h^{-2} R_h \Delta^2 v$, and obtain

$$\|A_h^{5/2}\widetilde{v}_h\|_h = \|A_h^{1/2} R_h \Delta^2 v\|_h = \|\nabla(R_h \Delta^2 v)\| \le C|v|_5,$$

and the first part of (8.14) follows, using Lemma 1.6, with $A_h\psi = \chi$, from

$$\begin{aligned}(\nabla(\widetilde{v}_h - R_h v), \nabla\chi) &= (R_h\Delta^2 v, \psi)_h - ((R_h - I)\Delta v, \chi) - \epsilon_h(R_h\Delta v, \chi) - (\Delta^2 v, \psi) \\ &= \epsilon_h(R_h\Delta^2 v, \psi) - ((R_h - I)\Delta v, \chi) - \epsilon_h(R_h\Delta v, \chi) \\ &\quad + ((R_h - I)\Delta^2 v, \psi) \le Ch|v|_5\|\nabla\chi\|, \quad \text{for } \chi \in S_h.\end{aligned}$$

For $q = 1$ we take, as in the proof of Theorem 8.2, $\widetilde{v}_h = -A_h^{-1} R_h \Delta v$, recalling from (8.9) and (8.10) that (8.14) then holds. □

In order to produce optimal order convergence for initial data only in L_2, assuming Q_h appropriate, one may modify the Crank-Nicolson scheme by taking the first two steps by the backward Euler method, which has a smoothing effect, to obtain the following result. The proof is analogous to those of Theorems 8.1 and 8.3, and uses the appropriate combination of Lemmas 8.1 and 8.2, cf. [7], Theorem 7.4.

Theorem 8.5. *Let U^n be the defined by* (8.1) *for $n = 1, 2$, and by* (8.11) *for $n \ge 3$, and let u be the solution of* (1.1)*. Then, if Q_h satisfies* (1.10)*, we have*

$$\|U^n - u(t_n)\| \le C(h^2 t_n^{-1} + k^2 t_n^{-2})\|v\|, \quad \text{if } v_h = P_h v, \quad \text{for } n \ge 1.$$

We remark that if the mesh ratio condition $k \le Ch^2$ and the inverse assumption $\|\nabla\chi\| \le Ch^{-1}\|\chi\|$, for $\chi \in S_h$ hold, then the use of the two preliminary backward Euler steps above is not needed, and also, since $k^2 t_n^{-2} \le k t_n^{-1}$, the error bound may be written as $\|U^n - u(t_n)\| \le Ch^2 t_n^{-1}\|v\|$. In fact, under these assumptions, the spectrum of kA_h is bounded above and one easily shows that (8.12) holds also with $q = 0$, which implies our claim. Similarly, if instead $k \le Ch^{5/3}$, then the spectrum of kA_h is bounded above by $Ch^{-1/3}$, and one easily finds that one backward Euler step suffices to show $\|U^n - u(t_n)\| \le Ch^2 t_n^{-3/2}\|v\|$.

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