SOME NEW ERROR ESTIMATES OF A SEMIDISCRETE FINITE VOLUME ELEMENT METHOD FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATION WITH NONSMOOTH INITIAL DATA

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Abstract

A semidiscrete finite volume element(FVE) approximation to parabolic integro-differential equation(PIDE) is analyzed in a two-dimensional convex polygonal domain. Optimal order L^2-error estimates are derived for both smooth and nonsmooth initial data. More precisely, for homogeneous equations, an elementary energy technique and duality argument is used to derive optimal L^2-error estimate of order $O\left(t^{-1}h^2\right)$ for positive time when the given initial function is only in L^2.

Key words. Parabolic equation, integro-differential equation, optimal-order error estimate, smooth and nonsmooth initial data.

AMS Subject Classifications. 65M12, 65M60, 65N40.

1 Introduction

The aim of this paper is to analyze a semidiscrete FVE method for solving initial-boundary value problems for an integro-differential equation of the form

$$\begin{aligned} u_t - \nabla \cdot (\mathcal{A}\nabla u) &= -\int_0^t \nabla \cdot (\mathcal{B}\nabla u(s))ds + f(x,t) \text{ in } \Omega \times J, \\ u &= 0 \text{ on } \partial\Omega \times J, \\ u(\cdot,0) &= u_0 \text{ in } \Omega. \end{aligned} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain with boundary $\partial\Omega$, $J = (0,T]$ with $T < \infty$ and $u_t = \partial u/\partial t$. Further, $\mathcal{A} = \{a_{i,j}(x)\}$ is a symmetric and uniformly positive definite matrix of size 2×2 in Ω and $\mathcal{B} = \{b_{i,j}(x,t,s)\}$ is a 2×2 matrix. The nonhomogeneous term $f = f(x,t)$ and the coefficients $a_{ij}(x)$, $b_{ij}(x;t,s)$ are assumed to be smooth for our purpose. For the sake of simplicity, we shall denote $Au = -\nabla \cdot (\mathcal{A}\nabla u)$ and $B(t,s)u(s) = -\nabla \cdot (\mathcal{B}\nabla u(s))$.

PIDE of the above type arise naturally in many applications, such as, for instance, in heat conduction in materials with memory ([25]), nonlocal reactive flows in porous media ([9]-[10]) and non-Fickian flow of fluid in porous media ([14]). One very important

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characteristic of these models is that they all express the conservation of a certain quantity (mass, momentum, heat, etc.) in any moment for any subdomain. This in many application is the most desirable feature of the approximation method when it comes to numerical solution of the corresponding initial-boundary value problem. For references to studies of existence, uniqueness and regularity of such problems, one may refer to [31].

To put our work into proper perspective, we first give a brief account of the development of the finite element methods for such problems. Over the last decade, various numerical methods based on finite element approximations in space and special quadrature in time have been developed and studied for this type of problems (see, e.g. [18],[22],[23],[27],[29], [30] and [32]). The crucial tools used in the analysis are the Ritz and Ritz-Volterra projection which are instrumental in deriving optimal-order error estimates in various Sobolev norms. In [29], the authors have studied this type of problem for both smooth and nonsmooth initial data cases. In particular, for a homogeneous equation with nonsmooth initial data, an optimal-order L^2-error estimate is proved via a semigroup theoretic approach. Subsequently, using energy method the authors of [23] have derived convergence of order $O(\frac{h^2}{t})$ for L^2-norm and $O\left(\frac{h^2}{t}\log(\frac{1}{h})\right)$ for L^∞-norm for the homogeneous equation, when the initial function is in $H_0^1(\Omega)\cap H^2(\Omega)$. Recently, in [24], the authors have carried over the analysis of the case $(B(t,s)=0)$ in [19] to a time dependent PIDE. They have proved optimal-order error estimates by energy techniques and a duality argument for the homogeneous equation with both smooth and nonsmooth initial data. In both papers, [19] and [24], negative norm estimates are used in a crucial way in their analysis. In the absence of the memory term i.e., when $B(t,s)\equiv 0$, the error estimates for finite element methods for both smooth and nonsmooth data cases are described in [2], [17], [26] [28] and the references cited therein.

In the recent years, the numerical methods for problem (1.1) by means of FVE discretizations were considered in [12] and [13]. The interest in such methods is due to certain conservation features of FVE methods that are desirable in many applications. In [12] and [13], the authors have studied FVE approximation of such problem in the framework of the standard Petrov-Galerkin formulation and have obtained L^2-error estimate of the form (cf. p. 305 in [13])

$$\begin{aligned}\|u(t)-u_h(t)\| \quad &\leq \quad Ch^2(\|u_0\|_{3,p}+\|u(t)\|_{3,p}\\ &\qquad +\int_0^t(\|u(s)\|_{3,p}+\|u_t(s)\|_{3,p})ds), \quad p>1, \end{aligned} \tag{1.2}$$

where u and u_h represent the solution of (1.1) and its FVE approximation, respectively. Note that the estimate (1.2) is optimal with respect to the approximation property, but its regularity requirement on the exact solution seems to be too high when compared with that for finite element methods. This is primarily due to the fact that the bounds in the L^2-norm of a new variant of the Ritz-Volterra projection (*so called* Petrov-Volterra projection introduced in [12], [13]) are not optimal with respect to the regularity of the solution.

In this paper, we analyze FVE method for the problem (1.1) and derive optimal-order L^2-error estimates for both smooth and nonsmooth initial data. For the homogeneous problem with smooth initial data, we are able to show an L^2 error estimate which is optimal with respect to the order of convergence as well as the regularity of the solution. This is exactly the result known for finite element methods (cf. [23]). More precisely we prove an optimal order L^2-error estimate for $f=0$ and initial data $u_0\in H^2(\Omega)\cap H_0^1(\Omega)$. This technique, quite new and promising, is based on an improved estimates for a new variant of the Ritz-Volterra projection (see, Theorems 3.1-3.2).

The main concern of this paper is to prove an optimal L^2 error estimate for homogeneous equation ($f = 0$) with nonsmooth initial data. This is motivated by the fact that the solutions of a homogeneous linear parabolic equation have the so-called *smoothing property*. That is, the solution is sufficiently smooth for positive time t, even when the initial data are not. In quantitative form, this may be expressed by the inequality

$$\|u(t)\|_\alpha \leq Ct^{-\alpha/2}\|u_0\|, \quad t \in J \tag{1.3}$$

which is valid for some $\alpha \geq 0$. Here $\|\cdot\|_\alpha$ is a Sobolev norm. However, this is not the case with parabolic integro-differential equations as they have a limited smoothing property. This fact is proved in [29], where the inequality (1.3) is shown to be valid only for $\alpha \leq 2$. Since the smoothing property plays a significant role in the error analysis in the semidiscrete solution, an attempt has been made in this paper to achieve optimal order of convergence in L^2-norm for the FVE method when the initial data u_0 is only in $L^2(\Omega)$. More important, our analysis uses only energy techniques and a duality argument.

The proposed techniques have several attractive features. Unlike the analysis of [19] and [24], we don't require error estimates in negative-indexed Sobolev norms while dealing with L^2-error estimates with nonsmooth initial data. Thus, these results hold for convex polygonal domains with corners, unlike [19] and [24]. Since FVE method is thought of as a perturbation of the Galerkin finite element method, the proposed technique can easily be adopted to finite element method as well. Thus, we can reproduce some known results in the finite element method for parabolic problems. However, to the best our knowledge error estimate for nonsmooth initial data using FVE method have not been established earlier.

The previous work on the theoretical framework and the basic tools for the analysis of the finite volume element methods for elliptic and parabolic problems are described in [3],[4], [8], [6], [7], [11], [15], [16], [20], [21] and references therein.

The outline of this paper is as follows. In Section 2, we introduce some notations, formulate FVE approximations in piecewise linear finite element spaces defined on a triangulation and recall some basic estimates from the literature. Moreover, the Ritz-Volterra projection is introduced and related estimates are obtained in Section 3. Section 4 is devoted to the error estimates for smooth initial data. Finally, error estimates with nonsmooth initial data are carried out in Section 5.

Throughout this paper C denotes a generic positive constant which does not depend on the mesh parameter h but may depend on T.

2 Notations and preliminaries.

Let $H_0^1(\Omega) = \{\phi \in H^1(\Omega) \mid \phi = 0 \text{ on } \partial\Omega\}$. Further, let $A(\cdot,\cdot)$ and $B(t,s;\cdot,\cdot)$ be the bilinear forms on $H_0^1(\Omega) \times H_0^1(\Omega)$ given by

$$A(u,v) = \int_\Omega \mathcal{A}(x)\nabla u \cdot \nabla v dx; \quad B(t,s;u(s),v) = \int_\Omega \mathcal{B}(x,t,s)\nabla u(s) \cdot \nabla v dx. \tag{2.4}$$

For the purpose of finite volume element approximations we now consider the following weak formulation: Find $u : \bar{J} \to H_0^1(\Omega)$ such that

$$(u_t, v) + A(u,v) = \int_0^t B(t,s;u(s),v)ds + (f,v) \quad \forall v \in H_0^1(\Omega), \quad t \in J, \tag{2.5}$$

with $u(0) = u_0$.

Here and below, we denote $(\cdot,\cdot)$ and $\|\cdot\|$ by L^2 inner product and the induced norm on $L^2(\Omega)$. Further, we shall use the standard notation for Sobolev spaces $W^{m,p}(\Omega)$ with $1 \le p \le \infty$. The norm on $W^{m,p}(\Omega)$ is defined by

$$\|u\|_{m,p,\Omega} = \|u\|_{m,p} = \left(\int_\Omega \sum_{|\alpha|\le m} |D^\alpha u|^p dx \right), \quad 1 \le p < \infty$$

with the standard modification for $p = \infty$. When $p = 2$, we write $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and denote the norm by $\|\cdot\|_m$. Further, $H^{-1}(\Omega)$ denotes the space of all bounded linear functionals on $H_0^1(\Omega)$. For a functional $f \in H^{-1}(\Omega)$, its action on a function $u \in H_0^1(\Omega)$ is denoted by (f, u), which represents the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. To avoid confusion, we use $(\cdot,\cdot)$ to denote both the $L^2(\Omega)$-inner product and the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

2.1 A priori estimates

In the following lemmas, we state some a priori bounds for the solution u satisfying (1.1) under appropriate regularity assumption on the initial function u_0. For a proof, one may refer to [23], [24] and [19].

Lemma 2.1 *Let u satisfy (1.1). If $u_0 \in L^2(\Omega)$ and $f \in L^2(\Omega)$ then*

$$\|u(t)\|^2 + \int_0^t \|u(s)\|_1^2 ds \le C \left(\|u_0\|^2 + \int_0^t \|f(s)\|^2 ds \right).$$

Moreover, when $u_0 \in H_0^1(\Omega)$ and $f \in L^2(\Omega)$, we have

$$\|u(t)\|_1^2 + \int_0^t \{\|u_s(s)\|^2 + \|u(s)\|_2^2\} ds \le C \left(\|u_0\|_1^2 + \int_0^t \|f(s)\|^2 ds \right).$$

Lemma 2.2 *Let u satisfy (1.1). If $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $f \in L^2(\Omega)$ then*

$$\|u_t(t)\|^2 + \int_0^t \|u_s(s)\|_1^2 ds \le C \left(\|u_t(0)\|^2 + \int_0^t \|f(s)\|^2 ds \right).$$

Lemma 2.3 *Let u satisfy (1.1) with $f = 0$, and let $0 \le i, j, k \le 2$.*
If $0 \le k + 2j - i \le 2$, then

$$t^i \left\| \frac{\partial^j u}{\partial t^j}(t) \right\|_k^2 \le C \|u_0\|_{k+2j-i}^2.$$

Further, if $0 \le k + 2j - i - 1 \le 2$, then

$$\int_0^t s^i \left\| \frac{\partial^j u}{\partial s^j}(s) \right\|_k^2 ds \le C \|u_0\|_{k+2j-i-1}^2.$$

2.2 Finite volume element approximation

Let T_h be a quasi-uniform triangulation of Ω such that $\bar{\Omega} = \cup_{K\in T_h} K$, where K is a closed triangle element. Let N_h be the set of all nodes or vertices of T_h, i.e.,

$$N_h = \{p \;:\; p \text{ is a vertex of element } K \in T_h \text{ and } p \in \bar{\Omega}\}.$$

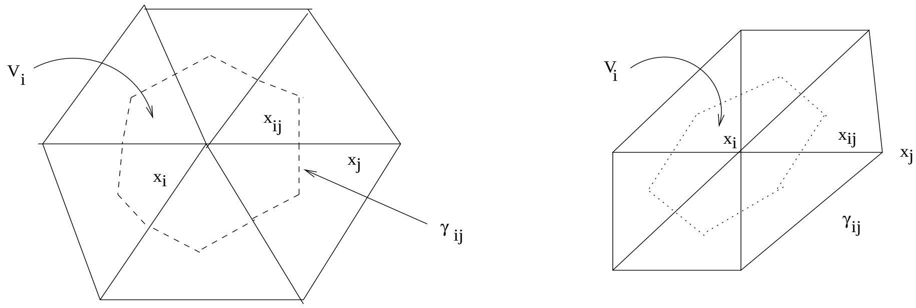

Figure 1: *Control volumes with barycenter as internal point and interface γ_{ij} of V_i and V_j.*

Further, we denote $N_h^0 = N_h \cap \Omega$. For a vertex $x_i \in N_h$, let $\Pi(i)$ be the index set of those vertices that, along with x_i, are in some element of T_h.

Based on the triangulation T_h, we now introduce a dual mesh T_h^* as follows: In each element $K \in T_h$ consisting of vertices x_i, x_j and x_k, select a point $q \in K$, select a point x_{ij} on the edge connection x_i and x_j and connect q with x_{ij} by straight lines $\gamma_{ij,K}$. Then for a vertex x_i we let V_i be the polygon whose edges are $\gamma_{ij,K}$ in which x_i is a vertex of the element K. We call this V_i a *control volume* centered at x_i. Further, we note that $\cup_{x_i \in N_h} V_i = \bar{\Omega}$. Thus, the dual mesh T_h^* is then defined as the collection of these *control volumes*. A *control volume* centered at a vertex x_i is given in Figure 1.

We call the control volume mesh T_h^* regular or quasi-uniform if there exists a positive constant $C > 0$ such that

$$C^{-1}h^2 \leq \text{meas}(V_i) \leq Ch^2 \quad \text{for all} \quad V_i \in T_h^*,$$

where h is the maximum diameter of all elements $K \in T_h$.

There are various ways to introduce a regular dual mesh T_h^* depending on the choices of the point q in an element $K \in T_h$ and the points x_{ij} on its edges. In this paper, we choose q to be the barycenter of an element $K \in T_h$, and the points x_{ij} are chosen to be the midpoints of the edges of K. In addition, if T_h is locally regular, i.e., there is a constant C such that

$$Ch_K^2 \leq \text{meas}(K) \leq h_K^2,$$

where $h_K = \text{diam}(K)$ for all elements $K \in T_h$. Then the dual mesh T_h^* is also locally regular. For the purpose of finite volume element approximation let S_h be the standard linear finite element space defined on the triangulation T_h,

$$S_h = \{v \in C(\Omega) \ : \ v|_K \ \text{is linear for all} \ K \in T_h \ \text{and} \ v|_{\partial\Omega} = 0\},$$

and its dual volume element space S_h^*,

$$S_h^* = \{v \in L^2(\Omega) \ : \ v|_V \ \text{is constant for all} \ V \in T_h^* \ \text{and} \ v|_{\partial\Omega} = 0\}.$$

Obviously, $S_h = \text{span}\{\phi_i(x) \ : \ x_i \in N_h^0\}$ and $S_h^* = \text{span}\{\psi_i(x) \ : \ x_i \in N_h^0\}$, where ϕ_i are the standard nodal basis functions associated with the node x_i, and ψ_i are the characteristic functions of the volume V_i. Let $I_h \ : \ C(\Omega) \to S_h$ and $I_h^* \ : \ C(\Omega) \to S_h^*$ be the usual interpolation operators, i.e.,

$$I_h u = \sum_{x_i \in N_h} u_i \phi_i(x) \quad \text{and} \quad I_h^* u = \sum_{x_i \in N_h} u_i \psi_i(x),$$

where $u_i = u(x_i)$.

The finite volume element approximation is then defined to be the function $u_h : \bar{J} \to S_h$ such that

$$(u_{ht}, I_h^*\chi) + A(u_h, I_h^*\chi) = \int_0^t B(t,s;u_h(s), I_h^*\chi)ds + \ (f, I_h^*\chi) \ \ \forall \chi \in S_h. \tag{2.6}$$

Here $u_h(0) = \tilde{P}_h u_0$, where $\tilde{P}_h u_0$ is the L^2-projection of u_0 onto S_h defined by

$$(\tilde{P}_h u_0, I_h^*\chi) = (u_0, I_h^*\chi), \quad \forall \chi \in S_h, \tag{2.7}$$

the bilinear forms $A(\cdot,\cdot)$ and $B(t,s;\cdot,\cdot)$ in (2.6) are defined by

$$A(u,v) = -\sum_{x_i \in N_h} v_i \int_{\partial V_i} \mathcal{A}(x)\nabla u \cdot \mathbf{n} dS_x$$
$$B(t,s;u,v) = -\sum_{x_i \in N_h} v_i \int_{\partial V_i} \mathcal{B}(x,t,s)\nabla u \cdot \mathbf{n} dS_x$$

for $(u,v) \in ((H_0^1 \cap H^2) \cup S_h) \times S_h^*$, and $\mathbf{n}$ is the outer-normal vector of the involved integration domain. Note that when $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ the bilinear forms $A(\cdot,\cdot)$ and $B(t,s;\cdot,\cdot)$ are given by (2.4).

In order to describe features of the bilinear forms defined in (2.5) and (2.6), we define some discrete norms on S_h and S_h^*,

$$|u_h|_{0,h}^2 = (u_h, u_h)_{0,h}, \ \ |u_h|_{1,h}^2 = \sum_{x_i \in N_h} \sum_{x_j \in \Pi(i)} \text{meas}(V_i)((u_{hi} - u_{hj})/d_{ij}^2,$$
$$\|u_h\|_{1,h}^2 = |u_h|_{0,h}^2 + |u_h|_{1,h}^2, \quad \||u_h\|| = (u_h, I_h^* u_h),$$

where $(u_h, v_h)_{0,h} = \sum_{x_i \in N_h} \text{meas}(V_i) u_{hi} v_{hi} = (I_h^* u_h, I_h^* v_h)$ and $d_{ij} = d(x_i, x_j)$ is the Euclidean distance between x_i and x_j.

The discrete norms $|\cdot|_{0,h}$ and $\|\cdot\|_{1,h}$ are equivalent to the usual norms $\|\cdot\|$ and $\|\cdot\|_1$, respectively on S_h. Some properties of the bilinear forms are stated below without proof. For a proof, see e.g [1], [11], and [13].

Lemma 2.4 *There exist positive constants C_1 and C_2 such that for all $v_h \in S_h$, we have*

$$C_1 |v_h|_{0,h} \le \|v_h\| \le C_2 |v_h|_{0,h},$$
$$C_1 \||v_h\|| \le \|v_h\| \le C_2 \||v_h\||,$$
$$C_1 \|v_h\|_{1,h} \le \|v_h\|_1 \le C_2 \|v_h\|_{1,h}.$$

Lemma 2.5 *There exist positive constant C such that, for all $\phi_h, \psi_h \in S_h$, we have*

$$|A(\phi_h, I_h^*\psi_h)| \ \le \ C\|\phi_h\|_1 \|\psi_h\|_1,$$
$$|B(t,s;\phi_h,\psi_h)| \ \le \ C\|\phi_h\|_1 \|\psi_h\|_1,$$

and

$$A(\phi_h, I_h^*\phi_h) \ge c\|\phi_h\|_1^2,$$

for some $c > 0$.

Lemma 2.6 *If the matrix $\mathcal{A}(x)$ is constant over each element $K \in T_h$ then we have*

$$A(u_h, \chi) = A(u_h, I_h^* \chi), \quad \forall u_h, \chi \in S_h.$$

Following the line of arguments of Lemma 2.3 on the discrete level, it is easy to derive the following stability estimates for FVE solution u_h satisfying (2.6).

Lemma 2.7 *Let u_h satisfy (2.6) with $f = 0$. Then we have*

$$\begin{aligned} \|u_h(t)\|^2 + \int_0^t \|u_h(s)\|_1^2 ds &\le C\|u_h(0)\|^2, \\ \int_0^t s\|u_{hs}(s)\|^2 + t\|u_h(t)\|_1^2 ds &\le C\|u_h(0)\|^2, \\ t^2\|u_{ht}(t)\|^2 + \int_0^t s^2\|u_{hs}(s)\|_1^2 ds &\le C\|u_h(0)\|^2. \end{aligned}$$

The following lemma gives the key feature of the bilinear forms in the finite volume element method. For a proof, see [11] or [6].

Lemma 2.8 *Let $\phi \in H_0^1(\Omega)$. Then we have*

$$\begin{aligned} A(\phi, \chi) &= A(\phi, I_h^* \chi) + \sum_{K \in T_h} \int_{\partial K} (A\nabla\phi \cdot \mathbf{n})(\chi - I_h^* \chi) dS \\ &\quad - \sum_{K \in T_h} \int_K (\nabla \cdot A\nabla\phi)(\chi - I_h^* \chi) dx, \quad \forall \chi \in S_h. \end{aligned}$$

The above identity holds true when $A(\cdot, \cdot)$ is replaced by $B(t, s; \cdot, \cdot)$.

Remark. We note that the above identity is proved in [11], [6] for $\phi, \chi \in S_h$. In fact, identities in Lemma 2.8 holds true even if $\phi \in H_0^1(\Omega)$.

3 Ritz-Volterra projection and related estimates

Following Lin *et al.* [18], we define the Ritz-Volterra projection $W_h : H_0^1(\Omega) \to S_h$ by

$$A(W_h u - u, I_h^* \chi) = \int_0^t B(t, s; (W_h u - u)(s), I_h^* \chi) ds, \ \forall \chi \in S_h. \tag{3.1}$$

Below, we shall prove a lemma which is of frequent use in our subsequent analysis.

Lemma 3.1 *For a function $\phi \in H^r(\Omega)$ (r=0,1), we have*

$$|(\phi, \chi - I_h^* \chi)| \le Ch^{1+r} \|\phi\|_r \|\chi\|_1 \ \forall \chi \in S_h. \tag{3.2}$$

Further, for $\phi \in H_0^1(\Omega)$, we have

$$|A(\phi, \chi - I_h^* \chi)| \le Ch\|\phi\|_1 \|\chi\|_1, \quad \forall \chi \in S_h. \tag{3.3}$$

The second inequality also holds true when $A(\cdot, \cdot)$ is replaced by $B(t, s; \cdot, \cdot)$.

Proof. We borrow the proof of (3.2) from [7]. To show (3.3), we have ([6])

$$\begin{aligned} A(\phi,\chi - I_h^*\chi) &= -\sum_{K\in T_h}\int_K (\nabla\cdot\mathcal{A}\nabla\phi)(\chi - I_h^*\chi)dx \\ &\quad + \sum_{K\in T_h}\int_{\partial K}((\mathcal{A}-\bar{\mathcal{A}}_K)\nabla\phi\cdot\mathbf{n})(\chi - I_h^*\chi)dS. \end{aligned} \tag{3.4}$$

Here, $\bar{\mathcal{A}}_K$ is a function designed in a piecewise manner such that for any edge E of a triangle $K \in T_h$ and $x \in E$, $\bar{\mathcal{A}}_K(x) = \mathcal{A}(x_c)$, where x_c is the middle point of E. Applying Cauchy-Schwartz inequality and using the fact that $\|\chi - I_h^*\chi\| \le Ch\|\chi\|_1$ and $|\mathcal{A}(x) - \bar{\mathcal{A}}_K| \le h\|\mathcal{A}\|_{1,\infty}$, we obtain

$$|A(\phi,\chi - I_h^*\chi)| \le Ch\|\phi\|_1\|\chi\|_1,$$

and this completes the proof. □

Set $\rho = W_hu - u$. We now establish H^1-error estimate for $W_hu - u$ and its temporal derivative.

Theorem 3.1 *Let W_hu be defined by (3.1). Then we have*

$$\|\rho(t)\|_1 \le Ch\left(\|u(t)\|_2 + \int_0^t \|u(s)\|_2 ds\right),$$

$$\|\rho_t(t)\|_1 \le Ch\left(\|u(t)\|_2 + \|u_t(t)\|_2 + \int_0^t \|u(s)\|_2 ds\right).$$

Proof. With $\phi_h = W_hu - I_hu$, we have

$$\begin{aligned} c\|\rho\|_1^2 &\le A(\rho,\rho) \\ &= A(\rho, I_hu - u) + A(\rho, W_hu - I_hu) \\ &= A(\rho, I_hu - u) + A(\rho,\phi_h - I_h^*\phi_h) + \int_0^t B(t,s;\rho(s), I_h^*\phi_h)ds. \end{aligned}$$

An application of (3.3) yields

$$c\|\rho\|_1^2 \le Ch(\|u\|_2 + \|u\|_1)\|\rho\|_1 + C\left(\int_0^t \|\rho\|_1 ds\right)(\|\rho\|_1 + h\|u\|_2),$$

where for the last term on the right we have used the fact that $\|\phi_h\|_1 \le C(\|\rho\|_1 + h\|u\|_2)$. Kicking back $\|\rho\|_1$ we get

$$\|\rho\|_1^2 \le C\left(h^2\|u\|_2^2 + \int_0^t \|\rho\|_1^2 ds\right).$$

Now applying Gronwall's lemma we obtain the first inequality. To estimate $\|\rho_t\|_1$, we differentiate (3.1) with respect to time t to get

$$A(\rho_t, I_h^*\chi) = B(t,t,\rho(t), I_h^*\chi) + \int_0^t B_t(t,s;\rho(s), I_h^*\chi)ds. \tag{3.5}$$

As before, with $\phi_h = W_hu_t - I_hu_t$ we obtain

$$\begin{aligned} c\|\rho_t\|_1^2 &\le A(\rho_t,\rho_t) \\ &= A(\rho_t, I_hu_t - u_t) + A(\rho_t,\phi_h - I_h^*\phi_h) + B(t,t,\rho(t), I_h^*\phi_h) \\ &\quad + \int_0^t B_t(t,s;\rho(s), I_h^*\phi_h)ds. \end{aligned}$$

Now apply (3.3), the estimate of $\|\rho\|_1$ and standard kickback argument to obtain the second inequality. $\square$

Next, we derive L^2 estimates for $\rho = W_h u - u$ and its temporal derivative in the following theorem.

Theorem 3.2 *Let $W_h u$ be defined by (3.1). Then we have*

$$\|\rho(t)\| \leq Ch^2 \left(\|u(t)\|_2 + \int_0^t \|u(s)\|_2 ds \right),$$
$$\|\rho_t(t)\| \leq Ch^2 \left(\|u(t)\|_2 + \|u_t(t)\|_2 + \int_0^t \|u(s)\|_2 ds \right).$$

Proof. The proof will proceed by duality argument. For $t \in (0, T)$ let $\psi(t) \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of

$$\begin{aligned} A\psi &= \rho \quad \text{in } \Omega, \\ \psi &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3.6}$$

satisfying the following regularity estimate (recall that Ω is convex)

$$\|\psi\|_2 \leq C\|\rho\|. \tag{3.7}$$

Multiplying (3.6) by ρ and then taking L^2 inner-product over Ω, we obtain

$$\begin{aligned} \|\rho\|^2 &= A(\rho, \psi - I_h\psi) + A(\rho, I_h\psi - I_h^*(I_h\psi)) \\ &+ \int_0^t B(t, s; \rho(s), I_h^*(I_h\psi) - I_h\psi)ds + \int_0^t B(t, s; \rho(s), I_h\psi - \psi)ds \\ &+ \int_0^t B(t, s; \rho(s), \psi)ds = I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

In view of Theorem 3.1, I_1 and I_4 are bounded as

$$|I_1| + |I_4| \leq Ch^2 \left(\|u\|_2 + \int_0^t \|u\|_2 ds \right) \|\psi\|_2.$$

For I_2 and I_3, an application of Lemma 3.3 and Theorem 3.1 yields

$$\begin{aligned} |I_2| + |I_3| &\leq Ch \left(\|\rho\|_1 + \int_0^t \|\rho\|_1 ds \right) \|\psi\|_1 \\ &\leq Ch^2 \left(\|u\|_2 + \int_0^t \|u\|_2 ds \right) \|\psi\|_1. \end{aligned}$$

Finally, I_5 is estimated as

$$|I_5| \leq |\int_0^t (\rho(s), B^*(t, s)\psi)|ds \leq C \left(\int_0^t \|\rho\| ds \right) \|\psi\|_2,$$

where $B^*(t, s)$ is the adjoint of $B(t, s)$. Now putting these estimates together and with an aid of (3.7) we obtain

$$\|\rho\| = Ch^2 \left(\|u\|_2 + \int_0^t \|u\|_2 ds \right) + C \int_0^t \|\rho\| ds.$$

Finally, an application of Gronwall's lemma yields the first estimate. To estimate $\|\rho_t\|$, we again use duality argument, (3.5) and the estimate of $\|\rho\|$ to complete the proof. $\square$

Remark. (i) The estimates in Theorem 3.2 are optimal with respect to the order of convergence as well as the regularity requirement on the solution. This improves upon the earlier result of [12] and [13] by requiring less regularity on the solution.

(ii) In the absence of integral term (when $B(t,s)=0$), as a consequence of Theorems 3.1 and 3.2, error estimates associated with the Petrov-Ritz projection $R_h : H_0^1 \to S_h$ defined by

$$A(R_h u - u, I_h^* \chi) = 0, \ \ \forall\, \chi \in S_h$$

can easily be obtained. Thus, we immediately have

$$\|R_h u - u\| + h\|R_h u - u\|_1 \le Ch^j \|u\|_j, \ \ u \in H_0^1(\Omega) \cap H^j(\Omega), \ \ j = 1, 2. \tag{3.8}$$

Below, we shall prove a lemma which is very crucial for the error estimate in the case of nonsmooth initial data to be discussed in section 5.

Define $\hat{\rho}(t) = \int_0^t \rho(\tau) d\tau$. Then, integrating by parts we rewrite (3.1) as

$$\begin{aligned} A(\hat{\rho}_t(t), I_h^*\chi) &= \int_0^t B(t,s;\hat{\rho}_s(s), I_h^*\chi) ds \\ &= B(t,t,\hat{\rho}, I_h^*\chi) - \int_0^t B_s(t,s;\hat{\rho}(s), I_h^*\chi) ds. \end{aligned}$$

Integrate from 0 to t to obtain

$$A(\hat{\rho}(t), I_h^*\chi) = \int_0^t B(s,s,\hat{\rho}(s), I_h^*\chi) ds - \int_0^t \int_0^s B_s(s,\tau;\hat{\rho}(\tau), I_h^*\chi) d\tau ds. \tag{3.9}$$

Lemma 3.2 *Let $\hat{\rho}$ satisfy (3.9). Then we have*

$$\|\hat{\rho}\| + h\|\hat{\rho}\|_1 \le Ch^2 \|u_0\|.$$

Proof. With $\phi_h = W_h \hat{u} - I_h \hat{u}$, we have

$$\begin{aligned} c\|\hat{\rho}\|_1^2 &\le A(\hat{\rho}, \hat{\rho}) \\ &= A(\hat{\rho}, I_h\hat{u} - \hat{u}) + A(\hat{\rho}, W_h\hat{u} - I_h\hat{u}) \\ &\le A(\rho, I_h\hat{u} - \hat{u}) + A(\hat{\rho}, \phi_h - I_h^*\phi_h) + \int_0^t B(s,s;\hat{\rho}(s), I_h^*\phi_h) ds \\ &- \int_0^t \int_0^s B_s(s,\tau;\hat{\rho}(\tau), I_h^*\phi_h) d\tau ds, \end{aligned}$$

where $\hat{u}(t) = \int_0^t u(s) ds$. Then proceeding as in the estimate of $\|\rho\|_1$ in Theorem 3.1 we obtain

$$\|\hat{\rho}\|_1 \le Ch \left(\|\hat{u}\|_2 + \int_0^t \|\hat{u}\|_2 ds \right). \tag{3.10}$$

Now it remains to estimate $\|\hat{u}\|_2$. From (1.1) with $f=0$, we have

$$\begin{aligned} Au &= -u_t + \int_0^t B(t,s)\hat{u}_s(s) ds \\ &= -u_t + B(t,t)\hat{u}(t) - \int_0^t B_s(t,s)\hat{u}(s) ds. \end{aligned}$$

Integrating from 0 to t and then using elliptic regularity and Lemma 2.1 we obtain

$$\|\hat{u}\|_2 \le \|u_0\| + \|u(t)\| + C\int_0^t \|\hat{u}\|_2 ds \le C\|u_0\| + C\int_0^t \|\hat{u}\|_2 ds.$$

Now application of Gronwall's lemma yields

$$\|\hat{u}\|_2 \le C\|u_0\|. \tag{3.11}$$

Combine (3.10) and (3.11) to obtain $\|\hat{\rho}\|_1$. Next, using (3.9), the proof technique of $\|\rho\|$ in Theorem 3.2 and (3.11), the estimate of $\|\hat{\rho}\|$ can be easily obtained. This completes the rest of the proof. $\square$

4 Error estimates for problems with smooth initial data

In this section, we estimate the error of the semidiscrete finite volume element method for problems with smooth initial data. In particular, optimal-order L^2-error estimate is obtained when $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$.

As usual we split the error $e(t) = u(t) - u_h(t)$ as $e(t) = (W_h u - u_h) - (W_h u - u) = \theta - \rho$. The estimate of $\|\rho\|$ is already established, so it is enough to estimate $\|\theta\|$. Using (2.5), (2.6) and (3.1), it is easy to verify that θ satisfies an error equation of the form

$$(\theta_t, I_h^*\chi) + A(\theta, I_h^*\chi) = \int_0^t B(t,s;\theta(s), I_h^*\chi)ds - (\rho_t, I_h^*\chi), \quad \forall \chi \in S_h. \tag{4.12}$$

Further, integrating (2.5) and (2.6) from 0 to t and then using (3.9) and (2.7) we obtain an error equation in $\hat{\theta}$ as

$$\begin{aligned}(\hat{\theta}_t, I_h^*\chi) \quad + \quad & A(\hat{\theta}, I_h^*\chi) = \int_0^t B(s,s;\hat{\theta}(s), I_h^*\chi)ds \\ & - \int_0^t\int_0^s B_\tau(s,\tau;\hat{\theta}(\tau), I_h^*\chi)d\tau ds - (\rho, I_h^*\chi), \quad \chi \in S_h,\end{aligned} \tag{4.13}$$

where $\hat{\theta}(t) = \int_0^t \theta(s)ds$. Below, we shall prove a sequence of lemmas that will lead us to the desired result.

Lemma 4.1 *Let $\hat{\theta}$ satisfy (4.13) and $u_h(0) = \tilde{P}_h u_0$, where $\tilde{P}_h$ is defined by (2.7). Then there is a positive constant C such that*

$$\|\hat{\theta}(t)\|^2 + \int_0^t \|\hat{\theta}(s)\|_1^2 ds \le C\int_0^t \|\rho(s)\|^2 ds.$$

Proof. Choose $\chi = \hat{\theta}$ in (4.13) to have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\hat{\theta}, I_h^*\hat{\theta}) \quad + \quad & A(\hat{\theta}, I_h^*\hat{\theta}) = \int_0^t B(s,s;\hat{\theta}(s), I_h^*\hat{\theta}(t))ds \\ & - \int_0^t\int_0^s B_\tau(s,\tau;\hat{\theta}(\tau), I_h^*\hat{\theta})d\tau ds - (\rho, I_h^*\hat{\theta}).\end{aligned}$$

Integrating from 0 to t and using standard kickback argument yields

$$\|\hat{\theta}(t)\|^2 + \int_0^t \|\hat{\theta}(s)\|_1^2 ds \quad \le \quad C\int_0^t \|\rho(s)\|^2 ds + C\int_0^t\int_0^s \|\hat{\theta}(\tau)\|_1^2 d\tau ds.$$

Finally, apply Gronwall's lemma to complete the rest of the proof. $\square$

Lemma 4.2 *Let $\hat{\theta}$ satisfy (4.13) and $u_h(0) = \tilde{P}_h u_0$. Then there is a positive constant C such that*

$$\int_0^t \|\theta(s)\|^2 ds + \|\hat{\theta}(t)\|_1^2 \le C \int_0^t \|\rho(s)\|^2 ds.$$

Proof. Take $\chi = \theta$ in (4.13) and integrate from 0 to t to have

$$\begin{aligned}\int_0^t (\theta, I_h^*\theta)ds \quad + \quad \frac{1}{2}A(\hat{\theta}, I_h^*\hat{\theta}) &= \int_0^t \int_0^s B(\tau,\tau;\hat{\theta}(\tau), I_h^*\theta(s))d\tau ds \\ &- \int_0^t \int_0^s \int_0^\tau B'_\tau(\tau,\tau';\hat{\theta}(\tau'), I_h^*\theta(s))d\tau' d\tau ds - (\rho,\theta) = I_1 + I_2 + I_3.\end{aligned}$$

For I_1, we note that

$$\begin{aligned} I_1 &= \int_0^t \int_\tau^t B(\tau,\tau;\hat{\theta}(\tau), I_h^*\hat{\theta}_s(s))ds d\tau \\ &= \int_0^t B(\tau,\tau;\hat{\theta}(\tau), I_h^*\hat{\theta}(t))d\tau ds - \int_0^t B(\tau,\tau;\hat{\theta}(\tau), I_h^*\hat{\theta}(\tau))d\tau.\end{aligned}$$

Similarly, we rewrite the term I_2. Now use standard kickback argument to obtain

$$\int_0^t \|\theta(s)\|^2 ds + \|\hat{\theta}\|_1^2 \le C \int_0^t \|\hat{\theta}\|_1^2 ds + C \int_0^t \|\rho\|^2 ds$$

Finally, an application of Lemma 4.1 completes the rest of the proof. □

Lemma 4.3 *Let θ satisfy (4.12) and $u_h(0) = \tilde{P}_h u_0$. Then there is a positive constant C independent of h such that*

$$t\|\theta(t)\|^2 + \int_0^t s\|\theta(s)\|_1^2 ds \le C \int_0^t \{\|\rho(s)\|^2 + s^2\|\rho_s(s)\|^2\} ds.$$

Proof. Take $\chi = t\theta$ in (4.12) and integrate by parts to have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\{t(\theta, I_h^*\theta)\} + tA(\theta, I_h^*\theta) &= \frac{1}{2}(\theta, I_h^*\theta) + tB(t,t;\hat{\theta}(t), I_h^*\theta(t)) \\ &- \int_0^t tB_s(t,s;\hat{\theta}(s), I_h^*\theta(t))ds - t(\rho_t, I_h^*\theta).\end{aligned}$$

Integrating from 0 to t and applying standard kickback argument, we obtain

$$\begin{aligned} t\|\theta(t)\|^2 + c\int_0^t s\|\theta(s)\|_1^2 ds &\le C\int_0^t \|\hat{\theta}(s)\|_1^2 ds + C\int_0^t \int_0^s \|\hat{\theta}(\tau)\|_1^2 d\tau ds \\ &+ C\int_0^t \{\|\theta\|^2 + s^2\|\rho_s\|^2\} ds\end{aligned}$$

and then use Lemma 4.1 and Lemma 4.2 to complete the proof. □

The main result of this section is given in the following theorem.

Theorem 4.1 *Let u and u_h, respectively, satisfy (1.1) and (2.6) with $f \equiv 0$. Then for $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_h(0) = \tilde{P}_h u_0$, we have*

$$\|e(t)\| \le Ch^2\|u_0\|_2.$$

Proof. By triangle inequality, we write

$$t^{1/2}\|e(t)\| \le t^{1/2}\|\rho(t)\| + t^{1/2}\|\theta(t)\|.$$

By Theorem 3.2 and a priori estimates in Lemma 2.3, the first term on the right is bounded by

$$t^{1/2}\|\rho(t)\| \le Ch^2 t^{1/2}\|u_0\|_2.$$

For the second term, we use Lemma 4.3, Theorem 3.2 and a priori estimates in Lemma 2.3 to have

$$\begin{aligned} t^{1/2}\|\theta\| &\le C\left(\int_0^t \{\|\rho(s)\|^2 + s^2\|\rho_s(s)\|^2\} ds\right)^{1/2} \\ &\le Ch^2 t^{1/2}\|u_0\|_2. \end{aligned}$$

Altogether these estimates yield the desired result and this completes the proof. □

Remark. From the proof of Theorem 4.1, it is clear that we can choose $u_h(0)$ as the L^2 projection of u_0 into S_h defined by (2.7) instead of the elliptic projection $R_h u_0$ or Ritz-Volterra projection $W_h u_0$. Note that the result presented in Theorem 4.1 is optimal with respect to the approximation property as well as the regularity of the solution. Similar result for finite element methods is established in [29], [23] and [24].

5 Error estimates for nonsmooth initial data.

In this section we establish one of the the main results of the paper, namely an error estimate for problems with nonsmooth initial data. More precisely, an optimal-order L^2-error estimate is obtained when $u_0 \in L^2(\Omega)$.

The following lemma is useful in our subsequent analysis.

Lemma 5.1 *For all $\chi_1, \chi_2 \in S_h$, we have*

$$A(\chi_1, \chi_2 - I_h^*\chi_2)| \le Ch^2(\|\chi_1\|_1 + h^{-1/2}\|\hat{u} - \chi_1\|_1)\|\chi_2\|_1.$$

The above estimate also holds true when $A(\cdot,\cdot)$ is replaced by $B(t,s;\cdot,\cdot)$.

Proof. From (3.4), we have

$$\begin{aligned} A(\chi_1, \chi_2 - I_h^*\chi_2) &= -\sum_{K\in T_h}\int_K (\nabla\cdot\mathcal{A}\nabla\chi_1)(\chi_2 - I_h^*\chi_2)dx \\ &+ \sum_{K\in T_h}\int_{\partial K}((\mathcal{A} - \bar{\mathcal{A}}_K)\nabla\chi_1\cdot\mathbf{n})(\chi_2 - I_h^*\chi_2)dS = I_1 + I_2. \end{aligned}$$

Since the dual mesh is formed by the barycenters, we have

$$\int_K (\chi - I_h^*\chi)dx = 0 \quad \forall \chi \in T_h,$$

and hence, apply Cauchy-Schwarz's inequality to have

$$|I_1| = \left|\sum_{K\in T_h}\int_K \left(\nabla\cdot\mathcal{A}\nabla\chi_1 - (\nabla\cdot\mathcal{A}\nabla\chi_1)_K\right)(\chi_2 - I_h^*\chi_2)\,dx\right| \le Ch^2\|\chi_1\|_1\|\chi_2\|_1,$$

where $(\nabla \cdot \mathcal{A}\nabla\chi_1)_K = \frac{1}{\text{area}(K)}\int_K(\nabla \cdot A\nabla\chi_1)$ is the average value of $(\nabla \cdot \mathcal{A}\nabla\chi_1)$ on K. Since $\nabla\hat{u}\cdot\mathbf{n}$ is continuous across any edge $E \in T_h$, we may rewrite I_2 as

$$I_2 = \sum_{K\in T_h}\int_{\partial K}\left((\mathcal{A}-\bar{\mathcal{A}}_K)\nabla(\hat{u}-\chi_1)\cdot\mathbf{n}\right)(\chi_2 - I_h^*\chi_2)dS,$$

and hence using the fact that $|\mathcal{A}(x)-\bar{\mathcal{A}}_K| \le h\|\mathcal{A}\|_{1,\infty}$, Cauchy-Schwarz inequality and trace results, we obtain

$$|I_2| \le Ch\sum_{K\in T_h}\|\nabla(\hat{u}-\chi_1)\|_{L^2(\partial K)}\|\chi_2 - I_h^*\chi_2\|_{L^2(\partial K)} \le Ch^{3/2}\|\hat{u}-\chi_1\|_1\|\chi_2\|_1.$$

Combining these estimates we complete the proof. $\Box$

Below, we shall prove a sequence of lemmas which will be used to derive the error estimates for problems with nonsmooth initial data.

Lemma 5.2 *Let u and u_h be the solution of (1.1) and (2.6), respectively. Then for $u_0 = 0$, we have*

$$\int_0^t \|u(s)-u_h(s)\|_1^2 ds \le Ch^2\left(\|f(0)\|^2 + \int_0^t \|f\|^2 ds\right).$$

Proof. Set $\chi = R_h e$ in the error equation

$$(e_t, I_h^*\chi) + A(e, I_h^*\chi) = \int_0^t B(t,s;e(s),I_h^*\chi)ds \tag{5.1}$$

to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|||e|||^2 &+ A(e,e) = (e_t, u - I_h^*(R_hu)) + A(e, u - I_h^*(R_hu)) \\ &- \int_0^t B(t,s;e(s),I_h^*(R_hu) - I_h^*u_h)ds + (e_t, I_h^*u_h - u_h) \\ &+ A(e, I_h^*u_h - u_h) = I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned}$$

By (3.8), (3.2) and (3.3), we have

$$\begin{aligned}|I_1| + |I_2| &\le |(e_t, u - R_hu)| + |(e_t, R_hu - I_h^*(R_hu))| \\ &\quad + |A(e, u - R_hu)| + |A(e, R_hu - I_h^*(R_hu))| \\ &\le Ch^2(\|e_t\|\|u\|_2 + \|e_t\|_1\|u\|_1) + Ch\|e\|_1(\|u\|_2 + \|u\|_1).\end{aligned}$$

Again, in view of (3.2) and (3.3), I_4 and I_5 can be estimated as

$$|I_4| + |I_5| \le C(h^2\|e_t\|_1 + h\|e\|_1)\|u_h\|_1.$$

To estimate I_3, we first rewrite it as

$$\begin{aligned}I_3 &= \int_0^t B(t,s;e(s),I_h^*(R_hu) - R_hu)ds + \int_0^t B(t,s;e(s),R_hu - u)ds \\ &+ \int_0^t B(t,s;e(s),e)ds + \int_0^t B(t,s;e(s),u_h - I_h^*u_h)ds.\end{aligned}$$

Apply (3.3) and (3.8) to obtain

$$|I_3| \le Ch\left(\int_0^t \|e(s)\|_1 ds\right)(\|u\|_1 + \|u\|_2 + \|u_h\|_1) + C\left(\int_0^t \|e(s)\|_1 ds\right)\|e\|_1.$$

Combing these estimates, it now leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|e\|^2 &+ A(e,e) \le Ch^2\|e_t\|\|u\|_2 + Ch^2\|e_t\|_1(\|u\|_1 + \|u_h\|_1)\\ &+ Ch\|e\|_1\|u\|_1 + h\left(\int_0^t \|e\|_1 ds\right)(\|u_1\| + \|u\|_2 + \|u_h\|_1)\\ &+ C\left(\int_0^t \|e(s)\|_1 ds\right)\|e\|_1.\end{aligned}$$

Integrate from 0 to t, use the fact that $e(0) = 0$ and then apply standard kickback argument to obtain

$$\begin{aligned}\int_0^t \|e(s)\|_1^2 ds &\le Ch^2\left[\int_0^t (\|u_h\|_1^2 + \|u\|_2^2 + \|e_t\|^2 + \|e_t\|_1^2)ds\right]\\ &\quad + C\int_0^t\int_0^s \|e(\tau)\|_1^2 d\tau ds.\end{aligned}$$

The desired estimate now easily follows from Lemmas 2.1-2.2 and its discrete analogue, and Gronwall's lemma. $\square$

Define $\hat{e}(t) = \int_0^t e(s)ds$. Then, as a consequence of Lemma 4.2, Lemma 3.2 and a priori estimates we have

Lemma 5.3 *Assume that $u_0 \in L^2(\Omega)$. Then there is a positive constant C independent of h such that*

$$\|\hat{e}(t)\|_1 \le Ch\|u_0\|.$$

In order to obtain optimal L^2-error estimate with nonsmooth data, it is convenient to prove an estimate of $\|\hat{e}\|$. For this purpose, we now consider the following backward problems. For fixed time $t > 0$ and given any $\bar{f} \in L^2(\Omega)$, let $v(s) \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of the following backward problem

$$\begin{aligned} v_s - Av &= -\int_s^t B^*(\tau, s)v(\tau)d\tau + \bar{f}, \quad s \le t, \\ v(t) &= g,\end{aligned} \tag{5.2}$$

where $B^*(\tau, s)$ is the adjoint of $B(\tau, s)$.

The associated weak solution is then defined to be the function $v : [0, t) \to H_0^1(\Omega)$ such that

$$(\phi, v_s) - A(\phi, v) = -\int_s^t B(\tau, s; \phi, v(\tau))d\tau + (\phi, \bar{f}), \quad \forall\phi \in H_0^1(\Omega),\ s \le t \tag{5.3}$$

with $v(t) = g$. Further, its FVE approximation is defined to be the function $v_h : [0, t) \to S_h$ such that

$$(I_h^*\chi, v_{hs}) - A(I_h^*\chi, v_h) = -\int_s^t B(\tau, s; I_h^*\chi, v_h(\tau))d\tau + (I_h^*\chi, \bar{f}), \tag{5.4}$$

$\forall \chi \in S_h,\ s \leq t$ with $v_h(t) = g_h$, where g_h is a suitable approximation of g in S_h to be defined later.

Remark: With a simple change of variables in the proofs of Lemmas 2.1-2.3 and using backward Gronwall's Lemma, it is easy to obtain a priori bounds for the backward solutions v and v_h.

Lemma 5.4 *Assume that $u_0 \in L^2(\Omega)$ and $f \equiv 0$. Then there is a generic constant C independent of h such that the following estimate holds*

$$\|\hat{e}(t)\| \leq Ch^2 \|u_0\|, \quad \forall t > 0. \tag{5.5}$$

Proof. Let $w(s) \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of the backward problem (5.2) with $\bar{f} = \hat{e}$ and $g = 0$. Then, with a simple change of variables in the proofs of Lemmas 2.1-2.3 and its discrete analogue, Lemma 5.2 and using backward Gronwall's Lemma, it is an easy exercise to check that the solution $w(s)$ and its FVE solution $w_h(s)$, which may be stated in a manner similar to (5.3)-(5.4), satisfy the following estimate

$$\int_0^t \{\|w_s - w_{hs}\|^2 + \|w_s - w_{hs}\|_1^2 + h^{-2}\|w - w_h\|_1^2 + \|w\|_2^2\} ds \leq C \int_0^t \|\hat{e}\|^2 ds. \tag{5.6}$$

We take an L^2-inner product of (5.2) with e and use (5.1) to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}\|\hat{e}(s)\|^2 \quad = \quad & \frac{d}{ds}(e, I_h^* w_h) + (e, w_s - w_{hs}) - (e, w_{hs} - I_h^* w_{hs}) - A(e, w - w_h) \\
& - A(e, w_h - I_h^* w_h) + \int_s^t B(\tau, s; e(s), (w - w_h)(\tau)) d\tau \\
& + \int_s^t B(\tau, s; e(s), (w_h - I_h^* w_h)(\tau)) d\tau + \int_s^t B(\tau, s; e(s), I_h^* w_h(\tau)) d\tau \\
& - \int_0^s B(s, \tau; e(\tau), I_h^* w_h(s)) d\tau.
\end{aligned}$$

With $\eta = u - I_h^*(R_h u)$ and $\zeta = I_h^* u_h - u_h$, we rewrite the above equation as

$$\begin{aligned}
& \frac{1}{2}\frac{d}{ds}\|\hat{e}(s)\|^2 = \frac{d}{ds}(e, I_h^* w_h) + (\eta, w_s - w_{hs}) - A(\eta, w - w_h) \\
& + \int_s^t B(\tau, s; \eta(s), (w - w_h)(\tau)) d\tau + (e, w_{hs} - I_h^* w_{hs}) - A(e, w_h - I_h^* w_h) \\
& + \int_s^t B(\tau, s; e(s), w_h(\tau) - I_h^* w_h(\tau)) d\tau + (\zeta, w_s - w_{hs}) - A(\zeta, w - w_h) \\
& + \int_s^t B(\tau, s; \zeta(s), (w - w_h)(\tau)) d\tau.
\end{aligned}$$

Multiply both side by s and integrate from 0 to t to have

$$\begin{aligned}
\frac{1}{2}t\|\hat{e}(t)\|^2 &= \frac{1}{2}\int_0^t \|\hat{e}(s)\|^2 ds + \int_0^t (e, I_h^* w_h) ds + \int_0^t s(\eta, w_s - w_{hs}) ds \\
&- \int_0^t sA(\eta, w - w_h) ds + \int_0^t \int_s^t sB(\tau, s; \eta, (w - w_h))(\tau) d\tau ds \\
&+ \int_0^t s(e, w_{hs} - I_h^* w_{hs}) ds - \int_0^t sA(e, w_h - I_h^* w_h) ds \\
&- \int_0^t \int_0^s sB(s, \tau; e(\tau), I_h^* w_h(s) - w_h(s)) d\tau ds + \int_0^t s(\zeta, w_s - w_{hs}) ds \\
&- \int_0^t sA(\zeta, w - w_h) ds + \int_0^t \int_s^t sB(\tau, s; \zeta(s), (w - w_h)(\tau)) d\tau ds \\
&= \frac{1}{2}\int_0^t \|\hat{e}(s)\|^2 ds + \sum_{i=1}^{10} I_i.
\end{aligned}$$

Since $\hat{e}(0) = 0 = I_h^* w(t)$, we obtain using (5.6)

$$\begin{aligned}
|I_1| &= |-\int_0^t (\hat{e}, I_h^* w_{hs}) ds| \leq C\int_0^t \|\hat{e}\|\|w_{hs}\| ds \\
&\leq \left(\int_0^t \|\hat{e}\|^2 ds\right)^{1/2} \left(\int_0^t \|w_{hs}\|^2 ds\right)^{1/2} \\
&\leq C\int_0^t \|\hat{e}(s)\|^2 ds.
\end{aligned}$$

For I_2, an application of (3.2), (3.8), (5.6) and a priori estimates yields

$$\begin{aligned}
|I_2| &= |\int_0^t s(u - R_h u, w_s - w_{hs}) ds + \int_0^t s(R_h u - I_h^*(R_h u), w_s - w_{hs}) ds| \\
&\leq Ch^4 \int_0^t (s^2\|u\|_2^2 + s\|u\|_1^2) ds + C\int_0^t \|w_s - w_{hs}\|_1^2 ds \\
&\leq Ch^4 t\|u_0\|^2 + C\int_0^t \|\hat{e}\|^2 ds.
\end{aligned}$$

Similarly, for I_3 and I_4, using (3.8), (3.3) and (5.6) we obtain

$$\begin{aligned}
|I_3| + |I_4| &\leq C\int_0^t s\|u - R_h u\|_1 \|w - w_h\|_1 ds + h\int_0^t s\|R_h u - I_h^*(R_h u)\|_1 \|w - w_h\|_1 ds \\
&\leq Ch^4 \int_0^t (s^2\|u\|_2^2 + s^2\|u\|_1^2) ds + Ch^{-2}\int_0^t \|w - w_h\|_1^2 ds \\
&\leq Ch^4 t\|u_0\|^2 + C\int_0^t \|\hat{e}\|^2 ds.
\end{aligned}$$

Apply (3.2), (5.6) and a priori estimates to have

$$\begin{aligned}
|I_5| &= Ch^4 \int_0^t s\|e\|_1^2 ds + C\int_0^t \|w_{hs}\|_1^2 ds \\
&\leq Ch^4 t\|u_0\|^2 + C\int_0^t \|\hat{e}\|^2 ds.
\end{aligned}$$

For I_7, with a simple change of variables and integrating by parts, we note that

$$\int_0^t \int_s^t sB(\tau,s;e(s),(w_h - I_h^* w_h)(\tau))d\tau ds = \int_0^t \int_0^s \tau B(s,\tau;\hat{e}_\tau(\tau),(w_h - I_h^* w_h)(s))d\tau ds$$

$$= \int_0^t sB(s,s;\hat{e}(s),(w_h - I_h^* w_h)(s))ds - \int_0^t \int_0^s \tau B_\tau(s,\tau;\hat{e}(\tau),(w_h - I_h^* w_h)(s))d\tau ds$$

$$- \int_0^t \int_0^s B(s,\tau;\hat{e}(\tau),(w_h - I_h^* w_h)(s))d\tau ds.$$

Similarly, we rewrite the term I_6 as

$$I_6 = \int_0^t sA(\hat{e}, w_{hs} - I_h^* w_{hs})ds + \int_0^t A(\hat{e}, w_h - I_h^* w_h)ds,$$

where we have used the fact that $w_h(t) = 0 = \hat{e}(0)$. Thus, applying (3.3), Lemma 5.3 and (5.6), I_6 and I_7 are bounded by

$$\begin{aligned} |I_6| + |I_7| &\leq Cth^4\|u_0\|^2 + C\int_0^t (\|w_h\|_1^2 + \|w_{hs}\|_1^2)ds \\ &\leq Cth^4\|u_0\|^2 + C\int_0^t \|\hat{e}\|^2 ds. \end{aligned}$$

Finally, using (3.2), (3.3) and (5.6) we obtain

$$\begin{aligned} |I_8| + |I_9| + |I_{10}| &\leq Ch^4 \int_0^t s\|u_h\|_1^2 ds + C\int_0^t (\|w_s - w_{hs}\|_1^2 + h^{-2}\|w - w_h\|_1^2)ds \\ &\leq Ch^4 t\|u_0\|^2 + C\int_0^t \|\hat{e}\|^2 ds. \end{aligned}$$

Altogether now leads to

$$t\|\hat{e}(t)\| \leq Ch^4 t\|u_0\|^2 + C\int_0^t \|\hat{e}(s)\|^2 ds. \tag{5.7}$$

It now remains to estimate $\int_0^t \|\hat{e}\|^2 ds$. Multiply (5.2) by $\hat{e}$ and integrate by parts with respect to x to have

$$\begin{aligned} \|\hat{e}(s)\|^2 &= \frac{d}{ds}(\hat{e}, I_h^* w_h) + (\hat{e}, w_s - w_{hs}) - A(\hat{e}, w - w_h) \\ &+ \int_s^t B(\tau,s;\hat{e}(s),w(\tau) - w_h(\tau))d\tau + (\hat{e}, w_{hs} - I_h^* w_{hs}) - A(\hat{e}, w_h - I_h^* w_h) \\ &- \int_0^s B(\tau,\tau;\hat{e}(\tau), I_h^* w_h(s))d\tau + \int_0^s \int_0^\tau B_{\tau'}(\tau,\tau';\hat{e}(\tau'), I_h^* w_h(s))d\tau' d\tau \\ &+ \int_s^t B(\tau,s;\hat{e}(s), w_h(\tau))d\tau. \end{aligned} \tag{5.8}$$

Here, we have used the relation

$$\begin{aligned} (e, I_h^*\chi) + A(\hat{e}, I_h^*\chi) &= \int_0^t B(s,s;\hat{e}(s), I_h^*\chi)ds \\ &\quad - \int_0^t \int_0^s B_\tau(s,\tau;\hat{e}(\tau), I_h^*\chi)d\tau ds \end{aligned}$$

which is obtained by integrating (5.1) from 0 to t and using (2.7). Now integrate (5.8) from 0 to t and use the fact that $\hat{e}(0) = 0 = I_h^* w_h(t)$ to have

$$\begin{aligned}
\int_0^t \|\hat{e}(s)\|^2 ds &= \int_0^t (\hat{\eta}, w_s - w_{hs}) ds - \int_0^t A(\hat{\eta}, w - w_h) ds \\
&- \int_0^t \int_s^t B(\tau, s; \hat{\eta}(s), (w - w_h)(\tau)) d\tau ds - \int_0^t \int_0^s B(\tau, \tau; \hat{e}(\tau), I_h^* w_h(s)) d\tau ds \\
&+ \int_0^t \int_0^s \int_0^\tau B_{\tau'}(\tau, \tau'; \hat{e}(\tau'), I_h^* w_h(s)) d\tau' d\tau ds + \int_0^t (\hat{e}(s), w_{hs} - I_h^* w_{hs}) ds \\
&- \int_0^t A(\hat{e}, w_h - I_h^* w_h) ds + \int_0^t \int_s^t B(\tau, s; \hat{e}(s), w_h(\tau)) d\tau ds + \int_0^t (\hat{\zeta}, w_s - w_{hs}) ds \\
&- \int_0^t A(\hat{\zeta}, w - w_h) ds - \int_0^t \int_s^t B(\tau, s; \hat{\zeta}(s), (w - w_h)(\tau)) d\tau ds \\
&= \sum_{i=1}^{11} J_i,
\end{aligned}$$

where $\hat{\eta} = \hat{u} - I_h^*(R_h \hat{u})$ and $\hat{\zeta} = I_h^* \hat{u}_h - \hat{u}_h$. Let us estimate each term separately. For J_1, use of (3.8), (3.11), (5.1) and (5.6) yields

$$\begin{aligned}
|J_1| &\leq \int_0^t \{|(\hat{u} - R_h \hat{u}, w_s - w_{hs})| + |(R_h \hat{u} - I_h^*(R_h \hat{u}), w_s - w_{hs})|\} ds \\
&\leq C(\epsilon) h^4 \int_0^t \|\hat{u}\|_2^2 ds + \epsilon \int_0^t \|w_s - w_{hs}\|^2 ds \\
&\leq C(\epsilon) t h^4 \|u_0\|^2 + \epsilon C \int_0^t \|\hat{e}\|^2 ds.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|J_2| + |J_3| &\leq C(\epsilon) t h^4 \|u_0\|^2 + \epsilon h^{-2} \int_0^t \|w - w_h\|_1^2 ds \\
&\leq C(\epsilon) t h^4 \|u_0\|^2 + \epsilon C \int_0^t \|\hat{e}\|^2 ds.
\end{aligned}$$

For J_4, we note that

$$\begin{aligned}
J_4 &= -\int_0^t \int_0^s B(\tau, \tau; \hat{e}(\tau), I_h^* w_h(s) - w_h(s)) d\tau ds \\
&- \int_0^t \int_0^s B(\tau, \tau; \hat{e}(\tau), w_h(s) - w(s)) d\tau ds - \int_0^t \int_0^s B(\tau, \tau; \hat{e}(\tau), w(s)) d\tau ds,
\end{aligned}$$

and hence, using (3.3), Lemma 5.3 and (5.6) we obtain

$$\begin{aligned}
|J_4| &\leq C \int_0^t \int_0^s \{h \|\hat{e}(\tau)\|_1 (\|w_h(s)\|_1 + h^{-1} \|w(s) - w_h(s)\|_1) + \|\hat{e}(\tau)\| \|w(s)\|_2\} d\tau ds \\
&\leq C t h^4 \|u_0\|^2 + C \int_0^t \int_0^s \|\hat{e}(\tau)\|^2 d\tau ds.
\end{aligned}$$

Similarly,

$$|J_5| \leq C t h^4 \|u_0\|^2 + C \int_0^t \int_0^s \|\hat{e}(\tau)\|^2 d\tau ds.$$

Using (5.1)-(3.3), Lemma 5.3 and (5.6) we obtain

$$\begin{aligned}|J_6|+|J_7| &\le C(\epsilon)h^2\int_0^t \|\hat{e}\|_1^2 ds+\epsilon\int_0^t\{\|w_{hs}\|_1^2+\|w_h\|_1^2\}ds\\ &\le Cth^4\|u_0\|^2+\epsilon C\int_0^t\|\hat{e}\|^2ds.\end{aligned}$$

By changing the order of integration, rewrite the term J_8 as

$$J_8=\int_0^t\int_0^s B(\tau,s;\hat{e}(s),(w_h-w)(\tau))dsd\tau+\int_0^t\int_0^s B(\tau,s;\hat{e}(s),w(\tau))dsd\tau.$$

In view of Lemma 5.3 and (5.6), we obtain

$$|J_8| \le Cth^4\|u_0\|^2+C\int_0^t\int_0^s\|\hat{e}(\tau)\|^2d\tau ds.$$

Finally, using (3.2)-(3.3) and (5.6), we have

$$\begin{aligned}|J_9|+|J_{10}|+|J_{11}| &\le C(\epsilon)th^4\|u_0\|^2+\epsilon\int_0^t(\|w_s-w_{hs}\|_1^2+h^{-2}\|w-w_h\|_1^2)ds\\ &\le C(\epsilon)th^4\|u_0\|^2+\epsilon C\int_0^t\|\hat{e}(s)\|^2ds.\end{aligned}$$

Putting these estimate together and choosing ϵ appropriately, we arrive at

$$\int_0^t\|\hat{e}(s)\|^2ds\le Cth^4\|u_0\|^2+C\int_0^t\int_0^s\|\hat{e}(\tau)\|^2d\tau ds.$$

An application of Gronwall's lemma yields

$$\int_0^t\|\hat{e}\|^2ds\le Cth^4\|u_0\|^2,$$

and this combine with (5.7) completes the rest of the proof. □

Remark. Defining the error $\bar{e}=v-v_h$ associated with the backward problem (5.3) and its FVE approximation (5.4), set $\tilde{\bar{e}}(s)=-\int_s^t\bar{e}(\tau)d\tau,\ s\le t$. Then, for $g\in L^2(\Omega)$ and $\bar{f}=0$, analogous to Lemma 5.3 and Lemma 5.4, it is easy to show that

$$\|\tilde{\bar{e}}\|_j\le Ch^{2-j}\|g\|,\ \ j=0,1. \tag{5.9}$$

We conclude this section by showing our main result in the following theorem.

Theorem 5.1 *Let u and u_h be solutions of (1.1) and (1.4), respectively with $f=0$. Assume that $u_0\in L^2(\Omega)$ and the matrix $\mathcal{A}$ is constant over each element $K\in T_h$. Then there is a generic positive constant C independent of h such that*

$$\|e(t)\|\le Ct^{-1}h^2\|u_0\|,\quad t\in J.$$

Proof. Using (5.3) and (5.4) with $\bar{f} \equiv 0$ and Lemma 2.6, we first note that

$$
\begin{aligned}
&\frac{d}{ds}\left\{s^2[(u,v)-(u_h,v_h)]\right\} = 2s\left\{(u,v)-(u_h,v_h)\right\} \\
&+\int_0^s s^2 B(s,\tau;u(\tau),v(s))d\tau - \int_s^t s^2 B(\tau,s;u(s),v(\tau))d\tau \\
&-\int_0^s s^2 B(s,\tau;u_h(\tau),I_h^* v_h(s)-v_h(s))d\tau + \int_s^t s^2 B(\tau,s;I_h^* u_h(s)-u_h(s),v_h(\tau))d\tau \\
&-\int_0^s s^2 B(s,\tau;u_h(\tau),v_h(s))d\tau + \int_s^t s^2 B(\tau,s;u_h(s),v_h(\tau))d\tau \\
&-s^2(u_{hs},v_h-I_h^* v_h) - s^2(u_h-I_h^* u_h, v_{hs}).
\end{aligned}
$$

Integrate the above equation from 0 to t. Then, with $g_h = P_h g$, where P_h is the standard L^2-projection onto S_h defined by $(P_h g,\chi)=(g,\chi),\ \chi\in S_h$, we have

$$
\begin{aligned}
t^2(e(t),g) &= 2\int_0^t s\left\{(u(s),v(s))-(u_h(s),v_h(s))\right\}ds \\
&\quad -\int_0^t\int_0^s s^2 B(s,\tau;u_h(\tau),I_h^* v_h(s)-v_h(s))d\tau \\
&\quad +\int_0^t\int_s^t s^2 B(\tau,s;I_h^* u_h(s)-u_h(s),v_h(\tau))d\tau \\
&\quad -\int_0^t s^2(u_{hs},v_h-I_h^* v_h)ds - \int_0^t s^2(u_h-I_h^* u_h,v_{hs})ds \\
&= 2I_1+I_2+I_3+I_4+I_5. \qquad (5.10)
\end{aligned}
$$

Integrating by parts, we note that

$$
\begin{aligned}
I_2 &= -\int_0^t\int_0^s s^2 B(s,\tau;\hat{u}_{h\tau}(\tau),(I_h^* v_h - v_h)(s))d\tau ds \\
&= -\int_0^t s^2 B(s,s;\hat{u}_h(s),(I_h^* v_h-v_h)(s))ds + \int_0^t\int_0^s s^2 B_\tau(s,\tau;\hat{u}_h(\tau),(I_h^* v_h-v_h)(s))d\tau ds \\
&= I_2^1 + I_2^2.
\end{aligned}
$$

For I_2^1, apply Lemma 5.1 with $\chi_1=\hat{u}_h$ and $\chi_2 = v_h$, Lemma 5.3 and a priori estimate to obtain

$$
\begin{aligned}
|I_2^1| &\le Ch^2\int_0^t s^2\left(\|\hat{u}_h\|_1 + h^{-1/2}\|\hat{e}\|_1\right)\|v_h(s)\|_1 ds \\
&\le Cth^2\|u_0\|\|g\|.
\end{aligned}
$$

The term I_2^2 is treated in a similar manner and hence

$$|I_2| \le Cth^2\|u_0\|\|g\|.$$

Similarly, defining $\tilde{v}_h(s) = -\int_s^t v_h(\tau)d\tau,\ s\le t$, we rewrite the term I_3 as

$$
\begin{aligned}
I_3 &= \int_0^t\int_s^t s^2 B(\tau,s;I_h^* u_h(s)-u_h(s),\tilde{v}_{h,\tau}(\tau))d\tau ds \\
&= -\int_0^t s^2 B(s,s;I_h^* u_h(s)-u_h(s),\tilde{v}_h(s))ds - \int_0^t\int_s^t s^2 B_\tau(\tau,s;I_h^* u_h(s)-u_h(s),\tilde{v}_h(\tau))d\tau ds \\
&= I_3^1 + I_3^2.
\end{aligned}
$$

As before, again an application of Lemma 5.1(analogous result for the backward problem), (5.9) and a priori bounds for the discrete solution yields

$$\begin{aligned} |I_3^1| &\leq Ch^2 \int_0^t s^2 \left(\|v_h\|_1 + h^{-1/2} \|\tilde{\bar{e}}\|_1 \right) \|u_h(s)\|_1 ds \\ &\leq Cth^2 \|u_0\| \|g\|. \end{aligned}$$

I_3^2 is treated in a similar fashion and hence,

$$|I_3| \leq Cth^2 \|u_0\| \|g\|.$$

For I_4 and I_5, apply (3.2) and a priori estimates to have

$$\begin{aligned} |I_4| + |I_5| &\leq Ch^2 \left(\int_0^t s^2 \|v_h\|_1^2 ds \right)^{1/2} \left(\int_0^t s^2 \|u_{hs}\|_1^2 ds \right)^{1/2} \\ &\quad + Ch^2 \left(\int_0^t s^2 \|u_h\|_1^2 ds \right)^{1/2} \left(\int_0^t s^2 \|v_{hs}\|_1^2 ds \right)^{1/2} \\ &\leq Cth^2 \|u_0\| \|g\|. \end{aligned}$$

It now remains to estimate the term I_1. We first rewrite I_1 as

$$\begin{aligned} I_1 &= \int_0^t s(e(s), v) ds - \int_0^t s(e(s), \bar{e}(s)) ds + \int_0^t s(u, \bar{e}(s)) ds \\ &= I_1^1 + I_1^2 + I_1^3. \end{aligned}$$

For I_1^1, we integrate by parts to have

$$I_1^1 = \int_0^t s(\hat{e}_s, v) ds = t(\hat{e}, g) - \int_0^t (\hat{e}, v) ds - \int_0^t s(\hat{e}, v_s) ds,$$

and hence, by Cauchy-Schwarz's inequality, Lemma 5.4 and Lemma 2.3(with time reverse) we obtain

$$\begin{aligned} |I_1^1| &\leq t\|\hat{e}\| \|g\| + \int_0^t \|\hat{e}\| \|v\| ds + \int_0^t s\|\hat{e}\| \|v_s\| ds \\ &\leq Cth^2 \|u_0\| \|g\|. \end{aligned}$$

Since $\tilde{\bar{e}}(t) = 0$, using (5.9) and a priori estimates in Lemma 2.3, we estimate I_1^3 as

$$\begin{aligned} |I_1^3| &\leq \int_0^t \|\tilde{\bar{e}}\| \|u\| ds + \int_0^t s\|\tilde{\bar{e}}\| \|u_s\| ds \\ &\leq Cth^2 \|u_0\| \|g\|. \end{aligned}$$

Similarly, for I_1^2, integration by parts lead to

$$I_1^2 = -\int_0^t s(\hat{e}_s, \bar{e}) = -t(\hat{e}, \bar{e}(t)) + \int_0^t (\hat{e}, \bar{e}) ds + \int_0^t s(\hat{e}, \bar{e}_s) ds.$$

Apply Cauchy-Schwarz's inequality, Lemma 5.4 and a priori estimates in Lemma 2.3(with time reverse) to obtain

$$|I_1^2| \leq Cth^2 \|u_0\| \|g\|.$$

Altogether these estimates yield the desired result and this completes the proof. □

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