Title: A Definition for Large-Eddy-Simulation Approximations of the Navier-Stokes Equations

Running title: A definition of LES

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Abstract: This paper proposes a mathematical definition of Large-Eddy-Simulation (LES) for three-dimensional turbulent incompressible viscous flows. It is proposed to call LES approximation of the Navier–Stokes equations any sequence of finite-dimensional approximations of the velocity and pressure fields that converge (possibly up to subsequences) in the Leray class to a suitable weak solution. This definition introduces a discretization scale h and a cutoff scale for the large eddies ε. A significant byproduct of this definition is that it yields a mathematical rule to select the ratio h/ε. The relevance of the definition is demonstrated on several examples, where it yields results that are in agreement with the intuitive idea of LES.

1. Introduction

1.1. **Introductory comments.** At the present time, computer predictions of turbulence phenomena by so-called Direct Numerical Simulation (DNS) of the Navier-Stokes equations are still considered a formidable task for Reynolds numbers larger than a few thousands. Since the times of Boussinesq and Reynolds, numerous turbulence models based on time-averaged or space-averaged quantities (Reynolds Averaged Navier–Stokes models, k-ϵ models, *etc.*) have been developed and then used in engineering applications as a means of overcoming, though often with limited success, the lack of sufficient computer resources required by DNS. During the past forty years a new class of turbulence models, collectively known as Large-Eddy-Simulation (LES) models [23], has emerged in the literature. These models are founded on the observation that representing the whole range of flow scales may not be important in many engineering applications as one is generally interested only in the large scale features of the flow. With this objective in mind, LES modelers have devised artifacts for representing the interactions between the unreachable small scales and the large ones. Although an extended variety of LES models is now available, see e.g. [6, 12, 19] for reviews, no satisfactory mathematical theory for LES has yet been proposed (for preliminary attempts of formalization see [15, 18, 19]). More surprisingly, no mathematical definition of LES has been stated either (to the best of our knowledge). For the time being, only formal definitions like that proposed in [8] are available: "We define a large eddy simulation as any simulation of a turbulent flow in which the large-scale motions are explicitly resolved while the small-scale motions are represented approximately by a model (in engineering nomenclature) or parameterization (in the geosciences)."

The objective of the present paper is to go beyond qualitative statements like the one above by proposing a mathematical definition of LES approximations of the Navier–Stokes equations, see Definition 2.1. This definition introduces a discretization scale h and a cutoff scale for the large eddies ε. The issue of determining the scale h with respect to the cutoff scale ε is often dealt with by resorting to heuristic arguments and has never been thoroughly addressed [9]. A significant byproduct of our definition is that it yields a mathematical rule to select the ratio h/ε. This will be shown on several examples.

The paper is organized as follows: in the remainder of the Introduction, we recall the notion of suitable weak solutions of the Navier-Stokes equations as it will be included into our definition of LES approximations. We state in Section 2 the definition that we propose for LES approximations and we discuss some of its features. In particular, we introduce the concept of pre-LES-models. In Section 3, we list existing models that fall into the category of the pre-LES-models. We proceed in Section 4 by providing examples of LES approximations and for each of these examples we show how to use our definition of LES to determine the relationship between the discretization parameter h and the cutoff scale parameter ε. Concluding remarks are finally reported in Section 5.

1.2. **Suitable weak solutions.** It is generally accepted that the Navier–Stokes equations stand as a reasonable model to predict the behavior of turbulent incompressible flows of viscous fluids. Upon denoting by $\Omega \subset \mathbb{R}^3$ the open smooth connected domain occupied by the fluid, $]0,T[$ some time interval, $\mathbf{u}$ the velocity field, and p the pressure, the problem is formulated as follows:

$$
\begin{cases}
\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\nabla^2\mathbf{u} = \mathbf{f} \quad \text{in } Q_T,\\
\nabla\cdot\mathbf{u} = 0 \quad \text{in } Q_T,\\
\mathbf{u}|_\Gamma = 0 \quad \text{or } \mathbf{u} \text{ is periodic},\\
\mathbf{u}|_{t=0} = u_0,
\end{cases}
\tag{1.1}
$$

where $Q_T = \Omega \times (0,T)$, Γ is the boundary of Ω, $\mathbf{u}_0$ the solenoidal initial data, $\mathbf{f}$ a source term, ν the viscosity, and the density is chosen equal to unity. We assume that (1.1) has been nondimensionalized, i.e. ν is the inverse of the Reynolds number.

Henceforth, we restrict ourselves to weak solutions of (1.1). To implicitly account for boundary conditions, we introduce

$$
\mathbf{X} = \begin{cases}
\mathbf{H}^1_0(\Omega), & \text{If Dirichlet conditions},\\
\{\mathbf{v} \in \mathbf{H}^1(\Omega),\ \mathbf{v} \text{ periodic}\}, & \text{If periodic conditions},\\
\text{Closed subspace of } \mathbf{H}^1(\Omega), & \text{If mixed boundary conditions},
\end{cases}
\tag{1.2}
$$

and we look for weak solutions of (1.1) such that $\mathbf{u} \in L^2(0,T;\mathbf{X}) \cap L^\infty(0,T;\mathbf{L}^2(\Omega))$ and $p \in H^{-1}(0,T,L^2(\Omega))$. Moreover, we set

$$
\mathbf{V} = \{\mathbf{v} \in \mathbf{X},\ \nabla\cdot\mathbf{v} = 0\},
\tag{1.3}
$$

$$
\mathbf{H} = \overline{\mathbf{V}}^{L^2}
\tag{1.4}
$$

In mathematical terms, the turbulence question is an elusive one. Since the bold definition of turbulence by Leray in the 1930's [24], calling "*solution turbulente*" any weak solution of the Navier–Stokes equations, progress has been frustratingly slow. The major obstacle is the question of uniqueness of solutions in three dimensions, a question not yet solved owing to the possibility that the occurrence of so-called vorticity bursts reaching scales smaller than the Kolmogorov scale cannot be excluded.

If weak solutions are not unique, a fundamental question is then to distinguish the physically relevant solutions. A possible piece of the maze might be the notion of suitable weak solutions proposed by Scheffer [31]:

Definition 1.1 (Scheffer)**.** *A weak solution to the Navier-Stokes equation* $(\mathbf{u}, p)$ *is suitable if* $\mathbf{u} \in L^2(0,T;\mathbf{X}) \cap L^\infty(0,T;\mathbf{L}^2(\Omega))$, $p \in L^{\frac{3}{2}}(Q_T)$ *and the local energy balance*

$$
\partial_t(\frac{1}{2}\mathbf{u}^2) + \nabla\cdot((\frac{1}{2}\mathbf{u}^2 + p)\mathbf{u}) - \nu\nabla^2(\frac{1}{2}\mathbf{u}^2) + \nu(\nabla\mathbf{u})^2 - \mathbf{f}\cdot\mathbf{u} \le 0
\tag{1.5}
$$

is satisfied in the distributional sense.

Suitable solutions are known to exist in general (at least for periodic and Dirichlet boundary conditions). It is remarkable that, to date, it is for the class of suitable weak solutions that the best partial regularity result has been proved, as stated in the so-called Caffarelli-Kohn-Nirenberg (CKN) Theorem, see Caffarelli *et al.* [2], Lin [25], Scheffer [31]. This theorem states that the one-dimensional Hausdorff measure of singular points of suitable solutions is zero. In other words, suitable weak solutions are almost classical (*i.e.*, almost smooth). Whether these solutions are indeed classical is still far from being clear (see *e.g.* Scheffer [32]). Nevertheless, the result of the CKN Theorem is our principal motivation for incorporating the notion of suitable weak solutions in the definition of LES models. Note that, by analogy with nonlinear conservative laws, the condition (1.5) can be viewed as an entropy-like condition in the sense that it is expected to select physical solutions of (1.1). Duchon and Robert [7] have given an explicit form of the distribution $D(\mathbf{u})$ that is missing in the left-hand side of (1.5) to reach equality. The distribution $D(\mathbf{u})$ is zero if the flow $\mathbf{u}$ is smooth, but it may be nontrivial for nonregular flows. Suitable solutions are those which satisfy $D(\mathbf{u}) \geq 0$, i.e., if singularities appear, only those that dissipate energy are admissible.

2. The main definition

The definition we propose is based on two criteria. First, we believe that Large-Eddy-Simulation approximations should be defined as solutions of finite-dimensional problems which can be implemented on digital computers. In other words, a reasonable definition should fill the gap between the current so-called LES modeling theories, which almost always deal with infinite-dimensional settings (*i.e.* time and space are continuous), and their various numerical implementations. The existence of this gap has repeatedly been acknowledged in the literature without being seriously addressed; see e.g. Ferziger [9]: "In general, there is a close connection between the numerical methods and the modeling approach used in simulation; this connection has not been sufficiently appreciated by many authors." Second, LES approximations should select physical solutions of the Navier-Stokes equations under the appropriate limiting process. As a result, we propose the following

Definition 2.1. *A sequence $(\mathbf{u}_\gamma, p_\gamma)_{\gamma>0}$ with $\mathbf{u}_\gamma \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{X})$ and $p_\gamma \in \mathcal{D}'(]0,T[,L^2(\Omega))$ is said to be a LES approximation to (1.1) if*

- (i) *There are two finite-dimensional vectors spaces $\mathbf{X}_\gamma \subset \mathbf{X}$ and $M_\gamma \subset L^2(\Omega)$ such that $\mathbf{u}_\gamma \in \mathcal{C}^1([0,T];\mathbf{X}_\gamma)$ and $p_\gamma \in \mathcal{C}^0([0,T];M_\gamma)$ for all $T > 0$.*
- (ii) *The sequence converges (up to subsequences) to a weak solution of (1.1), say $\mathbf{u}_\gamma \rightharpoonup \mathbf{u}$ weakly in $L^2(0,T;\mathbf{X})$ and $p_\gamma \to p$ in $\mathcal{D}'(]0,T[,L^2(\Omega))$.*
- (iii) *The weak solution (u,p) is suitable.*

At this point, we want to emphasize that two parameters are actually hidden in the above definition. Since $\mathbf{X}_\gamma$ and M_γ are finite-dimensional, there is a discretization parameter h associated with the size of the smallest scale that can be represented in $\mathbf{X}_\gamma$, roughly $\dim(\mathbf{X}_\gamma) = \mathcal{O}((L/h)^3)$ where $L = \mathrm{diam}(\Omega)$. As we shall see in examples below, the definition also implicitly involves a small parameter ε associated with a regularization process of the Navier–Stokes equations. This parameter is the lengthscale of the smallest eddies that are allowed to be nonlinearly active in the flow (large eddies). The parameter γ in the above definition is any combination of the two parameters h and ε such that (ii) and (iii) hold.

In practice, the construction of a LES model will be decomposed into the following three steps:

1. Construction of what we hereafter call the pre-LES-model: This step consists of regularizing the Navier-Stokes equations by introducing a parameter ε representing the large eddy scale beyond which the nonlinear effects are dampened. The purpose of the regularization process is to yield a well-posed problem for all times. Moreover, the limit solution of the pre-LES-model must be a weak solution to the Navier–Stokes equations as $\varepsilon \to 0$ and should be suitable.

(2) Discretization of the pre-LES-model: This step introduces the meshsize parameter h and the finite-dimensional spaces $\mathbf{X}_\gamma$, M_γ.
(3) Determination of a (possibly maximal) relationship between ε and h: The parameters ε and h should be selected in such a way that the discrete solution is ensured to converge to a suitable solution of the Navier-Stokes equations when $\varepsilon \to 0$ and $h \to 0$.

The novelty in the proposed definition is that enforcing the limit solution $(\mathbf{u}, p)$ to be suitable yields a constraint on the limiting processes $\lim_{\varepsilon\to 0}$ and $\lim_{h\to 0}$. Because of this constraint, we prefer to use the neutral parameter γ than either one of the two, and whenever we write $\gamma \to 0$, it will be understood that $\varepsilon \to 0$ and $h \to 0$ in some manner yet to be specified.

Note that the definition is independent of the boundary conditions as long as (1.1) has a suitable weak solution.

A first question that comes to mind is the following: Is a sequence of Direct Numerical Solutions (DNS) a LES approximation as defined above? To clarify this issue let us consider the construction proposed by Hopf [17]. Let $\mathbf{X}_h \subset \mathbf{X}$ and $M_h \subset L^2(\Omega)$ be two finite-dimensional vector spaces and consider the following Galerkin approximation

$$(2.1) \qquad \begin{cases} (\partial_t \mathbf{u}_h, \mathbf{v}) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) - (p_h, \nabla\cdot\mathbf{v}) + \nu(\nabla\mathbf{u}_h, \nabla\mathbf{v}) \\ \qquad\qquad\qquad\qquad = (\mathbf{f}, \mathbf{v}) \quad \forall t \in [0, T],\ \forall \mathbf{v} \in \mathbf{X}_h, \\ (q, \nabla\cdot\mathbf{u}_h) = 0, \quad \forall t \in [0, T],\ \forall q \in M_h, \\ (\mathbf{u}_h|_{t=0}, \mathbf{v}) = (\mathbf{u}_0, \mathbf{v}),\ \forall \mathbf{v} \in \mathbf{X}_h, \end{cases}$$

where $(\cdot, \cdot)$ stands for the $\mathbf{L}^2$-scalar product, b is a trilinear form accounting for the nonlinear advection term and is skew-symmetric with respect to its second and third arguments. For instance $b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) = (\mathbf{u}_h\cdot\nabla\mathbf{u}_h, \mathbf{v}) - \frac{1}{2}(\mathbf{u}_h\nabla\cdot\mathbf{u}_h, \mathbf{v})$ and $b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) = ((\nabla\times\mathbf{u}_h)\times\mathbf{u}_h, \mathbf{v})$ are admissible candidates.

The above construction is usually referred to as DNS in the Computational Fluid Dynamics literature. Owing to standard a priori estimates uniform in h, it is clear that the approximate solution complies with items (i) and (ii) of the above definition; see, e.g., Lions [26, 27] or Temam [35]. However, it is not yet known whether such a construction yields a suitable solution at the limit when $h \to 0$, i.e., surprisingly item (iii) is not known to hold. In other words, it may happen that the solution sets spanned by the limits of DNS approximations and LES approximations are not identical. The answer to this question seems to be intimately linked with the regularity question, which, we recall, is the object of one the seven prizes offered by the Clay institute. Definition 2.1 is based on the conjecture that the LES solution set is strictly smaller than the DNS solution set in general.

3. Pre-LES-models

We now briefly survey regularization techniques that we will consider as pre-LES-models. These models do not comply with item (i) of Definition 2.1 since both $\mathbf{u}_\varepsilon$ and p_ε have values in infinite-dimensional vector spaces, but they all satisfy items (ii) and (iii).

3.1. **Hyperviscosity.** Lions [26, 27] proposed the following hyperviscosity perturbation of the Navier–Stokes equations:

$$
(3.1)\qquad \begin{cases} \partial_t \mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}_\varepsilon + \nabla p_\varepsilon - \nu \nabla^2 \mathbf{u}_\varepsilon + \varepsilon^{2\alpha}(-\nabla^2)^\alpha \mathbf{u}_\varepsilon = \mathbf{f} \quad \text{in } Q_T, \\ \nabla \cdot \mathbf{u}_\varepsilon = 0 \quad \text{in } Q_T, \\ \mathbf{u}_\varepsilon|_\Gamma,\ \partial_n \mathbf{u}_\varepsilon|_\Gamma,\ \dots\ ,\ \partial_n^{\alpha-1} \mathbf{u}_\varepsilon|_\Gamma = 0, \quad \text{or } \mathbf{u}_\varepsilon \text{ is periodic} \\ u|_{t=0} = u_0, \end{cases}
$$

where $\varepsilon > 0$ and α is an integer. The appealing aspect of this perturbation is that it yields a wellposed problem in the classical sense when $\alpha \geq \frac{5}{4}$ in three space dimensions. More precisely, upon denoting by $d \geq 2$ the space dimension, the following result (see [16, 26, 27]) holds

Theorem 3.1. *Assume* $\mathbf{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$ *and* $\mathbf{u}_0 \in \mathbf{H}^\alpha(\Omega) \cap \mathbf{X}$. *Problem (3.5) has a unique solution* $\mathbf{u}_\varepsilon$ *in* $L^\infty(0,T;\mathbf{H}^\alpha(\Omega)\cap\mathbf{X})$ *for all times* $T > 0$ *if* $\alpha \geq \frac{d+2}{4}$. *Up to subsequences,* $\mathbf{u}_\varepsilon$ *converges to a weak solution* $\mathbf{u}$ *of (1.1), weakly in* $L^2(0,T;\mathbf{X})$. *Moreover, if periodic boundary conditions are enforced, the limit solution* $(\mathbf{u},p)$ *is suitable.*

It is remarkable that hyperviscosity models are frequently used in computer simulations for oceanic and atmospheric flows [1].

3.2. **Leray mollification.** A simple construction yielding a suitable solution has indeed been proposed by Leray [24] before this very notion was introduced in the literature. Leray proved the existence of weak solutions by using a, now very popular, mollification[1] technique.

Assume that Ω is the three-dimensional torus $(0,2\pi)^3$. Denoting by $B(0,\varepsilon) \subset \mathbb{R}^3$ the ball of radius ε centered at 0, consider a sequence of mollifying functions $(\phi_\varepsilon)_{\varepsilon>0}$ satisfying:

$$
(3.2)\qquad \phi_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R}^3), \quad \operatorname{supp}(\phi_\varepsilon) \subset B(0,\varepsilon), \quad \int_{\mathbb{R}^3} \phi_\varepsilon(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1.
$$

Defining the convolution product $\phi_\varepsilon * \mathbf{v}(\mathbf{x}) = \int_{\mathbb{R}^3} \mathbf{v}(\mathbf{y})\phi_\varepsilon(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}$, Leray suggested to regularize the Navier–Stokes equations as follows:

$$
(3.3)\qquad \begin{cases} \partial_t \mathbf{u}_\varepsilon + (\phi_\varepsilon * \mathbf{u}_\varepsilon) \cdot \nabla \mathbf{u}_\varepsilon + \nabla p_\varepsilon - \nu \nabla^2 \mathbf{u}_\varepsilon = \phi_\varepsilon * \mathbf{f}, \\ \nabla \cdot \mathbf{u}_\varepsilon = 0, \\ \mathbf{u}_\varepsilon \text{ is periodic}, \\ \mathbf{u}_\varepsilon|_{t=0} = \phi_\varepsilon * \mathbf{u}_0. \end{cases}
$$

The following holds (see [24] and [7]):

Theorem 3.2. *For all* $\mathbf{u}_0 \in \mathbf{H}$, $\mathbf{f} \in \mathbf{H}$, *and* $\varepsilon > 0$, *(3.3) has a* unique $\mathcal{C}^\infty$ *solution. The velocity is bounded uniformly in* $L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})$ *and one subsequence converges weakly in* $L^2(0,T;\mathbf{V})$. *The limit solution as* $\varepsilon \to 0$ *is a suitable weak solution of the Navier–Stokes equations.*

Hence the above construction complies with items (ii) and (iii) of Definition 2.1. The mollification technique can be extended to account for the homogeneous Dirichlet boundary condition as done in [2]. The limit solution is suitable in this case as well.

[1]This term seems to have been coined latter by K.O. Friedrichs

Remark 3.1. (i) Roughly speaking, the convolution process removes scales that are smaller than ε. Hence, by using $\phi_\varepsilon * \mathbf{u}_\varepsilon$ as the advection velocity, scales smaller than ε are not allowed to be nonlinearly active. (ii) In (3.3) the regularization of $\mathbf{f}$ and $\mathbf{u}_0$ is not essential. Uniqueness and convergence to a suitable solution is obtained without it. Leray regularized $\mathbf{f}$ and $\mathbf{u}_0$ to obtain a $\mathcal{C}^\infty$ solution.

3.3. **NS-α and Leray-α models.** Introduce the so-called Helmholtz filter $\overline{(\cdot)}$: $v \mapsto \overline{v}$ such that

$$\overline{\mathbf{v}} := (I - \varepsilon^2 \nabla^2)^{-1} \mathbf{v}, \tag{3.4}$$

where either homogeneous Dirichlet boundary conditions or periodic boundary conditions are enforced depending on the setting considered. The so-called Navier–Stokes–alpha model introduced in Chen *et al.* [4] and Foias *et al.* [10, 11] consists of the following:

$$\begin{cases} \partial_t \mathbf{u}_\varepsilon + \overline{\mathbf{u}_\varepsilon} \cdot \nabla \mathbf{u}_\varepsilon + (\nabla \overline{\mathbf{u}_\varepsilon})^T \cdot \mathbf{u}_\varepsilon - \nu \nabla^2 \mathbf{u}_\varepsilon + \nabla \pi_\varepsilon = \mathbf{f}, \\ \nabla \cdot \overline{\mathbf{u}_\varepsilon} = 0, \\ \mathbf{u}_\varepsilon|_\Gamma = 0,\ \overline{\mathbf{u}_\varepsilon}|_\Gamma = 0, \quad \text{or } \mathbf{u}_\varepsilon,\ \text{and } \overline{\mathbf{u}_\varepsilon} \text{ are periodic}, \\ \mathbf{u}_\varepsilon|_{t=0} = \mathbf{u}_0, \end{cases} \tag{3.5}$$

Once again, regularization yields uniqueness as stated in the following

Theorem 3.3 (Foias, Holm and Titi [10, 11])**.** *Assume* $\mathbf{f} \in \mathbf{H}$, $\mathbf{u}_0 \in \mathbf{V}$. *Problem (3.5) with the Helmholtz filter (3.4) has a unique solution* $\mathbf{u}_\varepsilon$ *in* $\mathcal{C}^0([0,T);\mathbf{V})$ *with* $\partial_t \mathbf{u}_\varepsilon \in L^2(]0,T[;\mathbf{H})$. *The solution* $\overline{\mathbf{u}_\varepsilon}$ *is uniformly bounded in* $L^\infty(0,+\infty;\mathbf{H}) \cap L^2(0,+\infty;\mathbf{V})$ *and one subsequence converges weakly in* $L^2_{\text{loc}}(0,+\infty;\mathbf{V})$ *to a weak Navier–Stokes solution as* $\varepsilon \to 0$.

Here again, it is a simple matter to show that when periodic boundary conditions are enforced $\mathbf{u}_\varepsilon$ converges, up to subsequences, to a suitable solution.

A variant of the above regularization technique consists of replacing the term $(\nabla \overline{\mathbf{u}_\varepsilon})^T \cdot \mathbf{u}_\varepsilon$ in (3.5) by $\nabla \frac{1}{2} \mathbf{u}_\varepsilon^2$. The resulting model then falls in the class of the Leray regularization in the sense that the momentum equation is the same as that in (3.3) but for the advection velocity $\phi_\varepsilon * \mathbf{u}_\varepsilon$ which is replaced by $\overline{\mathbf{u}_\varepsilon}$. This model is called the Leray-α model; see also [13].

3.4. **Nonlinear viscosity models.** Recalling that the Navier–Stokes equations are based on Newton's linear hypothesis, Ladyženskaja and Kaniel proposed to modify the incompressible Navier–Stokes equations to take into account possible large velocity gradients, [20, 21, 22].

Ladyženskaja introduced a nonlinear viscous tensor $\mathbf{T}_{ij}(\nabla \mathbf{u})$, $1 \le i, j \le 3$ satisfying the following conditions:

L1) $\mathbf{T}$ is continuous and there exists $\mu \ge \frac{1}{4}$ such that

$$\forall\ \boldsymbol{\xi} \in \mathbb{R}^{3\times 3}, \quad |\mathbf{T}(\boldsymbol{\xi})| \le c(1 + |\boldsymbol{\xi}|^{2\mu})|\boldsymbol{\xi}|. \tag{3.6}$$

L2) $\mathbf{T}$ satisfies the coercivity property:

$$\forall\ \boldsymbol{\xi} \in \mathbb{R}^{3\times 3}, \quad \mathbf{T}(\boldsymbol{\xi}) : \boldsymbol{\xi} \ge c|\boldsymbol{\xi}|^2(1 + c'|\boldsymbol{\xi}|^{2\mu}). \tag{3.7}$$

L3) $\mathbf{T}$ possesses the following monotonicity property: There exists a constant $c > 0$ such that for all solenoidal fields $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ in $\mathbf{W}^{1,2+2\mu}(\Omega)$ either coinciding on the boundary Γ or being periodic,

$$\int_\Omega (\mathbf{T}(\nabla\boldsymbol{\xi}) - T(\nabla\boldsymbol{\eta})) : (\nabla\boldsymbol{\xi} - \nabla\boldsymbol{\eta}) \geq c \int_\Omega |\nabla\boldsymbol{\xi} - \nabla\boldsymbol{\eta}|^2. \tag{3.8}$$

The three above conditions are satisfied if

$$\mathbf{T}(\boldsymbol{\xi}) = \beta(|\boldsymbol{\xi}|^2)\boldsymbol{\xi}, \tag{3.9}$$

provided the viscosity function $\beta(\tau)$ is a positive monotonically-increasing function of $\tau \geq 0$ and for large values of τ the following inequality holds

$$c\tau^\mu \leq \beta(\tau) \leq c'\tau^\mu,$$

with $\mu \geq \frac{1}{4}$ and c, c' are some strictly positive constants.

Introducing a positive constant $\varepsilon > 0$, the modified Navier–Stokes equations take the form

$$\begin{cases} \partial_t \mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}_\varepsilon + \nabla p_\varepsilon - \nabla \cdot (\nu \nabla \mathbf{u}_\varepsilon + \varepsilon^{2\mu+1} \mathbf{T}(\nabla \mathbf{u}_\varepsilon)) = \mathbf{f}, \\ \nabla \cdot \mathbf{u}_\varepsilon = 0 \\ \mathbf{u}_\varepsilon|_\Gamma = 0, \quad \text{or } \mathbf{u}_\varepsilon \text{ is periodic}, \\ \mathbf{u}_\varepsilon|_{t=0} = \mathbf{u}_0. \end{cases} \tag{3.10}$$

The main result from [22, 21] (see [20] for a similar result where monotonicity is also assumed) is the following theorem

Theorem 3.4. *Assume $\mathbf{u}_0 \in \mathbf{H}$ and $\mathbf{f} \in L^2(]0,+\infty[;\mathbf{L}^2(\Omega))$. Provided conditions* L1*,* L2*, and* L3 *hold, then (3.10) has a unique weak solution, for all $R > 0$, in $L^{2+2\mu}(]0,T[;\mathbf{W}^{1,2+2\mu}(\Omega) \cap \mathbf{V}) \cap C^0([0,T];\mathbf{H})$.*

This result states that a small appropriate amount of nonlinear viscosity is actually sufficient to ascertain that the energy cascade stops, which automatically translates into uniqueness of the solution for arbitrary times. Moreover, for periodic boundary conditions, $(\mathbf{u}_\varepsilon, p_\varepsilon)$ converges, up to subsequences, to a suitable solution of (1.1) as $\varepsilon \to 0$.

Possibly one of the most popular models for Large-Eddy-Simulation is that proposed by Smagorinsky [33], which corresponds to setting $\mathbf{T}(\nabla\mathbf{u}) = |\mathbf{D}|\mathbf{D}$. For this model $\beta(\tau) = \tau^\mu$ with $\mu = \frac{1}{2}$.

In conclusion, perturbing the Navier–Stokes equations with a term like Smagorinsky's model solves the uniqueness question and ensures that the limit solution are suitable. Note however that, at this point, item (i) of Definition 2.1 does not hold; i.e., the above arguments do not give guidelines on the way ε should be linked to a discretization parameter like the meshsize h.

4. Discretization

The purpose of this section is to illustrate Definition 2.1. We introduce discrete versions of some of the pre-LES-models described above. In each case we determine the relationship between the parameters h and ε so that the resulting approximate solutions converge to suitable solutions when the parameters go to zero. For the sake of simplicity, we restrict ourselves to periodic boundary conditions and spectral approximation techniques, but we stress again that Definition 2.1 is not restricted to this simplified setting.

4.1. **The discrete hyperviscosity model.** We turn our attention to the hyperviscosity model introduced in Section 3.1. We construct a Galerkin-Fourier approximation, and we show how Definition 2.1 enforces strong constraints on the relation linking the hyperviscosity coefficient ε to the polynomial degree of the approximation.

For any $\mathbf{z} \in \mathbb{C}^\ell$, $1 \leq \ell \leq 3$, we denote by $|\mathbf{z}|$ the Euclidean norm and by $|\mathbf{z}|_\infty$ the maximum norm. We denote by $\overline{\mathbf{z}}$ the conjugate of $\mathbf{z}$. Recall that Sobolev spaces $H^s(\Omega), s \geq 0$, can be equivalently defined in terms of Fourier series as follows

$$H^s(\Omega) = \left\{ u = \textstyle\sum_{\mathbf{k}\in\mathbb{Z}^3} u_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}},\ u_{\mathbf{k}} = \overline{u}_{-\mathbf{k}},\ \sum_{\mathbf{k}\in\mathbb{Z}^3} (1+|\mathbf{k}|^2)^s |u_{\mathbf{k}}|^2 < +\infty \right\}.$$

In other words, the set of trigonometric polynomials $\exp(i\mathbf{k}\cdot\mathbf{x})$, $\mathbf{k} \in \mathbb{Z}^3$, is complete and orthogonal in $H^s(\Omega)$ for all $s \geq 0$. The scalar product in $L^2(\Omega)$ is denoted by $(u, v) = (2\pi)^{-3} \int_\Omega u\overline{v}$ and the dual of $H^s(\Omega)$ by $H^{-s}(\Omega)$. We introduce the closed subspace $\dot{H}^s(\Omega)$ of $H^s(\Omega)$ composed of the functions of zero mean value.

Let N be a positive integer, henceforth referred to as the cutoff wave number. We introduce the set of trigonometric polynomials of partial degree less than or equal to N:

$$\mathbb{P}_N = \left\{ p(\mathbf{x}) = \textstyle\sum_{|\mathbf{k}|_\infty \leq N} c_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, c_{\mathbf{k}} = \overline{c}_{-\mathbf{k}} \right\},$$

and denote by $\dot{\mathbb{P}}_N$ the subspace of $\mathbb{P}_N$ composed of the trigonometric polynomials of zero mean value.

Finally, to approximate the velocity and the pressure fields we introduce the following finite-dimensional vector spaces:

$$\mathbf{X}_N = \dot{\mathbb{P}}_N^3, \quad \text{and} \quad M_N = \dot{\mathbb{P}}_N. \tag{4.1}$$

The construction of the hyperviscosity model involves the definition of a large eddy scale ε_N and a hyperviscosity kernel $Q(\mathbf{x})$. We choose to build the kernel in such a way that it acts only on the high wave numbers of the velocity field, namely $N_i \leq |\mathbf{k}|_\infty \leq N$, N_i standing for an intermediate cutoff wave number yet to be fixed. This idea is similar to the spectral viscosity technique that Tadmor [34, 29, 3] developed for nonlinear scalar conservation laws.

We start by introducing the hyperviscosity parameter α such that

$$\alpha > \frac{5}{4}. \tag{4.2}$$

We then introduce θ, with $0 < \theta < 1$, from which we define the large scale parameter N_i and the corresponding large eddy cutoff scale ε_N

$$N_i = N^\theta, \qquad \varepsilon_N = \frac{1}{N_i} \tag{4.3}$$

We also introduce a hyperviscosity kernel $Q(\mathbf{x})$ as follows:

$$Q(\mathbf{x}) = (2\pi)^{-3} \sum_{N_i \leq |\mathbf{k}|_\infty \leq N} |\mathbf{k}|^{2\alpha} e^{i\mathbf{k}\cdot\mathbf{x}} \tag{4.4}$$

The kernel Q is such that for all $v_N \in X_N$

$$Q * v_N(\mathbf{x}) = \int_\Omega v_N(\mathbf{y}) Q(\mathbf{x}-\mathbf{y}) d\mathbf{y} = \sum_{N_i \leq |\mathbf{k}|_\infty \leq N} |\mathbf{k}|^{2\alpha} v_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \tag{4.5}$$

When α is an integer, $Q*(\cdot)$ is the α-th power of the restriction of the Laplace operator on the space spanned by the Fourier modes comprised between N_i and N.

Note that when θ increases, the vanishing viscosity amplitude decreases and the range of wavenumbers on which the kernel $Q(\mathbf{x})$ is active shrinks.

The spectral hyperviscosity model consists of the following:

$$\text{(4.6)} \qquad \begin{cases} \text{Seek } \mathbf{u}_N \in \mathcal{C}^1([0,T];\mathbf{X}_N) \text{ and } p_N \in \mathcal{C}^0([0,T];M_N) \text{ such that} \\ (\partial_t \mathbf{u}_N, \mathbf{v}) + (\mathbf{u}_N \cdot \nabla \mathbf{u}_N, \mathbf{v}) - (p_N, \nabla \cdot \mathbf{v}) + \nu(\nabla \mathbf{u}_N, \nabla \mathbf{v}) \\ \qquad\qquad + \varepsilon_N^{2\alpha}(Q * \mathbf{u}_N, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{X}_N, \forall t \in (0,T], \\ (\nabla \cdot \mathbf{u}_N, q) = 0, \qquad \forall q \in M_N, \forall t \in (0,T], \\ (\mathbf{u}_N, \mathbf{v})|_{t=0} = (\mathbf{u}_0, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{X}_N, . \end{cases}$$

One interesting feature of the above technique is the following

Proposition 4.1. *The hyperviscosity perturbation is spectrally small, i.e.,*

$$\text{(4.7)} \qquad \varepsilon_N^{2\alpha} \|Q * v_N\|_{L^2} \lesssim N^{-\theta s} \|v_N\|_{H^s}, \qquad \forall v_N \in H^s(\Omega), \ \forall s \geq 2\alpha.$$

The interpretation of the above result is that the consistency error induced by the hyperviscosity is arbitrarily small if the exact solution of the Navier–Stokes equation is smooth.

The main result of this section is the following:

Theorem 4.1. *Let* $\mathbf{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$ *and* $\mathbf{u}_0 \in \mathbf{H}^\alpha(\Omega) \cap \mathbf{V}$. *Assume that (4.2) and (4.3) hold. Moreover, assume that*

$$\text{(4.8)} \qquad 0 < \theta < \begin{cases} \frac{4\alpha - 5}{4\alpha} & \text{If } \alpha \leq \frac{3}{2}, \\ \frac{2(\alpha-1)}{2\alpha+3} & \text{Otherwise.} \end{cases}$$

then, up to subsequences, the solution u_N *to (4.6) converges weakly in* $L^2(0,T;\mathbf{V})$ *and strongly in any* $L^r(0,T;\mathbf{L}^s(\Omega))$, *with* $2 \leq r < \frac{4s}{3(s-2)} < +\infty$, *to a suitable solution of (1.1) as* N *goes to infinity.*

The above result, proved in [16], can be interpreted as follows. Since the hyperviscosity term is meant to be a perturbation of the Navier–Stokes equations, one would want this term to be as small as possible. In fact, one would like θ to be as large as possible; the choice $\theta = 1$ is that which minimizes the impact of the hyperviscosity perturbation. But, if θ is too close to one, the hyperviscosity term cannot play the role it is assigned, i.e., the limit solution cannot be guaranteed to be suitable (see item (iii)). It is shown in [16] that a sufficient condition for the limit solution to be suitable is that the bound from above in (4.8) holds. In this sense we claim that Definition 2.1 is constructive.

To implement the above algorithm, proceed as follows: pick α, take $\theta < \frac{4\alpha-5}{4\alpha}$ or $\theta < \frac{2(\alpha-1)}{2\alpha+3}$, depending on the value of α, and finally set $N_i = c_1 N^\theta$ and $\varepsilon_N = c_2 N_i^{-1}$. The parameter α and the constants c_1, c_2 are coefficients that can be tuned, provided $\alpha > \frac{5}{4}$ and c_1, c_2 of order one. Some admissible values of the parameters α and θ are shown in Table 1.

Remark 4.1. With the polynomial degree N we can associate the mesh size $h_N = N^{-1}$. Then we observe that the condition (4.8), i.e. $\theta < 1$, means that $h_N \ll \varepsilon_N = N^{-\theta}$. In other words, the scales filtered by the hyperviscosity are significantly larger than the grid size.

α	$\frac{3}{2}$	2	3	4	5
θ	$< \frac{1}{6}$	$< \frac{2}{7}$	$< \frac{4}{9}$	$< \frac{6}{11}$	$< \frac{8}{13}$

TABLE 1. Admissible values of the parameters α and θ.

4.2. **The discrete Leray model.** For the sake of simplicity, we restrict the analysis to the periodic case, i.e. Ω is assumed to be the three-dimensional torus, and we reuse the Fourier setting introduced in Section 4.1. Let N be an integer and set

$$\mathbf{X}_N = \dot{\mathbb{P}}^3_N, \quad \text{and} \quad M_N = \dot{\mathbb{P}}_N. \tag{4.9}$$

Introduce θ such that $0 < \theta < 1$ and set

$$N_i = N^\theta. \tag{4.10}$$

We also consider the truncation operator $P_{N_i} : \mathbf{H}^s(\Omega) \longrightarrow \mathbb{P}^3_{N_i}$ such that

$$P_{N_i} : \mathbf{H}^s(\Omega) \ni \sum_{\mathbf{k}\in\mathbb{Z}^3} \mathbf{v}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} = \mathbf{v} \longmapsto \sum_{|\mathbf{k}|_\infty \le N_i} \mathbf{v}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \in \mathbb{P}^3_{N_i}$$

and let Q_{N_i} denote the operator $Q_{N_i} = I - P_{N_i}$. In the Fourier setting, a simple way of regularizing the nonlinear term consists of removing the high frequencies from the advection velocity. Thus, instead of using $\mathbf{u}_N$ as advection field, we use $P_{N_i}\mathbf{u}_N$. The discrete Leray model then reads:

$$\begin{cases} \text{Seek } \mathbf{u}_N \in \mathcal{C}^1([0,T];\mathbf{X}_N) \text{ and } p_N \in \mathcal{C}^0([0,T];M_N) \text{ such that} \\ \text{for all } t \in (0,T], \text{ for all } \mathbf{v} \in \mathbf{X}_N, \text{ and for all } q \in M_N, \\ (\partial_t \mathbf{u}_N, \mathbf{v}) + (P_{N_i}\mathbf{u}_N\cdot\nabla\mathbf{u}_N, \mathbf{v}) - (p_N, \nabla\cdot\mathbf{v}) + \nu(\nabla\mathbf{u}_N, \nabla\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \\ (\nabla\cdot\mathbf{u}_N, q) = 0, \\ (\mathbf{u}_N, \mathbf{v})|_{t=0} = (\mathbf{u}_0, \mathbf{v}). \end{cases} \tag{4.11}$$

Theorem 4.2. *Let $\mathbf{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$ and $\mathbf{u}_0 \in \mathbf{H}$. Assume that (4.10) hold. If*

$$\theta < \frac{2}{3}, \tag{4.12}$$

the solution $\mathbf{u}_N$ to (4.11) converges weakly, up to subsequences, in $L^2(0,T;\mathbf{V})$ and strongly in any $L^r(0,T;\mathbf{L}^q(\Omega))$, with $2 \le r < \frac{4s}{3(s-2)} < +\infty$, to a suitable solution of (1.1) as N goes to infinity.

Proof. We just sketch the proof since the details are similar to those given in the proof of Theorem 5.1 in [16]. The main difficulty revolves around the handling of the nonlinear term when proving that the limit solution is suitable. As usual, the basic a priori estimates are

$$\|\mathbf{u}_N\|^2_{L^2} + \int_0^T \|\nabla P_{N_i}\mathbf{u}_N\|^2_{L^2} + \|\nabla Q_{N_i}\mathbf{u}_N\|^2_{L^2} \le c. \tag{4.13}$$

Let $\phi \in \mathcal{D}(Q_T)$ and use $P_N(\phi \mathbf{u}_N)$ to test the momentum equation in (4.11). The nonlinear term gives

$$\begin{aligned}(P_{N_i}\mathbf{u}_N\cdot\nabla\mathbf{u}_N, P_N(\mathbf{u}_N\phi)) &= (P_{N_i}\mathbf{u}_N\cdot\nabla\mathbf{u}_N, \mathbf{u}_N\phi) + R_1\\ &= -(\tfrac{1}{2}|\mathbf{u}_N|^2 P_{N_i}\mathbf{u}_N, \nabla\phi) + R_1,\\ &= -(\tfrac{1}{2}|\mathbf{u}_N|^2 \mathbf{u}_N, \nabla\phi) + R_1 + R_2,\end{aligned}$$

where

$$\begin{aligned}R_1 &= (P_{N_i}\mathbf{u}_N\cdot\nabla\mathbf{u}_N, P_N(\mathbf{u}_N\phi) - \mathbf{u}_N\phi),\\ R_2 &= (\tfrac{1}{2}|\mathbf{u}_N|^2 Q_{N_i}\mathbf{u}_N, \nabla\phi).\end{aligned}$$

The first residual is handled as follows:

$$\begin{aligned}|R_1| &\le \|P_{N_i}\mathbf{u}_N\|_{L^\infty}\|\nabla\mathbf{u}_N\|_{L^2}\|P_N(\mathbf{u}_N\phi) - \mathbf{u}_N\phi\|_{L^2},\\ &\lesssim N_i{}^{\frac{3}{2}} N^{-1}\|P_{N_i}\mathbf{u}_N\|_{L^2}\|\mathbf{u}_N\|_{H^1}\|\mathbf{u}_N\phi\|_{H^1},\\ &\lesssim N^{\frac{3}{2}\theta-1}\|\mathbf{u}_N\|_{L^2}\|\mathbf{u}_N\|_{H^1}^2\|\phi\|_{W^{1,\infty}},\end{aligned}$$

where $\lesssim$ means that the inequality holds up to a constant independent of N. Then, it is clear that $\int_0^T |R_1| \to 0$ as $N \to \infty$ owing to (4.12). In a similar fashion, we have that $\int_0^T |R_2| \to 0$ as $N \to \infty$. The rest of the proof follows as that of Theorem 5.1 in [16]. □

Remark 4.2. Denoting by $\varepsilon_N = N_i^{-1}$ the regularization scale (the large eddy scale) and $h_N = N^{-1}$ the discretization scale, Theorem 4.2 shows that if $\varepsilon_N^{\frac{3}{2}} \gg h_N$, then the pair $(\mathbf{u}_N, p_N)$ is a LES approximation in the sense of Definition 2.1.

4.3. **The discrete Leray-α model.** Still keeping the Fourier framework, we now consider the following discrete version of the Leray-α model introduced at the end of Section 3.3; see also [5, 14]:

$$\begin{cases}\text{Seek } \mathbf{u}_N \in \mathcal{C}^1([0,T];\mathbf{X}_N) \text{ and } p_N \in \mathcal{C}^0([0,T];M_N) \text{ such that}\\ \text{for all } t \in (0,T], \text{ for all } \mathbf{v} \in \mathbf{X}_N, \text{ and for all } q \in M_N,\\ (\partial_t\mathbf{u}_N, \mathbf{v}) + (\bar{\mathbf{u}}_N\cdot\nabla\mathbf{u}_N, \mathbf{v}) - (p_N, \nabla\cdot\mathbf{v}) + \nu(\nabla\mathbf{u}_N, \nabla\mathbf{v}) = (\mathbf{f}, \mathbf{v}),\\ (\bar{\mathbf{u}}_N, \mathbf{v}) + \varepsilon_N^2(\nabla\bar{\mathbf{u}}_N, \nabla\mathbf{v}) = (\mathbf{u}_N, \mathbf{v}),\\ (\nabla\cdot\mathbf{u}_N, q) = 0,\\ (\mathbf{u}_N, \mathbf{v})|_{t=0} = (\mathbf{u}_0, \mathbf{v}),\end{cases} \tag{4.14}$$

where ε_N is the scale of the smallest eddies that we authorize to be nonlinearly active:

$$\varepsilon_N = N_i^{-1} = N^{-\theta}, \qquad 0 < \theta < 1. \tag{4.15}$$

Note that the system (4.14) is similar to (4.11) except for the advective velocity $P_{N_i}\mathbf{u}_N$ which is now replaced by the regularized velocity $\bar{\mathbf{u}}_N$.

Theorem 4.3. *Let* $\mathbf{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$ *and* $\mathbf{u}_0 \in \mathbf{H}$*. Assume that (4.15) holds. If*

$$\theta < \frac{2}{3}, \tag{4.16}$$

the solution $\mathbf{u}_N$ *to (4.14) converges weakly, up to subsequences, in* $L^2(0,T;\mathbf{V})$ *and strongly in any* $L^r(0,T;\mathbf{L}^s(\Omega))$*, with* $2 \le r < \frac{4s}{3(s-2)} < +\infty$*, to a suitable solution of (1.1) as* N *goes to infinity.*

Proof. Using the fact that $\nabla\cdot\bar{\mathbf{u}}_N = 0$ and taking $\mathbf{v} = P_N(\mathbf{u}_N\phi)$, the nonlinear term becomes

$$(\bar{\mathbf{u}}_N\cdot\nabla\mathbf{u}_N, P_N(\mathbf{u}_N\phi)) = (\bar{\mathbf{u}}_N\cdot\nabla\mathbf{u}_N, \mathbf{u}_N\phi) + R = -(\tfrac{1}{2}|\mathbf{u}_N|^2\bar{\mathbf{u}}_N, \nabla\phi) + R,$$

with

$$\begin{aligned}|R| &= |(\bar{\mathbf{u}}_N\cdot\nabla\mathbf{u}_N, P_N(\mathbf{u}_N\phi) - \mathbf{u}_N\phi)| \\ &\lesssim \|\bar{\mathbf{u}}_N\|_{L^\infty}\|\nabla\mathbf{u}_N\|_{L^2}\|P_N(\mathbf{u}_N\phi) - \mathbf{u}_N\phi\|_{L^2} \\ &\lesssim N^{-1}\|\bar{\mathbf{u}}_N\|_{L^\infty}\|\mathbf{u}_N\|^2_{H^1}\|\phi\|_{W^{1,\infty}}.\end{aligned}$$

Using $\|\bar{\mathbf{u}}_N\|^2_{L^\infty} \lesssim \|\nabla\bar{\mathbf{u}}_N\|_{L^2}\|\Delta\bar{\mathbf{u}}_N\|_{L^2}$ and the bounds $\varepsilon_N\|\nabla\bar{\mathbf{u}}_N\|_{L^2} \lesssim \|\mathbf{u}_N\|_{L^2}$ and $\varepsilon_N^2\|\Delta\bar{\mathbf{u}}_N\|_{L^2} \lesssim \|\mathbf{u}_N\|_{L^2}$ (from the Helmholtz problem), an estimate of the residual is

$$\begin{aligned}|R| &\lesssim \varepsilon_N^{-\frac{3}{2}}N^{-1}\|\mathbf{u}_N\|_{L^2}\|\mathbf{u}_N\|^2_{H^1}\|\phi\|_{W^{1,\infty}} \\ &\lesssim N^{\frac{3}{2}\theta-1}\|\mathbf{u}_N\|_{L^2}\|\mathbf{u}_N\|^2_{H^1}\|\phi\|_{W^{1,\infty}}.\end{aligned}$$

It follows that $\int_0^T |R_1| \to 0$ as $N \to \infty$ provided that $\theta < \frac{2}{3}$. In other words, the discrete solution $(\mathbf{u}_N, p_N)$ converges to a suitable weak solution of the Navier-Stokes equations whenever $\varepsilon_N^{\frac{3}{2}} \gg h_N$. □

4.4. **The other discrete models.** We have not yet been able to show that discrete counterparts of the other pre-LES-models introduced in Section 3, namely the NS-α model and the nonlinear viscosity models, satisfy the LES definition.

Regarding the nonlinear viscosity models, we are still facing technical issues to prove that the discrete solutions actually converge to suitable weak solutions of the Navier-Stokes equations. In particular, the difficulty lies in the fact that the Fourier analysis is not the proper tool to work with L^p spaces when $p \neq 2$. However, we believe that these issues, being purely technical, will eventually be overcome in the near future.

In the case of the NS-α model, the presence of the term $(\nabla\overline{\mathbf{u}_\varepsilon})^T \cdot \mathbf{u}_\varepsilon$ poses some difficulties when passing to the limit. Whether these difficulties are either technical or fundamental is not yet clear.

Surprisingly we have also observed that the Nonlinear Galerkin Method [30] can be reinterpreted in the light of Definition 2.1 as a LES technique. Indeed, following a similar approach as above, it can be shown that the discrete solution of a slightly modified version of the Nonlinear Galerkin Method yields a suitable solution in the limit under appropriate conditions, (the modified version in question consists of replacing $\partial_t\mathbf{u}_N$ by $\partial_t P_{N_i}\mathbf{u}_N$ in (4.11)). This interpretation of the Nonlinear Galerkin Method will be the subject of a forthcoming paper.

5. Conclusions

A definition for LES approximations of the Navier-Stokes equations has been proposed in this paper. The proposed definition introduces two parameters, i.e. a discretization scale h and a cutoff scale for the large eddies ε, and prescribes that the ratio h/ε should be chosen so that the limit solution is suitable. We have applied this rule for the selection of h/ε to three examples and we have shown that h should be chosen much smaller than ε in order to ensure that the approximate solutions converge to a suitable solution of the Navier-Stokes equations; see Remarks 4.1

and 4.2. This result clearly challenges what is often suggested in the literature and commonly done in practice, namely to take ε of the same order as h. Otherwise, we have illustrated the relevance of the definition by showing that it applies to several existing LES models.

Whether the mathematical framework proposed herein really provides for models of turbulence is far from being clear, and we do not make any claim in this respect. This issue actually pertains to Large-Eddy-Simulation models in general. We identify two obstacles in the way. First, although the notion of turbulence is quite intuitive and is a daily experience, the very concept of turbulence has yet to be mathematically defined. In particular, we are not aware of any other mathematical definition of turbulence than that proposed by Leray, who identified "*solutions turbulentes*" and weak solutions. Second, there is still a possibility that Galerkin-based solutions could eventually be shown to be suitable. In this event, the framework we proposed for LES would be either irrelevant or should be adapted to the inviscid Euler equations. Actually, many colleagues tried to convince us that our definition of LES should apply to the Euler equations. We would like to agree, but the major obstacle in the way is that existence in the large of weak solutions to these equations is not yet known. In this spirit, it would probably be interesting to see whether the present definition could be extended to apply to the notion of "dissipative" solutions introduced by P.L Lions [28, Def 4.1, p. 154]. This last argument shows again that the regularity of the solutions to the Navier–Stokes (or Euler) equations is far from being a question of pure academic interest.

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