On the static and dynamic behavior of fluid saturated composite porous solids; a homogenization approach

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Abstract

The macroscopic description of the dynamical behavior of a porous solid composed of two nonwelded solid phases saturated by a single-phase fluid is derived using two-space homogenization techniques for periodic structures. The pore size is assumed to be small compared to the macroscopic scale under consideration. At the microscopic scale the two solids are described by the linear elastic equations, and the fluid by the linearized Navier-Stokes equations, with appropriate boundary conditions at the solid-solid and solid-fluid interfaces. The nonwelded interface between the two solid phases is represented by displacement and/or velocity discontinuities proportional to the stresses across the interface, while the stresses are assumed to be continuous. After performing the homogenization procedure, constitutive relations, Darcy's and Biot's type dynamic equations for the saturated composite porous material are obtained.

Keywords: composite porous solids, homogenization.

1 Introduction

The study of the deformation and wave propagation in porous saturated media is a subject of interest in many fields such as geophysics, rock physics, material science and ocean acoustics, among others.

The fundamental concepts about the stress-strain relations and the dynamics of deformable porous single-phase solids fully saturated by a fluid were established in the works of M. Biot [1],[2],[3]. This formulation, widely accepted by the researchers in this field, assumes that the quantities measured at the macroscopic scale can be described using the concepts of the continuum mechanics. In that context, the validity of Lagrange's equations and the existence of macroscopic strain and kinetic energy densities and a dissipation function are assumed.

When the porous matrix is composed by two (or more) different solid phases, a more precise modelization is required. Following Biot's approach Leclaire et al. [4] developed a phenomenological model to describe wave propagation in a porous solid matrix where the pore space is filled with ice and water, assuming no interaction between the solid and ice particles. This formulation, valid for uniform porosity, has been extended by Carcione and Tinivella [5] to include the interaction between the solid and ice particles and grain cementation with decreasing temperature. This model has been recently generalized to the case of variable porosity [7]. These models are useful for different applications in geophysics, such as seismic amplitude and velocity analysis in shaley sandstones [6] and in permafrost areas [8],[9] and for the detection of gas-hydrate concentrations in ocean-bottom sediments from seismic data [5]. This subject has also received attention recently for the evaluation of the freezing conditions of foods by ultrasonic techniques [10], [11].

The equations governing the macroscopic behaviour of porous media can also be obtained by means of *homogenization methods*, which consist on passing from the microscopic description at the pore and grain scales to the macroscopic scale. Important contributions to the solution of this problem were given by Sanchez-Palencia

[12] and Bensoussan et al.[13], who developed the so called *two-space* homogenization technique. This method provides a systematic procedure for deriving macroscopic dynamical equations starting from the equations which govern the behaviour of the medium at the microscale. It was successfully applied by different authors to obtain a theoretical justification of Darcy's law and Biot's equations for single phase media [14] [15].

Following these ideas, the aim of this paper is to apply the homogenization procedure to obtain a rigorous description of the macroscopic behaviour of porous saturated composite media. We restrict the analysis to the range of small deformations and for the case of Newtonian fluids, under the assumption of spatial periodicity. At the solid-fluid interfaces the usual non sliding boundary condition was assumed. At the nonwelded contact between the two solid phases continuity of stresses and velocity-displacement discontinuities (proportional to the stresses across the interface) are assumed. These kind of conditions, representing purely elastic, viscous or visco-elastic slip at the interface, are supported by experimental research on fractures (see references in [16]). Using this approach the constitutive relations, a form of Darcy's law and the equations of motion for this type of saturated composite porous material are obtained.

2 Local description

Let us consider a porous medium consisting of a skeleton composed of two nonwelded solid phases, referred to by the subscripts or superscripts 1 and 3, saturated by a fluid phase indicated by the subscript or superscript f. In this work we will restrict the analysis to a simplified scheme in which one of the solids envelops the other and the fluid is only in contact with this one. Figure 1 shows two extreme configurations of this model.

The porous medium will be considered to be periodic and composed of a large number of periods, with l and L denoting the length of the period and the macroscopic length, respectively, with a ratio $\epsilon = \frac{l}{L} << 1$. The microscopic and macroscopic behavior will be described by the two dependent spatial variables x and $y = \frac{x}{\epsilon}$. In this way the properties of the medium vary rapidly on the small scale y and slowly on the large scale x, so they are considered to be functions of (x, y) [15].

Let Ω denote one period of our composite porous medium consisting in two nonwelded solid parts Ω_1 and Ω_3 and a fluid part Ω_f, so that

$$\Omega = \Omega_1 \cup \Omega_3 \cup \Omega_f,$$

with boundaries

$$\Gamma^{1f} = \partial\Omega_1 \cap \partial\Omega_f, \quad \Gamma^{13} = \partial\Omega_1 \cap \partial\Omega_3, \quad \Gamma^{je} = \partial\Omega_3 \cap \partial\Omega, \quad j = 1, f, 3,$$

so that (see Figure 1):

$$\partial\Omega_f = \Gamma^{1f} \cup \Gamma^{fe}, \quad \partial\Omega_1 = \Gamma^{1f} \cup \Gamma^{13} \cup \Gamma^{1e}, \quad \partial\Omega_3 = \Gamma^{13} \cup \Gamma^{3e}.$$

We assume that all parts are connected, that Ω_1 is completely surrounded by Ω_3 and that Γ^{13} does not intersect Γ^{1f}.

We will analyze the behaviour of the composite porous medium under a monochromatic oscillation of angular temporal frequency ω. Thus all field variables will be understood to be defined in the space-frequency domain. We also assume that at the local level the two solid phases are linear elastic and the fluid is compressible and viscous Newtonian with constant viscosity η, density ρ_f and bulk modulus B_f [14],[15]. We further assume that at this level the fluid motion is slow enough to be described by the linearized Navier-Stokes equations and the transient Reynolds number (or equivalently the dimensionless viscosity $\eta/(\omega\rho_f\epsilon^2)$) is of order unity so that the fluid viscosity is scaled by ϵ^2 [14]. Let u_j and $\sigma_j, j = 1, 3, f$ denote the displacement vectors and stress tensors of the three phases, respectively and set $v_j = i\omega u_j$. The local variables are assumed to be zero outside their domain of definition.

The local equations are:

$$\text{solid 1:} \quad \nabla \cdot \sigma_1 = -\omega^2 \rho_1 u_1, \quad \text{in} \quad \Omega_1, \tag{2.1}$$

$$\sigma_1 = a_1 : e(u_1), \quad \text{in} \quad \Omega_1, \tag{2.2}$$

$$\text{solid 3:} \quad \nabla \cdot \sigma_3 = -\omega^2 \rho_3 u_3, \quad \text{in} \quad \Omega_3, \tag{2.3}$$

$$\sigma_3 = a_3 : e(u_3), \quad \text{in} \quad \Omega_3, \tag{2.4}$$

$$\text{fluid:} \quad \nabla \cdot \sigma_f = i\omega \rho_f v_f, \quad \text{in} \quad \Omega_f, \tag{2.5}$$

$$\sigma_f = -p_f I + \tau_f, \quad \text{in} \quad \Omega_f, \tag{2.6}$$

$$\tau_f = 2\eta\epsilon^2 e(v_f), \quad \text{in} \quad \Omega_f, \tag{2.7}$$

$$i\omega p_f = B_f \nabla \cdot v_f, \quad \text{in} \quad \Omega_f. \tag{2.8}$$

Here ρ_1, ρ_3, a_1 and a_3 are respectively the mass densities and fourth-order elastic tensors associated with the two solid phases, depending on the space variable and

Ω-periodic. Also, e denotes the linear strain tensor, i.e.,

$$e_{lm}(v_f) = \frac{1}{2}\left(\frac{\partial v_{f,l}}{\partial x_m} + \frac{\partial v_{f,m}}{\partial x_l}\right).$$

Here and in what follows if a, e and u are respectively, fourth, second and first order tensors, then $a : e$ denotes the index contraction operation $a_{klst}e_{st}$, with the usual Einstein's convention of summing on repeated indices. Similarly $e \cdot u$ denotes the index contraction $e_{st}u_s$.

Next, with $\nu_{jk}, j, k = 1, 3, f$ denoting the unit outer normal at the interface Γ^{jk}, the boundary conditions among the different solid and fluid phases are

$$\sigma_1 \cdot \nu_{1f} = \sigma_f \cdot \nu_{1f}, \quad \text{on } \Gamma^{1f}, \tag{2.9}$$

$$\sigma_1 \cdot \nu_{13} = \sigma_3 \cdot \nu_{13}, \quad \text{on } \Gamma^{13}, \tag{2.10}$$

$$\sigma_1 \cdot \nu_{13} = P \cdot [u] + Q \cdot [i\omega u], \quad \text{on } \Gamma^{13}, \tag{2.11}$$

$$v_1 = v_f, \quad \text{on } \Gamma^{1f}. \tag{2.12}$$

In (2.11) the symbol $[\cdot]$ indicates the jump discontinuity in the corresponding variable at the Γ^{13} interface, i.e.

$$[u] = \left(u_1 \cdot \nu_{13} + u_3 \cdot \nu_{31}, u_1 \cdot \chi_{13}^{(1)} + u_3 \cdot \chi_{31}^{(1)}, u_1 \cdot \chi_{13}^{(2)} + u_3 \cdot \chi_{31}^{(2)}\right), \tag{2.13}$$

where $\nu_{13}, \chi_{13}^{(1)}, \chi_{13}^{(2)}$ denote the unit outward vector and two unit tangent vectors on Γ^{13} such that $\{\nu_{13}, \chi_{13}^{(1)}, \chi_{13}^{(2)}\}$ is an orthonormal set on Γ^{13}, and similarly for $\nu_{31}, \chi_{31}^{(1)}, \chi_{31}^{(2)}$. The boundary condition (2.11) is a generalization (stated in tensor form) of that given in [17], [18], [16] and models a nonwelded contact between two solid phases assuming that the stresses across Γ^{13} are continuous but displacements and/or particle velocities across Γ^{13} are discontinuous. Is is also assumed that displacement and/or velocity discontinuities are proportional to the stresses across the interface, with the proportionality factors being the second order interface tensors P and Q known as specific stiffness and specific viscosity tensors, respectively. The tensors P and Q, which are dependent on the space variable and Ω-periodic, are referred to the local basis $\left\{\nu_{13}, \chi_{13}^{(1)}, \chi_{13}^{(2)}\right\}$. For $Q = 0$ equation (2.11) represents a purely elastic contact, while for $P = 0$ we obtain a pure viscous slip.

3 The homogenization procedure

Next, following Sanchez-Palencia [12] and Auriault et al.[14], we expand the unknowns vectors u_1, u_3, u_f in asymptotic power series of ϵ in the form

$$u_i^\epsilon = u_i(x,y) = u_i^{(0)}(x,y) + \epsilon u_i^{(1)}(x,y) + \epsilon^2 u_i^{(2)}(x,y) + \cdots \quad i = 1, f, 3, \tag{3.1}$$

where the functions $u_i^{(n)}(x,y), n = 0, 1, \cdots$ are Ω-periodic. The same expansion is also used for σ_1, σ_3 and p_f. Then we substitute the expansions (3.1) into equations (2.1)–(2.12) describing the local behavior, remembering that the spatial derivatives take the form

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \epsilon^{-1}\frac{\partial}{\partial y}.$$

Similarly,

$$e = e_x + \epsilon^{-1} e_y, \quad \nabla = \nabla_x + \epsilon^{-1}\nabla_y, \quad \Delta = \Delta_x + \epsilon^{-2}\Delta_y.$$

First, from (2.2) and (2.4),

$$\sigma_j = a_j : (e_x + \epsilon^{-1} e_y)(u_j^{(0)} + \epsilon u_j^{(1)} + \cdots) \tag{3.2}$$

$$= \epsilon^{-1} a_j : e_y(u_j^{(0)}) + a_j : \left(e_x(u_j^{(0)}) + e_y(u_j^{(1)}) \right) + \cdots$$

$$= \epsilon^{-1} \sigma_j^{(-1)} + \sigma_j^{(0)} + \cdots \qquad \text{in} \quad \Omega_j,\ j = 1, 3. \tag{3.3}$$

Next, from (2.1) and (3.2)

$$\epsilon^{-2} \nabla_y \cdot \sigma_j^{(-1)} + \epsilon^{-1} \left(\nabla_x \cdot \sigma_j^{(-1)} + \nabla_y \cdot \sigma_j^{(0)} \right) \tag{3.4}$$

$$+ \epsilon^{(0)} \left(\nabla_x \cdot \sigma_j^{(0)} + \nabla_y \cdot \sigma_j^{(1)} \right) + \cdots = -\rho_j \omega^2 \left(u_j^{(0)} + \epsilon u_j^{(1)} + \cdots \right)$$

$$\text{in} \quad \Omega_j, \quad j = 1, 3.$$

Also, from (2.6)-(2.7),

$$\sigma_f^{(0)} + \epsilon \sigma_f^{(1)} + \cdots = -p_f^{(0)} I + \epsilon \left(-p_f^{(1)} I + 2\eta e(v_f^{(0)}) \right) + \cdots$$

$$= -p_f^{(0)} I + \epsilon \left(-p_f^{(1)} I + \tau_f^{(1)} + \cdots \right) \qquad \text{in} \quad \Omega_f.$$

Next we use (2.9)–(2.12) to obtain the boundary conditions for the local problems.

First, from (2.9) and (2.12),

$$\epsilon^{-1} \sigma_1^{(-1)} + \sigma_1^{(0)} + \epsilon \sigma_1^{(1)} + \cdots \tag{3.5}$$

$$= -p_f^{(0)} I + \epsilon \left(-p_f^{(1)} I + 2\eta e_y(v_f^{(0)}) \right) + \cdots \quad \text{on} \quad \Gamma^{1f}.$$

Also, from (2.10) and (2.11)

$$\left(\epsilon^{-1} \sigma_1^{(-1)} + \sigma_1^{(0)} + \epsilon \sigma_1^{(1)} + \cdots \right) \cdot \nu_{13} \tag{3.6}$$

$$= P \cdot \left[u^{(0)} \right] + Q \cdot \left[i\omega u^{(0)} \right] + \epsilon \left(P \cdot \left[u^{(1)} \right] + Q \cdot \left[i\omega u^{(1)} \right] \right) + \cdots \quad \text{on} \quad \Gamma^{13}.$$

4 Solution of the local equations at the lowest order

Let us consider the local equations for the solid at the lowest order: from (3.4) at ϵ^{-2} and (3.2), (3.5) and (3.6) at ϵ^{-1} we obtain the elliptic system

$$\nabla_y \cdot \sigma_1^{(-1)} = 0 \quad \text{in}\, \Omega_1, \tag{4.1}$$

$$\sigma_1^{(-1)} = a_1 : e_y(u_1^{(0)}) \quad \text{in}\, \Omega_1, \tag{4.2}$$

$$\sigma_1^{(-1)} \cdot \nu_{1f} = 0 \quad \text{on}\, \Gamma^{1f}, \tag{4.3}$$

$$\sigma_1^{(-1)} \cdot \nu_{13} = 0 \quad \text{on}\, \Gamma^{13}. \tag{4.4}$$

It follows from (4.1)–(4.4) that

$$u_1^{(0)}(x,y) = u_1^{(0)}(x), \tag{4.5}$$

and from (4.2) we see that

$$\sigma_1^{(-1)} = 0. \tag{4.6}$$

With identical argument, for the solid phase 3 we get

$$u_3^{(0)}(x,y) \;= u_3^{(0)}(x), \tag{4.7}$$

$$\sigma_3^{(-1)} \;= 0. \tag{4.8}$$

Next, from the fluid equations (2.5)–(2.8) we get

$$\begin{aligned} &\eta\epsilon^2 \left(\Delta_x + \epsilon^{-2}\Delta_y)(v_f^{(0)} + \epsilon v_f^{(1)} + \epsilon^2 v_f^{(2)} + \cdots\right) \\ &= \left(\nabla_x + \epsilon^{-1}\nabla_y\right)\left(p_f^{(0)} + \epsilon p_f^{(1)} + \epsilon^2 p_f^{(2)} + \cdots\right) \\ &+ i\omega\rho_f \left(v_f^{(0)} + \epsilon v_f^{(1)} + \cdots\right) \quad \text{in} \quad \Omega_f, \end{aligned} \tag{4.9}$$

and

$$i\omega\left(p_f^{(0)}+\epsilon p_f^{(1)}+\cdots\right)= \tag{4.10}$$

$$B_f\left(\nabla_x\cdot+\epsilon^{-1}\nabla_y\cdot\right)\left(v_f^{(0)}+\epsilon v_f^{(1)}+\epsilon^2 v_f^{(2)}+\cdots\right)\quad\text{in}\quad\Omega_f.$$

Thus, from (4.9) at ϵ^{-1} we get

$$p_f^{(0)}(x,y)=p_f^{(0)}(x)\quad\text{in}\quad\Omega_f. \tag{4.11}$$

The analysis in this section shows that the zero order terms $p_f^{(0)}, u_1^{(0)}$ and $u_3^{(0)}$ only depend on the macroscopic variable x.

5 Solution of the local equations for the solid phases. The next order.

Here we will find expressions for the solid displacements at order one, i.e., $u_1^{(1)}, u_3^{(1)}$. First, from 3.4) at ϵ^{-1} and (4.6) we see that

$$\nabla_y\cdot\sigma_1^{(0)}=0\quad\text{in }\Omega_1. \tag{5.1}$$

Also, from (3.2) at $\epsilon^{(0)}$,

$$\sigma_1^{(0)}=a_1:\left(e_x(u_1^{(0)})+e_y(u_1^{(1)})\right)\quad\text{in}\quad\Omega_1. \tag{5.2}$$

From (3.5)–(3.6) at $\epsilon^{(0)}$ we get the boundary conditions

$$\sigma_1^{(0)}\cdot\nu_{1f}=-p_f^{(0)}(x)\nu_{1f}\quad\text{on }\Gamma^{1f}, \tag{5.3}$$

$$\sigma_1^{(0)}\cdot\nu_{13}=P\cdot\left[u^{(0)}\right]+Q\cdot\left[i\omega u^{(0)}\right]\quad\text{on }\Gamma^{13}. \tag{5.4}$$

With identical argument we get the following equations for the solid phase 3:

$$\nabla_y \cdot \sigma_3^{(0)} = 0 \quad \text{in } \Omega_3, \tag{5.5}$$

$$\sigma_3^{(0)} = a_3 : \left(e_x(u_3^{(0)}) + e_y(u_3^{(1)}) \right) \quad \text{in } \Omega_3, \tag{5.6}$$

$$\sigma_3^{(0)} \cdot \nu_{31} = -P \cdot \left[u^{(0)} \right] - Q \cdot \left[i\omega u^{(0)} \right] \quad \text{on } \Gamma^{13}. \tag{5.7}$$

Let us formulate (5.1)–(5.4) and (5.5)–(5.7) in variational form. Set

$$\mathcal{W}_\Omega^j = \left\{ \alpha \in [H^1(\Omega_j)]^3, \quad \alpha \text{ is complex valued and } \ \Omega - \text{periodic} \right\}, j = 1, 3.$$

Then a weak form of (5.1)–(5.4) can be stated as follows: find $u_1^{(1)} \in \mathcal{W}_\Omega^1$ such that

$$\begin{aligned} \left(a_1 : e_y(u_1^{(1)}), e_y(\alpha) \right)_{\Omega_1} &= - \left(a_1 : e_x(u_1^{(0)}), e_y(\alpha) \right)_{\Omega_1} \\ &+ \int_{\Gamma^{1f}} p_f^{(0)} \nu_{1f} \, \overline{\alpha} dS + \int_{\Gamma^{13}} P \cdot \left[u^{(0)} \right] + Q \cdot \left[i\omega u^{(0)} \right] \overline{\alpha} dS, \quad \alpha \in \mathcal{W}_\Omega^1. \end{aligned} \tag{5.8}$$

In (5.8) $(\cdot, \cdot)_{\Omega_1}$ denotes the complex inner product in $L^2(\Omega_1)$, $\overline{\alpha}$ denotes the complex conjugate of α and dS is the surface measure in the corresponding surface.

Equation (5.8) can be rewritten in the equivalent form: find $u_1^{(1)} \in \mathcal{W}_\Omega^1$ such that

$$\begin{aligned} \left(a_1 : e_y(u_1^{(1)}), e_y(\alpha) \right)_{\Omega_1} = p_f^{(0)}(x) & \int_{\Gamma 1f} \nu_{1f,l} \cdot \overline{\alpha}_l dS \\ & - e_{x,lm}(u_1^{(0)}) \int_{\Omega_1} a_{1,rtlm}(x,y) e_{y,rt}(\overline{\alpha}) dy \\ & + \left(u_{1,t}^{(0)} - u_{3,t}^{(0)} \right) \Bigg[\int_{\Gamma^{13}} \left(P_{r1} + i\omega Q_{r1} \right) \nu_{13,t} \overline{\alpha}_r dS \\ & + \int_{\Gamma^{13}} \left(P_{r2} + i\omega Q_{r2} \right) \chi_{13,t}^{(1)} \overline{\alpha}_r dS + \int_{\Gamma^{13}} \left(P_{r3} + i\omega Q_{r3} \right) \chi_{13,t}^{(2)} \overline{\alpha}_r dS \Bigg], \\ & \qquad \alpha \in \mathcal{W}_\Omega^1. \end{aligned} \tag{5.9}$$

Similarly, a weak form for $u_3^{(1)}$ is given by: find $u_3^{(1)} \in \mathcal{W}_\Omega^3$ such that

$$\begin{aligned}\Big(a_3 : e_y(u_3^{(1)}), e_y(\alpha)\Big)_{\Omega_3} &= -e_{x,lm}(u_3^{(0)}) \int_{\Omega_3} a_{3,rtlm}(x,y) e_{y,rt}(\overline{\alpha}) dy \\ &+ \left(u_{1,t}^{(0)} - u_{3,t}^{(0)}\right) \bigg[\int_{\Gamma^{13}} \left(P_{r1} + i\omega Q_{r1}\right) \nu_{13,t} \overline{\alpha}_r dS \\ &+ \int_{\Gamma^{13}} \left(P_{r2} + i\omega Q_{r2}\right) \chi_{13,t}^{(1)} \overline{\alpha}_r dS + \int_{\Gamma^{13}} \left(P_{r3} + i\omega Q_{r3}\right) \chi_{13,t}^{(2)} \overline{\alpha}_r dS \bigg], \\ &\qquad \alpha \in \mathcal{W}_\Omega^3. \end{aligned} \tag{5.10}$$

The solution of (5.9) and (5.10) can be obtained by superposition as follows. For $j = 1, 3$ and $A = P$ or $A = Q$, let $Z^{1f,l}$, $V^{j,rtlm}$, $W^{j,1,rt,(A)}$, $W^{j,2,rt,(A)}$ and $W^{j,3,rt,(A)}$ be the solutions of

$$\begin{aligned} &\big(a_j : e_y(Z^{1f,l}), e_y(\alpha)\big)_{\Omega_1} = \int_{\Gamma 1f} \nu_{1f,l} \overline{\alpha}_l dS, \qquad \alpha \in \mathcal{W}_\Omega^1 \quad \text{(l not summed)}, \\ &\big(a_j : e_y(V^{j,rtlm}), e_y(\overline{\alpha})\big)_{\Omega_j} = -\int_{\Omega_j} a_{j,rtlm}(x,y) e_{y,rt}(\overline{\alpha}) dy, \\ &\qquad\qquad \alpha \in \mathcal{W}_\Omega^j \text{ (r,t not summed)}, \\ &\big(a_j : e_y(W^{j,1,rt,(A)}), e_y(\alpha)\big)_{\Omega_j} = \int_{\Gamma^{13}} A_{r1} \nu_{13,t} \overline{\alpha}_r dS, \qquad \alpha \in \mathcal{W}_\Omega^j \text{ (r not summed)}, \\ &\big(a_j : e_y(W^{j,2,rt,(A)}), e_y(\overline{\alpha})\big)_{\Omega_j} = \int_{\Gamma^{13}} A_{r2} \chi_{13,t}^{(1)} \overline{\alpha}_r dS, \qquad \alpha \in \mathcal{W}_\Omega^1 \text{ (r not summed)}, \\ &\big(a_j : e_y(W^{j,3,rt,(A)}), e_y(\alpha)\big)_{\Omega_j} = \int_{\Gamma^{13}} A_{r3} \chi_{13,t}^{(2)} \overline{\alpha}_r dS, \qquad \alpha \in \mathcal{W}_\Omega^j \text{ (r not summed)}. \end{aligned}$$

Then the solutions of (5.9) and (5.10) defined up to vectors u_1^*, u_3^* which are functions of x alone are given by

$$\begin{aligned} u_1^{(1)}(x,y) &= p_f^{(0)}(x) \sum_l Z^{1,f,l} + e_{x,lm}(u_1^{(0)}(x)) \sum_{rt} V^{1,rtlm} \\ &+ \left(u_{1,t}^{(0)}(x) - u_{3,t}^{(0)}(x)\right) \sum_n \sum_r \left(W^{1,n,rt,(P)} + i\omega W^{1,n,rt,(Q)}\right) + u_1^*, \end{aligned} \tag{5.11}$$

and

$$u_3^{(1)}(x,y) = e_{x,lm}(u_3^{(0)}(x)) \sum_{rt} V^{3,rtlm} - \left(u_{1,t}^{(0)}(x) - u_{3,t}^{(0)}(x)\right) \sum_n \sum_r \left(W^{3,n,rt,(P)} + i\omega W^{3,n,rt,(Q)}\right) + u_3^*. \tag{5.12}$$

Let the third order tensors ξ_1, ξ_3, the first order tensor β_{1f} and the second order tensors $\delta_1^{(P)}, \delta_3^{(P)}, \mu_1^{(Q)}$ and $\mu_3^{(Q)}$ be defined by

$$\begin{aligned} &\xi_{j,lms}(x,y) = \sum_{rt} V_s^{j,rtlm}, \quad j = 1,3, \quad \beta_{1f,s}(x,y) = \sum_l Z_s^{1f,l}, \\ &\delta_{1,st}^{(P)}(x,y) = \sum_n \sum_r W_s^{1,n,rt,(P)}, \quad \mu_{1,st}^{(Q)}(x,y) = \sum_n \sum_r W_s^{1,n,rt,(Q)}, \\ &\delta_{3,st}^{(P)}(x,y) = -\sum_n \sum_r W_s^{3,n,rt,(P)}, \quad \mu_{3,st}^{(Q)}(x,y) = -\sum_n \sum_r W_s^{3,n,rt,(Q)}. \end{aligned} \tag{5.13}$$

Then the solutions $u_1^{(1)} = (u_{1,s}^{(1)})$, $u_3^{(1)} = (u_{3,s}^{(1)})$ in (5.11)and (5.12) can be written in the form

$$\begin{aligned} u_{1,s}^{(1)}(x,y) = {} & \beta_{1f,s}(x,y) p_f^{(0)}(x) + \xi_{1,lms}(x,y) e_{x,lm}(u_1^{(0)}(x)) + \\ & + \delta_{1,st}^{(P)}(x,y) \left(u_{1,t}^{(0)}(x) - u_{3,t}^{(0)}(x)\right) \\ & + \mu_{1,st}^{(Q)}(x,y) \left(i\omega u_{1,t}^{(0)}(x) - i\omega u_{3,t}^{(0)}(x)\right) + u_{1,s}^*, \end{aligned} \tag{5.14}$$

$$\begin{aligned} u_{3,s}^{(1)}(x,y) = {} & \xi_{3,lms}(x,y) e_{x,lm}(u_3^{(0)}(x)) + \delta_{3,st}^{(P)} \left(u_{1,t}^{(0)}(x) - u_{3,t}^{(0)}(x)\right) \\ & + \mu_{3,st}^{(Q)} \left(i\omega u_{1,t}^{(0)}(x) - i\omega u_{3,t}^{(0)}(x)\right) + u_{3,s}^*. \end{aligned} \tag{5.15}$$

6 The constitutive relations

First, from (4.10) at $\epsilon^{(0)}$ we get

$$i\omega p_f^{(0)} B_f^{-1} = \nabla_x \cdot v_f^{(0)} + \nabla_y \cdot v_f^{(1)} \quad \text{in } \Omega_f. \tag{6.1}$$

Next, it follows from the adherence condition (2.12) that

$$v_f^{(n)} = i\omega u_1^{(n)} \quad \text{on } \Gamma^{1f}, \quad n = 0, 1, 2, \cdots . \tag{6.2}$$

As the fluid velocity field v_f depends upon both x and y, to extract its slowly varying part we will average it over the fast variable y. In general we define a volume average of $\theta(y)$ over Ω, where θ is defined to be zero outside its domain of definition, in the form

$$\langle \theta \rangle = \frac{1}{|\Omega|} \int_\Omega \theta(y) dy, \tag{6.3}$$

and we also define the surface average

$$\langle\langle \theta \rangle\rangle = \frac{1}{|\Gamma^{13}|} \int_{\Gamma^{13}} \theta(y) dS. \tag{6.4}$$

We introduce the coefficient

$$\phi = \frac{|\Omega_f|}{|\Omega|}, \tag{6.5}$$

which gives a measure of the *porosity* of the medium. Then, averaging over Ω in (6.1) and using periodicity and (6.2) we conclude that

$$\begin{aligned} \nabla_x \cdot \left\langle v_f^{(0)} \right\rangle &= \phi i\omega B_f^{-1} p_f^{(0)} - \frac{1}{|\Omega|} \int_{\partial\Omega_f} v_f^{(1)} \cdot \nu_{f1} dS \\ &= \phi i\omega B_f^{-1} p_f^{(0)} + \frac{1}{|\Omega|} \int_{\Gamma^{1f}} i\omega u_1^{(1)} \cdot \nu_{1f} dS \\ &= \phi i\omega B_f^{-1} p_f^{(0)} + \frac{1}{|\Omega|} i\omega \int_{\Omega_1} \nabla_y \cdot u_1^{(1)} dy \\ &\quad -\phi_{13} \frac{1}{|\Gamma^{13}|} i\omega \int_{\Gamma^{13}} u_1^{(1)} \cdot \nu_{13} dS, \end{aligned} \tag{6.6}$$

where

$$\phi_{13} = \frac{|\Gamma^{13}|}{|\Omega|}.$$

Next, subtract the identity

$$i\omega\phi I e_x(u_1^{(0)}) = i\omega\phi\nabla_x \cdot u_1^{(0)},$$

from (6.6) and use (5.14)-(5.15) for expressing the terms $u_j^{(1)}, j = 1, 3$ in the resulting equation. Then, defining the scalar J, the second order tensor γ_1 and the first order tensors $M^{(P)}$ and $N^{(Q)}$ by

$$\begin{aligned}
J &= \phi B_f^{-1} + \langle \nabla_y \cdot \beta_{1f} \rangle - \phi_{13} \langle\langle \beta_{1f} \cdot \nu_{13} \rangle\rangle\,, \\
\gamma_1 &= -\langle \nabla_y \cdot \xi_1 \rangle + \phi I + \phi_{13} \langle\langle \xi_1 \cdot \nu_{13} \rangle\rangle\,, \\
M^{(P)} &= -\left\langle \nabla_y \cdot \delta_1^{(P)} \right\rangle + \phi_{13} \left\langle\left\langle \delta_1^{(P)} \cdot \nu_{13} \right\rangle\right\rangle, \\
N^{(Q)} &= -\left\langle \nabla_y \cdot \mu_1^{(Q)} \right\rangle + \phi_{13} \left\langle\left\langle \mu_1^{(Q)} \cdot \nu_{13} \right\rangle\right\rangle,
\end{aligned} \tag{6.7}$$

and introducing the absolute average fluid displacement in the form

$$U_f^{(0)} = \frac{1}{\phi} \left\langle u_f^{(0)} \right\rangle \tag{6.8}$$

we obtain the following equation for the macroscopic fluid pressure $p_f^{(0)}$

$$\begin{aligned}
\phi p_f^{(0)} &= \frac{\phi^2}{J} \nabla_x \cdot U_f^{(0)} + \frac{\phi}{J} (\gamma_1 - \phi I) : e_x(u_1^{(0)}) \\
&+ \frac{M^{(P)}\phi}{J} \cdot \left(u_1^{(0)} - u_3^{(0)}\right) + \frac{N^{(Q)}\phi}{J} \cdot \left(i\omega u_1^{(0)} - i\omega u_3^{(0)}\right) + \frac{\phi\phi_{13}}{J} u_1^* \cdot \langle\langle \nu_{13} \rangle\rangle\,.
\end{aligned} \tag{6.9}$$

Next, from (3.4) at ϵ^0 we have

$$\sigma_j^{(0)} = a_j : e_x(u_j^{(0)}) + a_j : e_y(u_j^{(1)}), \quad j = 1, 3. \tag{6.10}$$

Using (5.14) and averaging over Ω in (6.10) we conclude that the macroscopic stress in solid phase 1 takes the form

$$\left\langle \sigma_1^{(0)} \right\rangle = C_1 : e_x(u_1^{(0)}) + \langle a_1 : e_y(\beta_{1f}) \rangle \, p_f^{(0)} + \left\langle a_1 : e_y(\delta_1^{(P)}) \right\rangle \cdot \left(u_1^{(0)} - u_3^{(0)} \right)$$
$$+ \left\langle a_1 : e_y(\mu_1^{(Q)}) \right\rangle \cdot \left(i\omega u_1^{(0)} - i\omega u_3^{(0)} \right), \tag{6.11}$$

where C_1 is the fourth order tensor given by

$$C_1 = \langle a_1 : (I + e_y(\xi_1)) \rangle \,. \tag{6.12}$$

Thus, using (6.9) in (6.11) we obtain

$$\left\langle \sigma_1^{(0)} \right\rangle = \left[C_1 + \langle a_1 : e_y(\beta_{1f}) \rangle \frac{1}{J} \left(\gamma_1 - \phi I \right) \right] : e_x(u_1^{(0)}) \tag{6.13}$$
$$+ \langle a_1 : e_y(\beta_{1f}) \rangle \frac{\phi}{J} \nabla_x \cdot U_f^{(0)} + \langle a_1 : e_y(\beta_{1f}) \rangle \frac{\phi_{13}}{J} u_1^* \cdot \langle\langle \nu_{13} \rangle\rangle$$
$$+ \left[\left\langle a_1 : e_y(\delta_1^{(P)}) \right\rangle + \frac{M^{(P)}}{J} \langle a_1 : e_y(\beta_{1f}) \rangle \right] \cdot \left(u_1^{(0)} - u_3^{(0)} \right)$$
$$+ \left[\left\langle a_1 : e_y(\mu_1^{(Q)}) \right\rangle + \frac{N^{(Q)}}{J} \langle a_1 : e_y(\beta_{1f}) \rangle \right] \cdot \left(i\omega u_1^{(0)} - i\omega u_3^{(0)} \right).$$

Similarly, using (5.15) in (6.10) for the macroscopic stress in solid phase 3 we obtain

$$\left\langle \sigma_3^{(0)} \right\rangle = C_3 : e_x(u_3^{(0)}) + \left\langle a_3 : e_y(\delta_3^{(P)}) \right\rangle \cdot \left(u_1^{(0)} - u_3^{(0)} \right) \tag{6.14}$$
$$+ \left\langle a_3 : e_y(\mu_3^{(Q)}) \right\rangle \cdot \left(i\omega u_1^{(0)} - i\omega u_3^{(0)} \right),$$

where the fourth order tensor C_3 is defined by

$$C_3 = \langle a_3 : (I + e_y(\xi_3)) \rangle \,. \tag{6.15}$$

Remark: equations (6.9), (6.13) and (6.14) are the macroscopic constitutive relations for $\left\langle \sigma_1^{(0)} \right\rangle$ and $\left\langle \sigma_3^{(0)} \right\rangle$ and the fluid pressure $p_f^{(0)}$. The coefficients in these relations contain information about the size and geometry of the interface Γ^{13} between the two solid phases and explicitly show how the microscopic displacement and/or particle velocity discontinuities affect the macroscopic stresses and fluid pressure.

7 The equations of motion

7.1 Derivation of Darcy's law

Now we shall obtain a general relation between the relative velocity of the fluid and the macroscopic pressure gradient through a generalized permeability tensor, taking into account the elastic deformation of the solid in contact with the fluid.

First, from (4.9) at $\epsilon^{(0)}$ we get

$$\eta \Delta_y v_f^{(0)} = \nabla_y \cdot \tau_f^{(1)}(v_f^{(0)}) = \nabla_y p_f^{(1)} + \nabla_x p_f^{(0)} + i\omega\rho_f v_f^{(0)} \quad \text{in} \quad \Omega_f, \tag{7.1}$$

and from (4.10) at ϵ^{-1}

$$\nabla_y \cdot v_y^{(0)} = 0 \quad \text{in} \quad \Omega_f. \tag{7.2}$$

We now introduce a vector field $\widetilde{v}_f^{(0)}$ representing the zero order relative velocity of the fluid in the frequency domain, given by

$$\widetilde{v}_f^{(0)} = v_f^{(0)} - i\omega u_1^{(0)}(x), \quad \text{in} \quad \Omega_f \tag{7.3}$$

so that

$$\widetilde{v}_f^{(0)} = 0 \quad \text{on } \Gamma^{1f}. \tag{7.4}$$

In terms of this relative flow equations (7.1)–(7.2) become

$$-\nabla_y \cdot \tau_f^{(1)}(\widetilde{v}_f^{(0)}) + \nabla_y p_f^{(1)} + i\omega\rho_f \widetilde{v}_f^{(0)} = f^{(0)}(x,\omega) \quad \text{in } \Omega_f, \tag{7.5}$$

$$\nabla_y \cdot \widetilde{v}_f^{(0)} = 0 \quad \text{in } \Omega_f, \tag{7.6}$$

where

$$f^{(0)}(x,\omega) = -\left[\nabla_x p_f^{(0)} + i\omega\rho_f v_1^{(0)}\right]. \tag{7.7}$$

Let us solve the cell problem (7.5)- (7.6) for $\widetilde{v}_f^{(0)}$ with the boundary condition (7.4). Let

$$\mathcal{V}_{\Omega_f} = \left\{\varphi \in [H^1(\Omega_f)]^3,\ \nabla_y \cdot \varphi = 0,\, \varphi = 0 \text{ on } \Gamma^{1f},\right. \tag{7.8}$$

$$\left.\varphi \text{ is complex valued and }, \Omega - \text{periodic}\right\}, \tag{7.9}$$

provided with the natural (complex) inner product in $L^2(\Omega_f)$, denoted by $(\cdot,\cdot)_{\Omega_f}$. Then a variational formulation of (7.5), (7.6) and (7.4) can be stated as follows: Find $\widetilde{v}_f^{(0)} \in \mathcal{V}_{\Omega_f}$ such that

$$\left(\eta\nabla\widetilde{v}_f^{(0)}, \nabla\varphi\right)_{\Omega_f} + i\omega\left(\rho_f\widetilde{v}_f^{(0)}, \varphi\right)_{\Omega_f} = f^{(0)}(x,\omega)\cdot\int_{\Omega_f}\overline{\varphi}dy, \quad \varphi \in \mathcal{V}_{\Omega_f}. \tag{7.10}$$

It is known that (7.10) has a unique solution, which can be found as usual by solving the following set of problems [12]. Let V^s for s=1,2,3 be particular solutions of the problem

$$(\eta\nabla V^s, \nabla\varphi)_{\Omega_f} + i\omega\left(\rho_f V^s, \varphi\right)_{\Omega_f} = \int_{\Omega_f}\overline{\varphi}_s dy, \quad \varphi \in \mathcal{V}_{\Omega_f}, \tag{7.11}$$

and set the second order tensor $\mathcal{K}$ given by

$$\mathcal{K}(x,y,\omega) = (\mathcal{K})_{sj} = V_j^s. \tag{7.12}$$

Then by linearity we obtain the solution

$$\widetilde{v}_f^{(0)} = \mathcal{K}\cdot f^{(0)}(x,\omega). \tag{7.13}$$

Integrating (7.13) over Ω_f and dividing by $|\Omega|$ the resulting equation we obtain

$$\left\langle v_f^{(0)} \right\rangle - i\omega\phi u_1^{(0)} = - \langle \mathcal{K} \rangle \cdot \left[\nabla_x p_f^{(0)} + i\omega\rho_f v_1^{(0)} \right], \tag{7.14}$$

which is the form of Darcy's law for this system, being $\langle \mathcal{K}(x,\omega) \rangle$ its generalized permeability [14].

7.2 Dynamic equilibrium equations

In this section we will find a set of coupled differential equilibrium equations governing the macroscopic motion of the three phases.

To formulate the equation associated to the fluid phase, first we introduce the second order tensor H given by

$$H = \langle \mathcal{K} \rangle^{-1} = H_1 + iH_2. \tag{7.15}$$

Then, using (7.15) and (6.8) we can rewrite (7.14) in the form

$$\begin{aligned} -\nabla_x p_f^{(0)} = & -\omega^2 \left(\rho_f I - \frac{\phi H_2}{\omega} \right) \cdot u_1^{(0)} - \omega^2 \left(\frac{\phi H_2}{\omega} \right) \cdot U_f^{(0)} \\ & + i\omega\phi H_1 \cdot \left(U_f^{(0)} - u_1^{(0)} \right). \end{aligned} \tag{7.16}$$

Next, we obtain the corresponding equations for the solids starting from (3.4) at $\epsilon^{(0)}$

$$\nabla_x \cdot \sigma_1^{(0)} + \nabla_y \cdot \sigma_1^{(1)} = -\rho_1 \omega^2 u_1^{(0)} \quad \text{in } \Omega_1, \tag{7.17}$$

$$\nabla_x \cdot \sigma_3^{(0)} + \nabla_y \cdot \sigma_3^{(1)} = -\rho_3 \omega^2 u_3^{(0)} \quad \text{in } \Omega_3. \tag{7.18}$$

Also, (7.1) can be stated in the form

$$\nabla_y \cdot \sigma_f^{(1)} = \nabla_x p_f^{(0)} + i\omega\rho_f v_f^{(0)} \quad \text{in } \Omega_f. \tag{7.19}$$

Next, using (2.9), (2.11) and periodicity in the y-variable to cancel the outer boundary terms,

$$\int_{\Omega_1} \nabla_y \cdot \sigma_1^{(1)} dy + \int_{\Omega_f} \nabla_y \cdot \sigma_f^{(1)} dy = \int_{\partial\Omega_1} \sigma_1^{(1)} \cdot \nu_1 dS + \int_{\partial\Omega_f} \sigma_f^{(1)} \cdot \nu_f dS$$
$$= \int_{\Gamma^{13}} \sigma_1 \cdot \nu_{13} dS = \int_{\Gamma^{13}} \left(P \cdot \left[u^{(1)}\right] + Q \cdot \left[i\omega u^{(1)}\right]\right) dS. \quad (7.20)$$

Thus averaging (7.17) and (7.19) over Ω and adding the resulting equations we conclude that

$$\nabla_x \cdot \left\langle \sigma_1^{(0)} \right\rangle - \nabla_x \cdot \left\langle p_f^{(0)} I \right\rangle + \phi_{13} T^{13} = -\omega^2 \left\langle \rho_1 \right\rangle u_1^{(0)} + i\omega \rho_f \left\langle v_f^{(0)} \right\rangle, \quad (7.21)$$

where

$$T^{13} = \frac{1}{|\Gamma^{13}|} \int_{\Gamma^{13}} \left(P \cdot \left[u^{(1)}\right] + Q \cdot \left[i\omega u^{(1)}\right]\right) dS. \quad (7.22)$$

Next use (2.13) and the expressions for $u_1^{(1)}, u_3^{(1)}$ given in (5.14)-(5.15) to compute the boundary term T^{13} in (7.21). For this purpose it is convenient to define the first order tensors $\Lambda^{(P)}, \Lambda^{(Q)}$, the second order tensors $Z^{(P)}, Z^{(Q)}$, $E^{(P)}$, $E^{(Q)}, E^{(P,Q)}$, $E^{(Q,P)}$ and the third order tensors $\alpha_1^{(P)}, \alpha_3^{(P)}$ given by

$$\begin{aligned}
Z_{sl}^{(P)} &= P_{s1}\nu_{13,l} + P_{s2}\chi_{13,l}^{(1)} + P_{s3}\chi_{13,l}^{(2)}, \\
Z_{sl}^{(Q)} &= Q_{s1}\nu_{13,l} + Q_{s2}\chi_{13,l}^{(1)} + Q_{s3}\chi_{13,l}^{(2)}, \\
\alpha_{j,smr}^{(P)} &= Z_{sl}^{(P)}\xi_{j,lmr}, \qquad \alpha_{j,smr}^{(Q)} = Z_{sl}^{(Q)}\xi_{j,lmr}, \quad j = 1,3, \qquad (7.23) \\
\Lambda_s^{(P)} &= Z_{sl}^{(P)}\beta_{1f,l}, \qquad \Lambda_s^{(Q)} = Z_{sl}^{(Q)}\beta_{1f,l}, \\
E_{st}^{(P)} &= Z_{sl}^{(P)}\left(\delta_{1,lt}^{(P)} - \delta_{3,lt}^{(P)}\right), \quad E_{st}^{(P,Q)} = Z_{sl}^{(P)}\left(\mu_{1,lt}^{(Q)} - \mu_{3,lt}^{(Q)}\right) \\
E_{st}^{(Q,P)} &= Z_{sl}^{(Q)}\left(\delta_{1,lt}^{(P)} - \delta_{3,lt}^{(P)}\right), \quad E_{st}^{(Q)} = Z_{sl}^{(Q)}\left(\mu_{1,lt}^{(Q)} - \mu_{3,lt}^{(Q)}\right).
\end{aligned}$$

Thus, using these definitions in (7.22) and taking into account that $\left\langle p_f^{(0)} I \right\rangle = \phi I p_f^{(0)}$ equation (7.21) becomes

$$\nabla_x \cdot \left\langle \sigma_1^{(0)} \right\rangle - \phi \nabla_x p_f^{(0)} + \phi_{13} \Big[\left\langle \left\langle \alpha_1^{(P)} \right\rangle \right\rangle : e_x(u_1^{(0)}) + \left\langle \left\langle \alpha_3^{(P)} \right\rangle \right\rangle : e_x(u_3^{(0)}) \tag{7.24}$$
$$+ \left\langle \left\langle \Lambda^{(P)} \right\rangle \right\rangle p_f^{(0)} + \left\langle \left\langle E^{(P)} \right\rangle \right\rangle \cdot \left(u_1^{(0)} - u_3^{(0)} \right)$$
$$+ \left\langle \left\langle E^{(P,Q)} \right\rangle \right\rangle \cdot \left(i\omega u_1^{(0)} - i\omega u_3^{(0)} \right) + \left\langle \left\langle \alpha_1^{(Q)} \right\rangle \right\rangle : e_x(i\omega u_1^{(0)})$$
$$+ \left\langle \left\langle \alpha_3^{(Q)} \right\rangle \right\rangle : e_x(i\omega u_3^{(0)}) + \left\langle \left\langle \Lambda^{(Q)} \right\rangle \right\rangle i\omega p_f^{(0)} + \left\langle \left\langle E^{(Q,P)} \right\rangle \right\rangle \cdot \left(i\omega u_1^{(0)} - i\omega u_3^{(0)} \right)$$
$$-\omega^2 \left\langle \left\langle E^{(Q)} \right\rangle \right\rangle \cdot \left(u_1^{(0)} - u_3^{(0)} \right) + \left\langle \left\langle Z^{(P)} \right\rangle \right\rangle \cdot (u_1^* - u_3^*) + \left\langle \left\langle Z^{(Q)} \right\rangle \right\rangle \cdot (i\omega u_1^* - i\omega u_3^*) \Big]$$
$$= -\omega^2 \langle \rho_1 \rangle u_1^{(0)} - \omega^2 \rho_f \phi U_f^{(0)}.$$

Next using in (7.24) the expressions for $p_f^{(0)}$ and its gradient obtained in (6.9) and (7.16), respectively, we conclude that the equilibrium equation associated to the solid phase 1 takes the following form:

$$\nabla_x \cdot \left\langle \sigma_1^{(0)} \right\rangle + G_1^{(P)} : e_x(u_1^{(0)}) + G_3^{(P)} : e_x(u_3^{(0)}) + \phi_{13} \frac{\phi}{J} \left\langle \left\langle \Lambda^{(P)} \right\rangle \right\rangle \nabla_x \cdot U_f^{(0)} \tag{7.25}$$
$$+ D_1^{(Q)} : e_x(i\omega u_1^{(0)}) + D_3^{(Q)} : e_x(i\omega u_3^{(0)}) + \phi_{13} \frac{\phi}{J} \left\langle \left\langle \Lambda^{(Q)} \right\rangle \right\rangle \nabla_x \cdot \left(i\omega U_f^{(0)} \right)$$
$$= -\omega^2 \left[\langle \rho_1 I \rangle - \phi \left(\rho_f I - \phi \frac{H_2}{\omega} \right) - F^{(Q)} \right] \cdot u_1^{(0)} - \omega^2 F^{(Q)} \cdot u_3^{(0)}$$
$$-\omega^2 \phi \left(\rho_f I - \phi \frac{H_2}{\omega} \right) \cdot U_f^{(0)} - i\omega H_1 \phi^2 \cdot \left(U_f^{(0)} - u_1^{(0)} \right) - F^{(P)} \cdot \left(u_1^{(0)} - u_3^{(0)} \right)$$
$$-F^{(P,Q)} \cdot \left(i\omega u_1^{(0)} - i\omega u_3^{(0)} \right) - \phi_{13} \left\langle \left\langle Z^{(P)} \right\rangle \right\rangle \cdot (u_1^* - u_3^*)$$
$$-\phi_{13} \left\langle \left\langle Z^{(Q)} \right\rangle \right\rangle \cdot (i\omega u_1^* - i\omega u_3^*) - \frac{(\phi_{13})^2}{J} \left\langle \left\langle \Lambda^{(P)} \right\rangle \right\rangle u_1^* \cdot \langle\langle \nu_{13} \rangle\rangle$$
$$-\frac{(\phi_{13})^2}{J} \left\langle \left\langle \Lambda^{(Q)} \right\rangle \right\rangle i\omega u_1^* \cdot \langle\langle \nu_{13} \rangle\rangle .$$

In (7.25) we have introduced the second order tensors $F^{(P)}, F^{(P,Q)}$, $F^{(Q)}$ and the third order tensors $G_1^{(P)}, G_3^{(P)}, D_1^{(Q)}, D_3^{(Q)}$ by the formulae

$$
\begin{aligned}
G_1^{(P)} &= \phi_{13}\left(\left\langle\left\langle \alpha_1^{(P)}\right\rangle\right\rangle + \frac{1}{J}\left(\gamma_j - \phi I\right)\left\langle\left\langle \Lambda^{(P)}\right\rangle\right\rangle\right), \quad G_3^{(P)} = \phi_{13}\left\langle\left\langle \alpha_3^{(P)}\right\rangle\right\rangle, \\
D_1^{(Q)} &= \phi_{13}\left(\left\langle\left\langle \alpha_1^{(Q)}\right\rangle\right\rangle + \frac{1}{J}\left(\gamma_1 - \phi I\right)\left\langle\left\langle \Lambda^{(Q)}\right\rangle\right\rangle\right), \quad D_3^{(Q)} = \phi_{13}\left\langle\left\langle \alpha_3^{(Q)}\right\rangle\right\rangle, \\
F^{(P)} &= \phi_{13}\left(\left\langle\left\langle E^{(P)}\right\rangle\right\rangle + \frac{M^{(P)}}{J}\left\langle\left\langle \Lambda^{(P)}\right\rangle\right\rangle\right), \\
F^{(P,Q)} &= \phi_{13}\left(\left\langle\left\langle E^{(P,Q)}\right\rangle\right\rangle + \left\langle\left\langle E^{(Q,P)}\right\rangle\right\rangle + \frac{N^{(Q)}}{J}\left\langle\left\langle \Lambda^{(P)}\right\rangle\right\rangle + \frac{M^{(P)}}{J}\left\langle\left\langle \Lambda^{(Q)}\right\rangle\right\rangle\right), \\
F^{(Q)} &= \phi_{13}\left(\left\langle\left\langle E^{(Q)}\right\rangle\right\rangle + \frac{N^{(Q)}}{J}\left\langle\left\langle \Lambda^{(Q)}\right\rangle\right\rangle\right).
\end{aligned}
$$

In the same fashion, to obtain the equation of motion associated with the solid phase 3 we average (7.18) over Ω to conclude that

$$
\nabla_x \cdot \left\langle \sigma_3^{(0)} \right\rangle - \phi_{13} T^{13} = -\omega^2 \left\langle \rho_3 \right\rangle u_3^{(0)}. \tag{7.26}
$$

Finally, applying in (7.26) the argument leading to (7.25) we get the following macroscopic equation of motion for the solid phase 3:

$$
\begin{aligned}
\nabla_x \cdot \left\langle \sigma_3^{(0)} \right\rangle &- G_1^{(P)} : e_x(u_1^{(0)}) - G_3^{(P)} : e_x(u_3^{(0)}) - \phi_{13}\frac{\phi}{J}\left\langle\left\langle \Lambda^{(P)}\right\rangle\right\rangle \nabla_x \cdot U_f^{(0)} \\
&- D_1^{(Q)} : e_x(i\omega u_1^{(0)}) - D_3^{(Q)} : e_x(i\omega u_3^{(0)}) - \phi_{13}\frac{\phi}{J}\left\langle\left\langle \Lambda^{(Q)}\right\rangle\right\rangle \nabla_x \cdot \left(i\omega U_f^{(0)}\right) \\
&= -\omega^2\left[\left\langle \rho_3 I\right\rangle - F^{(Q)}\right] \cdot u_3^{(0)} - \omega^2 F^{(Q)} \cdot u_1^{(0)} + F^{(P)} \cdot \left(u_1^{(0)} - u_3^{(0)}\right) \\
&+ F^{(P,Q)} \cdot \left(i\omega u_1^{(0)} - i\omega u_3^{(0)}\right) + \phi_{13}\left\langle\left\langle Z^{(P)}\right\rangle\right\rangle \cdot \left(u_1^* - u_3^*\right) \\
&+ \phi_{13}\left\langle\left\langle Z^{(Q)}\right\rangle\right\rangle \cdot \left(i\omega u_1^* - i\omega u_3^*\right) + \frac{(\phi_{13})^2}{J}\left\langle\left\langle \Lambda^{(P)}\right\rangle\right\rangle u_1^* \cdot \left\langle\left\langle \nu_{13}\right\rangle\right\rangle \\
&+ \frac{(\phi_{13})^2}{J}\left\langle\left\langle \Lambda^{(Q)}\right\rangle\right\rangle i\omega u_1^* \cdot \left\langle\left\langle \nu_{13}\right\rangle\right\rangle.
\end{aligned} \tag{7.27}
$$

Equations (7.16), (7.25) and (7.27) are the equations of motion for our composite system. The equation of motion for the fluid (7.16) is the same that for the classic

Biot's theory, due to the restricted geometrical configuration being analyzed in which only one solid phase is in contact with the fluid. The equations of motion (7.25) and (7.27), expressed in terms of the macroscopic particle displacements $u_1^{(0)}, u_3^{(0)}$ and $U_f^{(0)}$, contain zero order jumps, first order spatial derivatives and mass terms relating such macroscopic variables, all of them with coefficients indicated by the superindices $^{(P)}$, $^{Q)}$ and $^{(P,Q)}$. The equation of motion (7.25) for the solid phase 1 also contains the classic viscous dissipative Darcy term related to the relative fluid flow between the solid phase 1 and the fluid.

8 Conclusions

We have obtained the constitutive relations and the equations of motion representing the macroscopic monochromatic motion of a fluid saturated porous solid in which the matrix is composed of two nonwelded solid phases by employing the two-space homogenization procedure. The analysis is carried over for the case in which the local Reynolds number (or equivalently, the dimensionless viscosity $\eta/(\omega\rho_f\epsilon^2)$ is of order unity. As expected, the behavior of the composite system is determined by the boundary conditions at the solid-solid interface, where it is assumed that at the microscopic level the stresses are continuous while the displacement and/or particle velocities are discontinuous. The macroscopic equations obtained display the static and dynamic interaction among the three phases. The jump in the microscopic displacements and/or velocities introduce jumps in the corresponding macroscopic displacements and/or velocities in the constitutive relations (6.11) and (6.14). Those microscopic jumps also introduce zero order jumps and first order terms in the macroscopic equations of motion (7.25) and (7.27) for the two solid phases, as well as dissipative terms related to the difference between the macroscopic particle velocities of the two solid phases. Since only one solid phase *sees* the fluid phase, the equation of motion for the fluid (7.16) reduces to that of the classic Biot theory as in references [14], [15].

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References

[1] M. A. Biot, J. Acoust. Soc. Am. 28 (1956) 168-171.

[2] M. A. Biot, J. Acoust. Soc. Am. 28(1956) 179-191.

[3] M. A.Biot, J. Appl. Physics 33 (4) (1962) 1482–1498.

[4] Ph. Leclaire, F. Cohen-Tenoudji and J. Aguirre Puente, J. Acoust. Soc. Amer. 96 (6) (1994) 3753–3767.

[5] J. M. Carcione and U. Tinivella, Geophysics 65 (1) (2000) 54–67.

[6] J. M. Carcione, B. Gurevich and F. Cavallini, Geophysical Prospecting 48 (2000) 539–557.

[7] J.E. Santos, C.L. Ravazzoli and J.M. Carcione, Journal of the Acoustical Society of America 115 (6) (2004) 2749-2760.

[8] J. L. Morack and J. C. Rogers, Arctic 3 (1981) 166–174.

[9] J. M. Carcione and G. Seriani, Geophysical Prospecting 46 (1998) 441–454.

[10] S. Lee, P. Cornillon and O. Campanella,2002 Annual Meeting and Food Expo., Anaheim (CA).

[11] S. Lee, L. Pyrak-Nolte, P. Cornillon and O. Campanella, to appear in Journal of the Science of Food and Agriculture.

[12] E. Sanchez-Palencia, Non-homogeneous media and vibration theory, Lecture Notes in Physics, Springer-Verlag, New York, 1980.

[13] A. Bensoussan,J. L. Lions and G. C. Papanicolaou, Asymptotic analysis for periodic structures, in: Studies in Mathematics and its Applications, 5, North-Holland, 1978.

[14] J. L. Auriault, L. Borne and R. Chambon, J. Acoust. Soc. Am. 77 (5) (1985) 1641-1650.

[15] R. Burridge and J. B. Keller, J. Acoust. Soc. Amer. 70 (4) (1981) 1140-1146.

[16] L. J. Pyrak-Nolte, L. R. Myer and N. G. Cook, J. Geophys. Research 95 (B7) (1990) 11345-11358.

[17] J.M. Carcione, Wave fields in real media: Wave propagation in anisotropic, anelastic and porous media, Handbook of Geophysical Exploration 31, Pergamon Press Inc., Amsterdam, 2001.

[18] M. Schoenberg, J. Acoust. Soc. Am., 68 (5) (1980) 1516–1521.

Figure 1

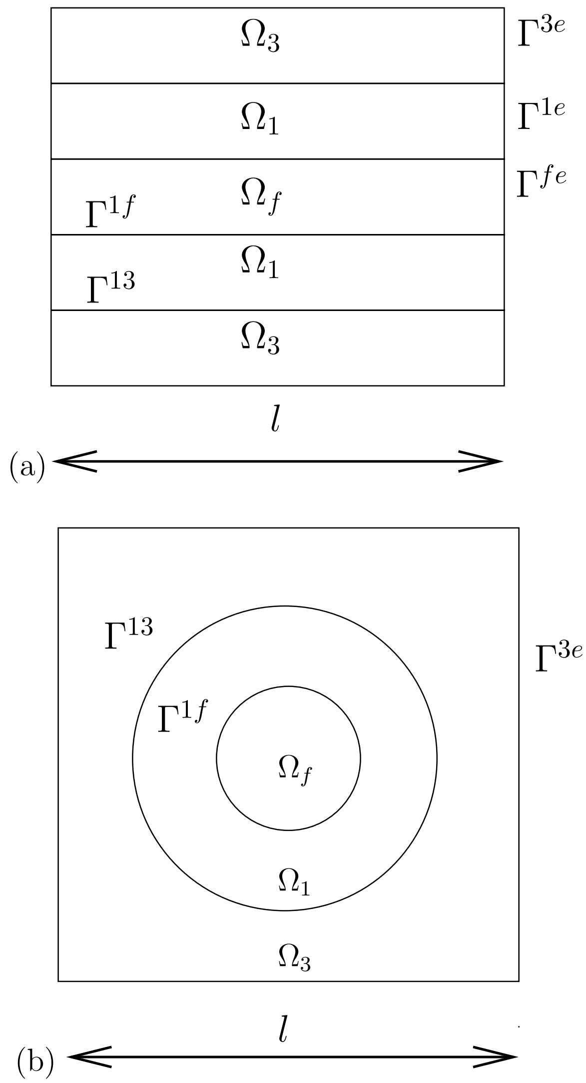
Ω_3
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Figure Captions

Figure 1: Some simple examples of the composite homogenization volume.