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A Finite Volume Element Method for a Nonlinear Elliptic Problem

P. Chatzipantelidis, V. Ginting and R. D. Lazarov*

Department of Mathematics, Texas A&M University, College Station, TX, 77843

Dedicated to Owe Axelsson on the occasion of his 70th birthday

SUMMARY

We consider a finite volume discretization of second order nonlinear elliptic boundary value problems on polygonal domains. For sufficiently small data, we show existence and uniqueness of the finite volume solution using a fixed point iteration method. We derive error estimates in H^1-, L_2- and $L_\infty-$ norm. In addition a Newton's method is analyzed for the approximation of the finite volume solution and numerical experiments are presented. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: finite volume element method, nonlinear elliptic equation, error estimates, fixed point iterations, Newton's method

1. INTRODUCTION

We analyze a finite volume element method for the discretization of second order nonlinear elliptic partial differential equations on a polygonal domain $\Omega \subset \mathbb{R}^2$. Namely, for a given function f we seek u such that

$$L(u)u \equiv -\nabla \cdot (A(u)\nabla u) = f \quad \text{in } \Omega, \quad \text{and} \quad u = 0, \quad \text{on } \partial\Omega, \tag{1.1}$$

with $A : \mathbb{R} \to \mathbb{R}$ sufficiently smooth such that there exist constants β_i, $i = 1, 2, 3$, satisfying

$$0 < \beta_1 \le A(x) \le \beta_2, \quad |A'(x)| \le \beta_3, \quad \text{for } x \in \mathbb{R}. \tag{1.2}$$

Finite volume approximations rely on the local conservation property expressed by the differential equation. Namely, integrating (1.1) over any region $V \subset \Omega$ and using Green's formula, we obtain

$$-\int_{\partial V} (A(u)\nabla u) \cdot n \, ds = \int_V f \, dx, \tag{1.3}$$

*Correspondence to: R. D. Lazarov, Department of Mathematics, Texas A&M University, College Station, TX, 77843 (e-mail:lazarov@math.tamu.edu)

where n denotes the unit exterior normal to ∂V.

There are various approaches in deriving finite volume approximations of nonlinear elliptic equations. One, often called finite volume element method, uses a finite element partition of Ω, where the solution space consists of continuous piecewise linear functions, a collection of vertex centered control volumes and a test space of piecewise constant functions over the control volumes, cf., e.g., [5, 20, 19]. A second approach, usually called finite volume difference method, uses cell-centered grids and approximates the derivatives in the balance equation by finite differences, cf., e.g., [16]. A third, uses mixed reformulation of the problem, [23]. The first approach is quite close to the finite element method. The second approach is closer to the classical finite difference method and extends it to more general than rectangular meshes. It is used mostly on PEBI or Voronoi type of meshes. The third approach is close to mixed and hybrid finite element methods and can deal for example with irregular quadrilateral and hexahedral cells. Finite volume discretizations for more general nonlinear convection–diffusion–reaction problems were studied by many authors, cf., e.g., [12, 17].

We shall use the standard notation for the Sobolev spaces W_p^s and $H^s = W_2^s$, cf., [1]. Namely, $L_p(V)$, $1 \le p < \infty$, denotes the p–integrable real–valued functions over $V \subset \mathbb{R}^2$, $(\cdot,\cdot)_V$ the inner product in $L_2(V)$, and $\|\cdot\|_{W_p^s(V)}$ the norm in the Sobolev space $W_p^s(V)$, $s \ge 0$. If $V = \Omega$ we suppress the index V, and if $p = 2$ we write $H^s = W_2^s$ and $\|\cdot\| = \|\cdot\|_{L_2}$. Further we shall denote with p' the adjoint of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$, $p > 1$.

It is well known that for domains with smooth boundary, for $f \in C^r$, with $r \in (0,1)$, there exists a unique solution $u \in C^{2+r}$, cf., e.g., [14]. Also for $\|f\|$ sufficiently small, there exists a unique solution $u \in H^2 \cap H_0^1$. However, here, since we assume the domain Ω to be polygonal, we do not expect the solution u to have such regularity. We shall assume that for $f \in L_2$, problem (1.1) has a solution $u \in W_q^2 \cap H_0^1$, with $4/3 < q \le 2$. Note that in order (1.3) to be well defined, $u \in H^{1+s}$ with $s > 1/2$. Using a standard Sobolev embedding we see that for $u \in W_q^2$, with $q > 4/3$ this is true.

We shall study approximations of (1.1) by the finite volume element method, which for brevity we shall refer to as the finite volume method below. The approximate solution will be sought in the piecewise linear finite element space

$$X_h \equiv X_h(\Omega) = \{\chi \in C(\Omega) : \ \chi|_K \ \text{ linear}, \ \forall K \in \mathcal{T}_h; \ \chi|_{\partial\Omega} = 0\},$$

where $\{\mathcal{T}_h\}_{0<h<1}$ is a family of quasi-uniform triangulations of Ω, h denotes the maximum diameter of the triangles of $\mathcal{T}_h$.

The discrete finite volume problem will satisfy a relation similar to (1.3) for V in a finite collection of subregions of Ω called control volumes, the number of which will be equal to the dimension of the finite element space X_h. These control volumes are constructed in the following way. Let z_K be the barycenter of $K \in \mathcal{T}_h$. We connect z_K with line segments to the midpoints of the edges of K, thus partitioning K into three quadrilaterals K_z, $z \in Z_h(K)$, where $Z_h(K)$ are the vertexes of K. Then with each vertex $z \in Z_h = \cup_{K \in \mathcal{T}_h} Z_h(K)$ we associate a control volume V_z, which consists of the union of the subregions K_z, sharing the vertex z (see Figure 1). We denote the set of interior vertexes of Z_h by Z_h^0.

The finite volume method is then to find $u_h \in X_h$ such that

$$-\int_{\partial V_z} (A(u_h)\nabla u_h) \cdot n \, ds = \int_{V_z} f \, dx, \quad \forall z \in Z_h^0. \tag{1.4}$$

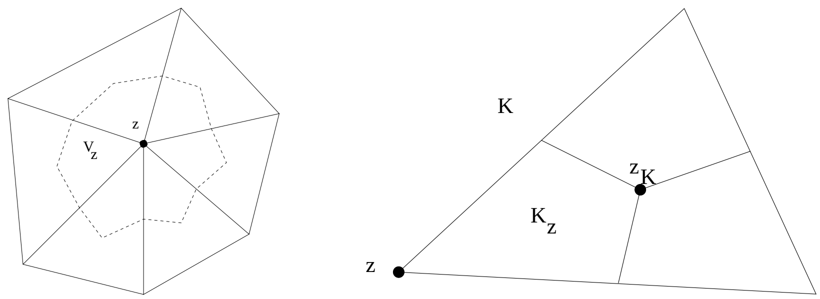

Figure 1. *Left:* A union of triangles that have a common vertex z; the dotted line shows the boundary of the corresponding control volume V_z. *Right:* A triangle K partitioned into the three subregions K_z.

The Galerkin finite element method for (1.1) is: Find $\underline{u}_h \in X_h$ such that

$$a(\underline{u}_h; \underline{u}_h, \chi) = (f, \chi), \quad \forall \chi \in X_h, \tag{1.5}$$

with $a(\cdot; \cdot, \cdot)$ the form defined by

$$a(v; w, \phi) = \int_\Omega A(v) \nabla w \cdot \nabla \phi \, dx.$$

It is known that the solution $\underline{u}_h$ of (1.5) satisfies

$$\begin{aligned} \|\underline{u}_h - u\| + h\|\nabla(\underline{u}_h - u)\| &\leq C(u, f)h^2 \\ \|\underline{u}_h - u\|_{L_\infty} &\leq C_p \inf_{\chi \in X_h} \|\nabla(u - \chi)\|_{W_p^1}, \text{ with } p > 2. \end{aligned} \tag{1.6}$$

Numerical methods for this type and more general problems has been considered by many authors, cf., e.g, [4, 13, 18, 21].

Here for sufficiently small data we shall derive similar results for the finite volume method. Li, in [20], considers a variation of the finite volume method under investigation here. The method differs in the construction of the control volumes. Instead of the barycenter z_K, the circumcenter is selected. For this finite volume method similar results with the finite element method, for the H^1-norm error estimate, are valid.

In Section 3, we establish existence of the finite volume solution u_h of (2.3), using a fixed point iteration method. In particular, in Theorem 3.1 we show that the iterations remain inside a fixed ball with a radius that depends only on f. Then in Theorem 3.2 we show that for a sufficiently small data, f, the fixed point iteration operator is Lipschitz continuous with Lipschitz constant less that 1.

In Section 4 we derive optimal order H^1–, L_2– and almost optimal L_∞–norm error estimates. Note that for the L_2 estimation we assume that A' is also Lipschitz continuous, $A'' \in L_1(\mathbb{R})$ and $f \in H^1$.

Also in Section 5 we analyze a Newton's method for the approximation of the finite volume solution u_h. We consider an inexact Newton iteration, a variant of the Newton iteration for nonlinear systems of equations, where the Jacobian of the system is solved approximately, cf., e.g., [2, 3, 11]. A similar approach for the finite element method is analyzed by Douglas and Dupont in [13]. As it is expected, one has to start the Newton iteration with an initial approximation u_h^0 sufficiently close to u_h. Also, following [13], we show that the Newton iterations converge to u_h with order 2. Finally in Section 6 numerical results are presented.

2. PRELIMINARIES–THE FINITE VOLUME METHOD

There has been a tendency of analyzing finite volume element method using the existing results from its finite element counterpart, cf., e.g., [7, 8, 9, 10]. The investigations recorded in all these references were concentrated on elliptic and/or parabolic problems with coefficients independent of the solution, i.e., the function A is only spatially varied. The finite volume element method is viewed as a perturbation of standard Galerkin finite element method with the help of an interpolation operator $I_h : C(\Omega) \to Y_h$, defined by

$$I_h v = \sum_{z \in Z_h^0} v(z)\Psi_z, \tag{2.1}$$

where

$$Y_h = \{\eta \in L_2(\Omega) : \ \eta|_{V_z} = \text{constant}, \ \forall z \in Z_h^0; \ \eta|_{V_z} = 0, \ \forall z \in \partial\Omega\},$$

and Ψ_z is characteristic function of V_z. We note that $I_h : X_h \to Y_h$ is a bijection and bounded with respect to the L_2-norm, i.e., there exist $c_1, c_2 > 0$, such that

$$c_1\|\chi\| \le \|I_h\chi\| \le c_2\|\chi\|, \quad \forall \chi \in X_h. \tag{2.2}$$

The finite volume problem (1.4) can be rewritten in a variational form. For an arbitrary $\eta \in Y_h$, we multiply the integral relation in (1.4) by $\eta(z)$ and sum over all $z \in Z_h^0$ to obtain the Petrov–Galerkin formulation, to find $u_h \in X_h$ such that

$$a_h(u_h; u_h, \eta) = (f, \eta), \quad \forall \eta \in Y_h, \tag{2.3}$$

where the form $a_h(\cdot;\cdot,\cdot) : X_h \times X_h \times Y_h \to \mathbb{R}$ is defined by

$$a_h(w; v, \eta) = -\sum_{z \in Z_h^0} \eta(z) \int_{\partial V_z} (A(w)\nabla v) \cdot n \, ds, \quad v, w \in X_h, \ \eta \in Y_h. \tag{2.4}$$

Obviously, $a_h(w; v, \eta)$ may be defined by (2.4) also for $v, w \in W_p^1(\Omega) \cap H_0^1(\Omega)$, $p > 2$, and using Green's formula we easily see that

$$a_h(w; v, \eta) = (L(w)v, \eta), \quad \text{for } v, w \in W_p^1(\Omega) \cap H_0^1(\Omega), \ \eta \in Y_h. \tag{2.5}$$

The bilinear form $a_h(w;\cdot,\cdot)$, with $w \in L_\infty$, of (2.4) may equivalently be written as

$$a_h(w; v, \eta) = \sum_K \Big\{ (L(w)v, \eta)_K + (A(w)\nabla v \cdot n, \eta)_{\partial K} \Big\}, \quad \forall v \in X_h, \ \eta \in Y_h. \tag{2.6}$$

Indeed, by integration by parts, we obtain, for $z \in Z_h^0$ and $K \in \mathcal{T}_h$,

$$\int_{K_z} L(w)v \, dx = -\int_{\partial K_z \cap \partial K} (A(w)\,\nabla v) \cdot n \, ds - \int_{\partial K_z \cap \partial V_z} (A(w)\,\nabla v) \cdot n \, ds, \tag{2.7}$$

and (2.6) hence follows by multiplication by $\eta(z)$ and by summation first over the triangles that have z as a vertex and then over the vertexes $z \in Z_h^0$. Also, we can easily see that I_h has the following properties, cf., e.g., [7],

$$\int_K I_h\chi \, dx = \int_K \chi \, dx, \ \ \forall \chi \in X_h, \ \text{ for any } K \in T_h, \tag{2.8}$$

$$\int_e I_h\chi\, ds = \int_e \chi\, ds, \quad \forall \chi \in X_h, \quad \text{for any side } e \text{ of } K \in T_h, \tag{2.9}$$

$$\|I_h\chi\|_{L_\infty(e)} \le \|\chi\|_{L_\infty(e)}, \quad \forall \chi \in X_h, \quad \text{for any side } e \text{ of } K \in T_h, \tag{2.10}$$

$$\|\chi - I_h\chi\|_{L_p(K)} \le h\|\nabla\chi\|_{L_p(K)}, \quad \forall \chi \in X_h, \quad 1 \le p < \infty. \tag{2.11}$$

In addition in [7, Lemma 6.1, Remark 6.1, Lemma 5.1] the following lemma was derived.

Lemma 2.1. *Let e be a side of a triangle $K \in T_h$. Then for $v \in W_p^1(K)$ there exists a constant $C_1 > 0$ independent of h such that*

$$|\int_e v(\chi - I_h\chi)\, ds| \le C_1 h\|\nabla v\|_{L_p(K)}\|\nabla\chi\|_{L_{p'}(K)}, \qquad \forall \chi \in X_h, \ \text{with } \frac{1}{p} + \frac{1}{p'} = 1. \tag{2.12}$$

Also, for $f \in W_p^i$, $i = 0, 1$ and $\chi \in X_h$,

$$|\varepsilon_h(f, \chi)| \le Ch^{i+j}\|f\|_{W_p^i}\|\chi\|_{W_{p'}^j}, \quad f \in W_p^i, \ i, j = 0, 1, \ \text{with } \frac{1}{p} + \frac{1}{p'} = 1, \tag{2.13}$$

where $\varepsilon_h : L_2 \times X_h \to \mathbb{R}$ is defined by

$$\varepsilon_h(f, \chi) = (f, \chi - I_h\chi). \tag{2.14}$$

Lemma 2.2. *Let $v \in W_q^2$, $4/3 < q \le 2$. The following identities hold.*

$$\sum_K \int_{\partial K} A(\bar{w})\nabla v \cdot n\, \chi\, ds = 0, \quad \sum_K \int_{\partial K} A(\bar{w})\nabla v \cdot n\, I_h\chi\, ds = 0, \quad \forall \chi \in X_h. \tag{2.15}$$

where $\bar{w}$ could be an element of X_h or the point value at the midpoint of the edge e of triangle K, of an element of X_h.

Proof. Note, that for $v \in W_q^2$, the trace $\nabla v \cdot n$ on ∂K exists for $q > 4/3$. The left identity is obvious by rewriting the sum as integrals of jump terms over the interior edges of $\mathcal{T}_h$. These jumps obviously vanish due to the continuity of $A(\bar{w})\nabla v \cdot n$ (in the trace sense). A similar argument gives the second identity. □

Our analysis will be based on the corresponding one for linear problems, cf., e.g, [7, 8]. There the error estimations are derived by bounding the error between the bilinear forms of the finite element, a, and the finite volume methods, a_h. This is shown to be $O(h)$ uniformly in X_h. Then for sufficiently small h the finite volume bilinear form a_h is coercive in X_h, which leads to the existence and uniqueness of the finite volume approximation.

In the nonlinear case a similar estimation for the error functional ε_a,

$$\varepsilon_a(w; v_h, \chi) = a(w; v_h, \chi) - a_h(w; v_h, I_h\chi) \quad \forall v_h, \chi \in X_h, \ w \in L_\infty, \tag{2.16}$$

shows that this error is not $O(h)$ uniformly in X_h, cf. Lemma 2.3. This is due to the fact that the bound of $\varepsilon_a(w_h; v_h, \chi)$, will depend on $\|w_h\|_{L_\infty}$. Inverse inequalities of the form, cf., e.g., [6],

$$\|\nabla\chi\|_{L_s} \le Ch^{2/s-2/t}\|\nabla\chi\|_{L_t}, \quad \forall \chi \in X_h, \ \text{with } 1 \le t \le s \le \infty, \tag{2.17}$$

which are true in a quasi-uniform mesh, give $\varepsilon_a = O(h^{1-2/t})$, uniformly in a ball of X_h with respect to W_t^1–norm, for $t > 2$.

In the sequel we derive estimations for ε_a.

Lemma 2.3. *There exists a constant $C_2 > 0$, independent of h, such that*

$$|\varepsilon_a(w_h; v_h, \chi)| \le C_2\beta_3 h \|\nabla w_h \cdot \nabla v_h\|_{L_p} \|\nabla\chi\|_{L_{p'}}, \quad \forall w_h, v_h, \chi \in X_h,\ \frac{1}{p} + \frac{1}{p'} = 1. \tag{2.18}$$

Proof. In view of Green's formula and (2.6), we may write ε_a in the following form:

$$\begin{aligned} \varepsilon_a(w_h; v_h, \chi) &= \sum_K \left\{(L(w_h)v_h, \chi - I_h\chi)_K + (A(w_h)\nabla v_h \cdot n, \chi - I_h\chi)_{\partial K}\right\} \\ &= \sum_K \{I_K + II_K\}. \end{aligned} \tag{2.19}$$

Applying Hölder's inequality to I_K, and using the fact that w_h and v_h are linear in K, and using (1.2) and (2.11), we have

$$|I_K| \le \beta_3 \|\nabla w_h \cdot \nabla v_h\|_{L_p(K)} \|\chi - I_h\chi\|_{L_{p'}(K)} \le \beta_3 h \|\nabla w_h \cdot \nabla v_h\|_{L_p(K)} \|\nabla\chi\|_{L_{p'}(K)}. \tag{2.20}$$

For the II_K, we break the integration over the boundary of each triangle K, into the sum of integrations over its sides, and thus may use (2.12), and follow the same steps as in estimating I_K. Hence,

$$|II_K| \le C_1 h |A(w_h)\nabla v_h|_{W_p^1(K)} \|\nabla\chi\|_{L_{p'}(K)} \le C_1\beta_3 h \|\nabla w_h \cdot \nabla v_h\|_{L_p(K)} \|\nabla\chi\|_{L_{p'}(K)}. \tag{2.21}$$

Finally, (2.20) and (2.21) establish the desired estimate for $C_2 = C_1 + 1$. □

The following lemma will be used in Section 4 to estimate the error in the L_2–norm. For this estimation we will need to assume that A' is Lipschitz continuous with constant L, i.e.

$$|A'(x) - A'(y)| \le L|x - y|, \quad \forall x, y \in \mathbb{R}. \tag{2.22}$$

Lemma 2.4. *Assume that A' is Lipschitz continuous and $v \in W_q^2 \cap H_0^1$, for $4/3 < q \le 2$. Then there exists a constant $C > 0$ independent of h such that for $w_h, v_h, \chi \in X_h$,*

$$\begin{aligned} |\varepsilon_a(w_h; v_h, \chi)| \le\, & C\big\{h^2 \|\nabla w_h\|_{L_\infty} (\|\nabla w_h \cdot \nabla v_h\| + \|v\|_{W_q^2}) \\ & + h\|\nabla w_h \cdot \nabla(v_h - v)\|_{L_q}\big\} \|\nabla\chi\|_{L_{q'}}, \end{aligned} \tag{2.23}$$

with $1/q + 1/q' = 1$.

Proof. Let w_K and w_e denote the average value of a function w over triangle K and the edge e, respectively. Since $v \in W_q^2$, Lemma 2.2 gives the identity

$$\big((A(w_h) - A(w_{h,e}))\nabla v \cdot n, \chi - I_h\chi\big)_{\partial K} = 0, \quad \forall \chi \in X_h.$$

Employing this identity, the fact that v_h is linear in K, Green's formula, and (2.8) we get

$$\begin{aligned} \varepsilon_a(w_h; v_h, \chi) = &\sum_K \big((A'(w_h) - A'(w_{h,K}))\nabla w_h \cdot \nabla v_h, \chi - I_h\chi\big)_K \\ &+ \sum_K \big((A(w_h) - A(w_{h,e}))\nabla(v_h - v) \cdot n, \chi - I_h\chi\big)_{\partial K} = \sum_K \{I_K + II_K\}. \end{aligned}$$

Using now Hölder's inequality, the fact that w_h is linear in K, and (2.11), we can bound I_K,

$$\begin{aligned} |I_K| &\le C \int_K |w_h - w_{h,K}|\, |\nabla w_h \cdot \nabla v_h|\, |\chi - I_h\chi|\, dx \\ &\le Ch^2 \|\nabla w_h\|_{L_\infty} \|\nabla w_h \cdot \nabla v_h\|_{L_2(K)} \|\nabla\chi\|_{L_2(K)}. \end{aligned} \tag{2.24}$$

For the estimation of II_K, we apply (2.12) and we get,

$$|II_K| \le Ch\,|(A(w_h) - A(w_{h,e}))\nabla(v_h - v)|_{W^1_q(K)}\,\|\nabla\chi\|_{L_{q'}(K)}. \tag{2.25}$$

Further, a simple calculation gives

$$|(A(w_h) - A(w_{h,e}))\nabla(v_h - v)|_{W^1_q(K)} \le C\big(\|\nabla w_h \cdot \nabla(v_h - v)\|_{L_q(K)} + h\|\nabla w_h\|_{L_\infty}\|v\|_{W^2_q(K)}\big).$$

Summing now over all triangles, the relation above, (2.24), (2.25) and using the fact that $q' > 2$, we obtain (2.23). □

Next we will derive a "Lipschitz"-type estimation for ε_a.

Lemma 2.5. *Let $v \in H^1 \cap L_\infty$, $w \in W^1_p$ with $p > 2$ and A' be Lipschitz continuous with constant L, cf.* (2.22). *There exists $C_2 > 0$ such that*

$$\begin{aligned} &|\varepsilon_a(v;\phi_h,\chi) - \varepsilon_a(w;\phi_h,\chi)| \\ &\quad\le C_2 h\|\nabla\phi_h\|_{L_\infty}(\beta_3 + L\|\nabla w\|_{L_p})\|\nabla(v-w)\|\,\|\nabla\chi\|, \quad \forall \phi_h,\chi \in X_h, \end{aligned} \tag{2.26}$$

where β_3 is the upper bound of A', cf., (1.2).

Proof. We can easily see that

$$\begin{aligned} \varepsilon_a(v;\phi_h,\chi) - \varepsilon_a(w;\phi_h,\chi) = \sum_K \bigg\{ &\int_K \operatorname{div}((A(v) - A(w))\nabla\phi_h)(\chi - I_h\chi)\,dx \\ &+ \int_{\partial K} (A(v) - A(w))\nabla\phi_h \cdot n(\chi - I_h\chi)\,ds \bigg\}. \end{aligned}$$

Also, since ϕ_h is linear in K, $\operatorname{div}(\nabla\phi_h) = 0$, therefore,

$$\operatorname{div}((A(v) - A(w))\nabla\phi_h) = \big\{A'(v)\nabla(v-w) + (A'(v) - A'(w))\nabla w\big\} \cdot \nabla\phi_h, \quad \text{in } K.$$

Then, this, (2.11), (2.12), the Hölder inequality

$$\|vw\|_{L_s} \le \|v\|_{L_t}\|w\|_{L_{\bar t}}, \text{ with } t > s, \quad \frac{s}{t} + \frac{s}{\bar t} = 1, \tag{2.27}$$

for $s = 2$ and $t = p$ and the Sobolev inequality, cf. e.g., [6, 4.x.11],

$$\|v\|_{L_s} \le \|\nabla v\|, \quad \forall s < \infty, \tag{2.28}$$

give for $C_2 = C_1 + 1$

$$\begin{aligned} |\varepsilon_a(v;\phi_h,\chi) - \varepsilon_a(w;\phi_h,\chi)| &\le C_2 h(\beta_3\|\nabla(v-w)\| + L\|\,|v-w|\,|\nabla w|\,\|)\|\nabla\chi\|\,\|\nabla\phi_h\|_{L_\infty} \\ &\le C_2 h(\beta_3\|\nabla(v-w)\| + L\|v-w\|_{L_{\bar p}}\|\nabla w\|_{L_p})\|\nabla\chi\|\,\|\nabla\phi_h\|_{L_\infty} \\ &\le C_2 h(\beta_3 + L\|\nabla w\|_{L_p})\|\nabla(v-w)\|\,\|\nabla\chi\|\,\|\nabla\phi_h\|_{L_\infty}. \qquad \square \end{aligned}$$

3. EXISTENCE OF FVE APPROXIMATIONS FOR SMALL DATA

In this section using a fixed point iteration we will show that a finite volume solution u_h of (2.3) exists and is in the ball

$$\mathcal{B}_M = \{\chi \in X_h : \ \|\nabla\chi\|_{L_p} \le M\}, \quad \text{with } p > 2,$$

where $M = M(f) > 0$, cf. Theorem 3.1. Further, if M is sufficiently small, i.e., an appropriate norm of f is small, the finite volume solution u_h is unique, cf., Corollary 3.3.

For a fixed $f \in L_2$, we consider the iteration map $T_h : X_h \to X_h$ given by

$$a_h(v_h; T_h v_h, \eta) = (f, \eta), \quad \forall \eta \in Y_h. \tag{3.1}$$

In view of the Sobolev imbedding, $\|v\|_{L_\infty} \le C\|v\|_{W_p^1}$ for $p > 2$, we shall employ the following inf–sup condition, cf., e.g., [6, Chapter 7]: There exist constants $\alpha = \alpha(A, \Omega) > 0$, $h_\alpha > 0$ and $\epsilon = \epsilon(A, \Omega) > 0$ such that for all $0 < h \le h_\alpha$ and $v_h \in X_h$ and $w \in L_\infty$,

$$\|\nabla v_h\|_{L_p} \le \alpha \sup_{0 \ne \chi \in X_h} \frac{a(w; v_h, \chi)}{\|\nabla \chi\|_{L_{p'}}}, \tag{3.2}$$

with $2 \le p \le 2 + \epsilon$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

In view of the identity $a(w_h; v_h, \chi) = a_h(w_h; v_h, I_h\chi) + \varepsilon_a(w_h; v_h, \chi)$ and the error estimate in Lemma 2.3 and (2.17),

$$|\varepsilon_a(w_h; v_h, \chi)| \le Ch\|\nabla w_h \cdot \nabla v_h\|_{L_p}\|\nabla \chi\|_{L_{p'}} \le Ch^{1-2/p}\|\nabla w_h\|_{L_p}\|\nabla v_h\|_{L_p}\|\nabla \chi\|_{L_{p'}},$$

there exists $h_M > 0$ such that for all $0 < h \le h_M \le h_\alpha$

$$\|\nabla v_h\|_{L_p} \le \alpha \sup_{0 \ne \chi \in X_h} \frac{a_h(w_h; v_h, I_h\chi)}{\|\nabla \chi\|_{L_{p'}}}, \quad \forall v_h \in X_h, w_h \in \mathcal{B}_M,\ 2 < p \le 2 + \epsilon. \tag{3.3}$$

Therefore, for $h < h_M$ and $v_h \in \mathcal{B}_M$, $T_h v_h$ is well defined. Note that (3.3) holds also for $p = 2$ and $w_h \in \tilde{\mathcal{B}}_M = \{\chi \in X_h : \ \|\nabla \chi\|_{L_{\tilde{p}}} \le M\}$, with $\tilde{p} > 2$.

In the following two theorems we will show that in a sufficiently small ball $\mathcal{B}_M$ and data f, there exists a unique solution $u_h \in X_h$ of (2.3).

Theorem 3.1. *There exists $h_M > 0$, such that for all $0 < h < h_M$, if $\|f\| \le M\alpha^{-1}$ then T_h maps $\mathcal{B}_M$ into itself for $2 < p < 2 + \epsilon$.*

Proof. Let $v_h \in \mathcal{B}_M$ then in view of (3.3) we have

$$\|\nabla T_h v_h\|_{L_p} \le \alpha \sup_{0 \ne \chi \in X_h} \frac{a_h(v_h; T_h v_h, I_h\chi)}{\|\nabla \chi\|_{L_{p'}}} \le \alpha \sup_{0 \ne \chi \in X_h} \frac{(f, I_h\chi)}{\|\nabla \chi\|_{L_{p'}}}. \tag{3.4}$$

Then, using (2.2) and the Sobolev inequality $\|v\| \le \|v\|_{W_p^1}$, for $p > 1$, cf. [6, 4.x.11], we get

$$\|\nabla T_h v_h\|_{L_p} \le \alpha\|f\|, \tag{3.5}$$

which gives the desired result. □

Next, we will show that the iteration map T_h is Lipschitz continuous. For M sufficiently small, T_h is a contraction in $\mathcal{B}_M$ in H^1-norm, which gives the uniqueness of the solution u_h of (2.3) and the convergence of the fixed point iteration, $v_h^{n+1} = T_h v_h^n \to u_h$, as $n \to \infty$.

Theorem 3.2. *Let A' be Lipschitz continuous with constant L, cf. (2.22). Then there exists a constant $C_L = C_L(A, \Omega) > 0$ and $h'_M > 0$, such that for $\|f\| \le M\alpha^{-1}$, $M < {C_L}^{-1}$ and all $0 < h \le h'_M$, T_h is a contraction, with constant $\ell = C_L M < 1$,*

$$\|\nabla(T_h v - T_h w)\| \le \ell\|\nabla(v - w)\|, \quad \forall v, w \in \mathcal{B}_M. \tag{3.6}$$

Proof. Let $v, w \in \mathcal{B}_M$. Then, in view of the definition of T_h, (3.1) we have

$$a_h(v; T_h v, \eta) - a_h(w; T_h w, \eta) = 0, \quad \forall \eta \in Y_h.$$

Therefore, we can easily see that for $\eta = I_h\chi$, $\chi \in X_h$,

$$\begin{aligned} a_h(v; T_h v - T_h w, I_h\chi) &= a_h(w; T_h w, I_h\chi) - a_h(v; T_h w, I_h\chi) \\ &= \varepsilon_a(v; T_h w, \chi) - \varepsilon_a(w; T_h w, \chi) + ((A(w) - A(v))\nabla T_h w, \nabla\chi). \end{aligned} \tag{3.7}$$

Using now the fact that for sufficiently small h, $T_h w \in \mathcal{B}_M$, cf., Theorem 3.1, the Hölder inequality (2.27) with $s = 2$ and $t = p$ and the Sobolev (2.28), the last term of the right–hand side of (3.7) can be bounded for any $\chi \in X_h$,

$$\begin{aligned} |(A(w) - A(v))\nabla T_h w, \nabla\chi)| &\le \beta_3 \|(w - v)|\nabla T_h w|\| \, \|\nabla\chi\| \\ &\le \beta_3 \|w - v\|_{L_{\bar{p}}} \|\nabla T_h w\|_{L_p} \|\nabla\chi\| \le \beta_3 M \|\nabla(v - w)\| \, \|\nabla\chi\|. \end{aligned} \tag{3.8}$$

Also, in view of Lemma 2.5 the remaining two terms in the right–handside of (3.7), give

$$|\varepsilon_a(v; T_h w, \chi) - \varepsilon_a(w; T_h w, \chi)| \le C_2 h^{1-2/p} M(\beta_3 + LM)\|\nabla(v - w)\| \, \|\nabla\chi\|. \tag{3.9}$$

Since, $a_h(v_h; \cdot, \cdot)$ is coercive for $v_h \in \mathcal{B}_M$ and h sufficiently small, choosing $\chi = T_h v - T_h w$ in the above relation and in (3.7) and (3.8) gives that there exists a constant $C_L = C_L(A, \Omega) > 0$ such that

$$\|\nabla(T_h v - T_h w)\| \le C_L M \|\nabla(v - w)\|.$$

Therefore, for $M < C_L^{-1}$, T_h is a contraction with constant $0 < \ell = C_L M < 1$. □

Finally, Theorems 3.1 and 3.2 give the following corollary,

Corollary 3.3. *Assume that A' is Lipschitz continuous with a constant L. Then there exist constants $C_L = C_L(A, \Omega) > 0$ and $h_0 > 0$ such that if $\|f\| \le \alpha^{-1} C_L^{-1}$, with $2 < p < 2 + \epsilon$ then for h sufficiently small the problem* (2.3)*, i.e., find $u_h \in X_h$ such that*

$$a_h(u_h; u_h, I_h\chi) = (f, I_h\chi), \quad \forall \chi \in X_h,$$

has a unique solution, with ϵ given in (3.2).

4. ERROR ESTIMATES

In this section we shall derive W_s^1–, with $2 \le s < p$, L_2– and L_∞-norm error estimates for the error $u_h - u$ for $f \in L_2$. We shall assume that the nonlinear problem (1.1) has a unique solution $u \in W_q^2 \cap H_0^1$, with $4/3 < q \le 2$. In Section 3 we show that a finite volume solution u_h of (2.3) exists and is unique.

First, we will derive an a priori error estimate in $\|\nabla \cdot\|_{L_s}$, $2 \le s < p$, norm. For $s = 2$ we get the usual H^1–norm error bound. But for $s > 2$ this estimate combined with a standard Sobolev imbedding gives an L_∞–norm error estimate, cf. Theorem 4.2.

Theorem 4.1. *Let u_h and u be the solutions of* (2.3) *and* (1.1)*, respectively, with $f \in L_2$. Then, if $\gamma = \alpha\beta_3 M < 1$ there exists a constant $C = C(u, f)$, independent of h, such that for $0 < h \le h_M$*

$$\|\nabla(u_h - u)\|_{L_s} \le C(u, f) h^{1+2/s-2/q}, \quad \text{with } 2 \le s < p < 2 + \epsilon, \quad \frac{4}{3} < q \le 2, \tag{4.1}$$

where α is the constant appeared in (3.2).

Proof. Using the triangle inequality we get

$$\|\nabla(u_h - u)\|_{L_s} \le \|\nabla(u - \chi)\|_{L_s} + \|\nabla(u_h - \chi)\|_{L_s}, \quad \forall \chi \in X_h. \tag{4.2}$$

In view of the approximation property of X_h,

$$\inf_{\chi \in X_h} \|\nabla(v - \chi)\|_{L_s} \le Ch^{1+2/s-2/q}\|v\|_{W_q^2}, \quad \text{with } 4/3 < q \le 2 \le s, \tag{4.3}$$

the first term of the right–handside of (4.2) is bounded as desired. Also, we can easily see that

$$a(u; u_h - \chi, \psi) = a(u; u_h - u, \psi) + a(u; u - \chi, \psi) \le a(u; u_h - u, \psi) + \beta_2\|\nabla(u - \chi)\|_{L_s}\|\nabla\psi\|_{L_{s'}},$$

with $1/s + 1/s' = 1$. Hence, in view of (3.2), we may write for $2 \le s < p$,

$$\begin{aligned} \|\nabla(u_h - \chi)\|_{L_s} &\le \alpha \sup_{0 \ne \psi \in X_h} \frac{a(u; u_h - \chi, \psi)}{\|\nabla\psi\|_{L_{s'}}} \\ &\le \alpha \sup_{0 \ne \psi \in X_h} \frac{a(u; u_h - u, \psi)}{\|\nabla\psi\|_{L_{s'}}} + \alpha\beta_2\|\nabla(u - \chi)\|_{L_s}. \end{aligned} \tag{4.4}$$

Then in view of (4.3), it suffices to estimate the first term of the right–handside in the relation above. We can easily see for any $\psi \in X_h$,

$$\begin{aligned} a(u; u_h - u, \psi) &= a(u; u_h, \psi) - (f, \psi) \\ &= \{a(u; u_h, \psi) - a(u_h; u_h, \psi)\} + \{\varepsilon_a(u_h; u_h, \psi) - \varepsilon_h(f, \psi)\} = I + II. \end{aligned} \tag{4.5}$$

Using then the fact that $u_h \in \mathcal{B}_M$, the Hölder inequality (2.27), with $t = p$, and the Sobolev inequality (2.28), we have for any $\chi, \psi \in X_h$,

$$\begin{aligned} |I| &= |a(u; u_h, \psi) - a(u_h; u_h, \psi)| \le \beta_3\|(u_h - u)|\nabla u_h|\|_{L_s}\|\nabla\psi\|_{L_{s'}} \\ &\le \beta_3\|u_h - u\|_{L_{\bar{p}}}\|\nabla u_h\|_{L_p}\|\nabla\psi\|_{L_{s'}} \le \beta_3 M\|\nabla(u_h - u)\|\,\|\nabla\psi\|_{L_{s'}} \\ &\le \beta_3 M(\|\nabla(u_h - \chi)\|_{L_s} + \|\nabla(u - \chi)\|_{L_s})\|\nabla\psi\|_{L_{s'}}. \end{aligned} \tag{4.6}$$

The remaining term II can be bounded using Lemma 2.3 and (2.13), the inverse inequality (2.17) and the Hölder inequality (2.27), with $t = 2q/(2-q)$ and $\bar{t} = st/(t-s)$,

$$|\varepsilon_h(f, \psi)| \le Ch\|f\|\|\nabla\psi\| \le Ch^{2-2/s'}\|f\|\,\|\nabla\psi\|_{L_{s'}} = Ch^{2/s}\|f\|\,\|\nabla\psi\|_{L_{s'}}, \tag{4.7}$$

and

$$\begin{aligned} |\varepsilon_a(u_h; u_h, \psi)| &\le Ch(\|\nabla u_h \cdot \nabla(u_h - u)\|_{L_s} + \|\nabla u_h \cdot \nabla u\|_{L_s})\|\nabla\psi\|_{L_{s'}} \\ &\le C\big(h^{1-2/p}M\|\nabla(u_h - u)\|_{L_s} + h\|\nabla u_h\|_{L_{\bar{t}}}\|\nabla u\|_{L_t}\big)\|\nabla\psi\|_{L_{s'}} \\ &\le C\big(h^{1-2/p}M\|\nabla(u_h - u)\|_{L_s} + h^{1+2/\bar{t}-2/p}M\|\nabla u\|_{L_t}\big)\|\nabla\psi\|_{L_{s'}}. \end{aligned} \tag{4.8}$$

Further, in view of the Sobolev imbedding, cf., e.g., [1]

$$\|v\|_{L_t} \le C\|v\|_{W_r^1}, \quad \forall v \in W_r^1,\ r \le 2, \text{ and } t \le 2r/(2-r), \tag{4.9}$$

and

$$1 + \frac{2}{\bar{t}} - \frac{2}{p} = 1 - \frac{2}{t} + \frac{2}{s} - \frac{2}{p} = 2 - \frac{2}{q} + \frac{2}{s} - \frac{2}{p} > 1 + \frac{2}{s} - \frac{2}{q},$$

relation (4.8) becomes

$$|\varepsilon_a(u_h;u_h,\psi)| \le C\big(h^{1-2/p}M\|\nabla(u_h-u)\|_{L_s} + h^{1+2/s-2/q}M\|u\|_{W_q^2}\big)\|\nabla\psi\|_{L_{s'}}. \tag{4.10}$$

Thus (4.4)–(4.10) and the fact that $1-2/q \le 0$, give

$$\begin{aligned}(1-\gamma)\|\nabla(u_h-\chi)\|_{L_s} \le (\gamma+\alpha\beta_2)\|\nabla(u-\chi)\|_{L_s} + Ch^{1-2/p}\|\nabla(u_h-u)\|_{L_s}\\ + Ch^{1+2/s-2/q}(\|u\|_{W_q^2}+\|f\|).\end{aligned} \tag{4.11}$$

Finally, for h sufficiently small, the estimation above, (4.2) and (4.3) give the desired estimate. □

Corollary 4.2. *Let u_h and u be the solutions of* (2.3) *and* (1.1)*, respectively, with $f \in L_2$. Then, if $\gamma = \alpha\beta_3 M < 1$ there exists a constant $C_s = C_s(u,f)$, independent of h, such that for $0 < h \le h_M$*

$$\|u-u_h\|_{L_\infty} \le C_s(u,f)h^{1+2/s-2/q}, \quad \text{with } 2 < s < p < 2+\epsilon. \tag{4.12}$$

Proof. In view of the Sobolev imbedding $\|v\|_{L_\infty} \le C_s\|\nabla v\|_{L_s}$, $s>2$ and Theorem 4.1 we can easily see that (4.12) holds. □

Note that the constant C_s in Corollary 4.2 blows-up as $s \to 2$. Later, in Theorem 4.5, we will show an almost optimal order L_∞ error estimate. Next, we will show that the finite volume solution u_h is also bounded in $\|\nabla\cdot\|_{L_{\bar q}}$, $2/q + 2/\bar q = 1$. This will be used later in the L_2–norm error estimation.

Theorem 4.3. *Let u_h and u be the solutions of* (2.3) *and* (1.1)*, respectively, with $u \in W_q^2 \cap H_0^1$, $4/3 < q \le 2$. Then $u_h \in W_{\bar q}^1$, uniformly for all $0 < h \le h_M$, i.e.,*

$$\|\nabla u_h\|_{L_{\bar q}} \le C(u,f), \quad \text{with } \frac{2}{q}+\frac{2}{\bar q} = 1. \tag{4.13}$$

Proof. We rewrite u_h by adding and subtracting $R_h u$ and $\Pi_h u$, where $R_h : H_0^1 \to X_h$ is the elliptic projection operator defined by

$$a(u;R_h u,\chi) = a(u;u,\chi), \quad \forall \chi \in X_h,$$

and $\Pi_h : C(\Omega) \to X_h$ the standard nodal interpolant. Thus

$$\begin{aligned}\|\nabla u_h\|_{L_{\bar q}} &\le \|\nabla(u_h - R_h u)\|_{L_{\bar q}} + \|\nabla R_h u\|_{L_{\bar q}}\\ &\le \|\nabla(u_h-R_h u)\|_{L_{\bar q}} + \|\nabla(R_h u - \Pi_h u)\|_{L_{\bar q}} + \|\nabla\Pi_h u\|_{L_{\bar q}}.\end{aligned} \tag{4.14}$$

In view of the approximation property, (4.3), Π_h satisfies

$$|\nabla(\Pi_h v - v)\|_{L_s} \le Ch^{1+2/s-2/q}\|v\|_{W_q^2}, \quad 4/3 < q \le 2 \le s. \tag{4.15}$$

Then, the last term in (4.14) can easily be estimated in view of (4.9) and (4.15), we have

$$\|\nabla\Pi_h u\|_{L_{\bar q}} \le C\|u\|_{W_q^2}. \tag{4.16}$$

Also, we can easily see that the identity

$$a(u;R_h u - u, R_h u - u) = a(u;R_h u - u, \Pi_h u - u),$$

gives

$$\|\nabla(R_h u - u)\| \le C\|\nabla(\Pi_h u - u)\|.$$

Thus, using the inverse inequality (2.17), (4.15) and the fact that $2 - 2/q = 1 - 2/\bar{q}$, we can bound the second term in (4.14) by

$$\begin{aligned} \|\nabla(R_h u - \Pi_h u)\|_{L_{\bar q}} &\le C h^{2/\bar q - 1}\|\nabla(R_h u - \Pi_h u)\| \\ &\le C h^{2/\bar q - 1}(\|\nabla(R_h u - u)\| + \|\nabla(\Pi_h u - u)\|) \le C\|u\|_{W_q^2} \end{aligned} \tag{4.17}$$

Finally, the first term, in (4.14) can be estimated similarly. From Theorem 4.1, (2.17) and the fact that $2 - 2/q = 1 - 2/\bar{q}$ we have

$$\begin{aligned} \|\nabla(u_h - R_h u)\|_{L_{\bar q}} &\le C h^{2/\bar q - 1}\|\nabla(u_h - R_h u)\| \\ &\le C h^{2/\bar q - 1}(\|\nabla(u_h - u)\| + \|\nabla(R_h u - u)\| + \|\nabla(\Pi_h u - u)\|) \\ &\le C(u, f). \end{aligned} \tag{4.18}$$

Combining now this with (4.14), (4.16) and (4.17), proves the theorem. □

For the proof of the L_2–norm error estimate we will employ a similar duality argument as the one used in [13]. Let us consider the following auxiliary problem. Let $\varphi \in H_0^1$ be such that

$$a(u; \varphi, v) + (A'(u)\nabla u \nabla\varphi, v) = (u - u_h, v), \quad \forall v \in H_0^1. \tag{4.19}$$

If $A(u)$ is Lipschitz continuous and $A'(u)\nabla u \in L_\infty$, then the solution φ of (4.19) satisfies the following elliptic regularity estimate,

$$\|\varphi\|_{W_{q_0}^2} \le C\|u_h - u\|, \quad \text{with } 4/3 < q_0 \le 2, \tag{4.20}$$

where q_0 depends on the biggest interior angle of Ω and the coefficients $A(u)$, $A'(u)\nabla u$. If Ω is convex then $q_0 = 2$, and if it is nonconvex then $q_0 < 2$.

Theorem 4.4. *Let u_h and u be the solutions of* (2.3) *and* (1.1)*, respectively, with $u \in W_q^2 \cap H_0^1 \cap W_\infty^1$, $4/3 < q \le 2$. Then, if u and A' are Lipschitz continuous, $A'' \in L_1(\mathbb{R})$, $f \in H^1$ and $\gamma = \beta_1^{-1}\beta_3 M < 1$ there exists a constant C, independent of h, such that for sufficiently small h,*

$$\|u_h - u\| \le C(u, f)h^{4 - 2/q - 2/q_0}. \tag{4.21}$$

Proof. Before we begin the proof we note the following Taylor expansions

$$\begin{aligned} A(u_h) - A(u) &= (u_h - u)\int_0^1 A'(u - t(u - u_h))\,dt \equiv (u_h - u)\bar{A}', \\ A(u_h) - A(u) - A'(u)(u_h - u) &= (u_h - u)^2\int_0^1 A''(u - t(u - u_h))(1 - t)\,dt \\ &\equiv (u_h - u)^2\bar{A}''. \end{aligned} \tag{4.22}$$

Then, in view of (4.19), we have

$$\begin{aligned} \|u - u_h\|^2 &= a(u; u - u_h, \varphi) + (A'(u)(u - u_h)\nabla u, \nabla\varphi) \\ &= a(u; u, \varphi) - a(u_h; u_h, \varphi) + ((A(u_h) - A(u))\nabla u_h, \nabla\varphi) \\ &\quad - ((A(u_h) - A(u))\nabla u, \nabla\varphi) + ((A(u_h) - A(u))\nabla u, \nabla\varphi) - (A'(u)(u_h - u)\nabla u, \nabla\varphi) \\ &= a(u; u, \varphi) - a(u_h; u_h, \varphi) + ((A(u_h) - A(u))\nabla(u_h - u), \nabla\varphi) \\ &\quad + ((A(u_h) - A(u) - A'(u)(u_h - u))\nabla u, \nabla\varphi). \end{aligned}$$

Further, using (2.3) and (4.22), the relation above gives for any $\chi \in X_h$,

$$\begin{aligned} \|u-u_h\|^2 &= a(u;u,\varphi-\chi) - a(u_h;u_h,\varphi-\chi) + \varepsilon_h(f,\chi) - \varepsilon_a(u_h;u_h,\chi) \\ &\quad + ((u_h-u)\bar{A}'\nabla(u_h-u) + (u_h-u)^2\bar{A}''\nabla u, \nabla\varphi) \\ &= \{a(u_h;u-u_h,\varphi-\chi) + ((u_h-u)\bar{A}'\nabla u, \nabla(\varphi-\chi)) + \varepsilon_h(f,\chi)\} \\ &\quad - \varepsilon_a(u_h;u_h,\chi) + \{(u_h-u)\bar{A}'\nabla(u_h-u) + ((u_h-u)^2\bar{A}''\nabla u, \nabla\varphi)\} \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{4.23}$$

Choosing now $\chi = \Pi_h\varphi$ in (4.23) and using (2.13) and Lemma 2.4 we get

$$\begin{aligned} |I_1| &\le C(\|\nabla(u_h-u)\| + \|\nabla u\|_{L_\infty}\|u_h-u\|)\|\nabla(\varphi-\Pi_h\varphi)\| + Ch^2\|f\|_{H^1}\|\nabla\Pi_h\varphi\|, \\ |I_2| &\le C\big\{h^2\|\nabla u_h\|_{L_\infty}(\|\,|\nabla u_h|^2\| + \|u\|_{W_q^2}) + h\|\nabla u_h\cdot\nabla(u_h-u)\|_{L_q}\big\}\|\nabla\Pi_h\varphi\|_{L_{q'}}. \end{aligned} \tag{4.24}$$

Since $2 < \bar{q} = 2q/(2-q)$, (4.16), the approximation property (4.15) and the fact that $2 \ge 3 - 2/q_0$, now give

$$|I_1| \le Ch^{2-2/q_0}(\|\nabla(u-u_h)\| + \|\nabla u\|_{L_\infty}\|u-u_h\| + h\|f\|_{H^1})\|\varphi\|_{W_{q_0}^2}. \tag{4.25}$$

Using then Theorem 4.1 and (4.20), we obtain

$$\begin{aligned} |I_1| &\le C(u)h^{2-2/q_0}\big\{\|\nabla(u_h-u)\| + h\|f\|_{H^1} + \|u_h-u\|\big\}\|u_h-u\| \\ &\le C(u,f)h^{4-2/q-2/q_0}\|u_h-u\| + C(u,f)h^{2-2/q_0}\|u_h-u\|^2. \end{aligned} \tag{4.26}$$

Also, using the fact that $q, q_0 > 4/3$ we get $q' \le 2q_0/(2-q_0)$, thus in view of (4.9) and (4.15),

$$\|\nabla\Pi_h\varphi\|_{L_{q'}} \le C\|\varphi\|_{W_{q_0}^2}.$$

Then this, the inverse inequality (2.17), the Hölder inequality (2.27), with $s=2$, $t=\bar{q}$ and $s=q$, $t=2$, and the fact that $2\bar{q}/(\bar{q}-2) \le \bar{q}$, for $q > 4/3$, give

$$\begin{aligned} |I_2| &\le C\big\{h^{2-2/\bar{q}}\|\nabla u_h\|_{L_{\bar{q}}}(\|\nabla u_h\|_{L_{\bar{q}}}\|\nabla u_h\|_{L_{2\bar{q}/(\bar{q}-2)}} + \|u\|_{W_q^2}) \\ &\qquad + h\|\nabla u_h\|_{L_{\bar{q}}}\|\nabla(u_h-u)\|)\big\}\|\nabla\Pi_h\varphi\|_{L_{q'}} \\ &\le C\|\nabla u_h\|_{L_{\bar{q}}}\big\{h^{2-2/\bar{q}}(\|\nabla u_h\|_{L_{\bar{q}}}^2 + \|u\|_{W_q^2}) + h\|\nabla(u_h-u)\|\big\}\|\varphi\|_{W_{q_0}^2}. \end{aligned}$$

Using, next Theorems 4.1 and 4.3 and (4.20), we obtain

$$|I_2| \le C(u,f)(h^{2-2/\bar{q}} + h\|\nabla(u_h-u)\|)\|u_h-u\| \le C(u,f)h^{3-2/q}\|u-u_h\|. \tag{4.27}$$

Next, we turn to the estimation of the term I_3 in (4.23). For this we use the Hölder inequality (2.27) with $t = q_0$; hence

$$|I_3| \le C\|\nabla(u_h-u)\|\,\|(u-u_h)\nabla\varphi\| \le C\|\nabla(u_h-u)\|\,\|u_h-u\|_{L_{q_0}}\|\nabla\varphi\|_{L_{\bar{q}_0}}. \tag{4.28}$$

Then the interpolation inequality, cf., e.g., [15, Appendix B],

$$\|v\|_{L_{q_0}} \le \|v\|^{1/2}\|v\|_{L_s}^{1/2}, \text{ with } s = 2q_0/(4-q_0),$$

and the Sobolev inequality (2.28) give

$$\|u_h-u\|_{L_{q_0}} \le C\|\nabla(u_h-u)\|^{1/2}\|u_h-u\|^{1/2}.$$

Therefore, using this and Theorem 4.1 in (4.28) give

$$|I_3| \le C\|\nabla(u-u_h)\|^{3/2}\|u-u_h\|^{1/2}\|\varphi\|_{W^2_{q_0}} \le (C\|\nabla(u-u_h)\|^3 + \frac{1}{2}\|u-u_h\|)\|u-u_h\|$$
$$\le C(u,f)h^{3(2-2/q)}\|u-u_h\| + \frac{1}{2}\|u-u_h\|^2.$$

We can easily see that $3(2-2/q) > 4-2/q-2/q_0$. Therefore, combining the relation above with (4.23), (4.26) and (4.27), we get

$$\|u-u_h\|^2 \le |I_1| + |I_2| + |I_3|$$
$$\le C(u,f)h^{4-2/q-2/q_0}\|u_h-u\| + C(u,f)h^{2-2/q_0}\|u_h-u\|^2 + C(u,f)h^{3-2/q}\|u-u_h\|$$
$$+ C(u,f)h^{3(2-2/q)}\|u-u_h\| + \frac{1}{2}\|u-u_h\|^2,$$

which for sufficiently small h gives the desired estimate. □

Theorem 4.5. *Let u_h and u be the solutions of* (2.3) *and* (1.1)*, respectively. Then, if Ω is convex, $\gamma = C_\Omega \beta_1^{-1}\beta_2\beta_3\|u\|_{W^1_p} < 1$, with $C_\Omega > 0$ a constant depending only on Ω, $u \in W^2_\infty$ and $f \in L_\infty$, then there exists a constant C independent of h, such that for sufficiently small h,*

$$\|u-u_h\|_{L_\infty} \le C(u,f)h^2\log(\frac{1}{h}). \tag{4.29}$$

Proof. Using again a triangle inequality we get

$$\|u_h-u\|_{L_\infty} \le \|\underline{u}_h-u\|_{L_\infty} + \|u_h-\underline{u}_h\|_{L_\infty},$$

where $\underline{u}_h$ is the Galerkin finite element approximation of u, i.e.,

$$a(\underline{u}_h;\underline{u}_h,\chi) = (f,\chi), \quad \forall\chi \in X_h. \tag{4.30}$$

In the case of the linear problem $-\text{div}\,(A(x)\nabla w) = f$, we have for $A \in W^2_\infty$, cf., eg., [6]

$$\|w_h-w\|_{L_\infty} \le Ch^2\log(\frac{1}{h})\|w\|_{W^2_\infty},$$

where w_h is the finite element approximation of w. Since $f \in L_\infty$ and $u \in W^2_\infty$, then $A(u) \in W^2_\infty$. Therefore,

$$\|R_hu-u\|_{L_\infty} \le C(u)h^2\log(\frac{1}{h}). \tag{4.31}$$

The estimation of $\|\underline{u}_h - R_hu\|_{L_\infty}$ was derived in [21], where it shown that

$$\|\underline{u}_h - R_hu\|_{L_\infty} \le \gamma\|\underline{u}_h-u\|_{L_\infty}, \tag{4.32}$$

with $\gamma = C_\Omega\beta_1^{-1}\beta_2\beta_3\|u\|_{W^1_p}$. Thus (4.31) and (4.32) give

$$(1-\gamma)\|\underline{u}_h-u\|_{L_\infty} \le C(u)h^2\log(\frac{1}{h}), \tag{4.33}$$

We turn now to the estimation of $\|\underline{u}_h-u_h\|_{L_\infty}$. Let $x_0 \in K_0 \in \mathcal{T}_h$ such that $\|\underline{u}_h-u_h\|_{L_\infty} = |(\underline{u}_h-u_h)(x_0)|$ and $\delta_{x_0} = \delta \in C_0^\infty(\Omega)$ a regularized Dirac δ–function satisfying

$$(\delta,\chi) = \chi(x_0), \quad \forall\chi \in X_h.$$

For such a function δ, cf., e.g., [6], we have

$$\operatorname{supp}\delta \subset B = \{x \in \Omega : |x - x_0| \le h/2\},$$
$$\int_\Omega \delta = 1, \quad 0 \le \delta \le Ch^{-2}, \quad \|\delta\|_{L_p} \le Ch^{2(1-p)/p}, \quad 1 < p < \infty.$$

Also let us consider the corresponding regularized Green's function $G \in H_0^1$, defined by

$$a(\underline{u}_h; G, v) = (\delta, v), \quad \forall v \in H_0^1. \tag{4.34}$$

Then, we have

$$\begin{aligned} \|\underline{u}_h - u_h\|_{L_\infty} &= (\delta, \underline{u}_h - u_h) = a(\underline{u}_h; G, \underline{u}_h - u_h) = a(\underline{u}_h; G_h, \underline{u}_h - u_h) \\ &= (f, G_h) - a(\underline{u}_h; u_h, G_h) \\ &= \varepsilon_h(f, G_h) - \varepsilon_a(u_h; u_h, G_h) + \{a(\underline{u}_h; u_h, G_h) - a(u_h; u_h, G_h)\}, \end{aligned} \tag{4.35}$$

where $G_h \in X_h$ is the finite element approximation of G, i.e.,

$$a(\underline{u}_h; G, \chi) = a(\underline{u}_h; G_h, \chi), \quad \forall \chi \in X_h.$$

Since $u \in W_\infty^2$, we have $u \in H^2$. Thus, in view of Theorem 4.3, $\|\nabla u_h\|_{L_\infty} \le C$. Further, using Lemma 2.4 and (2.13), (1.2) and Theorem 4.4, we obtain

$$\begin{aligned} \|\underline{u}_h - u_h\|_{L_\infty} &\le C\big\{h^2(\|f\|_{H^1} + \|\nabla u_h\|_{L_\infty}^2 \|\nabla u_h\| + \|\nabla u_h\|_{L_\infty}\|u\|_{H^2}) \\ &\quad + h\|\nabla u_h\|_{L_\infty}\|\nabla(u_h - u)\| + \|(\underline{u}_h - u_h)|\nabla u_h|\|\big\}\|\nabla G_h\| \\ &\le Ch^2(\|f\|_{H^1} + \|u\|_{H^2} + \|\underline{u}_h - u\|)\|\nabla G_h\|. \end{aligned} \tag{4.36}$$

The last term can be estimated by, cf., e.g., [13],

$$\|\underline{u}_h - u\| \le C(u, f)h^2. \tag{4.37}$$

In addition in view of [22, Lemma 3.1] we get

$$\|G_h\|_{H^1} \le C\|\nabla G\|_{L_2} \le C\frac{1}{(s-1)^{1/2}}\|\delta\|_{L_s}, \tag{4.38}$$

with $s \downarrow 1$. Choosing now $s = 1 + (\log(1/h))^{-1}$ we have

$$\|G_h\|_{H^1} \le C(\log(\frac{1}{h}))^{1/2}. \tag{4.39}$$

Combining now (4.35)–(4.39), we obtain

$$\|\underline{u}_h - u_h\|_{L_\infty} \le C(u, f)h^2 \log(\frac{1}{h})^{1/2}. \tag{4.40}$$

¿From this and (4.33) for $\gamma < 1$ we get the desired estimation (4.29).

5. NEWTON'S METHOD

In this section we shall analyze Newton's method for the computation of the finite volume solution u_h of (2.3). We consider an inexact Newton iteration, a variant of the Newton iteration for nonlinear systems of equations, where the Jacobian of the system is solved approximately, cf., e.f., [2, 3, 11]. Our analysis is based on a similar approach for the finite element method, studied by Douglas and Dupont in [13].

Also here, we will assume that (1.1) has a unique solution $u \in H^2 \cap H_0^1$. For $\phi \in H^1$ we define the bilinear form $N(\phi; \cdot, \cdot)$ on $H_0^1 \times H_0^1$ by

$$N(\phi; v, w) = a(\phi; v, w) + d(\phi; v, w), \tag{5.1}$$

where d is given by

$$d(\phi; v, w) = (A'(\phi)v\nabla\phi, \nabla w). \tag{5.2}$$

Further, let N_h be the corresponding finite volume form to N, defined for $\phi \in H^2 \cap H_0^1$ on $(H^2 \cap H_0^1) + X_h \times (H^2 \cap H_0^1) + X_h$ by

$$N_h(\phi; v, w) = a_h(\phi; v, w) + d_h(\phi; v, w), \tag{5.3}$$

where d_h is given by

$$d_h(\phi; v, w) = -\sum_K \int_K \operatorname{div}(A'(\phi)v\nabla\phi) I_h w \, dx + \int_{\partial K} (A'(\phi)v\nabla\phi) \cdot n I_h w \, ds. \tag{5.4}$$

For $u_h^0 \in X_h$, the Newton approximations to the solution u_h forms a sequence $\{u_h^k\}_{k=0}^{\infty}$ in X_h satisfying

$$N_h(u_h^k; u_h^{k+1} - u_h^k, \chi) = (f, I_h\chi) - a_h(u_h^k; u_h^k, I_h\chi), \quad \forall \chi \in X_h. \tag{5.5}$$

We will show that $u_h^k \to u_h$ in H^1–norm as $k \to \infty$, with order two, provided that u_h^0 is sufficiently close to u_h. For this we will assume that u_h converges to u sufficiently fast,

$$\|u - u_h\|_{L_\infty} + \sigma_h \|u - u_h\|_{H^1} \to 0, \quad \text{as } h \to 0, \tag{5.6}$$

where

$$\sigma_h \equiv \sup\{\|\chi\|_{L_\infty} / \|\chi\|_{H^1} : 0 \neq \chi \in X_h\}. \tag{5.7}$$

Since $\mathcal{T}_h$ is a quasi-uniform mesh, there exists a constant C, independent of h such that

$$|\sigma_h| \le C \log(\frac{1}{h}). \tag{5.8}$$

Further, let C_3 be another constant, independent of h, satisfying

$$\|u_h\|_{W_\infty^1} \le C_3. \tag{5.9}$$

Note that this assumption holds, for $u \in H^2$, cf. Section 3. In addition we assume that A'' is bounded and is Lipschitz continuous, i.e.,

$$|A''(x)| \le \beta_4, \quad |A''(x) - A''(y)| \le L_2|x - y|, \quad \forall x, y \in \mathbb{R}. \tag{5.10}$$

Next, we will show various auxilliary results that helps in the proof of Theorem 5.1. We start by stating the following lemma of Douglas and Dupont, [13].

Lemma 5.1. *Given $\tau > 0$, there exists positive constants δ, h_0 and C_4 such that the following holds. If $0 < h < h_0$, if $\phi \in W^1_\infty$ satisfies*

$$\|\phi\|_{W^1_\infty} \leq \tau \quad \text{and} \quad \sigma_h \|\phi - u\|_{H^1} \leq \delta,$$

and if G is a linear functional on H^1_0 with

$$|||G||| = \sup_{0 \neq \chi \in X_h} \frac{|G(\chi)|}{\|\chi\|_{H^1}},$$

then there exists a unique $v \in X_h$ satisfying the equations

$$N(\phi; v, \chi) = G(\chi), \quad w \in X_h. \tag{5.11}$$

Furthermore, v satisfies the bound

$$\|v\|_{H^1} \leq C_4 |||G|||. \tag{5.12}$$

We shall also use the error functional ϵ_N, defined by $\varepsilon_N = N - N_h$, and we derive similar estimates to ε_a, cf. Section 2.

Lemma 5.2. *For $\phi \in X_h$ the error functional ε_N satisfies*

$$|\varepsilon_N(\phi; \psi, \chi)| \leq Ch \|\nabla\phi\|_{L_\infty} (1 + \sigma_h \|\phi\|_{H^1}) \|\psi\|_{H^1} \|\chi\|_{H^1}, \quad \forall \chi, \psi \in X_h.$$

Proof. ¿From the definition of ε_N we can easily see that, $\varepsilon_N = \varepsilon_a + (d - d_h)$. Therefore in view of Lemma 2.3, it suffices to bound $d - d_h$. Following the proof of Lemma 2.3 we have,

$$\begin{aligned} &d(\phi; \psi, \chi) - d_h(\phi; \psi, \chi) \\ &\quad = \sum_K \left\{ (\mathrm{div}\big((A'(\phi)\psi)\nabla\phi\big), \chi - I_h\chi)_K + \big((A'(\phi)\psi)\nabla\phi\big) \cdot n, \chi - I_h\chi)_{\partial K} \right\} \\ &\quad = \sum_K \{ I_K + II_K \}. \end{aligned} \tag{5.13}$$

Applying Hölder's inequality to I_K, and using the fact that ϕ is linear in K, (1.2), (5.10) and (2.11), we have

$$\begin{aligned} |I_K| &\leq (\beta_3 \|\nabla\phi \cdot \nabla\psi\|_{L_2(K)} + \beta_4 \| \, |\nabla\phi|^2 \psi\|_{L_2(K)}) \|\chi - I_h\chi\|_{L_2(K)} \\ &\leq Ch(\beta_3 \|\nabla\phi \cdot \nabla\psi\|_{L_2(K)} + \beta_4 \| \, |\nabla\phi|^2 \psi\|_{L_2(K)}) \|\nabla\chi\|_{L_2(K)}. \end{aligned} \tag{5.14}$$

For the II_K, we break the integration over the boundary of each triangle K, into the sum of integrations over its sides, and thus may use (2.12), and follow the same steps as in estimating I_K. Hence,

$$\begin{aligned} |II_K| &\leq Ch |(A'(\phi)\psi)\nabla\phi|_{H^1(K)} \|\nabla\chi\|_{L_2(K)} \\ &\leq Ch(\beta_3 \|\nabla\phi \cdot \nabla\psi\|_{L_2(K)} + \beta_4 \| \, |\nabla\phi|^2 \psi\|_{L_2(K)}) \|\nabla\chi\|_{L_2(K)}. \end{aligned}$$

Then combining this with Lemma 2.3 and (5.14), we get

$$|\varepsilon_N(\phi; \psi, \chi)| \leq Ch\big(\|\nabla\phi\|_{L_\infty} \|\nabla\psi\| + \|\nabla\phi\|_{L_\infty} \|\psi\|_{L_\infty} \|\nabla\phi\|\big) \|\chi\|_{H^1}.$$

Finally, in view of the definition of σ_h we get the desired estimate. □

Next, we derive a "Lipchitz"–type estimation for ε_N.

Lemma 5.3. *Let $v, w, \phi, \chi \in X_h$ then*

$$|\varepsilon_N(v;\phi,\chi) - \varepsilon_N(w;\phi,\chi)| \le Ch\big\{\|\nabla(v-w)\cdot\nabla\phi\| + \|\nabla w\|_{L_\infty}\|(v-w)\nabla\phi\| \\ + \|(|\nabla v|^2 - |\nabla w|^2)\phi\| + \|\nabla w\|^2_{L_\infty}\|(v-w)\phi\|\big\}\|\nabla\chi\|. \tag{5.15}$$

Proof. Similarly as in the proof of the previous lemma, we can easily see that $\varepsilon_N = \varepsilon_a + (d - d_h)$. Thus in view of Lemma 2.5, it suffices to estimate $d(v;\phi,\chi) - d_h(w;\phi,\chi)$. Using a similar decomposition as in (5.13) and then applying (2.11) and (2.12) we get

$$|d(v;\phi,\chi) - d_h(w;\phi,\chi)| \le Ch\big\{\|\mathrm{div}\big((A'(v)\nabla v - A'(w)\nabla w)\phi\big)\| \\ + |(A'(v)\nabla v - A'(w)\nabla w)\phi|_{H^1}\big\}\|\nabla\chi\|. \tag{5.16}$$

Next, since $\phi \in X_h$, we have

$$\begin{aligned}
&\mathrm{div}((A'(v)\nabla v - A'(w)\nabla w)\phi) \\
&= (A''(v)|\nabla v|^2 - A''(w)|\nabla w|^2)\phi + (A'(v)\nabla v - A'(w)\nabla w)\cdot\nabla\phi \\
&= (A''(v)(|\nabla v|^2 - |\nabla w|^2)\phi + (A''(v) - A''(w))|\nabla w|^2\phi \\
&\quad + (A'(v)(\nabla v - \nabla w)\cdot\nabla\phi + (A'(v) - A'(w))\nabla w\cdot\nabla\phi.
\end{aligned} \tag{5.17}$$

Therefore, (5.16) gives

$$|d(v;\phi,\chi) - d_h(w;\phi,\chi)| \le Ch(\|\nabla(v-w)\cdot\nabla\phi\| + \|\nabla w\|_{L_\infty}\|(v-w)\nabla\phi\|)\|\nabla\chi\| \\ + Ch(\|(|\nabla v|^2 - |\nabla w|^2)\phi\| + \|\nabla w\|^2_{L_\infty}\|(v-w)\phi\|)\|\nabla\chi\|. \tag{5.18}$$

Finally, this estimation and Lemma 2.5 give the desired (5.15). □

Next, we show an error bound that we will employ in the proof of Theorem 5.1.

Lemma 5.4. *For $v_h, w_h, \chi \in X_h$, we have*

$$|\varepsilon_N(v_h; w_h - v_h, \chi) + \varepsilon_a(v_h; v_h, \chi) - \varepsilon_a(w_h; w_h, \chi)| \\ \le Ch\big(\sigma_h(\|\nabla v_h\|^2_{L_\infty} + \|\nabla(w_h + v_h)\|_{L_\infty}) + h^{-1}\big)\|w_h - v_h\|^2_{H^1}\|\chi\|_{H^1}. \tag{5.19}$$

Proof. In view of the definition of ε_N and ε_a we have

$$\begin{aligned}
&\varepsilon_N(v_h; w_h - v_h, \chi) + \varepsilon_a(v_h; v_h, \chi) - \varepsilon_a(w_h; w_h, \chi) \\
&= \sum_K \int_K \mathrm{div}\Big(A(v_h)\nabla(w_h - v_h) + A'(v_h)(w_h - v_h)\nabla v_h + A(v_h)\nabla v_h \\
&\qquad\qquad - A(w_h)\nabla w_h\Big)(\chi - I_h\chi)\,dx \\
&+ \sum_K \int_{\partial K}\Big(A(v_h)\nabla(w_h - v_h) + A'(v_h)(w_h - v_h)\nabla v_h + A(v_h)\nabla v_h \\
&\qquad\qquad - A(w_h)\nabla w_h\Big)\cdot n(\chi - I_h\chi)\,ds.
\end{aligned}$$

Then, since v_h, w_h are linear in $K \in \mathcal{T}_h$, we get

$$\begin{aligned}
&\mathrm{div}\Big(A(v_h)\nabla(w_h - v_h) + A'(v_h)(w_h - v_h)\nabla v_h + A(v_h)\nabla v_h - A(w_h)\nabla w_h\Big) \\
&= 2A'(v_h)\nabla v_h\cdot\nabla(w_h - v_h) + A''(v_h)(w_h - v_h)|\nabla v_h|^2 + A'(v_h)|\nabla v_h|^2 - A'(w_h)|\nabla w_h|^2 \\
&= A''(v_h)(w_h - v_h)|\nabla v_h|^2 + A'(v_h)|\nabla v_h|^2 - A'(w_h)|\nabla v_h|^2 \\
&\quad + A'(w_h)|\nabla v_h|^2 - A'(w_h)|\nabla w_h|^2 + 2A'(v_h)\nabla v_h\cdot\nabla(w_h - v_h).
\end{aligned}$$

We consider now similar Taylor expansions as in (4.22) and denoting this $\tilde{A}'$ and $\tilde{A}''$ the expressions corresponding to $\bar{A}'$ and $\bar{A}''$, where we substitute A with A'. Then the previous relation gives

$$\begin{aligned}
&\operatorname{div}\Big(A(v_h)\nabla(w_h - v_h) + A'(v_h)(w_h - v_h)\nabla v_h + A(v_h)\nabla v_h - A(w_h)\nabla w_h\Big)\\
&= -(w_h - v_h)^2\tilde{A}''|\nabla v_h|^2 - A'(w_h)\nabla v_h \cdot \nabla(w_h - v_h) - A'(w_h)\nabla w_h \cdot \nabla(w_h - v_h)\\
&\quad + 2A'(v_h)\nabla v_h \cdot \nabla(w_h - v_h)\\
&= -(w_h - v_h)^2\tilde{A}''|\nabla v_h|^2 + (A'(v_h) - A'(w_h))\nabla v_h \cdot \nabla(w_h - v_h)\\
&\quad + (A'(v_h) - A'(w_h))\nabla w_h \cdot \nabla(w_h - v_h) - A'(v_h)|\nabla(w_h - v_h)|^2\\
&= -(w_h - v_h)^2\tilde{A}''|\nabla v_h|^2 + (A'(v_h) - A'(w_h))\nabla(w_h + v_h) \cdot \nabla(w_h - v_h) - A'(v_h)|\nabla(w_h - v_h)|^2\\
&= -(w_h - v_h)^2\tilde{A}''|\nabla v_h|^2 - (w_h - v_h)\tilde{A}'\nabla(w_h + v_h) \cdot \nabla(w_h - v_h) - A'(v_h)|\nabla(w_h - v_h)|^2.
\end{aligned}$$

Finally, this combined with (2.11) and (2.12) give the desired estimate

$$\begin{aligned}
&|\varepsilon_N(v_h; w_h - v_h, \chi) + \varepsilon_a(v_h; v_h, \chi) - \varepsilon_a(w_h; w_h, \chi)|\\
&\le Ch\big(\|w_h - v_h\|_{L_\infty}\|\nabla v_h\|^2_{L_\infty} + \|w_h - v_h\|_{L_\infty}\|\nabla(w_h + v_h)\|_{L_\infty}\\
&\qquad + \|w_h - v_h\|_{L_\infty}\big)\|w_h - v_h\|_{H^1}\|\chi\|_{H^1}\\
&\le Ch\big(\sigma_h(\|\nabla v_h\|^2_{L_\infty} + \|(w_h + v_h)\|_{L_\infty}) + h^{-1}\big)\|w_h - v_h\|^2_{H^1}\|\chi\|_{H^1}. \qquad \Box
\end{aligned}$$

Next, we show that the Newton sequence obtained by (5.5), is well defined and it converge to the finite volume approximation u_h of (2.3) with order 2.

Theorem 5.1. *There exists positive constants h_0, δ and C_5 such that if $0 < h \le h_0$ and $\sigma_h\|u_h^0 - u_h\|_{H^1} \le \delta$ then $\{u_h^k\}_{k=0}^\infty$ exists and $\nu_k = \|u_h^k - u_h\|_{H^1}$ is a decreasing sequence satisfying*

$$\nu_{k+1} \le C_5\sigma_h\nu_k^2. \tag{5.20}$$

Proof. The proof is based on a similar result of Douglas and Dupont, [13], for the finite element method. First we show that for h_0 and δ are sufficiently small, and $\sigma_h\|u_h^k - u_h\|_{H^1} = \sigma_h\nu_k \le \delta$, with $0 < h \le h_0$, there exists a unique u_h^{k+1}, given by (5.5). It suffices to show that if

$$N_h(u_h^k; v, \chi) = 0, \quad \forall\chi \in X_h,$$

then $v \equiv 0$, or else $\|v\|_{H^1} \le 0$. For this we will employ Lemma 5.1 and demonstrate that $C_4|||G||| < \|v\|_{H^1}$, for an appropriately defined functional G. We can easily see that

$$N(u_h; v, \chi) = G(\chi),$$

where G is given by

$$G(\chi) = N(u_h; v, \chi) - N_h(u_h^k; v, \chi) = \{N(u_h; v, \chi) - N(u_h^k; v, \chi)\} + \varepsilon_N(u_h^k; v, \chi) = I + II,$$

Following the proof in [13] we have that that

$$|I| \le C\sigma_h\|u_h - u_h^k\|_{H^1}\|v\|_{H^1}\|\chi\|_{H^1} = C\sigma_h\nu_k\|v\|_{H^1}\|\chi\|_{H^1}. \tag{5.21}$$

For the estimation of II we use the inverse inequality, (2.17), (5.9), Lemma 5.2 and the fact that induction hypothesis and (5.6) give

$$\|u_h^k\|_{H^1} \le \nu_k + \|u_h\|_{H^1} \le \sigma_h^{-1}\delta + \|u_h\|_{H^1} \le C, \tag{5.22}$$

to get

$$\begin{aligned}|II| &\le C(\nu_k(1+\sigma_h\|u_h^k\|_{H^1}) + h(1+\sigma_h\|u_h^k\|_{H^1})\|u_h\|_{W^1_\infty})\|v\|_{H^1}\|\chi\|_{H^1} \\ &\le C\sigma_h\nu_k\|v\|_{H^1}\|\chi\|_{H^1} + Ch\sigma_h\|v\|_{H^1}\|\chi\|_{H^1}.\end{aligned} \tag{5.23}$$

Hence, since $\sigma_h \le C\log(1/h)$, (5.21) and (5.23) give for δ and h sufficiently small, $\|v\|_{H^1} \le C_0\sigma_h(\nu_k + h\log(1/h))\|v\|_{H^1} < \|v\|_{H^1}$; thus $v = 0$.

In order to show (5.20) we will employ again Lemma 5.1 for a different functional G. This time let

$$N(u_h; u_h^{k+1} - u_h, \chi) = G(\chi), \quad \forall \chi \in X_h,$$

where G is defined by

$$\begin{aligned}G(\chi) &= N(u_h; u_h^k - u_h, \chi) + N(u_h^k; u_h^{k+1} - u_h^k, \chi) \\ &\quad + N(u_h; u_h^{k+1} - u_h^k, \chi) - N(u_h^k; u_h^{k+1} - u_h^k, \chi) \\ &= \{N(u_h; u_h^k - u_h, \chi) + a(u_h; u_h, \chi) - a(u_h^k; u_h^k, \chi)\} \\ &\quad + \{\varepsilon_N(u_h^k; u_h^{k+1} - u_h^k, \chi) - \varepsilon_a(u_h; u_h, \chi) + \varepsilon_a(u_h^k; u_h^k, \chi)\} \\ &\quad + \{N(u_h; u_h^{k+1} - u_h^k, \chi) - N(u_h^k; u_h^{k+1} - u_h^k, \chi)\} = I + II + III.\end{aligned} \tag{5.24}$$

We will show that

$$|||G||| \le C\sigma_h\nu_k(\nu_k + \nu_{k+1}) + Ch\sigma_h\nu_{k+1}. \tag{5.25}$$

Then Lemma 5.1, and $\sigma_h\nu_k \le \delta$, give

$$\begin{aligned}\nu_{k+1} &\le C_4|||G||| \le C\sigma_h\nu_k(\nu_k + \nu_{k+1}) + Ch\sigma_h\nu_{k+1} \\ &\le C\sigma_h\nu_k^2 + C(\delta + h\log(\frac{1}{h}))\nu_{k+1}.\end{aligned} \tag{5.26}$$

Finally for sufficiently small δ and h, the desired estimate, (5.20), follows easily.

Let us turn now to the estimation of $|||G|||$, for G given by (5.24). The terms I and III are similar to the ones that appear in the analysis of the finite element method in [13], thus using the same arguments we get

$$|I + III| \le C\sigma_h\nu_k(\nu_k + \nu_{k+1})\|\chi\|_{H^1}. \tag{5.27}$$

Then, we can easily see that II can be rewritten in the following way,

$$\begin{aligned}II &= \varepsilon_N(u_h^k; u_h^{k+1} - u_h^k, \chi) - \varepsilon_N(u_h; u_h^{k+1} - u_h^k, \chi) \\ &\quad + \varepsilon_N(u_h; u_h^{k+1} - u_h^k, \chi) - \varepsilon_a(u_h; u_h, \chi) + \varepsilon_a(u_h^k; u_h^k, \chi) \\ &= \{\varepsilon_N(u_h^k; u_h^{k+1} - u_h^k, \chi) - \varepsilon_N(u_h; u_h^{k+1} - u_h^k, \chi)\} \\ &\quad + \varepsilon_N(u_h; u_h^{k+1} - u_h, \chi) \\ &\quad - \{\varepsilon_N(u_h; u_h^k - u_h, \chi) + \varepsilon_a(u_h; u_h, \chi) - \varepsilon_a(u_h^k; u_h^k, \chi)\} = II_1 + II_2 + II_3.\end{aligned} \tag{5.28}$$

Using Lemma 5.3, (5.9), inverse inequality, (2.17), (5.6) and (5.22), we can bound II_1 in the following way,

$$
\begin{aligned}
|II_1| &\le Ch\big\{(\|\nabla(u_h^k-u_h)\|_{L_\infty}+\|\nabla u_h\|_{L_\infty}\|u_h^k-u_h\|_{L_\infty})\|\nabla(u_h^{k+1}-u_h^k)\| \\
&\quad+\big(\|\nabla(u_h^k+u_h)\|_{L_\infty}\|\nabla(u_h^k-u_h)\| \\
&\qquad+\|\nabla u_h\|^2_{L_\infty}\|u_h^k-u_h\|\big)\|u_h^{k+1}-u_h^k\|_{L_\infty}\big\}\|\chi\|_{H^1} \\
&\le C\big((1+(\|u_h^k+u_h\|_{H^1}+h)\sigma_h)\nu_k(\nu_k+\nu_{k+1})\big)\|\chi\|_{H^1} \\
&\le C\sigma_h\nu_k(\nu_k+\nu_{k+1})\|\chi\|_{H^1}.
\end{aligned}
\tag{5.29}
$$

Further, using Lemma 5.2, (5.9) and (5.6), we can easily bound II_2,

$$
\begin{aligned}
|II_2| &\le Ch(\|\nabla u_h\|_{L_\infty}+\sigma_h\|\nabla u_h\|_{L_\infty}\|u_h\|_{H^1})\|u_h^{k+1}-u_h\|_{H^1}\|\chi\|_{H^1} \\
&\le Ch(1+\sigma_h)\nu_{k+1}\|\chi\|_{H^1}.
\end{aligned}
\tag{5.30}
$$

Finally using, Lemma 5.4 and the fact that $\|\nabla u_h\|_{L_\infty}\le C_3$ and $h\|\nabla u_h^k\|_{L_\infty}\le C\|u_h^k\|_{H^1}\le C$, II_3 can be estimated by

$$
\begin{aligned}
|II_3| &\le C(h\sigma_h\|\nabla u_h\|^2_{L_\infty}+h\sigma_h\|\nabla(u_h^k+u_h)\|_{L_\infty}+1)\|u_h^k-u_h\|^2_{H^1}\|\chi\|_{H^1} \\
&\le C(\sigma_h+1)\nu_k^2\|\chi\|_{H^1}.
\end{aligned}
\tag{5.31}
$$

Therefore combining (5.27) and (5.29)–(5.31), we get the desired (5.25). □

6. NUMERICAL IMPLEMENTATIONS

In this section we present procedures for implementing the finite volume method for the nonlinear problem. A series of numerical examples is given to further assess the theories that were preceedingly deduced. Following the previous mathematical works, we implement two iterative schemes to solve the nonlinear finite volume problems, namely the fixed point iteration and the Newton iteration. As will be clear in the following subsection, these two schemes are built in the finite dimensional setting, i.e., using the finite element space X_h. We denote $\{\phi_i\}_{i=1}^d$ to be the standard piecewise linear basis functions of X_h. Then the finite volume element solution may be written as

$$
u_h=\sum_{i=1}^d \alpha_i\,\phi_i \quad \text{for some} \quad \alpha=(\alpha_1,\alpha_2,\cdots,\alpha_d)^T
$$

6.1. Fixed Point Iteration vs Newton Iteration

To describe the schemes, we begin with several notations, noting that some of them have already been mentioned. Let Z_h be the collection of vertices z_i that belong to all triangles $K\in\mathcal{T}_h$ and $Z_h^0=\{z_i\in Z_h : z_i\notin\Gamma_D\}$. Let $I=\{i : z_i\in Z_h^0\}$, $I_K=\{m : z_m \text{ is a vertex of } K\}$, $T_{h,i}=\{K\in\mathcal{T}_h : i\in I_K\}$, and $I_i=\{m\in I : z_m \text{ is a vertex of } K\in T_{h,i}\}$. Let V_i be the control volume surrounding the vertex z_i.

Now we may write this finite volume problem as to find $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_d)^T$ such that

$$
F(\alpha)=0, \tag{6.1}
$$

where $F : \mathbb{R}^d \to \mathbb{R}^d$ is a nonlinear operator with

$$F_i(\alpha) = -\int_{\partial V_i} A(u_h)\,\nabla\, u_h \cdot n\, ds - \int_{V_i} f\, dx \qquad \forall\, i \in I. \tag{6.2}$$

The fixed point iteration is derived from the linearization of (6.1) on the coefficient $A(u)$ in (6.2). Thus, given an initial iterate α^0 (i.e., equivalently $u_h^0 = \sum_{i=0}^d \alpha_i^0 \phi_i$), for $k = 0, 1, 2, \cdots$ until convergence solve the linear system $M(\alpha^k)\,\alpha^{k+1} = q$, where $M(\alpha^k)$ is the resulting stiffness matrix evaluated at $u_h^k = \sum_{i=0}^d \alpha_i^k \phi_i$, whose entries are

$$M_{ij}^k = -\int_{\partial V_i} A(u_h^k)\,\nabla\,\phi_j \cdot n\, ds.$$

On the other hand, the classical Newton iteration relies on the first order Taylor expansion of $F(\alpha)$. It results in solving a linear system of the Jacobian of $F(\alpha)$. An inexact-Newton iteration is a variation of Newton iteration for nonlinear system of equations in that the system Jacobian is only solved approximately, cf. e.g., [2, 3, 11]. To be specific, given an initial iterate α^0, for $k = 0, 1, 2, \cdots$ until convergence do the following:
(a) Solve $F'(\alpha^k)\delta^k = -F(\alpha^k)$ until $\|F(\alpha^k) + F'(\alpha^k)\delta^k\| \le \beta_k\, \|F(\alpha^k)\|$;
(b) Update $\alpha^{k+1} = \alpha^k + \delta^k$.

In this algorithm $F'(\alpha^k)$ is the Jacobian matrix evaluated at iteration k. For iterative technique solving a linear system such as the Krylov method we only need the action of the Jacobian to a vector. It has been common practice to use the following finite difference approximation for such an action:

$$F'(\alpha^k)\, v \;\approx\; \frac{F(\alpha^k + \sigma v) - F(\alpha^k)}{\sigma}, \tag{6.3}$$

where σ is a small number computed as follows:

$$\sigma = \frac{\text{sign}(\alpha^k \cdot v)\,\sqrt{\epsilon}\,\max(|\alpha^k \cdot v|, \|v\|_1)}{v \cdot v},$$

with ϵ being the machine unit round-off number. We note that when $\beta_k = 0$ then we have recovered the classical Newton iteration. One common used relation is

$$\beta_k = 0.001 \left(\frac{\|F(\alpha^k)\|}{\|F(\alpha^{k-1})\|} \right)^2,$$

with $\beta_0 = 0.001$. Choosing β_k this way we avoid oversolving the Jacobian system when α^k is still considerably far from the exact solution.

Instead of using (6.3), we will present below an explicit construction of the Jacobian matrix. We note that we may decompose $F_i(\alpha)$ as follows:

$$F_i(\alpha) = \sum_{K \in T_{h,i}} F_{i,K}(\alpha), \quad \text{where} \quad F_{i,K}(\alpha) = -\int_{K \cap \partial V_i} A(u_h)\,\nabla u_h \cdot n\, ds - \int_{K \cap V_i} f\, dx.$$

From the above description it is apparent that $F_i(\alpha)$ is not fully dependent on all $\alpha_1, \alpha_2, \cdots, \alpha_d$. Consequently, $\frac{\partial F_i(\alpha)}{\partial \alpha_j} = 0$ for $j \notin I_i$. Next we want to find an explicit form of $\frac{\partial F_i(\alpha)}{\partial \alpha_j}$ for $j \in I_i$.

Now suppose the edge $\overline{z_i z_j}$ is shared by triangles $K_l, K_r \in T_{h,i}$. Then

$$\frac{\partial F_i}{\partial \alpha_j} = -\sum_{e=l,r} \int_{K_e \cap \partial V_i} \left(A'(u_h)\, \phi_j \nabla u_h \cdot n + A(u_h)\, \nabla \phi_j \cdot n\right) ds.$$

Furthermore,

$$\frac{\partial F_i}{\partial \alpha_i} = -\sum_{K \in T_{h,i}} \int_{K \cap \partial V_i} \left(A'(u_h)\, \phi_i \nabla u_h \cdot n + A(u_h)\, \nabla\, \phi_i \cdot n\right) ds.$$

From this derivation it is obvious that the Jacobian matrix is not symmetric but sparse. Computation of this Jacobian matrix is similar to computing the stiffness matrix resulting from standard finite volume element, in that each entry is formed by accumulation of element by element contribution. Once we have the matrix stored in memory, then its action to a vector is straightforward. Since it is a sparse matrix, devoting some amount of memory for entries storage is not very expensive.

6.2. Numerical Examples

In this subsection we present several numerical experiments to verify the theoretical investigations. We solve a set of Dirichlet boundary value problems in $\Omega = [0,1] \times [0,1]$. We compare the fixed point iteration and the Newton iteration. In both schemes, the iteration is stopped once $\|u_h^k - u_h^{k-1}\|_{L_\infty} < 10^{-10}$. In all examples below, the initial iteration is taken to be $\alpha = (0, 0, \cdots, 0)^T$.

The first example is solving $-\nabla \cdot (k(u)\nabla u) = f$ in Ω where the function f is chosen such that the known solution is $u(x,y) = (x - x^2)(y - y^2)$ The nonlinearity comes from the coefficient with $k(u) = \frac{1}{(1+u)^2}$. The results are listed in Table I. First column represents the mesh size. The domain is discretized into N numbers of rectangle in each direction. Each of these rectangle is divided into two triangles. Second and third columns correspond to the number of iterations performed until the stopping criteria is reached for fixed point iteration (FP) and Newton iteration (NW), respectively. The table shows that a superconvergence is observed in H^1-norm due to the smoothness of the solution. Number of iterations in both schemes do not depend on the the mesh size. The numerical results for the second example are presented in Table II. Here

Table I. Error of FVEM for nonlinear elliptic BVP, with $u = (x - x^2)(y - y^2)$ and $k(u) = 1/(1 + u)^2$

h	# iter		H^1-seminorm		L_2-norm		L_∞-norm	
	FP	NW	Error $\times 10^{-5}$	Rate	Error $\times 10^{-5}$	Rate	Error $\times 10^{-5}$	Rate
1/16	7	5	17.1931	-	3.73555	-	7.51200	-
1/32	7	5	4.31635	1.99	0.94094	1.99	1.88100	1.99
1/64	7	5	1.08075	1.99	0.23568	1.99	0.47000	2.00
1/128	7	5	0.27778	1.96	0.05894	2.00	0.01180	1.99

the exact solution is chosen to be $u = 40(x - x^2)(y - y^2)$ and $k(u) = 0.125(-u^3 + 4u^2 - 7u + 8)$ if $u < 1$ and $k(u) = 1/(1 + u)$ if $u \geq 1$. Again a superconvergence is observed for this example. Furthermore, number of iterations needed are slightly higher than the previous example, which may be due to larger source term f. In this case the Newton iteration is shown to converge faster than the fixed point iteration. Next we consider a problem with known solution $u(x, y) = x^{1.6}$

Table II. Error of FVEM for nonlinear elliptic BVP, with $u = 40(x - x^2)(y - y^2)$ and $k(u) = 0.125(-u^3 + 4u^2 - 7u + 8)$ if $u < 1$ and $k(u) = 1/(1 + u)$ if $u \geq 1$

N	# iter		H^1-seminorm		L_2-norm		L_∞-norm	
	FP	NW	Error $\times 10^{-2}$	Rate	Error $\times 10^{-2}$	Rate	Error $\times 10^{-2}$	Rate
1/16	16	10	33.65484	-	7.33022	-	13.3000	-
1/32	15	8	9.10047	1.89	1.98347	1.89	3.57150	1.90
1/64	15	7	2.32645	1.97	0.50708	1.97	0.91120	1.97
1/128	15	7	0.58451	1.99	0.12740	1.99	0.22880	1.99

with $k(u) = 1 + u$. Obviously, this solution is an element of $H^2(\Omega)$ but not in $H^3(\Omega)$. Also the resulting source term f only belongs to $L^2(\Omega)$. The results are presented in Table III. These experiments show that the H^1-norm of the error decreases at first order. The L_2-norm of the error decreases slower than second order. Again, this case shows that the Newton iteration is relatively faster than the fixed point iteration.

Table III. Error of FVEM for nonlinear elliptic BVP with $u(x, y) = x^{1.6}$ and $k(u) = 1 + u$

N	# iter		H^1-seminorm		L_2-norm		L_∞-norm	
	FP	NW	Error $\times 10^{-4}$	Rate	Error $\times 10^{-4}$	Rate	Error $\times 10^{-4}$	Rate
1/16	11	6	34.1671	-	3.71216	-	8.97536	-
1/32	11	7	17.5558	0.96	1.44873	1.36	3.53674	1.34
1/64	11	7	8.68644	1.02	0.53714	1.43	1.33414	1.40
1/128	11	8	4.20084	1.05	0.19272	1.48	0.48582	1.46

Tables IV and V illustrate Theorem 5.1. In this theorem, it has been shown that there exists a sequence of solutions in the Newton iteration such that their errors with respect to the finite volume solution u_h are a decreasing sequence. Using the notation in that theorem, $\nu_k = \|u_h^k - u_h\|_{H^1}$ is a decreasing sequence satisfying

$$\nu_{k+1} \leq C_5 \sigma_h \nu_k^2, \qquad k = 0, 1, 2, \cdots .$$

We would like to examine the numerical behavior of this sequence for a fixed mesh size h. It is obvious that given ν_0 we have

$$\nu_k \leq (C_5 \sigma_h)^{2^k - 1} \nu_0^{2^k}, \qquad k = 1, 2, \cdots ,$$

which after dividing by $\nu_0^{2^k}$ and taking logarithm on both sides give

$$|\log(\nu_k / \nu_0^{2^k})| \leq C_5 \sigma_h \, (2^k - 1), \qquad k = 1, 2, \cdots .$$

Hence we should expect that the sequence ν_k would decrease exponentially as $k \to \infty$.

Table IV. Results for case 2

k	$h = 1/32$		$h = 1/64$		$h = 1/128$	
	$\lvert \log(\nu_k/\nu_0^{2^k})\rvert$	m	$\lvert \log(\nu_k/\nu_0^{2^k})\rvert$	m	$\lvert \log(\nu_k/\nu_0^{2^k})\rvert$	m
1	1.13		1.13		1.13	
2	3.40	3.02	3.40	3.02	3.40	3.02
3	7.97	7.08	7.96	7.06	7.96	7.05
4	16.8	15.0	16.8	14.9	16.6	14.7

The Tables IV and V show the decreasing behavior of the sequence resulting from the Newton iteration for last two model problems described above. In each table, k represents the iteration level, h is the mesh size, and m is the value of row k divided by the value of row $k-1$.

For case 2 presented in Table IV, in which the problem has a piecewise continuous coefficient and larger source term, we see that the decreasing behavior of the sequence is approximately exponential, and it is independent of the mesh size. Similar trends are also evident for case 3 shown in Table V.

Table V. Results for case 3

k	$h = 1/32$		$h = 1/64$		$h = 1/128$	
	$\lvert \log(\nu_k/\nu_0^{2^k})\rvert$	m	$\lvert \log(\nu_k/\nu_0^{2^k})\rvert$	m	$\lvert \log(\nu_k/\nu_0^{2^k})\rvert$	m
1	1.17		1.32		1.45	
2	3.57	3.05	3.86	2.93	4.19	2.89
3	8.04	6.85	8.72	6.63	9.26	6.37
4	16.8	14.3	18.2	13.9	19.6	13.4
5	32.7	27.9	36.9	28.1	40.1	27.6

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