Overlapping Schwarz preconditioner for the mixed formulation of plane elasticity

Yanqiu Wang

November 13, 2003

Abstract

Recently a stable pair of finite element spaces for the mixed formulation of the plane elasticity system has been developed by Arnold and Winther. Here we construct a two-level overlapping Schwarz preconditioner for the resulting discrete system. Essentially, this reduces to finding an efficient preconditioner for the form $(\cdot,\cdot)+(\mathbf{div}\,\cdot,\mathbf{div}\,\cdot)$ in the symmetric tensor space $\boldsymbol{H}(\mathbf{div}\,,\Omega)$. The main difficulty comes from the well known complexity of building preconditioners for the $\mathbf{div}$ operator. We solve it by taking a decomposition similar to the Helmholz decomposition. Both additive and multiplicative preconditioners are studied, and the conditioner numbers are shown to be uniform with respect to the mesh size.

1 Introduction

The purpose of this paper is to present and analyze the overlapping Schwarz preconditioner for the mixed formulation of the plane elasticity system. Compared to the primal-based methods, mixed finite element methods have some well-known advantages [1, 14]. For example, the dual variable, which is usually the variable of primary interest, is computed directly as a fundamental unknown. Another important advantage, in the case of linear elasticity, is that the mixed formulations exhibit robustness in the computation of nearly incompressible materials. Mixed methods also have some obvious disadvantages, such as the necessity of constructing stable pairs of finite element spaces and the fact that the resulting discrete system is indefinite. For decades extensive research has been taken to explore the mixed formulation of the plane elasticity system (also known as the weak formulation of the Hellinger-Reissner principle). Most of them focused on developing stable pairs of mixed finite element spaces and several different solutions have been proposed [2, 3, 6, 21]. As stated in those papers, the crux of the difficulty is that the stress tensor in the Hellinger-Reissner principle has to be symmetric. Indeed, the symmetry condition of the stress tensor is so hard to satisfy that the authors of [2, 3, 21] had to resort to composite elements. Only recently did Arnold and Winther propose a new pair of mixed finite elements (the Arnold-Winther elements) which does not use composite elements [6]. In this paper we choose to use the Arnold-Winther elements. The Arnold-Winther finite element spaces consist of piecewise polynomials and satisfy the stability requirement. In [6] only the pure displacement boundary problems are considered, but we will show that the spaces also work for the pure traction boundary problems. To prove it, one only need to modify the interpolation operator given in [6] so that it preserves the essential boundary condition. Finally, we mention some alternative ways to circumvent the difficulty of constructing stable pairs of spaces while preserving symmetry for the stress tensor. One way is to reformulate the saddle-point problem by using Lagrangian functionals so that it does not require symmetric tensors [4]. Another way is to use least-square formulation so that it does not require the classical discrete inf-sup condition [7].

Throughout the paper, we adopt the convention that a Greek character denotes 2×2 symmetric tensor, a bold Latin character in lower case denotes a vector and a bold Latin character in upper case denotes an operator or a matrix. Let $\boldsymbol{\tau} = (\tau_{ij})_{1\le i,j\le 2}$ be a symmetric tensor, $\boldsymbol{v} = (v_1, v_2)^t$ be a vector and q be a scalar. Define $\operatorname{div} \boldsymbol{v} = \frac{\partial}{\partial x} v_1 + \frac{\partial}{\partial y} v_2$ and

$$\mathbf{div}\,\boldsymbol{\sigma} = \begin{pmatrix} \frac{\partial}{\partial x}\tau_{11} + \frac{\partial}{\partial y}\tau_{12} \\ \frac{\partial}{\partial x}\tau_{21} + \frac{\partial}{\partial y}\tau_{22} \end{pmatrix}, \qquad \mathbf{airy}\, q = \begin{pmatrix} \frac{\partial^2}{\partial y^2} q & -\frac{\partial^2}{\partial x \partial y} q \\ -\frac{\partial^2}{\partial x \partial y} q & \frac{\partial^2}{\partial x^2} q \end{pmatrix}.$$

Define the innerproduct between vectors and the innerproduct between matrices as:

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2, \qquad \boldsymbol{\sigma} : \boldsymbol{\tau} = \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_{ij} \tau_{ij}.$$

Let Ω be a convex polygon in $\mathbb{R}^2$. We will use the usual notation $L^2(\Omega)$ for the set of square integrable functions on Ω and $H^s(\Omega)$, where s is a real number, for the normal Sobolev space defined on Ω [15]. Denote $\|\cdot\|_{s,\Omega}$ the H^s-norm and $|\cdot|_{s,\Omega}$ the H^s-seminorm as defined in [15]. Define the spaces

$$\begin{aligned} \boldsymbol{L}^2(\Omega) &= \{\text{vectors } \boldsymbol{v} = (v_1, v_2)^t \text{ such that } v_i \in L^2(\Omega) \text{ for } i = 1, 2\}, \\ \boldsymbol{H}(\mathbf{div}\,, \Omega) &= \{\text{symmetric tensors } \boldsymbol{\tau} = (\tau_{ij})_{1\le i,j\le 2} \text{ such that } \tau_{ij} \in L^2(\Omega) \\ &\qquad \text{and } \mathbf{div}\,\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)\}, \\ \boldsymbol{H}_0(\mathbf{div}\,, \Omega) &= \{\boldsymbol{\tau} \in \boldsymbol{H}(\mathbf{div}\,, \Omega) \text{ such that } \boldsymbol{\tau}\boldsymbol{n}|_{\partial\Omega} = \mathbf{0}\}, \end{aligned}$$

where $\boldsymbol{n}$ is the outer normal vector on $\partial\Omega$. For simplicity, we will use the notation $(\cdot,\cdot)$ to denote the L^2-innerproduct and $\|\cdot\|$ to denote the L^2-norm over scalar, vector or tensor fields defined on the whole Ω. Define the norm on $\boldsymbol{H}(\mathbf{div}\,, \Omega)$ as $\|\boldsymbol{\tau}\|_{\boldsymbol{H}(\mathbf{div}\,,\Omega)} = [(\boldsymbol{\tau}, \boldsymbol{\tau}) + (\mathbf{div}\,\boldsymbol{\tau}, \mathbf{div}\,\boldsymbol{\tau})]^{1/2}$. Now we can state the mixed formulation of the plane elasticity problem rigorously. We only consider the pure traction boundary problem: Find $\boldsymbol{\sigma} \in \boldsymbol{H}_0(\mathbf{div}\,, \Omega)$ and $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ such that

$$\begin{cases} \int_\Omega \mathbf{A}\boldsymbol{\sigma} : \boldsymbol{\tau}\, d\boldsymbol{x} + \int_\Omega \mathbf{div}\,\boldsymbol{\tau} \cdot \boldsymbol{u}\, d\boldsymbol{x} = 0 & \forall\, \boldsymbol{\tau} \in \boldsymbol{H}_0(\mathbf{div}\,, \Omega), \\ \int_\Omega \mathbf{div}\,\boldsymbol{\sigma} \cdot \boldsymbol{v}\, d\boldsymbol{x} = \int_\Omega \boldsymbol{g} \cdot \boldsymbol{v}\, d\boldsymbol{x} & \forall\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega). \end{cases} \tag{1}$$

where the compliance tensor $\mathbf{A} : \boldsymbol{H}(\mathbf{div}\,,\Omega) \to \boldsymbol{H}(\mathbf{div}\,,\Omega)$ is bounded, symmetric and uniformly positive definite and $g \in \boldsymbol{L}^2(\Omega)$ is the body force. In order that the above problem be well posed, we need the compatibility condition on $\boldsymbol{g}$. Let

$$RM := span\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -y \\ x \end{pmatrix} \}$$

be the space of infinitesimal rigid motions. By Korn's inequality, one can see that for any $\boldsymbol{g} \in RM^{\perp_{\boldsymbol{L}^2(\Omega)}}$, which means the orthogonal complement of RM in $\boldsymbol{L}^2(\Omega)$, system (1) has unique solution in $\boldsymbol{H}_0(\mathbf{div}\,,\Omega) \times RM^{\perp_{\boldsymbol{L}^2(\Omega)}}$ [14].

The discretization of the system (1) leads to a symmetric indefinite linear system. Generally speaking, there were three main approaches toward solving large symmetric indefinite linear systems corresponding to certain mixed formulations. One can use the well-studied Uzawa-type method [10, 14, 17]. The second choice is the positive definite reformulation proposed by Bramble and Pasciak in [8] and [9]. The third choice is the preconditioned minimum residual method analyzed in [5, 22]. Our paper adopts the idea of preconditioned minimum residual method. An analysis similar to the one in [5] will show that the problem of constructing a preconditioner for the indefinite linear system derived from system (1) is essentially the same as the problem of constructing a preconditioner for the bilinear form $(\cdot,\cdot) + (\mathbf{div}\,\cdot, \mathbf{div}\,\cdot)$ in the tensor space $\boldsymbol{H}_0(\mathbf{div}\,,\Omega)$. We construct the preconditioner by the overlapping Schwarz method. For more background on this topic, one can refer to [16, 11, 12, 24, 25]. The main difficulty comes from the well-known complexity of building overlapping Schwarz preconditioners for the **div** operator and also the non-nested character of the finite element spaces.

In Section 2 we briefly present the mixed finite elements introduced in [6]. Furthermore some important observations on this finite element space are stated and proved. In Section 3 the details of the overlapping Schwarz preconditioner are explained and the condition number of the preconditioned system is analyzed. In Section 4 the main assumption used in the proof in Section 3 is proved. The results of numerical experiments illustrating the theory are given in Section 5. In Appendix A, we give a proof of the stability and approximation property of the mixed finite element spaces. In Appendix B, we construct a Clement-type interpolation operator which is used in Section 4.

2 Finite element discretization

First we present the Arnold-Winther elements. Let $\mathcal{T}$ be a quasi-uniform triangulation of Ω. On each triangular element $T \in \mathcal{T}$ define

$$\begin{aligned} &\boldsymbol{\Sigma}_T = \{\text{symmetric tensors } \boldsymbol{\tau} \in (P_3(T))^3 \text{ such that } \mathbf{div}\,\boldsymbol{\tau} \in (P_1(T))^2\}, \\ &\boldsymbol{V}_T = (P_1(T))^2, \end{aligned}$$

where $P_i(T)$ denotes the space consisting of polynomials of degree i or less. The degrees of freedom for $\boldsymbol{\Sigma}_T$ are

- the nodal values of the three components of $\boldsymbol{\tau}(x)$ (9 dofs)
- the moments of degree 0 and 1 of the two normal components of $\boldsymbol{\tau}$ on each edge of T (12 dofs)
- the moments of degree 0 of the three components of $\boldsymbol{\tau}$ on T (3 dofs)

and the degrees of freedom of $\boldsymbol{V}_T$ are given as the zero'th and first order moments. Figure 1 illustrates the degrees of freedom for $\boldsymbol{\Sigma}_T$. The finite element spaces on mesh

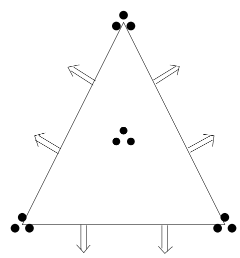

Figure 1: Finite element $\boldsymbol{\Sigma}_T$

$\mathcal{T}$ and domain Ω are defined as follows:

$$\begin{aligned}\boldsymbol{\Sigma}(\mathcal{T},\Omega) = \{&\boldsymbol{\tau} \text{ defined on } \Omega \text{ satisfying } \boldsymbol{\tau}|_T \in \boldsymbol{\Sigma}_T \text{ for each } T \in \mathcal{T},\\ &\boldsymbol{\tau} \text{ is continuous on the degrees of freedom on each vertex}\\ &\text{and each edge of } \mathcal{T} \text{ and } \boldsymbol{\tau n}|_{\partial\Omega} = \mathbf{0}.\}\\ \boldsymbol{V}(\mathcal{T},\Omega) = \{&\boldsymbol{v} \in \boldsymbol{L}_2(\Omega) \text{ such that } \boldsymbol{v}|_T \in \boldsymbol{V}_T \text{ for each } T \in \mathcal{T}\}.\end{aligned}$$

The definition of $\boldsymbol{\Sigma}(\mathcal{T},\Omega)$ clearly implies that $\boldsymbol{\Sigma}(\mathcal{T},\Omega) \subset \boldsymbol{H}_0(\mathbf{div}\,,\Omega)$ (see [6, 14]). Note that the boundary condition $\boldsymbol{\sigma n}|_{\partial\Omega} = 0$ implies two linear relations among the three components of $\boldsymbol{\sigma}$ on boundary nodes. Hence on the corner vertices where two boundary edges meet, we will have $\boldsymbol{\sigma} = \mathbf{0}$. This fact was noticed by Arnold and Winther in [6]. Another immediate observation is that, by Green's formula,

$$\mathbf{div}\,\boldsymbol{\sigma} \in RM^{\perp_{\boldsymbol{V}(\mathcal{T},\Omega)}}, \qquad \text{for all } \boldsymbol{\sigma} \in \boldsymbol{\Sigma}(\mathcal{T},\Omega).$$

We have the discrete elasticity problem: find $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}(\mathcal{T},\Omega)$ and $\boldsymbol{u} \in \boldsymbol{V}(\mathcal{T},\Omega)$ such that

$$\begin{cases}(\mathbf{A}\boldsymbol{\sigma},\boldsymbol{\tau}) + (\mathbf{div}\,\boldsymbol{\tau},\boldsymbol{u}) = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}(\mathcal{T},\Omega),\\ (\mathbf{div}\,\boldsymbol{\sigma},\boldsymbol{v}) = (\boldsymbol{g},\boldsymbol{v}) & \forall \boldsymbol{v} \in \boldsymbol{V}(\mathcal{T},\Omega).\end{cases} \tag{2}$$

Let $\boldsymbol{V}^\perp \subset \boldsymbol{\Sigma}$ be defined as $\{\boldsymbol{\tau} \in \boldsymbol{\Sigma} \,|\, (\mathbf{div}\,\boldsymbol{\tau},\, \boldsymbol{v}) = \mathbf{0}, \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}\}$. We say the pair of mixed finite element spaces $(\boldsymbol{\Sigma}(\mathcal{T},\Omega),\, \boldsymbol{V}(\mathcal{T},\Omega))$ is stable if there exists constants c and C independent of the mesh size such that

$$\begin{aligned}(\mathbf{A}\boldsymbol{\sigma},\boldsymbol{\sigma}) \geq c\|\boldsymbol{\sigma}\|^2_{\boldsymbol{H}(\mathbf{div}\,,\Omega)}, &\qquad \text{for all } \boldsymbol{\sigma} \in \boldsymbol{V}^\perp,\\ \sup_{\boldsymbol{\tau}\in\boldsymbol{\Sigma}(\mathcal{T},\Omega)} \frac{(\mathbf{div}\,\boldsymbol{\tau},\boldsymbol{v})}{\|\boldsymbol{\tau}\|_{\boldsymbol{H}(\mathbf{div}\,,\Omega)}} \geq C\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}, &\qquad \text{for all } \boldsymbol{v} \in RM^{\perp_{\boldsymbol{V}(\mathcal{T},\Omega)}}.\end{aligned}$$

The second condition is called the discrete inf-sup condition. In [6], the stability of this new pair of spaces is proved for the pure displacement boundary problem. A slight modification in the proof will show that it also works for the pure traction case (see Appendix A). It follows from the results given in Appendix A that for $\boldsymbol{g} \in RM^{\perp_{\boldsymbol{L}^2(\Omega)}}$, the discrete solution of (2) exists and provides a good approximation for the weak solution of (1).

Next we introduce the Argyris element which will play an important role in the analysis to be given later. Let $\mathbf{Q}_T$ denote the Argyris element [15] defined on T. It is a quintic element and the degrees of freedom are

- the nodal values (3 dofs), the first derivatives at the nodes (6 dofs) and the second derivatives at the nodes (9 dofs)
- the moments of degree 0 of $\frac{\partial}{\partial \boldsymbol{n}} q$ on the edges of T (3 dofs)

Define the space

$$\begin{aligned}\tilde{\mathbf{Q}}(\mathcal{T}, \Omega) = \{&q \text{ defined on } \Omega \text{ satisfying } q|_T \in \mathbf{Q}_T \text{ for each } T \in \mathcal{T},\\ &q \text{ is continuous on the degrees of freedom on each vertex}\\ &\text{and each edge of } \mathcal{T} \text{ and } \mathbf{airy}\, q \in \boldsymbol{\Sigma}(\mathcal{T}, \Omega)\}.\end{aligned}$$

Clearly $\tilde{\mathbf{Q}}(\mathcal{T}, \Omega) \subset H^2(\Omega)$.

It is well know that for any $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathbf{div}\,, \Omega)$ satisfying $\mathbf{div}\, \boldsymbol{\sigma} = \mathbf{0}$, there exists a $q \in H^2(\Omega)$ such that $\mathbf{airy}\, q = \boldsymbol{\sigma}$. Analogously on the discrete level we have the following exact sequence:

$$0 \longrightarrow P_1(\Omega) \stackrel{\subset}{\longrightarrow} \tilde{\mathbf{Q}}(\mathcal{T}, \Omega) \stackrel{\mathbf{airy}}{\longrightarrow} \boldsymbol{\Sigma}(\mathcal{T}, \Omega) \stackrel{\mathbf{div}}{\longrightarrow} \boldsymbol{V}(\mathcal{T}, \Omega).$$

The exactness of this sequence for discrete spaces without boundary conditions was proved on page 408 of [6] and the proof of the above exact sequence follows from their result. For the convenience of further analysis, we note the following lemmas:

Lemma 2.1. *For any $q \in \tilde{\mathbf{Q}}(\mathcal{T}, \Omega)$, $q|_{\partial\Omega}$ is a linear function.*

Proof. Since $(\mathbf{airy}\, q)\boldsymbol{n}|_{\partial\Omega} = \mathbf{0}$, we have

$$\left.\begin{pmatrix} \frac{\partial^2}{\partial y^2} q & -\frac{\partial^2}{\partial x \partial y} q \\ -\frac{\partial^2}{\partial x \partial y} q & \frac{\partial^2}{\partial x^2} q \end{pmatrix} \boldsymbol{n}\right|_{\partial\Omega} = \begin{pmatrix} \frac{\partial}{\partial \boldsymbol{s}} \frac{\partial}{\partial y} q \\ -\frac{\partial}{\partial \boldsymbol{s}} \frac{\partial}{\partial x} q \end{pmatrix} = \mathbf{0}.$$

By the continuity of the first and second derivatives of q on the vertices, it is obvious that $\nabla q|_{\partial\Omega} = \mathbf{const}$. □

Let $H_0^2(\Omega) = \{q \in H^2(\Omega) \text{ such that } q|_{\partial\Omega} = 0,\ \nabla q|_{\partial\Omega} = \mathbf{0}\}$ and define

$$\begin{aligned}\mathbf{Q}(\mathcal{T}, \Omega) = \{&q \text{ defined on } \Omega \text{ satisfying } q|_T \in \mathbf{Q}_T \text{ for each } T \in \mathcal{T},\\ &q \text{ is continuous on the degrees of freedom on each vertex}\\ &\text{and each edge of } \mathcal{T} \text{ and } q|_{\partial\Omega} = 0,\ \nabla q|_{\partial\Omega} = \mathbf{0}\}.\end{aligned}$$

Clearly $\mathbf{Q}(\mathcal{T}, \Omega) \subset H_0^2(\Omega)$. From the previous analysis we can derive

Lemma 2.2. *The following exact sequence holds:*

$$0 \longrightarrow \mathbf{Q}(\mathcal{T}, \Omega) \stackrel{\mathbf{airy}}{\longrightarrow} \boldsymbol{\Sigma}(\mathcal{T}, \Omega) \stackrel{\mathbf{div}}{\longrightarrow} \boldsymbol{V}(\mathcal{T}, \Omega). \tag{3}$$

3 Overlapping Schwarz preconditioner

In this section we develop an overlapping Schwarz preconditioner for the discrete problem (2) so that preconditioned minimal residual method can be used to solve this problem. For simplicity, denote $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\mathcal{T}, \Omega)$ and $\boldsymbol{V} = \boldsymbol{V}(\mathcal{T}, \Omega)$. Denote $\|\cdot\|_{\boldsymbol{\Sigma}}$ and $\|\cdot\|_{\boldsymbol{V}}$ to be the norms on $\boldsymbol{\Sigma}$ and $\boldsymbol{V}$ respectively, which are just $\|\cdot\|_{\boldsymbol{H}(\mathbf{div},\Omega)}$ and $\|\cdot\|_{\boldsymbol{L}^2(\Omega)}$. Let $\boldsymbol{\Sigma}^*$ and $\boldsymbol{V}^*$ be the dual spaces of $\boldsymbol{\Sigma}$ and $\boldsymbol{V}$ with dual norms $\|\cdot\|_{\boldsymbol{\Sigma}^*}$ and $\|\cdot\|_{\boldsymbol{V}^*}$. Define operators

$$\begin{cases} \mathcal{A}:\ \boldsymbol{\Sigma} \to \boldsymbol{\Sigma}^* & (\mathcal{A}\boldsymbol{\sigma}, \boldsymbol{\tau}) = (\mathbf{A}\boldsymbol{\sigma}, \boldsymbol{\tau}), \quad \text{for all } \boldsymbol{\tau} \in \boldsymbol{\Sigma}, \\ \mathcal{B}:\ \boldsymbol{\Sigma} \to \boldsymbol{V}^* & (\mathcal{B}\boldsymbol{\sigma}, \boldsymbol{v}) = (\mathbf{div}\,\boldsymbol{\sigma}, \boldsymbol{v}), \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}. \end{cases}$$

Let $\mathcal{B}^t : \boldsymbol{V} \to \boldsymbol{\Sigma}^*$ be the adjoint of $\mathcal{B}$. Equation (2) can be rewritten as

$$\mathcal{M}\begin{pmatrix} \boldsymbol{\sigma} \\ \boldsymbol{u} \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{B}^t \\ \mathcal{B} & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{\sigma} \\ \boldsymbol{u} \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}, \tag{4}$$

where $F \in \boldsymbol{\Sigma}^*$, $G \in \boldsymbol{V}^*$ and $\mathcal{M} : \boldsymbol{\Sigma} \times \boldsymbol{V} \to \boldsymbol{\Sigma}^* \times \boldsymbol{V}^*$. Let $\boldsymbol{V}/RM$ be the quotient space with the quotient norm $\|\cdot\|_{\boldsymbol{V}/RM}$. We have the following lemma:

Lemma 3.1. *If $(\boldsymbol{\sigma}, \boldsymbol{u})$ is a solution of the equation (4), then*

$$c_0(\|F\|_{\boldsymbol{\Sigma}^*} + \|G\|_{\boldsymbol{V}^*}) \le \|\boldsymbol{\sigma}\|_{\boldsymbol{\Sigma}} + \|\boldsymbol{u}\|_{\boldsymbol{V}/RM} \le c_1(\|F\|_{\boldsymbol{\Sigma}^*} + \|G\|_{\boldsymbol{V}^*}),$$

where c_0 and c_1 are positive and independent of h.

Proof. By the stability of the finite elements spaces $(\boldsymbol{\Sigma}, \boldsymbol{V})$ and Proposition 1.3 in [14],

$$\|\boldsymbol{\sigma}\|_{\boldsymbol{\Sigma}} + \|\boldsymbol{u}\|_{\boldsymbol{V}/RM} \le c_1(\|F\|_{\boldsymbol{\Sigma}^*} + \|G\|_{\boldsymbol{V}^*}),$$

where c_1 is independent of h. By the Schwartz inequality, the other direction comes from

$$\begin{aligned} \|F\|_{\boldsymbol{\Sigma}^*} + \|G\|_{\boldsymbol{V}^*} &= \sup_{\boldsymbol{\tau}\in\boldsymbol{\Sigma}} \frac{F(\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}}} + \sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{G(\boldsymbol{v})}{\|\boldsymbol{v}\|_{\boldsymbol{V}}} \\ &= \sup_{\boldsymbol{\tau}\in\boldsymbol{\Sigma}} \frac{(A\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathbf{div}\,\boldsymbol{\tau}, \boldsymbol{u})}{\|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}}} + \sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{(\mathbf{div}\,\boldsymbol{\sigma}, \boldsymbol{v})}{\|\boldsymbol{v}\|_{\boldsymbol{V}}} \\ &\le c(\|\boldsymbol{\sigma}\|_{\boldsymbol{\Sigma}} + \|\boldsymbol{u}\|_{\boldsymbol{V}}). \end{aligned}$$

□

Our purpose is to find a preconditioner for the operator $\mathcal{M}$. By lemma 3.1, we only need to find an operator $\mathcal{S} : \boldsymbol{\Sigma}^* \times \boldsymbol{V}^* \to \boldsymbol{\Sigma} \times \boldsymbol{V}$ such that $\|\mathcal{S}\|_{\mathcal{L}(\boldsymbol{\Sigma}^*\times\boldsymbol{V}^*, \boldsymbol{\Sigma}\times\boldsymbol{V})}$ and $\|\mathcal{S}^{-1}\|_{\mathcal{L}(\boldsymbol{\Sigma}\times\boldsymbol{V}, \boldsymbol{\Sigma}^*\times\boldsymbol{V}^*)}$ are bounded uniformly in h(see [5] for details). Indeed we can consider those $\mathcal{S}$ in the form $\mathcal{S} = \begin{pmatrix} \mathcal{S}_1 & 0 \\ 0 & \mathcal{S}_2 \end{pmatrix}$, where $\mathcal{S}_1 : \boldsymbol{\Sigma}^* \to \boldsymbol{\Sigma}$ and $\mathcal{S}_2 : \boldsymbol{V}^* \to \boldsymbol{V}$ and their inverses are bounded uniformly in h. Define the innerproduct

$$\boldsymbol{\Lambda}(\boldsymbol{\sigma}, \boldsymbol{\tau}) = (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathbf{div}\,\boldsymbol{\sigma}, \mathbf{div}\,\boldsymbol{\tau})$$

on $\mathbf{\Sigma}$. Consider the following problem: find $\boldsymbol{\sigma} \in \mathbf{\Sigma}$ such that

$$\mathbf{\Lambda}(\boldsymbol{\sigma}, \boldsymbol{\tau}) = F(\boldsymbol{\tau}), \quad \forall\, \boldsymbol{\tau} \in \mathbf{\Sigma}. \tag{5}$$

Clearly a good preconditioner for this problem will yield an ideal $\mathcal{S}_1$. Similarily, a good preconditioner for the problem: find $\boldsymbol{u} \in \boldsymbol{V}$ such that

$$(\boldsymbol{u}, \boldsymbol{v}) = G(\boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{V} \tag{6}$$

will yield an ideal $\mathcal{S}_2$. The space $\boldsymbol{V}$ consists of discontinuous linears on the triangles so the solution of (6) reduces to the inversion of a 3×3 block diagonal matrix. Hence the problem of finding $\mathcal{S}$ reduces to the problem of constructing $\mathcal{S}_1$. In the remainder of this paper we will focus on constructing a two-level Schwarz preconditioner for problem (5).

Let $\mathcal{T}_H$ be a quasi-uniform mesh on Ω with characteristic mesh size H and $\mathcal{T}_h$ be a quasi-uniform refinement of $\mathcal{T}_H$ with characteristic mesh size h. Let $\tilde{\Omega}_i$, $i = 1, \cdots, k$ be a non-overlapping decomposition of Ω whose boundaries align with the coarse mesh $\mathcal{T}_H$. Extend $\tilde{\Omega}_i$ by one or more layers of fine elements to get Ω_i, then we have an overlapping cover of Ω whose boundaries align with the fine mesh $\mathcal{T}_h$. Figure 2 illustrates how the subdomains are defined inside Ω and near the boundary of Ω. The bold line contour draws the boundary of $\tilde{\Omega}_i$ and the outmost dashed line contour draws the boundary of Ω_i. We have illustrated the case of one cell overlap although we may overlap many more cells in practice.

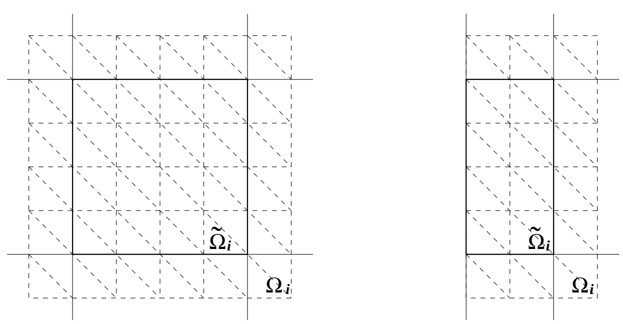

Figure 2: Subdomains $\tilde{\Omega}_i$ and Ω_i

Assume all $\tilde{\Omega}_i$ and Ω_i be convex polygons and denote the characteristic distance between $\partial\tilde{\Omega}_i \backslash \partial\Omega$ and $\partial\Omega_i \backslash \partial\Omega$ as δ. Furthermore, assume there exist a positive integer N_c such that for all $x \in \Omega$, x is included in at most N_c subdomains in $\{\Omega_i\}$. Define

$$\begin{aligned} \mathbf{Q}_0 &= \mathbf{Q}(\mathcal{T}_H, \Omega), & \mathbf{\Sigma}_0 &= \mathbf{\Sigma}(\mathcal{T}_H, \Omega), & \boldsymbol{V}_0 &= \boldsymbol{V}(\mathcal{T}_H, \Omega), \\ \mathbf{Q} &= \mathbf{Q}(\mathcal{T}_h, \Omega), & \mathbf{\Sigma} &= \mathbf{\Sigma}(\mathcal{T}_h, \Omega), & \boldsymbol{V} &= \boldsymbol{V}(\mathcal{T}_h, \Omega). \end{aligned}$$

For $i = 1, \cdots, k$ define $\mathbf{\Sigma}_i$, $\boldsymbol{V}_i$ and $\mathbf{Q}_i$ to be the subspaces of $\mathbf{\Sigma}$, $\boldsymbol{V}$ and $\mathbf{Q}$ respectively, which vanish outside Ω_i. Recalling how we defined the boundary conditions for $\mathbf{Q}(\mathcal{T},\Omega)$ and $\mathbf{\Sigma}(\mathcal{T},\Omega)$, it is clear that

$$\mathbf{Q}_i \subsetneq \mathbf{Q}(\mathcal{T}_h,\Omega_i), \qquad \mathbf{\Sigma}_i \subsetneq \mathbf{\Sigma}(\mathcal{T}_h,\Omega_i), \qquad \text{for all } i=1,\cdots,k.$$

Hence the space $\mathbf{\Sigma}_i$ does not correspond to a natural stress tensor approximation subspace with pure traction boundary condition.

Denote $\Psi(\mathcal{T})$ to be the set of all nodes in the mesh $\mathcal{T}$. We know that $\mathbf{Q}_0 \nsubseteq \mathbf{Q}$ and $\mathbf{\Sigma}_0 \nsubseteq \mathbf{\Sigma}$ since, for example, a function $\boldsymbol{\sigma} \in \mathbf{\Sigma}_0$ is not continuous at the points in $\Psi(\mathcal{T}_h)$ which are on the edges of the coarse grid. Hence we need to define interpolation operators. The easiest way to do this is to take the average of the degrees of freedom on those nodes where discontinuity occurs. For any point $v \in \Psi(\mathcal{T}_h)$, let $\Theta(v)$ be the set of all triangles in $\mathcal{T}_H$ which contain the vertex v and $|\Theta(v)|$ denote the number of triangles in $\Theta(v)$. We define $\tilde{q}$ and $\tilde{\boldsymbol{\tau}}$ as follows: on each element $T \in \mathcal{T}_h$, let $\tilde{q}|_T \in \mathbf{Q}_T$ and $\tilde{\boldsymbol{\tau}}|_T \in \mathbf{\Sigma}_T$ satisfy

$$\begin{aligned}
\mathbf{airy}\,\tilde{q}(v)|_T &= \left(\frac{1}{|\Theta(v)|}\sum_{T_v\in\Theta(v)} \mathbf{airy}\,q(v)|_{T_v}\right) - \mathbf{airy}\,q(v)|_T,\\
\tilde{\boldsymbol{\tau}}(v)|_T &= \left(\frac{1}{|\Theta(v)|}\sum_{T_v\in\Theta(v)} \boldsymbol{\tau}(v)|_{T_v}\right) - \boldsymbol{\tau}(v)|_T,
\end{aligned} \tag{7}$$

on each vertex v of T and vanish at all the other degrees of freedom. Define

$$\begin{aligned}
\mathcal{I}_0 q &= q + \tilde{q}, &&\text{for all } q \in \mathbf{Q}_0,\\
\mathbf{I}_0 \boldsymbol{\tau} &= \boldsymbol{\tau} + \tilde{\boldsymbol{\tau}}, &&\text{for all } \boldsymbol{\tau} \in \mathbf{\Sigma}_0.
\end{aligned}$$

It is not hard to see that $\mathcal{I}_0$ maps $\mathbf{Q}_0$ to $\mathbf{Q}$ and $\mathbf{I}_0$ maps $\mathbf{\Sigma}_0$ to $\mathbf{\Sigma}$. Therefore $\tilde{q} \in H_0^2(\Omega)$ and $\tilde{\boldsymbol{\tau}} \in \boldsymbol{H}_0(\mathbf{div}\,,\Omega)$. Furthermore, since $\tilde{q}$ vanishes on all the other degrees of freedom except for the second derivatives on each node, we can derive from a standard scaling argument that for all $q \in \mathbf{Q}_0$ and $i = 0,1,2$,

$$|q - \mathcal{I}_0 q|_{i,\Omega} = |\tilde{q}|_{i,\Omega} \le ch^{2-i}|\tilde{q}|_{2,\Omega} \le ch^{2-i}|q|_{2,\Omega}, \tag{8}$$

where c is independent of h and H.

The following lemma shows the relations between the spaces defined above.

Lemma 3.2. *The following commutative diagram of exact sequences holds:*

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbf{Q}_0 & \xrightarrow{\mathbf{airy}} & \mathbf{\Sigma}_0 & \xrightarrow{\mathbf{div}} & \boldsymbol{V}_0\\
 & & \downarrow\mathcal{I}_0 & & \downarrow\mathbf{I}_0 & & \downarrow id\\
0 & \longrightarrow & \mathbf{Q} & \xrightarrow{\mathbf{airy}} & \mathbf{\Sigma} & \xrightarrow{\mathbf{div}} & \boldsymbol{V}
\end{array} \tag{9}$$

For each $i = 1,\cdots,k$, we have the exact sequence

$$0 \longrightarrow \mathbf{Q}_i \xrightarrow{\mathbf{airy}} \mathbf{\Sigma}_i \xrightarrow{\mathbf{div}} \boldsymbol{V}_i. \tag{10}$$

Proof. Let T be a triangle and v_i, $i = 1, 2, 3$ be its three vertices. Denote the opposite edge to each vertex v_i as l_i. Let $\boldsymbol{\tau}_i$, $i = 1, 2, 3$ be given constant symmetric tensors. Define $q \in \mathbf{Q}_T$ as:

$$\begin{aligned} & & \mathbf{airy}\, q(v_i) = \boldsymbol{\tau}_i, \qquad & \text{for } i = 1,2,3; \\ q(v_i) = 0, \quad & \nabla q(v_i) = \mathbf{0}, & \int_{l_i} \frac{\partial}{\partial \boldsymbol{n}} q\, ds = 0, \qquad & \text{for } i = 1,2,3 \end{aligned}$$

and define $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_T$ as:

$$\begin{aligned} & \boldsymbol{\tau}(v_i) = \boldsymbol{\tau}_i, \qquad & \text{for } i = 1,2,3; \\ \int_{l_i} \boldsymbol{\tau n}\, ds = \int_{l_i} \boldsymbol{\tau n} s\, ds = \mathbf{0}, \quad & \int_T \boldsymbol{\tau}\, d\boldsymbol{x} = \mathbf{0}, \qquad & \text{for } i = 1,2,3. \end{aligned}$$

Let $\boldsymbol{n}$ and $\boldsymbol{s}$ denote the outer normal vector and the unit tangential vector on ∂T respectively. Simple calculation shows that $\frac{\partial^2}{\partial \boldsymbol{s}^2} q = \boldsymbol{n}^T(\mathbf{airy}\, q)\boldsymbol{n}$ and $\frac{\partial^2}{\partial \boldsymbol{n} \partial \boldsymbol{s}} q = -\boldsymbol{n}^T(\mathbf{airy}\, q)\boldsymbol{s}$. Hence by the definition of q and the integration by parts, all of $\int_{l_i} (\mathbf{airy}\, q)\, \boldsymbol{n} \cdot \boldsymbol{n}\, ds$, $\int_{l_i} (\mathbf{airy}\, q)\, \boldsymbol{n} \cdot \boldsymbol{s}\, ds$, $\int_{l_i} (\mathbf{airy}\, q)\, \boldsymbol{n} \cdot \boldsymbol{n} s\, ds$ and $\int_{l_i} (\mathbf{airy}\, q)\, \boldsymbol{n} \cdot \boldsymbol{s} s\, ds$ vanish. Consequently,

$$\int_{l_i} (\mathbf{airy}\, q)\, \boldsymbol{n}\, ds = \mathbf{0}, \qquad \int_{l_i} (\mathbf{airy}\, q)\, \boldsymbol{n} s\, ds = \mathbf{0}.$$

Since $\int_{l_i} \frac{\partial}{\partial \boldsymbol{s}} q\, ds = 0$ and $\int_{l_i} \frac{\partial}{\partial \boldsymbol{n}} q\, ds = 0$ implies $\int_{l_i} \frac{\partial}{\partial x} q\, ds = \int_{l_i} \frac{\partial}{\partial y} q\, ds = 0$, so by Green's formula,

$$\int_T \mathbf{airy}\, q\, d\boldsymbol{x} = \mathbf{0}.$$

We have shown that $\boldsymbol{\tau}$ and $\mathbf{airy}\, q$ are identical on all the degrees of freedom. Therefore $\boldsymbol{\tau} = \mathbf{airy}\, q$ and consequently $\mathbf{div}\, \boldsymbol{\tau} = \mathbf{0}$. We will use these results to prove (9) and (10).

By lemma 2.2, in order to prove (9), it is sufficient to prove the commutativity property. By the definition of $\mathcal{I}_0$ and $\mathbf{I}_0$, for all $q \in \mathbf{Q}_0$ and $\boldsymbol{\tau} = \mathbf{airy}\, q$ we have $\tilde{\boldsymbol{\tau}}(v)|_T = \mathbf{airy}\, \tilde{q}(v)|_T$ at each vertex v of each $T \in \mathcal{T}_h$, where $\tilde{\boldsymbol{\tau}}$ and $\tilde{q}$ were defined by (7). We can conclude that $\tilde{\boldsymbol{\tau}} = \mathbf{airy}\, \tilde{q}$, which implies $\mathbf{airy}\, \mathcal{I}_0 = \mathbf{I}_0 \mathbf{airy}$. For any $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_0$ we have $\mathbf{div}\, \tilde{\boldsymbol{\tau}} = \mathbf{0}$, which implies $\mathbf{div}\, \mathbf{I}_0 \boldsymbol{\tau} = \mathbf{div}\, \boldsymbol{\tau}$. That completes the proof for (9).

By the definition of $\mathbf{Q}_i$ and $\boldsymbol{\Sigma}_i$ for $i = 1, \cdots, k$, we can see that for each $q \in \mathbf{Q}_i$, $\mathbf{airy}\, q$ vanishes on the vertices of $\mathcal{T}_h$ on $\partial\Omega_i$ and for each $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_i$, $\boldsymbol{\tau}$ vanishes on the vertices of $\mathcal{T}_h$ on $\partial\Omega_i$. Hence by lemma 2.2 and the previous analysis, (10) is clear. □

By the commutative diagram (9), we immediately have the following lemma.

Lemma 3.3. *For any $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_0$, there exists a positive constant ω independent of h and H such that*

$$\boldsymbol{\Lambda}(\mathbf{I}_0 \boldsymbol{\tau}, \mathbf{I}_0 \boldsymbol{\tau}) \le \omega \boldsymbol{\Lambda}(\boldsymbol{\tau}, \boldsymbol{\tau}). \tag{11}$$

Proof. Since

$$\mathbf{\Lambda}(\mathbf{I}_0\boldsymbol{\tau},\mathbf{I}_0\boldsymbol{\tau}) = \|\mathbf{I}_0\boldsymbol{\tau}\|^2 + \|\mathbf{div}\,\mathbf{I}_0\boldsymbol{\tau}\|^2 = \|\mathbf{I}_0\boldsymbol{\tau}\|^2 + \|\mathbf{div}\,\boldsymbol{\tau}\|^2,$$

we only need to show that $\|\mathbf{I}_0\boldsymbol{\tau}\|^2 \le \omega\|\boldsymbol{\tau}\|^2$. This follows from a standard scaling argument, the definition of $\mathbf{I}_0$ and the quasi-uniformity of the mesh. □

Now we can define our preconditioner. Let $\mathbf{I}$ be the identity operator. For $i = 1, \cdots, k$, let $\mathbf{I}_i$ denote the natural imbedding of $\mathbf{\Sigma}_i$ into $\mathbf{\Sigma}$. Define $\mathbf{P}_i : \mathbf{\Sigma} \to \mathbf{\Sigma}_i$ as the $\boldsymbol{H}(\mathbf{div}\,,\Omega)$ adjoint of $\mathbf{I}_i$ and define operators $\mathbf{T}_i = \mathbf{I}_i\mathbf{P}_i$, for $i = 0, \cdots, k$. We also define the bilinear form $\mathbf{\Lambda}_i$ on $\mathbf{\Sigma}_i \times \mathbf{\Sigma}_i$ for each i by

$$\mathbf{\Lambda}_i(\boldsymbol{\sigma},\boldsymbol{\tau}) = \mathbf{\Lambda}(\boldsymbol{\sigma},\boldsymbol{\tau}), \quad \text{for all } \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathbf{\Sigma}_i.$$

The additive and multiplicative Schwarz preconditioners (denoted by $\mathbf{B}_a$ and $\mathbf{B}_m$ respectively) are defined by:

$$\begin{aligned}
\mathbf{B}_a\mathbf{\Lambda} &= \sum_{i=0}^{k} \mathbf{T}_i;\\
\mathbf{B}_m\mathbf{\Lambda} &= \mathbf{I} - (\mathbf{I}-\mathbf{T}_k)(\mathbf{I}-\mathbf{T}_{k-1})\cdots(\mathbf{I}-\mathbf{T}_0)^2\cdots(\mathbf{I}-\mathbf{T}_{k-1})(\mathbf{I}-\mathbf{T}_k)\\
&= \mathbf{I} - \mathbf{E}^*\mathbf{E}.
\end{aligned}$$

Note that the computation of the action of $\mathbf{B}_a$ or $\mathbf{B}_m$ on a function $F \in \mathbf{\Sigma}^*$ involves the solution of subspace problems and the application of the interpolation operator $\mathbf{I}_i$ and its $\boldsymbol{L}^2$-adjoint.

The proof of the following result is standard: [13, 24]

Theorem 3.1. *Assume that (11) holds (with $\omega \in (0,2)$ in the multiplicative case) and that:*

- **(A)** *For any $\boldsymbol{\sigma} \in \mathbf{\Sigma}$ there exists a decomposition $\boldsymbol{\sigma} = \sum_{i=0}^{k} I_i\boldsymbol{\sigma}_i$ and a constant C_A such that $\sum_{i=0}^{k} \mathbf{\Lambda}_i(\boldsymbol{\sigma}_i,\boldsymbol{\sigma}_i) \le C_A\mathbf{\Lambda}(\boldsymbol{\sigma},\boldsymbol{\sigma})$.*

Then, we have

$$\begin{aligned}
\frac{1}{C_A}\mathbf{\Lambda}(\boldsymbol{\sigma},\boldsymbol{\sigma}) &\lesssim \mathbf{\Lambda}(\mathbf{B}_a\mathbf{\Lambda}\boldsymbol{\sigma},\boldsymbol{\sigma}) \lesssim N_c\omega\mathbf{\Lambda}(\boldsymbol{\sigma},\boldsymbol{\sigma}),\\
\frac{2-\omega}{C_A\omega^2N_c^2}\mathbf{\Lambda}(\boldsymbol{\sigma},\boldsymbol{\sigma}) &\lesssim \mathbf{\Lambda}(\mathbf{B}_m\mathbf{\Lambda}\boldsymbol{\sigma},\boldsymbol{\sigma}) \lesssim \mathbf{\Lambda}(\boldsymbol{\sigma},\boldsymbol{\sigma}),
\end{aligned} \tag{12}$$

in which $\lesssim$ means "less than or equal to" up to a trivial constant.

Remark 3.1. *Theorem 3.1 indicates that the condition numbers of $\mathbf{B}_a\mathbf{\Lambda}$ and $\mathbf{B}_m\mathbf{\Lambda}$ are bounded above by constants depending only on C_A, ω and N_c. Hence if we can prove assumption (A) with C_A independent of h and k, then the condition numbers of $\mathbf{B}_a\mathbf{\Lambda}$ and $\mathbf{B}_m\mathbf{\Lambda}$ are also bounded by constants independent of h and k.*

4 Proof of assumption (A)

In this section we prove the assumption (A). The main idea of the proof is very similar to that used in the analysis given in [18]. It is based on the exact sequence (3) which divides $\mathbf{\Sigma}$ into two parts, one of which is divergence free. The decomposition in assumption (A) will be constructed seperately on the two different parts of $\mathbf{\Sigma}$.

First we introduce some operators. Denote $\mathbf{P}_{\boldsymbol{V}_0}$ to be the $\boldsymbol{L}^2$ orthogonal projection from $\boldsymbol{V}$ onto $\boldsymbol{V}_0$. Clearly,

$$\|\mathbf{P}_{\boldsymbol{V}_0}\boldsymbol{v}\| \leq \|\boldsymbol{v}\|, \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{V}. \tag{13}$$

Let $\Pi_{\mathbf{Q}}$ denote the natural interpolation operator onto $\mathbf{Q}$ associated with the degrees of freedom.Denote $C^1(\Omega)$ to be the space of continuous functions with continuous first derivatives. It is not hard to see that $\Pi_{\mathbf{Q}}q$ is well defined as long as $q \in C^1(\Omega)$, q has continuous second derivatives on each node of the fine mesh, $q|_{\partial\Omega} = 0$ and $\nabla q|_{\partial\Omega} = \mathbf{0}$.

We construct a partition of unity $\{\theta_i\}_{i=1}^k$ using the Argyris finite elements on the mesh $\mathcal{T}_h$ (without any boundary conditions). Specifically, we start with a smooth partition of unity, $\{\tilde{\theta}_i\}_{i=1}^k$ satisfying

$$(1)\ supp(\tilde{\theta}_i) \subset \overline{\Omega_i}\,; \qquad (2)\ |\tilde{\theta}_i|_{j,\infty} \leq C\delta^{-j}, \quad j = 0, 1, 2,$$

where $|\cdot|_{j,\infty}$ denotes the W^j_∞ seminorm. We then define θ_i to be the Argyris interpolant of $\tilde{\theta}_i$. It easily follows that $\{\theta_i\}_{i=1}^k$ is a partition of unity satisfying

$$\begin{aligned} &(1)\ \theta_i|_T \in P_5(T) \text{ for any } T \in \mathcal{T}_h; \qquad (2)\ supp(\theta_i) \subset \overline{\Omega_i}; \\ &(3)\ |\theta_i|_{j,\infty} \leq C\delta^{-j}, \quad j = 0, 1, 2. \end{aligned}$$

Clearly we have

$$\begin{aligned} \theta_i|_{\partial\Omega_i\backslash\partial\Omega} = 0, \quad & \nabla\theta_i|_{\partial\Omega_i\backslash\partial\Omega} = \mathbf{0}, \\ & \mathbf{airy}\,\theta_i(v) = \mathbf{0}, \qquad \text{for all } v \in \Psi(\mathcal{T}_h) \cap (\partial\Omega_i\backslash\partial\Omega). \end{aligned}$$

Hence for any $q \in \mathbf{Q}$, we have $\Pi_{\mathbf{Q}}(\theta_i q) \in \mathbf{Q}_i$. Furthermore, by the approximation property of the Argyris element (Theorem 6.1.1 in [15]) and the inverse inequality,

$$|\theta_i q - \Pi_{\mathbf{Q}}(\theta_i q)|^2_{2,\Omega} \leq c \sum_{T\in\mathcal{T}_h} (h^4|\theta_i q|_{6,T})^2 \leq c|\theta_i q|^2_{2,\Omega} \quad \text{for all } q \in \mathbf{Q}.$$

Note that we can apply the inverse inequality here since $\theta_i q|_T$ is a polynomial of degree less than or equal to 10. Therefore we have

$$|\Pi_{\mathbf{Q}}(\theta_i q)|_{2,\Omega} \leq c|\theta_i q|_{2,\Omega}, \qquad \text{for all } q \in \mathbf{Q}. \tag{14}$$

We also need an interpolation operator $\mathcal{P}_{\mathbf{Q}_0} : H^2_0(\Omega) \to \mathbf{Q}_0$ such that

$$|(\mathbf{I} - \mathcal{P}_{\mathbf{Q}_0})q|_{i,\Omega} \leq cH^{2-i}|q|_{2,\Omega}, \qquad \text{for all } q \in H^2_0(\Omega),\ i = 0, 1, 2. \tag{15}$$

We will prove in Appendix B that such an operator exists.

Finally, we can prove the key result of this paper:

Theorem 4.1. *Under the settings of the subdomains and the meshes defined above, assumption (A) holds with*

$$C_A = c(\frac{H^4}{\delta^4} + \frac{H^2}{\delta^2} + 1),$$

where c depends only on ω and N_c.

Proof. For $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$, define $\boldsymbol{\sigma}_0^g \in \boldsymbol{\Sigma}_0$ and a $\boldsymbol{u}_0 \in \boldsymbol{V}_0$ such that

$$\begin{cases} (\boldsymbol{\sigma}_0^g, \boldsymbol{\tau}) + (\mathbf{div}\, \boldsymbol{\tau}, \boldsymbol{u}_0) = 0, & \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_0, \\ (\mathbf{div}\, \boldsymbol{\sigma}_0^g, \boldsymbol{v}) = (\mathbf{P}_{\boldsymbol{V}_0} \mathbf{div}\, \boldsymbol{\sigma}, \boldsymbol{v}), & \forall \boldsymbol{v} \in \boldsymbol{V}_0. \end{cases}$$

For $i = 1, \cdots, k$ define $\boldsymbol{\sigma}_i^g \in \boldsymbol{\Sigma}(\mathcal{T}_h, \tilde{\Omega}_i)$ and a $\boldsymbol{u}_i \in \boldsymbol{V}(\mathcal{T}_h, \tilde{\Omega}_i)$ such that

$$\begin{cases} (\boldsymbol{\sigma}_i^g, \boldsymbol{\tau})_{\tilde{\Omega}_i} + (\mathbf{div}\, \boldsymbol{\tau}, \boldsymbol{u}_i)_{\tilde{\Omega}_i} = 0, & \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}(\mathcal{T}_h, \tilde{\Omega}_i), \\ (\mathbf{div}\, \boldsymbol{\sigma}_i^g, \boldsymbol{v})_{\tilde{\Omega}_i} = (\mathbf{div}\, \boldsymbol{\sigma} - \mathbf{P}_{\boldsymbol{V}_0} \mathbf{div}\, \boldsymbol{\sigma}, \boldsymbol{v})_{\tilde{\Omega}_i}, & \forall \boldsymbol{v} \in \boldsymbol{V}(\mathcal{T}_h, \tilde{\Omega}_i). \end{cases}$$

We need to show the above definitions are proper, i.e. the compatibility conditions are satisfied. Since $RM \subset \boldsymbol{V}_0 \subset \boldsymbol{V}$ and $\mathbf{div}\, \boldsymbol{\sigma} \in RM^{\perp \boldsymbol{V}}$, so clearly

$$(\mathbf{P}_{\boldsymbol{V}_0} \mathbf{div}\, \boldsymbol{\sigma}, \boldsymbol{v}) = (\mathbf{div}\, \boldsymbol{\sigma}, \boldsymbol{v}) = 0, \qquad \text{for all } \boldsymbol{v} \in RM.$$

Thus $\boldsymbol{\sigma}_0^g$ is well defined. Since the boundary of $\tilde{\Omega}_i$ aligns with the coarse mesh, it's obvious that

$$\int_{\tilde{\Omega}_i} (\mathbf{div}\, \boldsymbol{\sigma} - \mathbf{P}_{\boldsymbol{V}_0} \mathbf{div}\, \boldsymbol{\sigma}) \cdot \boldsymbol{v}\, d\boldsymbol{x} = 0 \qquad \text{for all } \boldsymbol{v} \in RM.$$

Therefore $\boldsymbol{\sigma}_i^g$ is also well defined for $i = 1, \cdots, k$.

The moments of degree 0 and 1 of the normal components of $\boldsymbol{\sigma}_i$ on each edge of the fine mesh on $\partial\tilde{\Omega}_i$ are zero. By the proof of lemma 3.2, we can extend $\boldsymbol{\sigma}_i^g$ to Ω_i by a divergence-free function in $\Omega_i \backslash \tilde{\Omega}_i$ which has nonzero degrees of freedom only on the nodes on $\partial\tilde{\Omega}_i$. The resulting function can be extended by zero outside of Ω_i and yields a function (still denoted by $\boldsymbol{\sigma}_i^g$) in $\boldsymbol{\Sigma}_i$. By construction, $\mathbf{div}\, \boldsymbol{\sigma}_i^g = \mathbf{0}$ in $\Omega \backslash \tilde{\Omega}_i$. Since the mesh is quasi-uniform, there exists a constant c independent of h such that for $i = 1, \cdots, k$,

$$\|\boldsymbol{\sigma}_i^g\|_{\boldsymbol{H}(\mathbf{div}, \Omega)} \leq c \|\boldsymbol{\sigma}_i^g\|_{\boldsymbol{H}(\mathbf{div}, \tilde{\Omega}_i)}.$$

By the above inequality and lemma 3.1,

$$\begin{aligned} \sum_{i=0}^{k} \boldsymbol{\Lambda}(\boldsymbol{\sigma}_i^g, \boldsymbol{\sigma}_i^g) &\leq c(\|\boldsymbol{\sigma}_0^g\|^2_{\boldsymbol{H}(\mathbf{div}, \Omega)} + \sum_{i=1}^{k} \|\boldsymbol{\sigma}_i^g\|^2_{\boldsymbol{H}(\mathbf{div}, \tilde{\Omega}_i)}) \\ &\leq c(\|\mathbf{P}_{\boldsymbol{V}_0} \mathbf{div}\, \boldsymbol{\sigma}\|^2 + \sum_{i=1}^{k} \|\mathbf{div}\, \boldsymbol{\sigma} - \mathbf{P}_{\boldsymbol{V}_0} \mathbf{div}\, \boldsymbol{\sigma}\|^2_{\boldsymbol{L}^2(\tilde{\Omega}_i)}) \\ &\leq c\|\mathbf{div}\, \boldsymbol{\sigma}\|^2. \end{aligned}$$

Next consider $\boldsymbol{\sigma}^a = \boldsymbol{\sigma} - \mathbf{I}_0\boldsymbol{\sigma}_0^g - \sum_{i=1}^k \boldsymbol{\sigma}_i^g$. Simple calculation shows that $\mathbf{div}\,\boldsymbol{\sigma}^a = \mathbf{0}$. By the finite overlapping assumption and lemma 3.3, we know that $\boldsymbol{\Lambda}(\boldsymbol{\sigma}^a, \boldsymbol{\sigma}^a) \leq c\boldsymbol{\Lambda}(\boldsymbol{\sigma}, \boldsymbol{\sigma})$ where c only depends on N_c and ω. Set

$$\begin{aligned}
\boldsymbol{\sigma}_0^a &= \mathbf{airy}\,\mathcal{P}_{\mathbf{Q}_0}\mathbf{airy}^{-1}\boldsymbol{\sigma}^a, \\
\boldsymbol{\sigma}_i^a &= \mathbf{airy}\,\Pi_{\mathbf{Q}}(\theta_i\mathbf{airy}^{-1}(\boldsymbol{\sigma}^a - \mathbf{I}_0\boldsymbol{\sigma}_0^a)), \qquad \text{for } i = 1, \cdots, k.
\end{aligned}$$

The above definitions are proper since $\mathbf{div}\,\boldsymbol{\sigma}^a = \mathbf{0}$ and $\mathbf{div}\,(\boldsymbol{\sigma}^a - \mathbf{I}_0\boldsymbol{\sigma}_0^a) = \mathbf{0}$. Clearly $\boldsymbol{\sigma}^a = \sum_{i=0}^k \mathbf{I}_i\boldsymbol{\sigma}_i^a$ while $\boldsymbol{\sigma}_i^a \in \boldsymbol{\Sigma}_i$ and $\mathbf{div}\,\boldsymbol{\sigma}_i^a = \mathbf{0}$ for $i = 0, \cdots, k$. By inequality (15),

$$\boldsymbol{\Lambda}(\boldsymbol{\sigma}_0^a, \boldsymbol{\sigma}_0^a) = \|\mathbf{airy}\,\mathcal{P}_{\mathbf{Q}_0}\mathbf{airy}^{-1}\boldsymbol{\sigma}^a\|^2 \leq c\|\boldsymbol{\sigma}^a\|^2 = c\boldsymbol{\Lambda}(\boldsymbol{\sigma}^a, \boldsymbol{\sigma}^a).$$

Let $\hat{q} = \mathbf{airy}^{-1}(\boldsymbol{\sigma}^a - \mathbf{I}_0\boldsymbol{\sigma}_0^a)$ and $q = \mathbf{airy}^{-1}\boldsymbol{\sigma}^a$. Then

$$\hat{q} = \mathbf{airy}^{-1}(\boldsymbol{\sigma}^a - \mathbf{I}_0\mathbf{airy}\,\mathcal{P}_{\mathbf{Q}_0}q) = (\mathbf{I} - \mathcal{P}_{\mathbf{Q}_0})q + (\mathbf{I} - \mathcal{I}_0)\mathcal{P}_{\mathbf{Q}_0}q.$$

By inequality (14), the assumptions on θ_i, inequality (8) and inequality (15),

$$\begin{aligned}
\sum_{i=1}^k \boldsymbol{\Lambda}(\boldsymbol{\sigma}_i^a, \boldsymbol{\sigma}_i^a) = \sum_{i=1}^k \|\mathbf{airy}\,\Pi_{\mathbf{Q}}(\theta_i\hat{q})\|^2 &\leq c\sum_{i=1}^k |\theta_i\hat{q}|_{2,\Omega_i}^2 \\
&\leq c\sum_{i=1}^k (\delta^{-4}|\hat{q}|_{0,\Omega_i}^2 + \delta^{-2}|\hat{q}|_{1,\Omega_i}^2 + |\hat{q}|_{2,\Omega_i}^2) \\
&\leq cN_c(\frac{H^4}{\delta^4} + \frac{H^2}{\delta^2} + 1)|q|_{2,\Omega}^2 \\
&\leq cN_c(\frac{H^4}{\delta^4} + \frac{H^2}{\delta^2} + 1)\|\boldsymbol{\sigma}^a\|^2.
\end{aligned}$$

Therefore we can conclude that $\sum_{i=0}^k \boldsymbol{\Lambda}(\boldsymbol{\sigma}_i^a, \boldsymbol{\sigma}_i^a) \leq c(\frac{H^4}{\delta^4} + \frac{H^2}{\delta^2} + 1)\boldsymbol{\Lambda}(\boldsymbol{\sigma}, \boldsymbol{\sigma})$, where c depends on ω and N_c.

Finally, define $\boldsymbol{\sigma}_i = \boldsymbol{\sigma}_i^g + \boldsymbol{\sigma}_i^a$ for $i = 0, \cdots, k$. Clearly $\boldsymbol{\sigma} = \sum_{i=0}^k \mathbf{I}_i\boldsymbol{\sigma}_i$ while $\boldsymbol{\sigma}_i \in \boldsymbol{\Sigma}_i$ and

$$\sum_{i=0}^k \boldsymbol{\Lambda}(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_i) \leq 2(\sum_{i=0}^k \boldsymbol{\Lambda}(\boldsymbol{\sigma}_i^g, \boldsymbol{\sigma}_i^g) + \sum_{i=0}^k \boldsymbol{\Lambda}(\boldsymbol{\sigma}_i^a, \boldsymbol{\sigma}_i^a)) \leq c(\frac{H^4}{\delta^4} + \frac{H^2}{\delta^2} + 1)\boldsymbol{\Lambda}(\boldsymbol{\sigma}, \boldsymbol{\sigma}),$$

where c depends only on ω and N_c. This completes the proof of lemma 4.1. □

Remark 4.1. *We have shown in the above theorem that C_A is of order $O(\frac{H^4}{\delta^4} + \frac{H^2}{\delta^2} + 1)$. Recall that for the classical second order elliptic problem, similar result has been proved with C_A of order $O(\frac{H^2}{\delta^2} + 1)$. In our proof the divergence free part is mapped to the fourth order Argyris finite element space, which brings $\frac{H^4}{\delta^4}$ to the result. It is not clear whether a sharper estimate can be proved for our problem.*

5 Numerical Results

Let Ω be the unit square $(0,1)\times(0,1)$. We solve both the problem (5) and the problem (2) by the preconditioned MINRES method. The overlapping Schwarz preconditioners $\mathbf{B}_a$ and $\mathbf{B}_m$ are used for problem (5) and the preconditioner $\mathcal{S}$ is used for problem (2), as described in Section 3. In the implementation of the MINRES solver, we use preconditioned Lanczos procedure to generate a tridiagonal matrix whose eigenvalues are good approximations of the eigenvalues of the original matrix. Therefore we can derive an estimate of the condition number of the original matrix. Note that for symmetric positive matrix we define the condition number as the ratio between the maximum eigenvalue and the minimum eigenvalue, while for symmetric indefinite matrix whose eigenvalues lie in $[a,b]\cup[c,d]$ where $a<b<0<c<d$, we define its condition number as $\frac{ad}{bc}$. For problem (2) the linear system is indeed singular and its kernel is RM, but we can avoid this kernel in the computation as long as the body force $\boldsymbol{g}$ satisfies the compatibility condition and the initial guess in the iterative method is perpendicular to RM.

To get the most accurate condition number estimates from the Lanczos procedure, we need to choose the test problem carefully. Indeed we experimented over several different test problems and finally chose the one which gave the largest condition number estimates. For problem (5) we set the exact solution to be $\boldsymbol{\sigma}=\begin{pmatrix} x(1-x) & 0 \\ 0 & y(1-y)\end{pmatrix}$. For problem (2), we set $\mu=0.5$, $\lambda=1$ and the body force $\boldsymbol{g}=\begin{pmatrix}1-3x^2\\ 2y-1\end{pmatrix}$, which satisfies the compatibility condition.

Table 1: Condition number estimates, $H/\delta=4$, $k=4$.

h	Problem (5)			Problem (2)		
	No Prec.	Additive	Multiplicative	No Prec.	Additive	Multiplicative
1/8	3.4e+5	5.12	1.06	1.1e+5	12.61	2.09
1/16	1.4e+6	5.01	1.06	1.8e+5	12.36	2.13
1/32	5.5e+6	4.96	1.06	6.3e+5	11.34	2.15

Table 2: Condition number estimates for problem (5), $H/\delta=2$.

h	Additive			Multiplicative		
	$k=4$	$k=8$	$k=16$	$k=4$	$k=8$	$k=16$
1/8	4.86	6.07	6.04	1.02	1.02	1.02
1/16	4.88	5.98	6.02	1.02	1.02	1.02

In Table 1, we report the condition numbers of the unpreconditioned systems and the preconditioned systems for both the problem (5) and the problem (2). The

Table 3: Condition number estimates for problem (5), $k=4$, $H=1/2$

h	δ	Additive	Multiplicative
1/4	1/4	4.85	1.01
1/8	1/8	5.12	1.06
1/16	1/16	8.35	1.52
1/32	1/32	18.63	2.75

coarse mesh and four overlapping subdomains are fixed. The results are uniform with respect to h.

For problem (5), we also computed the condition numbers for various values of k and h. The results are given in Table 2 and they are uniform with respect to both k and h.

Finally, in Table 3, we give a set of condition numbers for various δ. Note that larger values of δ yields better preconditioners.

A Stability of the finite element spaces

For simplicity, we extend our notation $\|\cdot\|_{s,\Omega}$ of the Sobolev norm and $|\cdot|_{s,\Omega}$ of the Sobolev semi-norm to the vector case $(H^s(\Omega))^2$ and the symmetric tensor case $(H^s(\Omega))^3$. The weak solution for system (1) exists and is unique in $\boldsymbol{H}_0(\mathbf{div}\,,\Omega)\times RM^{\perp_{\boldsymbol{L}^2(\Omega)}}$. For simplicity, let $\boldsymbol{\Sigma}=\boldsymbol{\Sigma}(\mathcal{T},\Omega)$ and $\boldsymbol{V}=\boldsymbol{V}(\mathcal{T},\Omega)$. Since $\mathbf{div}\,\boldsymbol{\Sigma}\subset\boldsymbol{V}$, clearly

$$\|\boldsymbol{\sigma}\|_{\boldsymbol{H}(\mathbf{div}\,,\Omega)}=\|\boldsymbol{\sigma}\|_{0,\Omega}\le c(\mathbf{A}\boldsymbol{\sigma},\boldsymbol{\sigma})^{1/2},\qquad \text{for all } \boldsymbol{\sigma}\in\boldsymbol{V}^{\perp}.$$

We only need the discrete inf-sup condition to show the stability of the finite element spaces. Indeed, we have

Lemma A.1. *There exists a constant c independent of the mesh size such that*

$$\sup_{\boldsymbol{\tau}\in\boldsymbol{\Sigma}}\frac{(\mathbf{div}\,\boldsymbol{\tau},\boldsymbol{v})}{\|\boldsymbol{\tau}\|_{\boldsymbol{H}(\mathbf{div}\,,\Omega)}}\ge c\|\boldsymbol{v}\|_{0,\Omega},\quad \forall\boldsymbol{v}\in RM^{\perp_{\boldsymbol{V}}}.$$

Proof. The results of Grisvard [19, 20] imply that when Ω is a convex polygon, the solution $(\boldsymbol{\sigma},\boldsymbol{u})$ of system (1) with $\boldsymbol{g}\in RM^{\perp_{\boldsymbol{L}^2(\Omega)}}$ has the regularity $\boldsymbol{u}\in(H^2(\Omega))^2$, $\boldsymbol{\sigma}\in\boldsymbol{H}_0(\mathbf{div}\,,\Omega)\cap(H^1(\Omega))^3$ and $\|\boldsymbol{\sigma}\|_{1,\Omega}\le c\|\boldsymbol{g}\|_{0,\Omega}$. Taking $\boldsymbol{g}=\boldsymbol{v}$ gives $\|\boldsymbol{v}\|_{0,\Omega}\le c\frac{(\mathbf{div}\,\boldsymbol{\sigma},\boldsymbol{v})}{\|\boldsymbol{\sigma}\|_{1,\Omega}}$. Thus, it suffices to construct an interpolation operator $\Pi_h:\boldsymbol{H}_0(\mathbf{div}\,,\Omega)\cap(H^1(\Omega))^3\to\boldsymbol{\Sigma}$ bounded in $\mathcal{L}((H^1(\Omega))^3,\boldsymbol{H}(\mathbf{div}\,,\Omega))$ such that $\mathbf{div}\,\Pi_h=\mathbf{P}_{\boldsymbol{V}}\mathbf{div}\,$, where $\mathbf{P}_{\boldsymbol{V}}:\boldsymbol{L}^2(\Omega)\to\boldsymbol{V}$ is the L^2 orthogonal projection.

In [6], such an interpolation was defined for the pure displacement case. We only need to do a slight modification to make it work for the pure traction problem. Let h be the characteristic mesh size. Let R_h be the interpolation operator from $L^2(\Omega)$ onto the space of C^0-quadratics with respect to the mesh $\mathcal{T}_h$ as defined by Scott and Zhang [23]. R_h preserves the homogeneous boundary condition on $H^1(\Omega)$ and $R_hp=p$ for all C^0-quadratic p defined on mesh $\mathcal{T}_h$. Define $\mathbf{R}_h$ mapping $(H^1(\Omega))^3$ to the space of symmetric tensors of C^0-quadratics with respect to the mesh $\mathcal{T}_h$ by

- for any corner x of the polygon Ω, $\mathbf{R}_h(\boldsymbol{\tau})(x) = \mathbf{0}$;
- for all the other degrees of freedom x, $\mathbf{R}_h(\boldsymbol{\tau})(x) = \begin{pmatrix} (R_h\tau_{11})(x) & (R_h\tau_{12})(x) \\ (R_h\tau_{21})(x) & (R_h\tau_{22})(x) \end{pmatrix}$.

Consider the triangles in $\mathcal{T}_h$ as closed subsets of Ω which contain their boundary. For a triangle $T \in \mathcal{T}_h$, define $S_T = \bigcup\{T_i | T_i \cap T \neq \emptyset, \ \ T_i \in \mathcal{T}_h\}$. Following the proof of [23], we will show that

1. for any $\boldsymbol{\tau} \in \boldsymbol{H}_0(\mathbf{div}\,, \Omega) \cap (H^1(\Omega))^3$, $(\mathbf{R}_h\boldsymbol{\tau})\boldsymbol{n}|_{\partial\Omega} = \mathbf{0}$;
2. (stability) for $j = 0, 1$ and $\boldsymbol{\tau} \in (H^1(\Omega))^3$, $\|\mathbf{R}_h\boldsymbol{\tau}\|_{j,T} \leq \sum_{i=0}^{1} h^{i-j}|\boldsymbol{\tau}|_{i,S_T}$;
3. (approximability) for $j = 0, 1$, $1 \leq m \leq 3$ and $\boldsymbol{\tau} \in \boldsymbol{H}_0(\mathbf{div}\,, \Omega) \cap (H^m(\Omega))^3$, $\|\mathbf{R}_h\boldsymbol{\tau} - \boldsymbol{\tau}\|_{j,T} \leq ch^{m-j}|\boldsymbol{\tau}|_{m,S_T}$.

The first result is obvious from the definition. The proof for stability is exactly the same as the proof of theorem 3.1 in [23]. We need to prove the approximability. By the Bramble-Hilbert lemma, there exists a symmetric tensor of quadratic polynomials $\boldsymbol{\rho} = (\rho_{ij})_{1\leq i,j\leq 2} \in (P_2)^3$ such that

$$\|\boldsymbol{\tau} - \boldsymbol{\rho}\|_{j,S_T} \leq Ch^{m-j}|\boldsymbol{\tau}|_{m,S_T}, \qquad 0 \leq j \leq m \leq 3.$$

Hence

$$\begin{aligned} \|\mathbf{R}_h\boldsymbol{\tau} - \boldsymbol{\tau}\|_{j,T} &\leq \|\boldsymbol{\tau} - \boldsymbol{\rho}\|_{j,T} + \|\mathbf{R}_h(\boldsymbol{\tau} - \boldsymbol{\rho})\|_{j,T} + \|\boldsymbol{\rho} - \mathbf{R}_h\boldsymbol{\rho}\|_{j,T} \\ &\leq c\sum_{i=0}^{1} h^{i-j}\|\boldsymbol{\tau} - \boldsymbol{\rho}\|_{i,S_T} + \|\boldsymbol{\rho} - \mathbf{R}_h\boldsymbol{\rho}\|_{j,T} \\ &\leq ch^{m-j}|\boldsymbol{\tau}|_{m,S_T} + \|\boldsymbol{\rho} - \mathbf{R}_h\boldsymbol{\rho}\|_{j,T}. \end{aligned}$$

By the definition of $\mathbf{R}_h$, $\boldsymbol{\rho} - \mathbf{R}_h\boldsymbol{\rho}$ has none-zero nodal values only at the corners of polygon Ω. Denote V_c the set of the corners of polygon Ω. Then $\|\boldsymbol{\rho} - \mathbf{R}_h\boldsymbol{\rho}\|_{j,T} \leq ch^{-j+1}\sum_{v\in V_c\cap T}|\boldsymbol{\rho}(v)|$, where $|\boldsymbol{\rho}(v)|^2 = \sum_{i,j=1}^{2}|\rho_{ij}(v)|^2$. Now we evaluate $|\boldsymbol{\rho}(v)|$ for each $v \in V_c$. It is easy to see that v is the intersection of two edges γ_1, γ_2 of mesh $\mathcal{T}_h$ and $\gamma_1, \gamma_2 \subset \partial\Omega\cap\partial S_T$. Denote $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ the outer normal vectors on γ_1, γ_2 respectively. Then by the boundary condition of $\boldsymbol{\tau}$ and the trace theorem,

$$\begin{aligned} h|\boldsymbol{\rho}(v)|^2 &\leq c\sum_{i=1}^{2}\|\boldsymbol{\rho}\boldsymbol{n}_i\|_{0,\gamma_i}^2 = c\sum_{i=1}^{2}\|(\boldsymbol{\tau} - \boldsymbol{\rho})\boldsymbol{n}_i\|_{0,\gamma_i}^2 \\ &\leq ch(h^{-2}\|(\boldsymbol{\tau} - \boldsymbol{\rho})\|_{0,S_T}^2 + |(\boldsymbol{\tau} - \boldsymbol{\rho})|_{1,S_T}^2) \\ &\leq ch^{2m-1}|\boldsymbol{\tau}|_{m,S_T}^2. \end{aligned}$$

Hence $|\boldsymbol{\rho}(v)| \leq ch^{m-1}|\boldsymbol{\tau}|_{m,S_T}$ and consequently $\|\boldsymbol{\rho} - \mathbf{R}_h\boldsymbol{\rho}\|_{j,T} \leq ch^{m-j}|\boldsymbol{\tau}|_{m,S_T}$. That completes the proof of approximability for $\mathbf{R}_h$ because of the limited overlap property of $\{S_T\}$.

Define $\Pi_h = \Pi_h^0(\mathbf{I} - \mathbf{R}_h) + \mathbf{R}_h$, where Π_h^0 is defined exactly the same as is defined in [6]. Π_h clearly preserves the boundary condition of $\boldsymbol{H}_0(\mathbf{div}\,,\Omega)$ and as shown in [6], we have $\mathbf{div}\,\Pi_h\boldsymbol{\tau} = \mathbf{P}_{\boldsymbol{V}}\mathbf{div}\,\boldsymbol{\tau}$ and $\|\Pi_h^0\boldsymbol{\tau}\|_{0,\Omega} \le c(\|\boldsymbol{\tau}\|_{0,\Omega} + h\|\boldsymbol{\tau}\|_{1,\Omega})$ for all $\boldsymbol{\tau} \in \boldsymbol{H}(\mathbf{div}\,,\Omega) \cap (H^1(\Omega))^3$. Finally, by the properties of Π_h^0 and $\mathbf{R}_h$, it is easy to see that

$$\|\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau}\|_{0,\Omega} \le ch^m\|\boldsymbol{\tau}\|_{m,\Omega}, \quad \text{for } \boldsymbol{\tau} \in \boldsymbol{H}_0(\mathbf{div}\,,\Omega) \cap (H^m(\Omega))^3,\ 1 \le m \le 3.$$

Consequently for $\boldsymbol{\tau} \in \boldsymbol{H}_0(\mathbf{div}\,,\Omega) \cap (H^1(\Omega))^3$,

$$\begin{aligned}\|\Pi_h\boldsymbol{\tau}\|^2_{\boldsymbol{H}(\mathbf{div}\,,\Omega)} &= \|\Pi_h\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}\,\Pi_h\boldsymbol{\tau}\|^2_{0,\Omega} \\ &= \|\Pi_h\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{P}_{\boldsymbol{V}}\mathbf{div}\,\boldsymbol{\tau}\|^2_{0,\Omega} \\ &\le c\|\boldsymbol{\tau}\|^2_{1,\Omega}.\end{aligned}$$

Therefore Π_h is bounded in $\mathcal{L}((H^1(\Omega))^3, \boldsymbol{H}(\mathbf{div}\,,\Omega))$ by a constant independent of the mesh size. □

B Construction of the operator $\mathcal{P}_{\mathbf{Q}_0}$

Here we construct an operator $\mathcal{P}_{\mathbf{Q}_0} : H_0^2(\Omega) \to \mathbf{Q}_0$ such that inequality (15) holds. Let $H_0^1(\Omega) = \{w \in H^1(\Omega) \text{ such that } w|_{\partial\Omega} = 0\}$. Note that for $q \in H_0^2(\Omega)$, we have $\nabla q \in (H_0^1(\Omega))^2$. By Scott and Zhang [23], there is a Clement type operator Π from $H^1(\Omega)$ to its continuous piecewise linear subspace based on the mesh $\mathcal{T}_H$ preserving the homogeneous boundary condition. Let $T \in \mathcal{T}_H$ and $S_T = \bigcup\{T_i | T_i \cap T \neq \emptyset,\ \ T_i \in \mathcal{T}_H\}$. Let v_i, $i = 1, 2, 3$ be the three vertices of T and l_i be the edge of T which is opposite to the vertex v_i. From the proof of Theorem 3.1 in [23], there exists a constant c independent of H and T such that

$$\sum_{i=1}^{3} |\Pi w(v_i)| \le c \sum_{m=0}^{1} H^{m-1}|w|_{m,S_T}, \qquad \text{for all } w \in H^1(\Omega). \tag{16}$$

Define the operator $\mathcal{P}_{\mathbf{Q}_0}$ as

for each vertex v_i in $\mathcal{T}_H$:

$$(\mathcal{P}_{\mathbf{Q}_0} q)(v_i) = \Pi q(v_i), \qquad \mathbf{airy}\,(\mathcal{P}_{\mathbf{Q}_0} q)(v_i) = \mathbf{0},$$

$$\frac{\partial}{\partial x}(\mathcal{P}_{\mathbf{Q}_0} q)(v_i) = \Pi(\frac{\partial}{\partial x} q)(v_i), \qquad \frac{\partial}{\partial y}(\mathcal{P}_{\mathbf{Q}_0} q)(v_i) = \Pi(\frac{\partial}{\partial y} q)(v_i);$$

for each edge l_i in $\mathcal{T}_H$:

$$\int_{l_i} \frac{\partial}{\partial \boldsymbol{n}}(\mathcal{P}_{\mathbf{Q}_0} q)\, ds = \int_{l_i} \frac{\partial}{\partial \boldsymbol{n}} q\, ds.$$

Clearly $\mathcal{P}_{\mathbf{Q}_0}$ is well-defined and maps $H_0^2(\Omega)$ to $\mathbf{Q}_0$. We will show that $\mathcal{P}_{\mathbf{Q}_0}$ is stable in the following sense:

$$|\mathcal{P}_{\mathbf{Q}_0} q|_{i,T} \le \sum_{m=0}^{2} H^{m-i}|q|_{m,S_T}, \quad \text{for } q \in H_0^2(\Omega),\ T \in \mathcal{T}_H,\ i = 0, 1, 2. \tag{17}$$

By the inverse inequality, we only need to prove inequality (17) for $i = 0$. Let ϕ_j, $j = 1, \cdots, 21$ be the basis of the Argyris element in T, that is, ϕ_j equals to 1 on the j'th degree of freedom while vanishing on all the other degrees of freedom. The Argyris element is almost affine but not affine, but by using the technique in the proof of Theorem 6.1.1 in [15], we can conclude that there exists a constant c which is independent of H and T such that $\|\phi_j\|_{0,T} \leq cH$ when the j'th degree of freedom is the nodal value at the vertex or the moment on the edge, while $\|\phi_j\|_{0,T} \leq cH^2$ when the j'th degree of freedom is the first derivative at the vertex. For each $q \in H_0^2(\Omega)$, we have $\mathcal{P}_{\mathbf{Q}_0} q|_T = \sum_{j=1}^{21} N_j(q)\phi_j$, where $N_j : H_0^2(\Omega) \to \mathbb{R}$ are defined as

$$N_j(q) = \begin{cases} \Pi q(v_i), & \text{when the j'th dof is the nodal value on } v_i\,; \\ \Pi(\frac{\partial}{\partial x} q)(v_i), \\ \Pi(\frac{\partial}{\partial y} q)(v_i), & \text{when the j'th dof is the first derivative on } v_i\,; \\ 0, & \text{when the j'th dof is the second derivative on } v_i\,; \\ \int_{l_i} \frac{\partial}{\partial \boldsymbol{n}} q\, ds, & \text{when the j'th dof is the moment on } l_i\,. \end{cases}$$

Thus

$$\begin{aligned} \|\mathcal{P}_{\mathbf{Q}_0} q\|_{0,T} \leq \sum_{j=1}^{21} |N_j(q)| \|\phi_j\|_{0,T} \leq c\Big(H \sum_{i=1}^{3} |\Pi q(v_i)| \\ + H^2 \sum_{i=1}^{3} |\Pi(\frac{\partial}{\partial x} q)(v_i)| + H^2 \sum_{i=1}^{3} |\Pi(\frac{\partial}{\partial y} q)(v_i)| + H \sum_{i=1}^{3} |\int_{l_i} \frac{\partial}{\partial \boldsymbol{n}} q\, ds| \Big). \end{aligned}$$

By the inequality (16),

$$\begin{aligned} \sum_{i=1}^{3} |\Pi(\frac{\partial}{\partial x} q)(v_i)| + \sum_{i=1}^{3} |\Pi(\frac{\partial}{\partial y} q)(v_i)| &\leq c \sum_{m=1}^{2} H^{m-2} |q|_{m,S_T}, \\ \sum_{i=1}^{3} |\Pi q(v_i)| &\leq c \sum_{m=0}^{1} H^{m-1} |q|_{m,S_T}. \end{aligned}$$

By the trace theorem, we have

$$\begin{aligned} |\int_{l_i} \frac{\partial}{\partial \boldsymbol{n}} q\, ds|^2 &\leq cH \int_{l_i} |\frac{\partial}{\partial \boldsymbol{n}} q|^2\, ds \\ &\leq cH(H^{-1} \|\nabla q\|_T^2 + H |\nabla q|_{1,T}^2) \\ &\leq C \sum_{m=1}^{2} H^{2m-2} |q|_{m,T}^2. \end{aligned}$$

The stability result (17) follows immediately from combining all the above inequalities.

Finally we prove the inequality (15). Let $q \in H_0^2(\Omega)$. By the Bramble-Hilbert lemma, there exists a linear polynomial p such that

$$\|q - p\|_{i,S_T} \le cH^{2-i}|q|_{2,S_T}, \quad \text{for } i = 0, 1, 2.$$

One important observation is that $\mathcal{P}_{\mathbf{Q}_0} p|_T = p|_T$. By the triangle inequality and inequality (17),

$$\begin{aligned} |q - \mathcal{P}_{\mathbf{Q}_0} q|_{i,T} &\le |q - p|_{i,T} + |\mathcal{P}_{\mathbf{Q}_0}(q - p)|_{i,T} \\ &\le cH^{2-i}|q|_{2,S_T}, \end{aligned}$$

where c is independent of H and T. Thus inequality (15) follows from the limited overlap property of $\{S_T\}$.

References

[1] D. N. Arnold, Mixed finite element methods for ellipitic problems, Comput. Methods Appl. Mech. Engrg. 82 (1990) 281–300.

[2] D. N. Arnold, F. Brezzi, J. J. Douglas, Peers: A new mixed finite element for plane elasticity, Japan J. Appl. Math. 1 (1984) 347–367.

[3] D. N. Arnold, J. Jim Douglas, C. P. Gupta, A family of higher order mixed finite element methods for plane elasticity, Numer. Math. 45 (1984) 1–22.

[4] D. N. Arnold, R. S. Falk, A new mixed formulation for elasticity, Numer. Math. 53 (1988) 13–30.

[5] D. N. Arnold, R. S. Falk, R. Winther, Preconditioning in H(div) and applications, Math. Comp. 66 (219) (1997) 957–984.

[6] D. N. Arnold, R. Winther, Mixed finite element for elasticity, Numer. Math. 92 (3) (2002) 401–419.

[7] J. H. Bramble, R. D. Lazarov, J. E. Pasciak, Least-squares methods for linear elasticity based on a discrete negative norm, Comput. Methods in Appl. Mech. Engrg. 152 (2001) 520–543.

[8] J. H. Bramble, J. E. Pasciak, A preconditioning technique for indefinite systems resulting from mixed approximations of elliptic problems, Math. Comp. 50 (1988) 1–17.

[9] J. H. Bramble, J. E. Pasciak, A domain decomposition technique for Stokes problems, Appl. Numer. Math. 6 (1990) 251–261.

[10] J. H. Bramble, J. E. Pasciak, A. T. Vassilev, Analysis of the inexact Uzawa algorithm for saddle point problems, SIAM J. Numer. Anal. 34 (3) (1997) 1072–1092.

[11] J. H. Bramble, J. E. Pasciak, J. Wang, J. Xu, Convergence estimates for product iterative methods with applications to domain decomposition, Math. Comp. 57 (195) (1991) 1–21.

[12] S. C. Brenner, Two-level additive Schwarz preconditioners for nonconforming finite element methods, Math. Comp. 65 (215) (1996) 897–921.

[13] S. C. Brenner, Two-level additive Schwarz preconditioners for plate elements, Wuhan University Journal of Natural Science 1 (1996) 658–667.

[14] F. Brezzi, M. Fortin, Mixed and hybrid finite element methods, Springer-Verlag, 1991.

[15] P. G. Ciarlet, The finite element method for elliptic problems, North-Holland, 1978.

[16] M. Dryja, O. B. Widlund, Domain decomposition algorithms with small overlap, SIAM J. Sci. Comput. 15 (3) (1994) 604–620.

[17] H. C. Elman, G. H. Golub, Inexact and preconditioned Uzawa Algorithms for Saddle Point Problems, SIAM J. Numer. Anal. 31 (6) (1994) 1645–1661.

[18] R. E. Ewing, J. Wang, Analysis of the Schwarz algorithm for mixed finite element methods, RAIRO Math. Modél. Anal. Numér. 26 (6) (1992) 739–756.

[19] P. Grisvard, Problémes aux limites dans les polygons, Mode d'emploi. Bulletin de la Direction des Etudes et Recherches Series C Mathematiques, Informatique 1 (1986) 21–59.

[20] P. Grisvard, Singularités en elasticité, Arch. Rat. Mech. Anal. 107 (1989) 157–180.

[21] C. Johnson, B. Mercier, Some equilbrium finite element methods for two-dimensional elasticity problems, Numer. Math. 30 (1978) 103–116.

[22] T. Rusten, R. Winther, A preconditioned iterative method for saddlepoint problem, SIAM J. Matrix Anal. Appl. 13 (3) (1992) 887–904.

[23] L. R. Scott, S. Zhang, Finite element interpolation of nonsmooth function satisfying boundary conditions, Math. Comp. 54 (190) (1990) 483–493.

[24] B. F. Smith, P. E. Bjorstad, W. D. Gropp, Domain decomposition, Cambridge, 1996.

[25] J. Xu, Iterative methods by space decomposition and subspace correction, SIAM Rev. 34 (4) (1992) 581–613.