ANALYSIS OF FINITE ELEMENT APPROXIMATION AND ITERATIVE METHODS FOR TIME-DEPENDENT MAXWELL PROBLEMS

A Dissertation

by

JUN ZHAO

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2002

Major Subject: Mathematics

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ABSTRACT

Analysis of Finite Element Approximation and

Iterative Methods for Time-Dependent Maxwell Problems. (August 2002)

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In this dissertation we are concerned with the analysis of the finite element method for the time-dependent Maxwell interface problem when Nedelec and Raviart-Thomas finite elements are employed and preconditioning of the resulting linear system when implicit time schemes are used.

We first investigate the finite element method proposed by Makridakis and Monk in 1995. After studying the regularity of the solution to time dependent Maxwell's problem and providing approximation estimates for the Fortin operator, we are able to give the optimal error estimate for the semi-discrete scheme for Maxwell's equations.

We then study preconditioners for linear systems arising in the finite element method for time-dependent Maxwell's equations using implicit time-stepping. Such linear systems are usually very large but sparse and can only be solved iteratively. We consider overlapping Schwarz methods and multigrid methods and extend some existing theoretical convergence results. For overlapping Schwarz methods, we provide numerical experiments to confirm the theoretical analysis.

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CHAPTER I

INTRODUCTION

We consider the following time dependent Maxwell problem in a dielectric medium:

$$\begin{aligned} \varepsilon \boldsymbol{E}_t - \mathbf{curl}\, \boldsymbol{H} &= \boldsymbol{J}, \text{ in } \Omega \times (0,T), \\ \mu^{-1}\boldsymbol{H}_t + \mathbf{curl}\, \boldsymbol{E} &= 0, \text{ in } \Omega \times (0,T), \end{aligned} \tag{1.1}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric field and magnetic field respectively, the current density $\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{x}, t)$ is the source term, and the permittivity ε and permeability μ describe properties of the material occupying $\Omega \subset \mathbb{R}^3$.

Computational electromagnetics is the numerical approximation of the solution of Maxwell's equations. These solutions describe dynamic effects in electromagnetics, i.e. changing magnetic flux density produces a change in electric fields and vice versa. A fundamental understanding of these phenomena is critical in the design of many devices such as radars, computer chips, optical fiber systems, and mobile phone systems.

The Finite-Difference Time-Domain Method (FDTD) for approximating (1.1), as first proposed by Yee in 1966 [68], is a fully explicit numerical scheme based on the regular cartesian mesh. Yee evaluated the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$ about a unit cell at centers of edges and faces respectively. For time derivatives, the leap-frog scheme was used to obtain $\boldsymbol{E}$ and $\boldsymbol{H}$ at alternate half time steps. Taflove was among the first to rigorously analyze Yee's FDTD algorithm [62, 63]. For later development of FDTD methods, we refer to [59].

Yee's FDTD method has two main disadvantages [46] even though it is conceptually simple and easy to program. One is the lack of flexibility and accuracy in

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handling problems involving complex geometries and inhomogeneous media. When $\partial\Omega$ is a smooth curved surface, we can only use a stair-step approximation, which leads to significant errors in certain problems. In [38], a computational example was given to show that a stair-step FDTD model may require a mesh size eight times as small as that for a piecewise linear, boundary-conforming mesh to yield the same accuracy. Such errors have also been analyzed rigorously by Cangellaris and Wright for the stair-step approximation of a planar surface tilted 45 degrees to the grid [22], while Holland analyzed errors associated with a slightly tilted planar scatterer [37]. Another disadvantage is the use of an explicit scheme, which requires that the time step be consistent with the spatial mesh size because of the so-called CFL condition. This is especially the case for interface problems where a very small spatial mesh size has to be used due to the low regularity of the solution to (1.1). In Section A, Chapter III, we point out that the singularity of the solution to (1.1) is determined by the singularity of the solution to certain interface Laplacian problems with coefficients ε and μ. Such singularities come from corners and edges on the interface where ε and μ are discontinuous and thus tetrahedra of small size have to be used in the region close to the interface. These two problems can be avoided by using finite element methods and implicit time stepping schemes.

The finite element method (FEM) has proven to be a powerful tool in numerical modeling of numerous physical problems. There has been a great deal of work in computational electromagnetics using (mixed) finite element methods based on variational forms and appropriate finite element approximation spaces. Possible choices of finite elements can be vectorial Lagrange nodal elements, Nedelec edge elements and Raviart-Thomas elements, whose definitions are given in Chapter II. In 1980, Nedelec *et al.* [50] studied an implicit scheme on the time domain. For the spatial domain, a pair of Lagrange and Raviart-Thomas finite elements was used to approximate the solution of (1.1). In 1994, P. A. Raviart *et al.* [5] reformulated Equation (1.1) as a constrained wave equation system with a Lagrange multiplier associated with the condition $\operatorname{div} \boldsymbol{B} = 0$, where $\boldsymbol{B}$ is the magnetic induction. Then they approximated both the field $\boldsymbol{B}$ and the Lagrange multiplier with a mixed finite element method using Taylor-Hood finite element spaces which consisted of piecewise linear continuous (vector-valued) functions.

However, when vector-valued Lagrange elements are used, the convergence in $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$ can not be guaranteed. For a detailed description of this phenomenon, we refer to [10, 11]. Also it is difficult to impose boundary conditions $\boldsymbol{E} \times \boldsymbol{n} = 0$ or $\boldsymbol{B} \cdot \boldsymbol{n} = 0$, which can be seen from various ways in handling these boundary conditions [5, 39, 50]. The introduction of Nedelec and Raviart-Thomas finite elements [49, 52] avoids the drawbacks caused by the use of nodal vector-valued elements. It is worth mentioning that the finite element method using the lowest order elements on cubes is equivalent to Yee's finite difference scheme in [68] on the structured mesh.

In 1995, Makridakis and Monk [45] analyzed finite element methods using Nedelec edge elements and Raviart-Thomas face elements. With estimates obtained in [47], they were able to provide error estimates for finite element methods for problem (1.1) with smooth coefficients μ and ε. However, their analysis does not cover interface problems, i.e. μ and ε being piecewise constants. In 2000, J. Zou *et al.* [23] gave a different approach for the interface problem. They first obtained an equation for the electric field by eliminating the magnetic field $\boldsymbol{H}$ in (2.13) and then introduced a Lagrange multiplier corresponding the normal continuity of $\varepsilon\boldsymbol{E}$ across the interface. Finally they posed a mixed variational formulation for the electric field only. The error estimates followed from the continuous and discrete inf-sup conditions [57].

It turns out that the approach in [45] works well for interface problems. One of the goals of this dissertation is to provide a theoretical analysis of the effectiveness of the semi-discrete scheme in [45]. By studying a Fortin operator, we provide an error estimate for the semi-discrete finite element method in Chapter III.

We now turn to the second disadvantage of Yee's FDTD method. The stability of the FDTD method critically depends on the finite difference scheme chosen for the time derivative [50]. When an explicit scheme is used, the CFL condition requires that the time step be comparable with the spatial mesh size [44], and thus the computation is slow. On the contrary, an *implicit scheme*, for example, the backward-Euler scheme, allows the time step to be relatively large. For a detailed description of implicit and explicit schemes for time discretization of equation (1.1) we refer to [44, 45, 50, 68].

However, for the implicit scheme one has to solve a linear system at each time step and thus a fast solver is highly desirable. Note that it is often possible to eliminate $\boldsymbol{H}$ in the coupled linear system without any matrix inversion (see Section D), which leads to a symmetric positive definite system for $\boldsymbol{E}$ corresponding to the bilinear form on $\boldsymbol{H}_0(\mathbf{curl};\Omega)$:

$$A(\boldsymbol{u},\boldsymbol{v}) \equiv (\alpha\boldsymbol{u},\boldsymbol{v}) + (\beta\,\mathbf{curl}\,\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v}), \tag{1.2}$$

where α and β are known functions determined by ε,μ, and the time step used in the implicit scheme. We propose solving the discretized linear system corresponding to (1.2) using the popular preconditioned conjugate gradient (PCG) method. The convergence rate of PCG method depends on the condition numbers of the matrix of the underlying linear system [33]. The remainder of the dissertation studies techniques for developing preconditioners for (1.2).

It is well known [19, 27, 29, 60, 67] that domain decomposition methods and multigrid methods provide excellent preconditioners for discrete systems resulting from second order elliptic problems. It turns out [4, 35, 34, 64] that the same ideas can be used to construct efficient preconditioners for the discrete system resulting from the bilinear form (1.2).

Schwarz methods provide efficient and easily parallelized preconditioners for the discrete system corresponding to (1.2). But the theoretical analysis is still less than complete. In [64, 65], Toselli analyzed the convergence of overlapping Schwarz methods in the case of convex domains. In [34], Hiptmair and Toselli gave a unified and simplified approach to Schwarz methods for problems in $\boldsymbol{H}(\mathbf{curl};\Omega)$ and $\boldsymbol{H}(\mathrm{div};\Omega)$.

Multigrid methods are natural extensions to domain decomposition methods. They are well known for their optimal work estimates and rapid convergence. Hiptmair originally adapted multigrid ideas to the discretization problem (1.2) and obtained convergence results for V-cycle multigrid in [35]. Various numerical results were also given in [35] to show the robustness of the V-cycle method. For the same problem but a different construction of smoothers, Arnold *et al.* [4] presented another proof of the convergence of the V-cycle multigrid in a multigrid framework which combines the regularity and the smoothing conditions together. In 2000, Hiptmair [36] analyzed multilevel methods of an eddy-current problem on a non-convex polyhedral domain using approximate Helmholtz-decompositions of the function space $\boldsymbol{H}(\mathbf{curl};\Omega)$ into an $\boldsymbol{H}^1$-regular subspace and gradients.

Because of the large kernel of the **curl** operator, the Helmholtz decomposition of an arbitrary vector field into solenoidal and irrotational components plays an important role in the abovementioned work. However, the solenoidal component is not in general $\boldsymbol{H}^1$-regular when Ω is a non-convex polyhedron and many estimates in [4, 35, 34, 64] fail in that case. In this dissertation, we overcome this difficulty by splitting the solenoidal component further into a sum of a $\boldsymbol{H}^1$-regular vector field and a gradient. The construction and estimates are given in Section A, Chapter IV.

In Chapter IV, we extend theoretical results in [34, 64, 65] on overlapping Schwarz methods to a more general case. By using the regular Helmholtz-type decomposition of vector fields in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ [8, 26], we provide a stable decomposition which is critical in the estimate of the condition number of the preconditioned system.

In Chapter V, we give a convergence proof based on the framework in [14, 15]. To do this, we introduce a new (base) innerproduct in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, and check all conditions used in the abstract theory. The idea is borrowed from [13] in which the negative one innerproduct was used as the base innerproduct to analyze multigrid methods for pseudo-differential operators of order minus one. For our problem, we impose the negative one norm on the gradient field of the orthogonal Helmholtz decomposition and get a base norm weaker than the usual norm in $\boldsymbol{L}^2(\Omega)$. The detailed description is given in Chapter V. This approach is different from the one used in [36].

This dissertation is organized as follows. Chapter II contains an introduction to Sobolev spaces, finite element spaces and Maxwell's equations. In Chapter III we provide error estimates of the semi-discrete scheme for time dependent Maxwell's equations. In Chapters IV and V, we analyze overlapping Schwarz preconditioners and multigrid methods for the linear system when fully discrete schemes are applied to time dependent Maxwell's equations.

Concerning notations, we use boldface type for vector fields, spaces of vector fields, and operators between vector fields. Following a popular convention, we denote by C, with or without a subscript, a generic constant whose value may differ at different occurrences but does not depend on h, the discretization parameter.

CHAPTER II

PRELIMINARIES

In this chapter, we give a brief introduction of time dependent Maxwell's equations and finite element spaces. We start with some basic definitions and properties of Sobolev spaces and finite element spaces. Then we discuss the existence and uniqueness of the solution of time dependent Maxwell's equations by using semigroup theory. Some basic definitions and properties of semigroup theory and its application to evolution equations are given in the appendix. We end this chapter by introducing the semi-discrete scheme and a fully discrete scheme for time dependent Maxwell's equations. The error analysis of the semi-discrete scheme will be given in Chapter III. The discrete systems resulting from the fully discrete scheme motivates our work in Chapters IV and V.

A. Sobolev spaces

Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^3$ with a Lipschitz continuous boundary $\partial\mathcal{D}$ in the sense of [1]. In particular, domains with polyhedral boundaries belong to this class.

Let $L^2(\mathcal{D})$ denote the space of square integrable functions. We denote $(L^2(\mathcal{D}))^3$ by $\boldsymbol{L}^2(\mathcal{D})$. For conciseness of notation, we denote by $\|\cdot\|$, the norms on both $L^2(\mathcal{D})$ and $\boldsymbol{L}^2(\mathcal{D})$.

The Sobolev spaces $H^r(\mathcal{D})$, $r \in \mathbb{R}$, is well defined on the Lipschitz domain $\mathcal{D}$. When $r = m$ is an non-negative integer, $H^m(\mathcal{D})$ is the space of all distributions u defined in $\mathcal{D}$ such that $D^\alpha u \in L^2(\mathcal{D})$ for all $|\alpha| \le m$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-integer with non-negative components and

$$D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}\,\partial x_3^{\alpha_3}} \quad \text{and} \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3.$$

The norm $\|\cdot\|_{m,\mathcal{D}}$ and the seminorm $|\cdot|_{m,\mathcal{D}}$ in $H^m(\mathcal{D})$ are given by

$$\|u\|_{m,\mathcal{D}}^2 \equiv \sum_{|\alpha|\le m} \|D^\alpha u\|^2 \quad \text{and} \quad |u|_{m,\mathcal{D}}^2 \equiv \sum_{|\alpha|=m} \|D^\alpha u\|^2.$$

When $r = m + \sigma$ for some nonnegative integer m and $\sigma \in (0,1)$, $H^r(\mathcal{D})$ is the space of $u \in H^m(\mathcal{D})$ such that, for all $|\alpha| = m$,

$$\int_{\mathcal{D}}\int_{\mathcal{D}} \frac{|D^\alpha u(\boldsymbol{x}) - D^\alpha u(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{3+2\sigma}}\, dx\, dy < \infty.$$

The seminorm $|\cdot|_{r,\mathcal{D}}$ and the norm $\|\cdot\|_{r,\mathcal{D}}$ in this case are given by

$$|u|_{r,\mathcal{D}}^2 \equiv \sum_{|\alpha|=m} \int_{\mathcal{D}}\int_{\mathcal{D}} \frac{|D^\alpha u(\boldsymbol{x}) - D^\alpha u(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{3+2\sigma}}\, dx\, dy$$

and

$$\|u\|_{r,\mathcal{D}}^2 \equiv \|u\|_{m,\mathcal{D}}^2 + |u|_{r,\mathcal{D}}^2.$$

When $\mathcal{D}$ is clear in the context, we will drop the subscript $\mathcal{D}$. For $r > 0$, we denote by $H_0^r(\mathcal{D})$ the closure of $C_0^\infty(\mathcal{D})$ in $H^r(\mathcal{D})$. For $r < 0$, we denote by $H^r(\mathcal{D})$ the dual space of $H_0^{-r}(\mathcal{D})$.

Using the derivative in the distribution sense, we can define operators **curl** and div on $\boldsymbol{L}^2(\mathcal{D})$. Indeed, let $\langle\cdot,\cdot\rangle$ denote the duality pairing between $C_0^\infty(\mathcal{D})$ (or $(C_0^\infty(\mathcal{D}))^3$) and its dual space. For any function $\boldsymbol{v} = (v_1, v_2, v_3)$ in $\boldsymbol{L}^2(\mathcal{D})$, we have, for all $\boldsymbol{w} = (w_1, w_2, w_3)$ in $C_0^\infty(\mathcal{D})^3$,

$$\begin{aligned}\langle \mathbf{curl}\,\boldsymbol{v}, \boldsymbol{w}\rangle &= \int_{\mathcal{D}} \boldsymbol{v}\cdot \mathbf{curl}\,\boldsymbol{w}\, dx \\ &= \int_{\mathcal{D}} \Big[v_1\big(\frac{\partial w_3}{\partial x_2} - \frac{\partial w_2}{\partial x_3}\big) + v_2\big(\frac{\partial w_1}{\partial x_3} - \frac{\partial w_3}{\partial x_1}\big) + v_3\big(\frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2}\big)\Big]\, dx\end{aligned}$$

and, for all ϕ in $C_0^\infty(\mathcal{D})$,

$$\langle \mathrm{div}\,\boldsymbol{v}, \phi\rangle = -\int_{\mathcal{D}} \boldsymbol{v}\cdot\nabla\phi\, dx = -\int_{\mathcal{D}} \big(v_1\frac{\partial \phi}{\partial x_1} + v_2\frac{\partial \phi}{\partial x_2} + v_3\frac{\partial \phi}{\partial x_3}\big)\, dx.$$

This leads to the following definition.

Definition II.1. *The space* $\boldsymbol{H}(\mathbf{curl}, \mathcal{D})$ *is the space of* $\boldsymbol{u}$ *in* $\boldsymbol{L}^2(\mathcal{D})$ *whose* **curl** *is also in* $\boldsymbol{L}^2(\mathcal{D})$ *and is equipped with the norm*

$$\|\boldsymbol{u}\|^2_{\boldsymbol{H}(\mathbf{curl},\mathcal{D})} \equiv \|\boldsymbol{u}\|^2 + \|\mathbf{curl}\,\boldsymbol{u}\|^2.$$

The space $\boldsymbol{H}(\mathrm{div}, \mathcal{D})$ *is the space of* $\boldsymbol{v}$ *in* $\boldsymbol{L}^2(\mathcal{D})$ *whose* div *is also in* $L^2(\mathcal{D})$ *and is equipped with the norm*

$$\|\boldsymbol{u}\|^2_{\boldsymbol{H}(\mathrm{div},\mathcal{D})} \equiv \|\boldsymbol{u}\|^2 + \|\mathrm{div}\,\boldsymbol{u}\|^2.$$

It is shown [32] that any function $\boldsymbol{u}$ in $\boldsymbol{H}(\mathbf{curl}, \mathcal{D})$ has a tangential trace $\boldsymbol{u}\times\boldsymbol{n}$ in $\boldsymbol{H}^{-1/2}(\partial\mathcal{D})$, and any function $\boldsymbol{v}$ in $\boldsymbol{H}(\mathrm{div}, \mathcal{D})$ has a normal trace $\boldsymbol{v}\cdot\boldsymbol{n}$ in $\boldsymbol{H}^{-1/2}(\partial\mathcal{D})$. This allows us to introduce “homogeneous” spaces:

$$\boldsymbol{H}_0(\mathbf{curl}, \mathcal{D}) = \{\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}, \mathcal{D}) \ \mid \boldsymbol{u} \times \boldsymbol{n} = 0 \text{ on } \partial\mathcal{D}\},$$
$$\boldsymbol{H}_0(\mathrm{div}, \mathcal{D}) = \{\boldsymbol{u} \in \boldsymbol{H}(\mathrm{div}, \mathcal{D}) \ \mid \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \partial\mathcal{D}\}.$$

When $\mathcal{D}$ is simply connected, each function $\boldsymbol{u}$ in $\boldsymbol{H}_0(\mathbf{curl}, \mathcal{D})$ admits a unique orthogonal Helmholtz decomposition [32]

$$\boldsymbol{u} = \boldsymbol{z} + \nabla\phi, \tag{2.1}$$

where $\boldsymbol{z} \in \boldsymbol{H}_0(\mathbf{curl}\,, \mathcal{D})$ and $\phi \in H^1_0(\mathcal{D})$ satisfies $\mathrm{div}\,\boldsymbol{z} = 0$. The vectors $\boldsymbol{z}$ and $\nabla\phi$ are called solenoidal and irrotational vector fields of the Helmholtz decomposition, respectively.

B. Finite element spaces

We suppose that Ω is a bounded and simply-connected polyhedron in $\mathbb{R}^3$. Further, neither slits nor cuts are allowed, i.e., $\partial\bar{\Omega} = \partial\Omega$.

Let $\mathcal{T}_h$ be a decomposition of Ω consisting of closed tetrahedra. For each τ in $\mathcal{T}_h$, let h_τ be the diameter of τ and ρ_τ the maximum diameter of all balls contained in τ. Throughout this dissertation we assume that the mesh $\mathcal{T}_h$ is shape regular and quasi-uniform, i.e., there exists a constant C such that

$$\max_{\tau\in\mathcal{T}_h} \frac{h_\tau}{\rho_\tau} \le C \quad \text{and} \quad \max_{\tau\in\mathcal{T}_h} h_\tau \le C \min_{\tau\in\mathcal{T}_h} h_\tau.$$

On each element τ of $\mathcal{T}_h$ and for each integer $l \ge 0$, we define the space $\mathbb{P}_l(\tau)$ of all polynomials of total degree $\le l$ and its subspace $\overset{\circ}{\mathbb{P}}_l(\tau)$ of all homogeneous polynomials of degree l.

Fix a positive integer k. The Lagrange finite element space $\overline{S}_h$ consists of all functions $p(\boldsymbol{x}) \in H^1(\Omega)$ such that $p|_\tau \in \mathbb{P}_k(\tau)$ for all τ in $\mathcal{T}_h$. Degrees of freedom are given by values of $p(\boldsymbol{x})$ at the following points:

$$\Sigma_k = \Big\{\boldsymbol{x} = \sum_{j=1}^{4} \lambda_j \boldsymbol{a}_j \;\Big|\; \sum_{j=1}^{4} \lambda_j = 1,\ \lambda_j \in \{0, 1/k, \dots (k-1)/k, 1\},\ 1 \le j \le 4\Big\},$$

where $\boldsymbol{a}_j$, $j = 1, \dots, 4$, are the four vertices of τ. In the lowest order case, $k = 1$, those points are just the four vertices of τ. The interpolation operator I_h onto $\overline{S}_h$ is defined by $(I_h\, p)(\boldsymbol{x}) = p(\boldsymbol{x})$ for all $\boldsymbol{x}$ in Σ_k.

The Nedelec finite element space $\overline{\boldsymbol{U}}_h$ [49, 52] is defined by

$$\overline{\boldsymbol{U}}_h = \{\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl};\Omega) \;\big|\; \boldsymbol{u} = \boldsymbol{a} + \boldsymbol{b} \text{ on } \tau,\ \boldsymbol{a} \in \mathbb{P}_{k-1}(\tau)^3,\ \boldsymbol{b} \in \overset{\circ}{\mathbb{P}}_k(\tau)^3, \quad \text{for all } \tau \in \mathcal{T}_h\},$$

where $\boldsymbol{b}$ satisfies $\boldsymbol{b} \cdot \boldsymbol{x} = 0$ on τ.

Definition II.2. *Let τ be a tetrahedron in $\mathbb{R}^3$ with edges denoted by $\{e\}$ and faces* *by* $\{f\}$ *and* $\boldsymbol{u}$ *a function in* $\boldsymbol{H}(\mathbf{curl};\Omega)$. *The degrees of freedom are the following moments:*

1. $\int_e (\boldsymbol{u}\cdot\boldsymbol{t})\, q\, ds$, *for all* $q \in \mathbb{P}_{k-1}$,
2. $\int_f (\boldsymbol{u}\times\boldsymbol{n})\cdot\boldsymbol{q}\, dS$, *for all* $\boldsymbol{q} \in \mathbb{P}^2_{k-2}$,
3. $\int_\tau \boldsymbol{u}\cdot\boldsymbol{q}\, dx$, *for all* $\boldsymbol{q} \in \mathbb{P}^3_{k-2}$.

Here $\boldsymbol{t}$ *is a unit vector directed along the edge* e.

The total number of degrees of freedom is $k(k+2)(k+3)/2$. In the lowest order case, $k=1$, there are six degrees of freedom, each of which corresponds to one edge of τ. Based on degrees of freedom given above, we define the interpolant $\boldsymbol{\Pi}_\tau\boldsymbol{u}$ such that $\boldsymbol{\Pi}_\tau\boldsymbol{u}$ and $\boldsymbol{u}$ have the same degrees of freedom on τ, and define the interpolation operator $\boldsymbol{\Pi}_h$ onto $\overline{\boldsymbol{U}}$ by $\boldsymbol{\Pi}_h\boldsymbol{u}|_\tau = \boldsymbol{\Pi}_\tau\boldsymbol{u}$ on all τ in $\mathcal{T}_h$. Due to the dependence on edge moments $\int_e(\boldsymbol{u}\cdot\boldsymbol{t})\, q\, ds$, we require certain regularity of $\boldsymbol{u}$ for $\boldsymbol{\Pi}_h\boldsymbol{u}$ to be well defined. The following lemma [3] makes the condition specific.

Lemma II.1. *For any* $p > 2$ *and for any tetrahedron* τ, *the operator* $\boldsymbol{\Pi}_\tau$ *is well defined and continuous on the space*

$$\{\boldsymbol{u}\in \boldsymbol{L}^p(\tau) \,\big|\, \mathbf{curl}\,\boldsymbol{u}\in\boldsymbol{L}^p(\tau) \text{ and } \boldsymbol{u}\times\boldsymbol{n}\in L^p(\partial\tau)^2\}.$$

The Raviart-Thomas finite element space $\overline{\boldsymbol{V}}$ [49, 57] is given by

$$\overline{\boldsymbol{V}}_h = \{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div};\Omega)\,\big|\,\boldsymbol{v}=\boldsymbol{a}+b\,\boldsymbol{x} \text{ on } \tau,\ \boldsymbol{a}\in\mathbb{P}_{k-1}(\tau)^3,\ b\in\mathring{\mathbb{P}}_{k-1}(\tau), \quad \text{for all } \tau\in\mathcal{T}_h\}.$$

Definition II.3. *Let* τ *be a tetrahedron in* $\mathbb{R}^3$ *with faces denoted by* $\{f\}$ *and let* $\boldsymbol{v}$ *be a function in* $\boldsymbol{H}(\mathrm{div};\Omega)$. *The degrees of freedom are the following moments:*

1. $\int_f (\boldsymbol{v}\cdot\boldsymbol{n})\, q\, dS$, *for all* $q\in\mathbb{P}_{k-1}$,
2. $\int_\tau \boldsymbol{u}\cdot\boldsymbol{q}\, dx$, *for all* $\boldsymbol{q}\in\mathbb{P}^3_{k-2}$.

The total number of degrees of freedom is $k(k+1)(k+3)/2$. In the lowest order case, $k=1$, there are four degrees of freedom, each of which corresponds to one face of τ. Based on degrees of freedom given above, we define the interpolant $\boldsymbol{r}_\tau \boldsymbol{v}$ such that $\boldsymbol{r}_\tau \boldsymbol{v}$ and $\boldsymbol{v}$ have the same degrees of freedom on τ and define the interpolation operator $\boldsymbol{r}_h$ onto $\overline{\boldsymbol{V}}_h$ by $\boldsymbol{r}_h \boldsymbol{v}|_\tau = \boldsymbol{r}_\tau \boldsymbol{v}$ on all τ in $\mathcal{T}_h$. The operator $\boldsymbol{r}_h$ is well defined for vector fields in $\boldsymbol{H}(\mathrm{div};\Omega) \cap \boldsymbol{L}^p(\Omega)$ for any $p > 2$ [57]. It can be shown [2, 41, 55] that for all $\boldsymbol{v}$ in $\boldsymbol{H}(\mathrm{div};\Omega) \cap \boldsymbol{H}^\alpha(\cup\Omega_i)$, the interpolation operator $\boldsymbol{r}_h$ satisfies

$$\|\boldsymbol{v} - \boldsymbol{r}_h \boldsymbol{v}\| \le C \begin{cases} h^\alpha |\boldsymbol{v}|_{\alpha,\cup\Omega_i} + h\|\mathrm{div}\,\boldsymbol{v}\|, & 0 < \alpha \le 1/2, \\ h^\alpha |\boldsymbol{v}|_{\alpha,\cup\Omega_i}, & \alpha > 1/2. \end{cases} \tag{2.2}$$

The finite element space $\overline{W}_h$ is the subspace of $L^2(\Omega)$ consisting of arbitrary piecewise polynomials of degree at most $k-1$. The degrees of freedom for $\overline{W}_h$ are simply

$$\int_\tau w\, p\, d\boldsymbol{x}, \quad \text{for all } p \in \mathbb{P}_{k-1}(\tau).$$

The interpolation operator is denoted by ω_h.

A space symbol without overline stands for the corresponding finite element subspace of functions with natural homogeneous boundary conditions. For example, $\boldsymbol{U}_h = \boldsymbol{H}_0(\mathbf{curl};\Omega) \bigcap \overline{\boldsymbol{U}}_h$. However, W_h is the subspace of $\overline{W}_h$ consisting functions with zero mean value.

All the interpolation operators are indispensable tools in the analysis because of the following commuting diagram property, which follows from definitions of interpolations operators and theorems of Green and Stokes.

Theorem II.1. *The diagram*

$$\begin{array}{ccccccc}
H_0^1(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}_0(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}_0(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2/\mathbb{R} \\
\big\downarrow I_h & & \big\downarrow \boldsymbol{\Pi}_h & & \big\downarrow \boldsymbol{r}_h & & \big\downarrow \omega_h \\
S_h & \xrightarrow{\nabla} & \boldsymbol{U}_h & \xrightarrow{\mathbf{curl}} & \boldsymbol{V}_h & \xrightarrow{\mathrm{div}} & W_h
\end{array}$$

commutes when all interpolation operators are applied on sufficiently smooth functions.

In analogy to the case of $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, each function $\boldsymbol{u}_h$ in $\boldsymbol{U}_h$ admits a unique orthogonal decomposition [32]

$$\boldsymbol{u}_h = \boldsymbol{z}_h + \nabla\phi_h, \tag{2.3}$$

where $\boldsymbol{z}_h \in \boldsymbol{U}_h$ is $\boldsymbol{L}^2$-orthogonal to ∇S_h and ϕ_h belongs to S_h. We will call this decomposition the discrete Helmholtz decomposition of $\boldsymbol{u}_h$.

C. Time-dependent Maxwell's equations

We consider the following Maxwell's equations:

$$\frac{\partial \boldsymbol{D}}{\partial t} - \mathbf{curl}\,\boldsymbol{H} = -\boldsymbol{J}, \tag{2.4}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \mathbf{curl}\,\boldsymbol{E} = 0, \tag{2.5}$$

$$\mathrm{div}\,\boldsymbol{D} = q, \tag{2.6}$$

$$\mathrm{div}\,\boldsymbol{B} = 0, \tag{2.7}$$

which hold for all $(t,\boldsymbol{x})$ in $(0,T)\times\Omega$. Here $\boldsymbol{E}$ and $\boldsymbol{D}$ are the electric field and induction respectively, $\boldsymbol{H}$ and $\boldsymbol{B}$ are the magnetic field and induction respectively, $\boldsymbol{J}$ is a known function specifying the applied current, and q is the electric charge.

The law of proportionality of fields and inductions is expressed by two constituent relations;

$$\boldsymbol{D} = \varepsilon\boldsymbol{E}, \quad \boldsymbol{B} = \mu\boldsymbol{H}, \tag{2.8}$$

where ε is the dielectric permittivity and μ is the magnetic permeability.

Equation (2.4) corresponds to the law of conservation of electric charge. For any fixed domain $\mathcal{D}$ in Ω, the change per unit time of the total electric charge contained in the interior of $\mathcal{D}$ is produced by the flux of charges through $\partial\mathcal{D}$. This is expressed by

$$\frac{\partial}{\partial t}\int_{\mathcal{D}} q\,dx = -\int_{\partial\mathcal{D}} \boldsymbol{J}\cdot\boldsymbol{n}\,dS, \tag{2.9}$$

where q is the density of the electric charge, $\boldsymbol{J}$ is the electric current, and $\boldsymbol{n}$ is the outward unit normal at the boundary of $\mathcal{D}$. Since the electric induction $\boldsymbol{D}$ satisfies $q = \operatorname{div}\boldsymbol{D}$, (2.9) shows that the vector $\partial\boldsymbol{D}/\partial t + \boldsymbol{J}$ has divergence zero on $\mathcal{D}$. Since $\mathcal{D}$ is an arbitrary domain in Ω, the magnetic field $\boldsymbol{H}$ satisfies [30]

$$\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J} = \mathbf{curl}\,\boldsymbol{H},$$

which is (2.4).

Equation (2.5) corresponds to the Faraday's law. This law says that the derivative with respect to the time of the flux of magnetic induction $\boldsymbol{B}$ across the surface Σ is the opposite to the circulation of electric field along the contour $\partial\Sigma$. This can be expressed by

$$\frac{d}{dt}\int_{\Sigma} \boldsymbol{B}\cdot\boldsymbol{n}\,dS + \int_{\partial\Sigma} \boldsymbol{E}\cdot d\boldsymbol{s} = 0.$$

Since Σ can be the closed boundary of any open set $\mathcal{D}$ in Ω, the above implies Equation (2.5).

Equations (2.4)—(2.7) are not sufficient in themselves; they must be supplemented with boundary conditions, e.g.,

$$\boldsymbol{E}\times\boldsymbol{n} = 0 \quad \text{and} \quad \boldsymbol{B}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega, \tag{2.10}$$

and initial conditions

$$\boldsymbol{E}(\boldsymbol{x},0) = \boldsymbol{E}_0(\boldsymbol{x}) \quad \text{and} \quad \boldsymbol{B}(\boldsymbol{x},0) = \boldsymbol{B}_0(\boldsymbol{x}), \quad \text{in} \quad \Omega, \tag{2.11}$$

where $\boldsymbol{B}_0 \in \boldsymbol{H}_0(\text{div};\Omega)$ satisfies

$$\text{div}\,\boldsymbol{B}_0 = 0, \quad \text{in} \quad \Omega. \tag{2.12}$$

Equation (2.7) is a consequence of (2.5) and the divergence-free condition (2.12). Therefore, instead of studying Equations (2.4)—(2.7), we need only study the following equations

$$\begin{aligned} \varepsilon\boldsymbol{E}_t - \mathbf{curl}\,(\mu^{-1}\boldsymbol{B}) &= -\boldsymbol{J}, \qquad \text{in} \quad \Omega\times(0,T), \\ \boldsymbol{B}_t + \mathbf{curl}\,\boldsymbol{E} &= 0, \qquad\quad \text{in} \quad \Omega\times(0,T), \end{aligned} \tag{2.13}$$

together with initial conditions (2.11) and boundary conditions (2.10).

The existence and uniqueness of the solution to (2.13) and (2.11) are consequences of the semigroup theory. We set $\mathcal{H} = \boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)$, and set the innerproduct to be

$$\left(\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{pmatrix}, \begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{pmatrix}\right)_{\mathcal{H}} = (\boldsymbol{u},\boldsymbol{u})_\varepsilon + (\boldsymbol{v},\boldsymbol{v})_{\mu^{-1}} \equiv (\varepsilon\boldsymbol{u},\boldsymbol{u}) + (\mu^{-1}\boldsymbol{v},\boldsymbol{v}).$$

Since ε and μ are piecewise positive constants, weighted innerproducts $(\cdot,\cdot)_\varepsilon$ and $(\cdot,\cdot)_{\mu^{-1}}$ are equivalent to the usual innerproduct $(\cdot,\cdot)$.

We will define an operator $\mathcal{A}$ and write (2.13) in the operator form. The domain $D(\mathcal{A})$ is given by

$$D(\mathcal{A}) = \left\{(\boldsymbol{u},\boldsymbol{v})^T \in \mathcal{H} \,\middle|\, \boldsymbol{u}\in\boldsymbol{H}_0(\mathbf{curl};\Omega),\ \text{and}\ \mathbf{curl}\,\mu^{-1}\boldsymbol{v}\in\boldsymbol{L}^2(\Omega)\right\},$$

and we define $\mathcal{A}$ by

$$\mathcal{A}\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{pmatrix}=\begin{pmatrix}-\varepsilon^{-1}\,\mathbf{curl}\,\mu^{-1}\boldsymbol{v}\\ \mathbf{curl}\,\boldsymbol{u}\end{pmatrix},\quad \begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{pmatrix}\in D(\mathcal{A}). \tag{2.14}$$

Using $\mathcal{A}$, we can rewrite equations (2.13) as

$$\frac{d}{dt}\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{B}\end{pmatrix}+\mathcal{A}\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{B}\end{pmatrix}=\begin{pmatrix}-\varepsilon^{-1}\boldsymbol{J}\\ 0\end{pmatrix}. \tag{2.15}$$

For properties of the operator $\mathcal{A}$, we have the following lemma which is shown in [30] under the assumption that Ω is regular. Our proof extends this result to the more general case when Ω is a Lipschitz domain.

Lemma II.2. *The domain $D(\mathcal{A})$ is dense in $\mathcal{H}$ and $\mathcal{A}$ is closed. We have*

$$\mathcal{A}^*=-\mathcal{A}\quad \textit{and}\quad D(\mathcal{A}^*)=D(\mathcal{A}).$$

Proof. Evidently $D(\mathcal{A})$ is dense in $\mathcal{H}$. To show $\mathcal{A}$ is closed, let $\boldsymbol{\Phi}_j\in D(\mathcal{A})$ such that $\boldsymbol{\Phi}_j=(\boldsymbol{u}_j,\boldsymbol{v}_j)$ and $\mathcal{A}\boldsymbol{\Phi}_j$ converge to $\boldsymbol{\Phi}=(\boldsymbol{u},\boldsymbol{v})$ and $\boldsymbol{\Psi}$ respectively. We have that $\boldsymbol{u}_j\to\boldsymbol{u}$, $\boldsymbol{v}_j\to\boldsymbol{v}$ in $\boldsymbol{L}^2(\Omega)$, and $\mathbf{curl}\,\boldsymbol{u}_j$ and $\mathbf{curl}\,\mu^{-1}\boldsymbol{v}_j$ converge in $\boldsymbol{L}^2(\Omega)$. But $\mathbf{curl}\,\boldsymbol{u}_j\to\mathbf{curl}\,\boldsymbol{u}$, $\mathbf{curl}\,\mu^{-1}\boldsymbol{v}_j\to\mathbf{curl}\,\mu^{-1}\boldsymbol{v}$ in the dual of $(C_0^\infty(\Omega))^3$. Therefore, $\mathbf{curl}\,\boldsymbol{u}$ and $\mathbf{curl}\,\mu^{-1}\boldsymbol{v}$ belong to $\boldsymbol{L}^2(\Omega)$ and satisfy that $\mathbf{curl}\,\boldsymbol{u}_j\to\mathbf{curl}\,\boldsymbol{u}$, $\mathbf{curl}\,\mu^{-1}\boldsymbol{v}_j\to\mathbf{curl}\,\mu^{-1}\boldsymbol{v}$ in $\boldsymbol{L}^2(\Omega)$. Moreover, $\boldsymbol{u}_j\to\boldsymbol{u}$ in $\boldsymbol{H}(\mathbf{curl};\Omega)$ implies that $\boldsymbol{u}_j\times\boldsymbol{n}|_{\partial\Omega}(=0)$ converges to $\boldsymbol{u}\times\boldsymbol{n}|_{\partial\Omega}$ in $\boldsymbol{H}^{-1/2}(\partial\Omega)$. It follows that $\boldsymbol{u}\times\boldsymbol{n}=0$ on $\partial\Omega$ and thus $\boldsymbol{\Phi}\in D(\mathcal{A})$.

Let $\boldsymbol{\Phi}\in D(\mathcal{A}^*)$. Then there is $\boldsymbol{\Phi}^*\in\mathcal{H}$ such that

$$(\mathcal{A}\boldsymbol{\Psi},\boldsymbol{\Phi})_{\mathcal{H}}=(\boldsymbol{\Psi},\boldsymbol{\Phi}^*)_{\mathcal{H}},\quad \text{for all }\boldsymbol{\Psi}\in D(\mathcal{A}), \tag{2.16}$$

and we have $\mathcal{A}^*\boldsymbol{\Phi}=\boldsymbol{\Phi}^*$. Let $\boldsymbol{\Phi}=(\boldsymbol{u},\boldsymbol{v})$, $\boldsymbol{\Psi}=(\boldsymbol{x},\boldsymbol{y})$, and $\boldsymbol{\Phi}^*=(\boldsymbol{u}^*,\boldsymbol{v}^*)$. Then (2.16) implies that

$$(\mathbf{curl}\,\mu^{-1}\boldsymbol{y},\boldsymbol{u}) - (\mathbf{curl}\,\boldsymbol{x},\boldsymbol{v})_{\mu^{-1}} = (\boldsymbol{x},\boldsymbol{u}^*)_\varepsilon + (\boldsymbol{y},\boldsymbol{v}^*)_{\mu^{-1}}. \tag{2.17}$$

Taking $\boldsymbol{x}$ and $\mu^{-1}\boldsymbol{y} \in (C_0^\infty(\Omega))^3$ in (2.17) gives that

$$(\mu^{-1}\boldsymbol{y}, \mathbf{curl}\,\boldsymbol{u}) - (\boldsymbol{x}, \mathbf{curl}\,\mu^{-1}\boldsymbol{v}) = (\boldsymbol{x},\boldsymbol{u}^*)_\varepsilon + (\boldsymbol{y},\boldsymbol{v}^*)_{\mu^{-1}}. \tag{2.18}$$

By choosing $\boldsymbol{x} = 0$ and $\boldsymbol{y} = 0$ respectively in the above, we have that $\mathbf{curl}\,\boldsymbol{u} = \boldsymbol{v}^*$ and $\mathbf{curl}\,\mu^{-1}\boldsymbol{v} = \varepsilon\boldsymbol{u}^*$ belong to $\boldsymbol{L}^2(\Omega)$. To get $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, we set $\boldsymbol{x} = 0$ and $\mu^{-1}\boldsymbol{y} \in (C^\infty(\overline{\Omega}))^3$ in (2.17) and have that

$$(\mu^{-1}\boldsymbol{y}, \mathbf{curl}\,\boldsymbol{u}) - (\mathbf{curl}\,\mu^{-1}\boldsymbol{y},\boldsymbol{u}) = (\mu^{-1}\boldsymbol{y}, \mathbf{curl}\,\boldsymbol{u}) - (\mu^{-1}\boldsymbol{y},\boldsymbol{v}^*)_{\mu^{-1}} = 0.$$

By Lemma 2.4, Chapter I of [32], we have that $\boldsymbol{u}$ belongs to $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ and thus $\boldsymbol{\Phi} \in D(\mathcal{A})$ and $\boldsymbol{\Phi}^* = -\mathcal{A}\boldsymbol{\Phi}$ by (2.18).

Conversely, if $\boldsymbol{\Phi} \in D(\mathcal{A})$, we have

$$(\mathcal{A}\boldsymbol{\Phi},\boldsymbol{\Psi})_{\mathcal{H}} = (\boldsymbol{\Phi},\mathcal{A}^*\boldsymbol{\Psi})_{\mathcal{H}} = -(\boldsymbol{\Phi},\mathcal{A}\boldsymbol{\Psi})_{\mathcal{H}}, \quad \text{for all } \boldsymbol{\Psi} \in D(\mathcal{A}^*),$$

from which it follows that $\boldsymbol{\Phi} \in D(\mathcal{A}^*)$ and $\mathcal{A}\boldsymbol{\Phi} = -\mathcal{A}^*\boldsymbol{\Phi}$. □

By the above lemma and Stone's theorem, $\mathcal{A}$ is a generator of a (C_0) unitary group on $\mathcal{H}$, and thus we have the following theorem [56], a proof of which is also given in the appendix for completeness. We will denote by $C^m([0,T];\boldsymbol{L}^2(\Omega))$ (or $C^m([0,T];\mathcal{H})$) the space of m times continuously differentiable functions from $[0,T]$ into the space $\boldsymbol{L}^2$ (or $\mathcal{H}$).

Theorem II.2. *Assume that* $J \in C^1([0,T];\boldsymbol{L}^2(\Omega))$ *and* $(\boldsymbol{E}_0,\boldsymbol{B}_0)^T \in D(\mathcal{A})$. *Then* (2.13) *and* (2.11) *have a unique solution* $(\boldsymbol{E},\boldsymbol{B})^T \in C^1([0,T];\mathcal{H})$. *Moreover, for each*

$t \in [0,T]$, $(\boldsymbol{E}(t), \boldsymbol{B}(t))^T$ belongs to $D(\mathcal{A})$ and satisfies

$$\|\boldsymbol{E}(t)\| + \|\boldsymbol{E}_t(t)\| + \|\boldsymbol{B}(t)\| + \|\boldsymbol{B}_t(t)\| \le C. \tag{2.19}$$

Remark II.1. *Since $\boldsymbol{E}(t)$ belongs to $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, a consequence of* (2.12) *and* (2.5) *is that $\boldsymbol{B}(t) \in \boldsymbol{H}_0(\mathrm{div};\Omega)$ satisfies* $\operatorname{div} \boldsymbol{B}(t) = 0$ *for all $t \in [0,T]$.*

D. Finite difference time-domain method

Let $(\boldsymbol{E}, \boldsymbol{B})$ in $\boldsymbol{H}_0(\mathbf{curl};\Omega) \times \boldsymbol{H}_0(\mathrm{div};\Omega)$ be the solution to (2.13) and (2.11). Then, $(\boldsymbol{E}, \boldsymbol{B})$ satisfies

$$\begin{aligned} (\varepsilon \boldsymbol{E}_t, \boldsymbol{u}) - (\mu^{-1}\boldsymbol{B}, \mathbf{curl}\,\boldsymbol{u}) &= -(\boldsymbol{J}, \boldsymbol{u}), \qquad \text{for all } \boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega), \\ (\mu^{-1}\boldsymbol{B}_t, \boldsymbol{v}) + (\mathbf{curl}\,\boldsymbol{E}, \mu^{-1}\boldsymbol{v}) &= 0, \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div};\Omega). \end{aligned} \tag{2.20}$$

On the other hand, the system (2.20) together with initial conditions (2.11) has a unique solution. Indeed, let $(\boldsymbol{E}, \boldsymbol{B})$ be a solution to the system (2.20) with $\boldsymbol{J} = \boldsymbol{E}_0 = \boldsymbol{B}_0 = 0$. Taking $\boldsymbol{u} = \boldsymbol{E}$ and $\boldsymbol{v} = \boldsymbol{B}$ in (2.20), we get $(\varepsilon \boldsymbol{E}_t, \boldsymbol{E}) + (\mu^{-1}\boldsymbol{B}_t, \boldsymbol{B}) = 0$. This shows

$$\frac{d}{dt}\Big[(\varepsilon \boldsymbol{E}, \boldsymbol{E}) + (\mu^{-1}\boldsymbol{B}, \boldsymbol{B})\Big] = 0,$$

and thus $\boldsymbol{E} = \boldsymbol{B} = 0$ follows from $\boldsymbol{E}_0 = \boldsymbol{B}_0 = 0$.

So far we have shown that the weak form (2.20) is equivalent to (2.13) with initial conditions (2.11). Using Nedelec elements and Raviart-Thomas elements introduced in Section B, we can naturally transfer (2.20) to the semi-discrete scheme of seeking $(\boldsymbol{E}_h(t), \boldsymbol{B}_h(t))$ in $\boldsymbol{U}_h \times \boldsymbol{V}_h$ satisfying, for any $0 < t \le T$,

$$\begin{aligned} (\varepsilon \boldsymbol{E}_{h,t}, \boldsymbol{u}_h) - (\mu^{-1}\boldsymbol{B}_h, \mathbf{curl}\,\boldsymbol{u}_h) &= -(\boldsymbol{J}, \boldsymbol{u}_h), \qquad \text{for all } \boldsymbol{u}_h \in \boldsymbol{U}_h, \\ (\mu^{-1}\boldsymbol{B}_{h,t}, \boldsymbol{v}_h) + (\mathbf{curl}\,\boldsymbol{E}_h, \mu^{-1}\boldsymbol{v}_h) &= 0, \qquad \text{for all } \boldsymbol{v}_h \in \boldsymbol{V}_h, \end{aligned} \tag{2.21}$$

with given initial approximations

$$\boldsymbol{E}_h(0) \approx \boldsymbol{E}_0 \quad \text{and} \quad \boldsymbol{E}_h(0) \approx \boldsymbol{E}_0. \tag{2.22}$$

We will discuss possible choices of initial approximations in the following chapter.

When we try to discretize time derivatives in (2.21), the stability of the resulting fully discrete scheme critically depends on the finite difference scheme chosen for time derivatives [50]. When the simple forward-Euler scheme is used, the so-called CFL condition requires that the time step be comparable with the spatial mesh size, and thus the computation is slow. An *implicit scheme*, e.g., the backward-Euler scheme, allows the time step to be relatively large. For a detailed description of implicit and explicit schemes for time discretization of equation (2.13) we refer [45, 50, 68].

Here we describe the simple backward-Euler scheme for (2.21), which is also a motivation for our work in Chapters IV and V. But this does not imply that other implicit schemes, e.g., the Crank-Nicolson and Padé schemes, are less important. The backward-Euler scheme reads as follows: Find $(\boldsymbol{E}^n, \boldsymbol{B}^n)$ in $\boldsymbol{U}_h \times \boldsymbol{V}_h$ satisfying, for any $n = 1, \ldots, T/\triangle t$,

$$\begin{aligned} \tfrac{1}{\triangle t}(\boldsymbol{E}^n - \boldsymbol{E}^{n-1}, \boldsymbol{u}_h)_\varepsilon - (\boldsymbol{B}^n, \mathbf{curl}\, \boldsymbol{u}_h)_{\mu^{-1}} &= (\boldsymbol{J}, \boldsymbol{u}_h), && \text{for all } \boldsymbol{u}_h \in \boldsymbol{U}_h, \\ \tfrac{1}{\triangle t}(\boldsymbol{B}^n - \boldsymbol{B}^{n-1}, \boldsymbol{v}_h)_{\mu^{-1}} + (\mathbf{curl}\, \boldsymbol{E}^n, \boldsymbol{v}_h)_{\mu^{-1}} &= 0, && \text{for all } \boldsymbol{v}_h \in \boldsymbol{V}_h, \end{aligned} \tag{2.23}$$

and

$$\boldsymbol{E}^0 = \boldsymbol{E}_h(0) \quad \text{and} \quad \boldsymbol{B}^0 = \boldsymbol{B}_h(0).$$

To solve (2.23) at time step n, we transfer $\boldsymbol{E}^{n-1}$ and $\boldsymbol{B}^{n-1}$ to the right hand side, eliminate $\boldsymbol{B}^n$, and obtain a linear system for $\boldsymbol{E}^n$

$$A(\boldsymbol{E}^n, \boldsymbol{u}) \equiv (\alpha \boldsymbol{E}^n, \boldsymbol{u}) + (\mu^{-1} \mathbf{curl}\, \boldsymbol{E}^n, \mathbf{curl}\, \boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{u}), \quad \text{for all } \boldsymbol{u} \in \boldsymbol{U}_h, \tag{2.24}$$

where α depends on ε and $\triangle t$, and $\boldsymbol{f}$ depends on $\boldsymbol{J}$, $\boldsymbol{E}^{n-1}$ and $\boldsymbol{B}^{n-1}$. To recover $\boldsymbol{B}^n$, we need only solve a linear system corresponding to the well-conditioned mass matrix. This needs to be done at each time step.

We propose solving the linear system (2.24) using the popular preconditioned conjugate gradient (PCG) method. The convergence rate of PCG method can be estimated in terms of the condition number of the preconditioned system [33]. Preconditioners for (2.24) constructed using domain decomposition and multigrid methods will be analyzed in Chapters IV and V.

We point out that the problem (2.24) also arises in eddy-current simulation [11] and elasticity and Stokes' equations with various boundary conditions [31, 49].

CHAPTER III

ANALYSIS OF THE FINITE ELEMENT METHOD

In this chapter, we analyze the semi-discrete scheme for the time-dependent Maxwell's equations proposed in [45]. We start with investigations of the regularity of the solution of time-dependent Maxwell's equations in Section A. Having the error estimates developed in Section B, we provide the error analysis for the semi-discrete scheme.

For various fully discrete schemes for time-dependent Maxwell's equations, we refer readers to [45]. Our analysis can be extended to fully discrete schemes without essential difficulties.

In this chapter, we assume that $\partial\Omega$ is connected.

A. Regularity

To study the regularity of the solution $\boldsymbol{E}$ and $\boldsymbol{B}$ of (2.13), as in [24], we introduce two more spaces $\boldsymbol{X}_N(\Omega;\varepsilon)$ for electric field and $\boldsymbol{X}_T(\Omega;\mu)$ for magnetic filed, which are given by

$$\boldsymbol{X}_N(\Omega;\varepsilon) = \{\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl};\Omega) \cap \boldsymbol{H}(\mathrm{div};\varepsilon;\Omega) \mid \boldsymbol{u} \times \boldsymbol{n} = 0 \text{ on } \partial\Omega\}$$

and

$$\boldsymbol{X}_T(\Omega;\mu) = \{\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl};\Omega) \cap \boldsymbol{H}(\mathrm{div};\mu;\Omega) \mid \mu\boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\},$$

where

$$\boldsymbol{H}(\mathrm{div};\xi;\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \mid \mathrm{div}\,(\xi\boldsymbol{v}) \in \boldsymbol{L}^2(\Omega)\}.$$

Note that our definition of $\boldsymbol{X}_T(\Omega;\mu)$ is slightly different from the one in [24] and allows us not to assume that μ is constant in a neighborhood of $\partial\Omega$.

When ε (or μ) is constant, we will drop ε (or μ) in the above notations. The norms in both $\boldsymbol{X}_N(\Omega;\xi)$ and $\boldsymbol{X}_T(\Omega;\xi)$ are defined by

$$\|\boldsymbol{u}\|_{\boldsymbol{X}}^2 = \|\boldsymbol{u}\|^2 + \|\mathbf{curl}\,\boldsymbol{u}\|^2 + \|\mathrm{div}\,\xi\boldsymbol{u}\|^2.$$

If $\boldsymbol{u} \in \boldsymbol{X}_N(\Omega;\varepsilon)$ and $\mu^{-1}\boldsymbol{v} \in \boldsymbol{X}_T(\Omega;\mu)$, then $(\boldsymbol{u},\boldsymbol{v})^T$ belongs to $D(\mathcal{A})$ and $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$.

By the following theorem, we need only study the regularity of vector fields in $\boldsymbol{X}_N(\Omega;\varepsilon)$ and $\boldsymbol{X}_T(\Omega;\mu)$ in order to study the regularity of the solution $\boldsymbol{E}$ and $\boldsymbol{B}$.

Theorem III.1. *Let $\mathcal{A}$ be as in (2.14) and $\boldsymbol{g} = (\boldsymbol{E}_0, \boldsymbol{B}_0)^T$. Suppose that $\boldsymbol{J}$ and $\mathrm{div}\,\boldsymbol{J}$ belong to $C^3([0,T];\boldsymbol{L}^2)$ and $C^2([0,T];\boldsymbol{L}^2)$ respectively. If $\boldsymbol{g}$, $-\mathcal{A}\boldsymbol{g} + (\boldsymbol{J}(0),0)^T$ and $\mathcal{A}^2\boldsymbol{g} - \mathcal{A}(\boldsymbol{J}(0),0)^T + (\boldsymbol{J}'(0),0)^T$ belong to $\boldsymbol{X}_N(\Omega;\varepsilon)\times\mu\boldsymbol{X}_T(\Omega;\mu)$, the solution $(\boldsymbol{E},\boldsymbol{B})^T$ to (2.13) and (2.11) is such that $\boldsymbol{E}$ and $\boldsymbol{E}_t$ belong to $\boldsymbol{X}_N(\Omega;\varepsilon)$, and $\mu^{-1}\mathbf{curl}\,\boldsymbol{E}$, $\mu^{-1}\mathbf{curl}\,\boldsymbol{E}_t$ and $\mu^{-1}\boldsymbol{B}$ belong to $\boldsymbol{X}_T(\Omega;\mu)$.*

Proof. By (2.19), we have that

$$\begin{aligned}
&\|\boldsymbol{E}(t)\| + \|\boldsymbol{B}(t)\| + \|\mathbf{curl}\,\boldsymbol{E}(t)\| + \|\mathbf{curl}\,\mu^{-1}\boldsymbol{B}(t)\| \\
&= \|\boldsymbol{E}(t)\| + \|\boldsymbol{B}(t)\| + \|\boldsymbol{B}_t(t)\| + \|\varepsilon\boldsymbol{E}(t) + \boldsymbol{J}(t)\| \le C.
\end{aligned}$$

Note that $\mathrm{div}\,(\varepsilon\boldsymbol{E}_t) = \mathrm{div}\,\boldsymbol{J}(t)$ by (2.13). Thus, we have

$$\|\mathrm{div}\,(\varepsilon\boldsymbol{E}(t))\|^2 \le \|\mathrm{div}\,(\varepsilon\boldsymbol{E}(0))\|^2 + C\int_0^t \|\mathrm{div}\,\boldsymbol{J}(t)\|^2 dt,\ t\in[0,T].$$

So far we have shown that $\boldsymbol{E} \in \boldsymbol{X}_N(\Omega;\varepsilon)$ and $\mu^{-1}\boldsymbol{B} \in \boldsymbol{X}_T(\Omega;\mu)$ satisfy

$$\|\boldsymbol{E}(t)\|_{\boldsymbol{X}} + \|\mu^{-1}\boldsymbol{B}(t)\|_{\boldsymbol{X}} \le C,\ t\in[0,T]. \tag{3.1}$$

By Corollary 1, we know that $(\boldsymbol{E},\boldsymbol{B})^T \in C^3([0,T];\mathcal{H})$. If we differentiate both sides of (2.15) with respect to t and repeat the above argument, we get that $\boldsymbol{E}_t \in \boldsymbol{X}_N(\Omega;\varepsilon)$ and $\mu^{-1}\boldsymbol{B}_t$ $(=\mu^{-1}\mathbf{curl}\,\boldsymbol{E}) \in \boldsymbol{X}_T(\Omega;\mu)$ satisfy (3.1). Similarly, differentiating both sides of (2.15) twice yields that $\mu^{-1}\boldsymbol{B}_{tt}(=\mu^{-1}\mathbf{curl}\,\boldsymbol{E}_t)$ belongs to $\boldsymbol{X}_T(\Omega;\mu)$. □

The regularity of vector fields in $\boldsymbol{X}_N(\Omega;\xi)$ and $\boldsymbol{X}_T(\Omega;\xi)$ has been studied by M. Costable *et al.* [24]. They began the analysis with the decomposition of vector fields in $\boldsymbol{X}_N(\Omega;\varepsilon)$ and $\boldsymbol{X}_T(\Omega;\mu)$ as a sum of a "regular" part in $\boldsymbol{H}^1(\Omega)$ and a "singular" part in the form of a gradient, which contains, in particular, all the jumps through the interfaces.

Lemma III.1. *Any vector field* $\boldsymbol{u}\in\boldsymbol{X}_N(\Omega;\varepsilon)$ *admits a decomposition*

$$\boldsymbol{u}=\boldsymbol{w}+\nabla\phi \tag{3.2}$$

where $\boldsymbol{w}\in\boldsymbol{H}^1(\Omega)\cap\boldsymbol{X}_N(\Omega)$ *and* $\phi\in H^1_0(\Omega)$ *satisfy*

$$\|\boldsymbol{w}\|_1+\|\phi\|_1\le C\|\boldsymbol{u}\|_{\boldsymbol{X}}. \tag{3.3}$$

Similarly, any vector field $\boldsymbol{v}\in\boldsymbol{X}_T(\Omega;\mu)$ *admits a decomposition* (3.2) *where* $\boldsymbol{w}\in\boldsymbol{H}^1(\Omega)\cap\boldsymbol{X}_T(\Omega)$ *and* $\phi\in H^1(\Omega)/\mathbb{R}$ *satisfy* (3.3).

Proof. The proof is an exact rewriting of the proof of Theorem 3.4 in [24]. However, since our $\boldsymbol{X}_T(\Omega;\mu)$ is different from the one in [24], we sketch the proof here.

Let $\boldsymbol{u}$ be as in the lemma. Since its **curl** is a divergence-free field in $\boldsymbol{L}^2$ and Ω is simply connected with one boundary component, we can apply Lemma 3.1 in [24] and find $\boldsymbol{w}$ in $\boldsymbol{H}^1(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{w}=\mathbf{curl}\,\boldsymbol{u}$ and $\boldsymbol{w}\cdot\boldsymbol{n}=0$ on $\partial\Omega$. Then, $\boldsymbol{u}-\boldsymbol{w}$ is a curl-free field. Since Ω is simply connected, there exists ϕ in $H^1(\Omega)$ such that $\boldsymbol{v}-\boldsymbol{w}=\nabla\phi$. □

Based on the above lemma, M. Costable *et al.* related the regularity of vector fields in $\boldsymbol{X}_N(\Omega;\xi)$ and $\boldsymbol{X}_T(\Omega;\xi)$ to the regularity of solutions of certain Laplacian interface problems. For example, for $\boldsymbol{u} \in \boldsymbol{X}_T(\Omega;\mu)$, we have $\boldsymbol{u} = \boldsymbol{w} + \nabla\phi$ where $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$ and $\phi \in H^1(\Omega)$ satisfies, for all $\psi \in H^1(\Omega)/\mathbb{R}$,

$$\int_\Omega \mu\nabla\phi\cdot\nabla\psi\,dx = \int_\Omega \mu(\boldsymbol{u}-\boldsymbol{w})\cdot\nabla\psi\,dx \equiv (f,\psi), \tag{3.4}$$

where f belongs to the dual space of $H^1(\Omega)/\mathbb{R}$. Often the solution ϕ to (3.4) is more regular than $H^1(\Omega)$ since the right hand side f is smoother than functions in the dual space of $H^1(\Omega)/\mathbb{R}$. Indeed, for any $\alpha \in (0,1/2)$, we have

$$\begin{aligned}
|(f,\psi)| &= |\int_\Omega \mu\boldsymbol{u}\cdot\nabla\psi\,dx - \int_\Omega \mu\boldsymbol{w}\cdot\nabla\psi\,dx| \\
&= |\int_\Omega \operatorname{div}(\mu\boldsymbol{u})\psi\,dx - \sum_i \mu_i \int_{\Omega_i} \boldsymbol{w}\cdot\nabla\psi\,dx| \\
&\le C\|\psi\| + C\sum_i \|\boldsymbol{w}\|_{\alpha,\Omega_i}\|\nabla\psi\|_{-\alpha,\Omega_i} \le C\|\psi\|_{1-\alpha},
\end{aligned}$$

and thus f belongs to $H^{-1+\alpha}(\Omega)$.

M. Costable *et al.* [24] studied the regularity of the solution to (3.4). They pointed out pointed out that the regularity of vector fields in $\boldsymbol{X}_N(\Omega;\varepsilon)$ and $\boldsymbol{X}_T(\Omega;\mu)$ can be very low (near $\boldsymbol{L}^2(\Omega)$). For a detailed description, we refer to [24] and references therein. Throughout this chapter we will make the following assumption.

Assumption III.1. *$\boldsymbol{X}_N(\Omega;\varepsilon)$ and $\boldsymbol{X}_T(\Omega;\mu)$ are continuously imbedded in $\boldsymbol{H}^s(\cup\Omega_i)$ for some $s \in (0,1]$.*

When ε (or μ) is constant, we have the following imbedding result [3].

Lemma III.2. *There exists a real number $r > 1/2$ such that $\boldsymbol{X}_N(\Omega)$ and $\boldsymbol{X}_T(\Omega)$ are continuously imbedded in $\boldsymbol{H}^r(\Omega)$.*

The main result of this section is the following theorem, which follows from Theorem III.1 and Assumption III.1.

Theorem III.2. *Under Assumption III.1 and assumptions in Theorem III.1, we have that* $\boldsymbol{E}(t)$, $\mathbf{curl}\,\boldsymbol{E}(t)$, $\boldsymbol{E}_t(t)$, $\mathbf{curl}\,\boldsymbol{E}_t(t)$ *and* $\boldsymbol{B}(t)$ *belong to* $\boldsymbol{H}^s(\cup\Omega_i)$ *for all* t *in* $[0,T]$.

B. Fortin operator $\boldsymbol{\pi}_h$

The Fortin operator $\boldsymbol{\pi}_h$ which we shall now define plays an important role in the error analysis of the semidiscrete scheme to Maxwell's equations.

For any $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, $\boldsymbol{\pi}_h\boldsymbol{u} \in \boldsymbol{U}_h$ satisfies

$$(\mathbf{curl}\,\boldsymbol{\pi}_h\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{w}_h)_{\mu^{-1}} = (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{w}_h)_{\mu^{-1}}, \qquad \text{for all } \boldsymbol{w}_h \in \boldsymbol{U}_h, \tag{3.5}$$

$$(\boldsymbol{\pi}_h\boldsymbol{u}, \nabla\psi_h) = (\boldsymbol{u}, \nabla\psi_h), \qquad \text{for all } \psi_h \in S_h. \tag{3.6}$$

If μ is constant, this operator has been widely studied (see e.g. [9, 31, 47, 48, 55]).

It is shown in [32, 51] that $\boldsymbol{\pi}_h$ is well defined. This is also an application of the general results on mixed finite element methods in [57]. In the proof, one key property [32] is that if $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ satisfies $\mathbf{curl}\,\boldsymbol{u} = 0$, then

$$\boldsymbol{u} = \nabla p \quad \text{for some } p \in H_0^1(\Omega). \tag{3.7}$$

Another key inequality [3] is that if $\boldsymbol{u}_h \in \boldsymbol{U}_h$ satisfies that $(\boldsymbol{u}_h, \nabla p_h) = 0$ for all $p_h \in S_h$, then

$$\|\boldsymbol{u}_h\| \le C\|\mathbf{curl}\,\boldsymbol{u}_h\|. \tag{3.8}$$

Note that both (3.8) and the existence of p in (3.7) are only valid when $\partial\Omega$ is connected.

Remark III.1. $\boldsymbol{\pi}_h$ *is also computable. In fact,* $\boldsymbol{\pi}_h\boldsymbol{u} = \boldsymbol{u}_h$ *where* $(\boldsymbol{u}_h, p_h)$ *is the* *solution of the following problem: Find* $\boldsymbol{u}_h \in \boldsymbol{U}_h$ *and* $p_h \in S_h$ *such that*

$$\begin{aligned}(\mathbf{curl}\, \boldsymbol{\pi}_h \boldsymbol{u}, \mathbf{curl}\, \boldsymbol{w}_h)_{\mu^{-1}} + (\nabla p_h, \boldsymbol{w}_h) &= (\mathbf{curl}\, \boldsymbol{u}, \mathbf{curl}\, \boldsymbol{w}_h)_{\mu^{-1}}, \quad \textit{for all } \boldsymbol{w}_h \in \boldsymbol{U}_h,\\ (\boldsymbol{\pi}_h \boldsymbol{u}, \nabla \psi_h) &= (\boldsymbol{u}, \nabla \psi_h), \qquad\qquad \textit{for all } \psi_h \in S_h.\end{aligned} \tag{3.9}$$

This was also pointed out in the remark of Proposition 1.1 in [57].

The following lemma extends previous results to the case that μ is piecewise constant and $\boldsymbol{u}$ is of lower regularity.

Lemma III.3. *Let* s *be as in Assumption III.1. Under Assumption III.1 and III.2 below, if* $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega)$ *satisfies both* $\boldsymbol{u}$ *and* $\mathbf{curl}\, \boldsymbol{u}$ *belong to* $\boldsymbol{H}^s(\cup\Omega_i)$, *we have that*

$$\|\boldsymbol{u} - \boldsymbol{\pi}_h \boldsymbol{u}\| + \|\mathbf{curl}\,(\boldsymbol{u} - \boldsymbol{\pi}_h \boldsymbol{u})\| \le C h^s (\|\boldsymbol{u}\|_{s,\cup\Omega_i} + \|\mathbf{curl}\, \boldsymbol{u}\|_{s,\cup\Omega_i}). \tag{3.10}$$

To show the above lemma, we need some approximation results in S_h and $\boldsymbol{U}_h$, which are stated below in Lemma III.4 and Lemma III.5 respectively. For the first lemma, we need the following assumption.

Assumption III.2. *There are no points on* Γ *that belong to more than two* $\overline{\Omega}_i$*'s.*

Lemma III.4. *Under Assumption III.2, for all* $\phi \in H_0^1(\Omega) \cap H^{1+\alpha}(\cup\Omega_i)$, $0 \le \alpha \le 1/2$, *there exists* $\phi_h \in S_h$ *such that*

$$|\phi - \phi_h|_{1,\Omega} \le C h^\alpha \|\phi\|_{1+\alpha,\cup\Omega_i}. \tag{3.11}$$

Remark III.2. *Lemma III.4 appeared in [12], and the proof follows the technique in [18], which requires certain geometry regularity of the interface* Γ.

However, without Assumption III.2, we can show (3.11) *for all* $\alpha \in [0, 1/2)$. *Indeed we can take* $\phi_h = P_h \phi$, *where* P_h *is the energy projection onto* S_h *under the innerproduct* $(\nabla\cdot, \nabla\cdot)$. *Since the interpolation space between* $H_0^1(\Omega)$ *and* $H_0^1(\Omega) \bigcap H^2(\Omega)$

is $H_0^1(\Omega)\bigcap H^{1+\alpha}(\Omega)$ *[6], we have*

$$|\phi-\phi_h|_{1,\Omega}\le Ch^\alpha\|\phi\|_{1+\alpha},$$

and thus (3.11) *follows from the equivalence of* $\|\cdot\|_{1+\alpha,\cup\Omega_i}$ *and* $\|\cdot\|_{1+\alpha,\Omega}$. *This is the only place we use Assumption III.2.*

Lemma III.5. *Suppose that* $\boldsymbol{u}\in\boldsymbol{H}(\mathbf{curl};\Omega)$ *and that* $\boldsymbol{\Pi}_h$ *is the interpolation operator onto* $\boldsymbol{U}_h$. *Then, we have the following estimates.*

1. *If* $\boldsymbol{u}\in\boldsymbol{H}^\alpha(\cup\Omega_i)$ *for some* $1/2<\alpha\le 1$ *and* $\mathbf{curl}\,\boldsymbol{u}\in\overline{\boldsymbol{V}}_h$, *then*

$$\|\boldsymbol{u}-\boldsymbol{\Pi}_h\boldsymbol{u}\|\le Ch^\alpha(|\boldsymbol{u}|_{\alpha,\cup\Omega_i}+\|\mathbf{curl}\,\boldsymbol{u}\|).$$

2. *If* $\boldsymbol{u}\in\boldsymbol{H}^1(\cup\Omega_i)$ *and* $\mathbf{curl}\,\boldsymbol{u}\in\boldsymbol{H}^\alpha(\cup\Omega_i)$ *for some* $\alpha>0$, *then*

$$\|\boldsymbol{u}-\boldsymbol{\Pi}_h\boldsymbol{u}\|+h\|\mathbf{curl}\,(\boldsymbol{u}-\boldsymbol{\Pi}_h\boldsymbol{u})\|\le Ch(|\boldsymbol{u}|_{1,\cup\Omega_i}+|\mathbf{curl}\,\boldsymbol{u}|_{\alpha,\cup\Omega_i}).$$

3. *If both* $\boldsymbol{u}$ *and* $\mathbf{curl}\,\boldsymbol{u}$ *belong to* $\boldsymbol{H}^\alpha(\cup\Omega_i)$ *for some* $\alpha\in(1/2,1]$, *then*

$$\|\boldsymbol{u}-\boldsymbol{\Pi}_h\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}\le Ch^\alpha(\|\boldsymbol{u}\|_\alpha+\|\mathbf{curl}\,\boldsymbol{u}\|_\alpha).$$

4. *If* $\boldsymbol{u}\in\boldsymbol{H}_0(\mathbf{curl};\Omega)$ *satisfies* $\mathbf{curl}\,\boldsymbol{u}\in\boldsymbol{H}^\alpha(\cup\Omega_i)$ *for some* $\alpha>0$, *then*

$$\inf_{\boldsymbol{u}_h\in\boldsymbol{U}_h}\|\mathbf{curl}\,\boldsymbol{u}-\mathbf{curl}\,\boldsymbol{u}_h\|\le Ch^\alpha\|\mathbf{curl}\,\boldsymbol{u}\|_{\alpha,\cup\Omega_i}.$$

Proof. Inequality (3) is given in Proposition 5.6 of [2] (see also [55]), and the inequality (4) is given in Theorem 4.8 of [3].

Inequality (1) is an extension of (2.4) in [4], and the proof follows same lines there. For completeness, we also give a proof here. First by Lemma II.1 and the Sobolev imbedding theorem, $\boldsymbol{\Pi}_h$ is well defined for vector fields in $\boldsymbol{H}^\alpha(\Omega)$, $\alpha>1/2$, whose **curl** in $\overline{\boldsymbol{V}}_h$. Secondly, on the reference tetrahedron $\widehat{\tau}$ of unit size, we have

$$\|\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{\Pi}}_{\widehat{\tau}}\widehat{\boldsymbol{u}}\| \le C(\|\widehat{\boldsymbol{u}}\|_\alpha + \|\mathbf{curl}\,\widehat{\boldsymbol{u}}\|_{\boldsymbol{L}^\infty}) \le C(\|\widehat{\boldsymbol{u}}\|_\alpha + \|\mathbf{curl}\,\widehat{\boldsymbol{u}}\|),$$

by the equivalence of all norms in $\overline{\boldsymbol{V}}(\widehat{\tau})$. Since $\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{\Pi}}_{\widehat{\tau}}\widehat{\boldsymbol{u}}$ vanishes for constant $\widehat{\boldsymbol{u}}$, a Bramble-Hilbert argument yields

$$\|\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{\Pi}}_{\widehat{\tau}}\widehat{\boldsymbol{u}}\| \le C(|\widehat{\boldsymbol{u}}|_\alpha + \|\mathbf{curl}\,\widehat{\boldsymbol{u}}\|),$$

Finally, if we scale this estimate to a general tetrahedron using Lemmas 5.2 and 5.5 of [2] and sum over all the tetrahedra in $\mathcal{T}_h$, we get

$$\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{\Pi}_h\boldsymbol{u}\|^2 &\le C\sum_\tau h_\tau \|\widehat{\boldsymbol{u}} - \boldsymbol{\Pi}_{\widehat{\tau}}\widehat{\boldsymbol{u}}\|^2_{0,\widehat{\tau}} \le C\sum_\tau h_\tau(|\widehat{\boldsymbol{u}}|^2_{\alpha,\widehat{\tau}} + |\mathbf{curl}\,\widehat{\boldsymbol{u}}|^2_{0,\widehat{\tau}}) \\
&\le C\sum_\tau h_\tau^{1+\alpha}|\boldsymbol{u}|^2_{\alpha,\tau} + h_\tau^2\|\mathbf{curl}\,\boldsymbol{u}\|^2_{0,\tau} \\
&\le Ch^{2\alpha}(|\boldsymbol{u}|^2_{\alpha,\tau} + \|\mathbf{curl}\,\boldsymbol{u}\|^2).
\end{aligned}$$

The proof of (2) is very similar. Again by Lemma II.1 and the Sobolev imbedding theorem, $\hat{\boldsymbol{\Pi}}_{\widehat{\tau}}$ is well defined for $\boldsymbol{H}^1$ vector fields whose **curl** belong to $\boldsymbol{H}^\alpha$. On the reference tetrahedron $\widehat{\tau}$, we have $\|\widehat{\boldsymbol{u}} - \hat{\boldsymbol{\Pi}}_{\widehat{\tau}}\widehat{\boldsymbol{u}}\|_{0,\widehat{\tau}} \le C(\|\widehat{\boldsymbol{u}}\|_{1,\widehat{\tau}} + \|\mathbf{curl}\,\widehat{\boldsymbol{u}}\|_{\alpha,\widehat{\tau}})$. A Bramble-Hilbert argument gives that

$$\|\widehat{\boldsymbol{u}} - \boldsymbol{\Pi}_{\widehat{\tau}}\widehat{\boldsymbol{u}}\|_{0,\widehat{\tau}} \le C(|\widehat{\boldsymbol{u}}|_{1,\widehat{\tau}} + |\mathbf{curl}\,\widehat{\boldsymbol{u}}|_{\alpha,\widehat{\tau}}).$$

Similarly, we have

$$\begin{aligned}
\|\mathbf{curl}\,(\widehat{\boldsymbol{u}} - \boldsymbol{\Pi}_{\widehat{\tau}}\widehat{\boldsymbol{u}})\|_{0,\widehat{\tau}} &\le \|\mathbf{curl}\,\widehat{\boldsymbol{u}}\|_{0,\widehat{\tau}} + \|\mathbf{curl}\,\boldsymbol{\Pi}_{\widehat{\tau}}\widehat{\boldsymbol{u}}\|_{0,\widehat{\tau}} \le \|\widehat{\boldsymbol{u}}\|_{1,\widehat{\tau}} + C\|\boldsymbol{\Pi}_{\widehat{\tau}}\widehat{\boldsymbol{u}}\|_{0,\widehat{\tau}} \\
&\le C(\|\widehat{\boldsymbol{u}}\|_{1,\widehat{\tau}} + \|\mathbf{curl}\,\widehat{u}\|_{\alpha,\widehat{\tau}}) \le C(|\widehat{\boldsymbol{u}}|_{1,\widehat{\tau}} + |\mathbf{curl}\,\widehat{u}|_{\alpha,\widehat{\tau}}).
\end{aligned}$$

A scaling argument using Lemma 5.2 and 5.5 of [2] gives that

$$\begin{aligned}
&\|\boldsymbol{u}-\boldsymbol{\Pi}_h\boldsymbol{u}\|^2+h^2\|\mathbf{curl}\,(\boldsymbol{u}-\boldsymbol{\Pi}_h\boldsymbol{u})\|^2\\
&\qquad\le C\sum_\tau h_\tau(\|\widehat{\boldsymbol{u}}-\boldsymbol{\Pi}_{\widehat{\tau}}\widehat{\boldsymbol{u}}\|_{0,\widehat{\tau}}^2+\|\mathbf{curl}\,(\widehat{\boldsymbol{u}}-\boldsymbol{\Pi}_{\widehat{\tau}}\widehat{\boldsymbol{u}})\|_{0,\widehat{\tau}}^2)\\
&\qquad\le C\sum_\tau h_\tau(|\widehat{\boldsymbol{u}}|_{1,\widehat{\tau}}^2+|\mathbf{curl}\,\widehat{\boldsymbol{u}}|_{\alpha,\widehat{\tau}}^2)\\
&\qquad\le C\sum_\tau h_\tau(h_\tau|\boldsymbol{u}|_{1,\widehat{\tau}}^2+h_\tau^{1+2\alpha}|\mathbf{curl}\,\widehat{\boldsymbol{u}}|_{\alpha,\widehat{\tau}}^2)\\
&\qquad\le Ch^2(|\boldsymbol{u}|_{1,\cup\Omega_i}^2+|\mathbf{curl}\,\boldsymbol{u}|_{\alpha,\cup\Omega_i}^2).
\end{aligned}$$

□

Now we can give the proof of Lemma III.3 concerning error estimates of $\boldsymbol{\pi}_h$.

Proof of Lemma III.3. Let $\boldsymbol{u}$ be as in Lemma III.3. From the first equation of (3.5) and the third estimate of Lemma III.5, we have that

$$\|\mathbf{curl}\,(\boldsymbol{u}-\boldsymbol{\pi}_h\boldsymbol{u})\|_{\mu^{-1}}=\inf_{\boldsymbol{u}_h\in\boldsymbol{U}_h}\|\mathbf{curl}\,(\boldsymbol{u}-\boldsymbol{u}_h)\|_{\mu^{-1}}\le Ch^s\|\mathbf{curl}\,\boldsymbol{u}\|_{s,\cup\Omega_i}. \tag{3.12}$$

In the following, we will bound $\|\boldsymbol{u}-\boldsymbol{\pi}_h\boldsymbol{u}\|$. Let $\boldsymbol{u}=\boldsymbol{v}+\nabla\psi$ be the Helmholtz decomposition of $\boldsymbol{u}$ where $\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl};\Omega)$ and $\psi\in H_0^1(\Omega)$ satisfy $\operatorname{div}\boldsymbol{v}=0$ and $\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}+\|\psi\|_1\le C\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$. Since $\operatorname{div}\boldsymbol{v}=0$, the second equation of (3.5) implies that $(\boldsymbol{\pi}_h\boldsymbol{v},\nabla\psi_h)=0$ for all $\psi_h\in S_h$. Using (3.8) on $\boldsymbol{\pi}_h\boldsymbol{v}$, we have $\|\boldsymbol{\pi}_h\boldsymbol{v}\|\le C\|\mathbf{curl}\,\boldsymbol{\pi}_h\boldsymbol{v}\|$ and thus

$$\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{\pi}_h\boldsymbol{u}\|&\le\|\boldsymbol{v}-\boldsymbol{\pi}_h\boldsymbol{v}\|+\|\nabla\psi-\boldsymbol{\pi}_h\nabla\psi\|\\
&\le\|\boldsymbol{v}\|+C\|\mathbf{curl}\,\boldsymbol{\pi}_h\boldsymbol{v}\|+\|\nabla\psi\|\\
&\le\|\boldsymbol{v}\|+C\|\mathbf{curl}\,\boldsymbol{v}\|+\|\boldsymbol{u}\|\le C\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)},
\end{aligned}\tag{3.13}$$

where we have used the first inequality of (3.12) on $\boldsymbol{v}$.

When $s > 1/2$, $\|\boldsymbol{u} - \boldsymbol{\pi}_h \boldsymbol{u}\|$ can be bounded as follows. Using the stability (3.13) of $\boldsymbol{\pi}_h$ on $\boldsymbol{u} - \boldsymbol{\Pi}_h \boldsymbol{u}$ and the third inequality in Lemma III.5, we have that

$$\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{\pi}_h \boldsymbol{u}\| &\leq \|\boldsymbol{u} - \boldsymbol{\Pi}_h \boldsymbol{u}\| + \|(\boldsymbol{\Pi}_h - \boldsymbol{\pi}_h)\boldsymbol{u}\| \\
&= \|\boldsymbol{u} - \boldsymbol{\Pi}_h \boldsymbol{u}\| + \|\boldsymbol{\pi}_h(\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u})\| \\
&\leq C\|\boldsymbol{u} - \boldsymbol{\Pi}_h \boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \leq Ch^s(\|\boldsymbol{u}\|_{s,\cup\Omega_i} + \|\mathbf{curl}\,\boldsymbol{u}\|_{s,\cup\Omega_i}).
\end{aligned}$$

The main difficulty comes from the case $s \leq 1/2$. Since $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ has zero divergence, by Lemma III.1, we can decompose $\boldsymbol{v} = \boldsymbol{z} + \nabla\phi$ where $\boldsymbol{z} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}_0(\mathbf{curl};\Omega)$ and $\phi \in H_0^1(\Omega)$ satisfy $\|\boldsymbol{z}\|_1 + \|\phi\|_1 \leq C\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$. Therefore, letting $p = \phi + \psi \in H_0^1(\Omega)$, we get a decomposition $\boldsymbol{u} = \boldsymbol{z} + \nabla p$ which satisfies

$$\|\boldsymbol{z}\|_1 + \|p\|_1 \leq C\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}. \tag{3.14}$$

Since $\boldsymbol{u} \in \boldsymbol{H}^s(\cup\Omega_i)$ and $\boldsymbol{z} \in \boldsymbol{H}^1(\Omega)$, ∇p belongs to $\boldsymbol{H}^s(\cup\Omega_i)$. By the second equation of (3.5) and Lemma III.4, we have

$$\begin{aligned}
\|\nabla p - \boldsymbol{\pi}_h \nabla p\| &\leq \inf_{p_h \in S_h} \|\nabla p - \nabla p_h\| \leq Ch^s\|p\|_{1+s,\cup\Omega_i} \\
&\leq Ch^s(\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{u}\|_{s,\cup\Omega_i}).
\end{aligned} \tag{3.15}$$

Once we have shown that

$$\|\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}\| \leq Ch^s(\|\boldsymbol{u}\| + \|\mathbf{curl}\,\boldsymbol{u}\|_{s,\cup\Omega_i}), \tag{3.16}$$

the desired estimate for $\|\boldsymbol{u} - \pi_h \boldsymbol{u}\|$ will follow from (3.15), (3.16) and the triangle inequality.

To show (3.16), by Lemma III.5, we first note that $\boldsymbol{\Pi}_h \boldsymbol{z}$ is well defined and satisfies

$$\|\boldsymbol{z} - \boldsymbol{\Pi}_h \boldsymbol{z}\| \leq Ch(\|\boldsymbol{z}\|_1 + \|\mathbf{curl}\,\boldsymbol{z}\|_{s,\cup\Omega_i}) \leq Ch(\|\boldsymbol{u}\| + \|\mathbf{curl}\,\boldsymbol{u}\|_{s,\cup\Omega_i}). \tag{3.17}$$

Then we decompose $\mathbf{\Pi}_h \boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z} = \boldsymbol{w} + \nabla q$ where $q \in H_0^1(\Omega)$ and $\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ satisfies $\operatorname{div} \boldsymbol{w} = 0$. By Lemma III.2, $\boldsymbol{w}$ belongs to $\boldsymbol{H}^r(\Omega)$ and satisfies

$$\begin{aligned}\|\boldsymbol{w}\|_r &\le C\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \le C\|\mathbf{\Pi}_h \boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \\ &= C\|\boldsymbol{\pi}_h(\mathbf{\Pi}_h \boldsymbol{z} - \boldsymbol{z})\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \le C\|\boldsymbol{z} - \mathbf{\Pi}_h \boldsymbol{z}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \\ &\le C(\|\boldsymbol{u}\| + \|\mathbf{curl}\, \boldsymbol{u}\|_{s,\cup\Omega_i}), \qquad (3.18)\end{aligned}$$

where we have used (3.12) and (3.13) on $\boldsymbol{z}$ and the second estimate of Lemma III.5. Since $\mathbf{curl}\, \boldsymbol{w}$ belongs to $\boldsymbol{V}_h$, by Lemma III.5, $\mathbf{\Pi}_h \boldsymbol{w}$ is well defined and satisfies

$$\|\boldsymbol{w} - \mathbf{\Pi}_h \boldsymbol{w}\| \le Ch^r(\|\boldsymbol{w}\|_r + \|\mathbf{curl}\, \boldsymbol{w}\|) \le Ch^r(\|\boldsymbol{u}\| + \|\mathbf{curl}\, \boldsymbol{u}\|_{s,\cup\Omega_i}). \qquad (3.19)$$

Note that

$$\mathbf{\Pi}_h \boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z} = \mathbf{\Pi}_h \boldsymbol{w} + \mathbf{\Pi}_h \nabla q = \mathbf{\Pi}_h \boldsymbol{w} + \nabla q_h$$

for some $q_h \in S_h$. Therefore, we have, by the second equation of (3.5),

$$\begin{aligned}\|\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}\|^2 &= (\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \boldsymbol{z} - \mathbf{\Pi}_h \boldsymbol{z}) + (\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \mathbf{\Pi}_h \boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}) \\ &= (\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \boldsymbol{z} - \mathbf{\Pi}_h \boldsymbol{z}) + (\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \mathbf{\Pi}_h \boldsymbol{w} + \nabla q_h) \\ &= (\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \boldsymbol{z} - \mathbf{\Pi}_h \boldsymbol{z}) - (\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \boldsymbol{w} - \mathbf{\Pi}_h \boldsymbol{w}) + (\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \boldsymbol{w}) \\ &\le \|\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}\|(\|\boldsymbol{z} - \mathbf{\Pi}_h \boldsymbol{z}\| + \|\boldsymbol{w} - \mathbf{\Pi}_h \boldsymbol{w}\|) + (\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \boldsymbol{w}). \qquad (3.20)\end{aligned}$$

To estimate the term $(\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \boldsymbol{w})$, we define $\boldsymbol{t} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ satisfying

$$\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\, \boldsymbol{t}) = \boldsymbol{w} \qquad \text{and} \qquad \operatorname{div} \boldsymbol{t} = 0 \text{ in } \Omega. \qquad (3.21)$$

Thanks to $\operatorname{div} \boldsymbol{w} = 0$, $\boldsymbol{t}$ is well defined. Since $\mathbf{curl}\, \boldsymbol{t} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, $\mu^{-1}\mathbf{curl}\, \boldsymbol{t}$ actually belongs to $\boldsymbol{X}_T(\Omega;\mu)$. Thus, by Assumption III.1, we have

$$\|\mathbf{curl}\, \boldsymbol{t}\|_{s,\cup\Omega_i} \le C\|\boldsymbol{w}\|. \qquad (3.22)$$

Using (3.21) and the first equation of (3.5), we have

$$(\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \boldsymbol{w}) = (\mathbf{curl}\,(\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}), \mathbf{curl}\,\boldsymbol{t})_{\mu^{-1}}$$
$$= (\mathbf{curl}\,(\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}), \mathbf{curl}\,(\boldsymbol{t} - \boldsymbol{\pi}_h \boldsymbol{t}))_{\mu^{-1}}.$$

Since $\mathbf{curl}\,\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{u}$ belongs to $\boldsymbol{H}^s(\cup\Omega_i)$, by (3.12) and (3.22), we conclude that

$$(\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \boldsymbol{w}) \le Ch^{2s}(\|\mathbf{curl}\,\boldsymbol{z}\|_{s,\cup\Omega_i} + \|\mathbf{curl}\,\boldsymbol{t}\|_{s,\cup\Omega_i})^2$$
$$\le Ch^{2s}(\|\boldsymbol{u}\| + \|\mathbf{curl}\,\boldsymbol{u}\|_{s,\cup\Omega_i})^2. \tag{3.23}$$

Finally, the combination of (3.17), (3.19), (3.20), (3.22) and (3.23) gives that

$$\|\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}\|^2 \le Ch^s \|\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}\|(\|\boldsymbol{u}\| + \|\mathbf{curl}\,\boldsymbol{u}\|_{s,\cup\Omega_i})$$
$$+ Ch^{2s}(\|\boldsymbol{u}\| + \|\mathbf{curl}\,\boldsymbol{u}\|_{s,\cup\Omega_i})^2,$$

from which (3.16) follows. □

C. Error analysis for the semidiscrete scheme (2.21)

We recall some equations in Section D of Chapter II. From the discussion given there, the system (2.13) is equivalent to

$$\begin{aligned} (\varepsilon \boldsymbol{E}_t, \boldsymbol{u}) - (\mu^{-1}\boldsymbol{B}, \mathbf{curl}\,\boldsymbol{u}) &= -(\boldsymbol{J}, \boldsymbol{u}), && \text{for all } \boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega), \\ (\mu^{-1}\boldsymbol{B}_t, \boldsymbol{v}) + (\mathbf{curl}\,\boldsymbol{E}, \mu^{-1}\boldsymbol{v}) &= 0, && \text{for all } \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div};\Omega). \end{aligned} \tag{3.24}$$

The semidiscrete approximation $(\boldsymbol{E}_h(t), \boldsymbol{B}_h(t))$ in $\boldsymbol{U}_h \times \boldsymbol{V}_h$ was defined, for $0 < t \le T$, by

$$(\varepsilon \boldsymbol{E}_{h,t}, \boldsymbol{u}_h) - (\mu^{-1}\boldsymbol{B}_h, \mathbf{curl}\,\boldsymbol{u}_h) = -(\boldsymbol{J}, \boldsymbol{u}_h), \qquad \text{for all } \boldsymbol{u}_h \in \boldsymbol{U}_h, \tag{3.25}$$

$$(\mu^{-1}\boldsymbol{B}_{h,t}, \boldsymbol{v}_h) + (\mathbf{curl}\,\boldsymbol{E}_h, \mu^{-1}\boldsymbol{v}_h) = 0, \qquad \text{for all } \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.26}$$

with $\boldsymbol{E}_h(0) \approx \boldsymbol{E}(0)$ and $\boldsymbol{B}_h(0) \approx \boldsymbol{B}(0)$. This scheme is uniquely solvable [45].

Some possible choices of $\boldsymbol{E}_h(0)$ and $\boldsymbol{B}_h(0)$ are as follows. We can define $\boldsymbol{B}_h(0) = \boldsymbol{r}_h\boldsymbol{B}(0)$ since $\boldsymbol{r}_h\boldsymbol{B}(0)$ is well defined according to (2.2). We note that $\boldsymbol{B}_h(t)$ is divergence free for all t in this case by the second equation of (3.25). We can define $\boldsymbol{E}_h(0) = \boldsymbol{\pi}_h\boldsymbol{E}(0)$ by solving (3.9). However, if $\boldsymbol{E}(0)$ is smooth enough, we can approximate $\boldsymbol{E}(0)$ by $\boldsymbol{\Pi}_h\boldsymbol{E}(0)$ and avoid the solution of (3.9).

In the following theorem, we give the $\boldsymbol{L}^2$-error estimate for the semidiscrete scheme (3.25).

Theorem III.3. *Let* $(\boldsymbol{E}, \boldsymbol{B})$ *be the solution to* (2.13) *and* (2.11) *and* $(\boldsymbol{E}_h, \boldsymbol{B}_h)$ *be the solution to* (2.21) *and* (2.22). *Let* s *be as in Assumption III.1. Under Assumptions III.1 and III.2 and the same assumptions as for Theorem III.2, we have that*

$$\begin{aligned}&\|\boldsymbol{E}(t) - \boldsymbol{E}_h(t)\| + \|\boldsymbol{B}(t) - \boldsymbol{B}_h(t)\| \\ &\qquad\qquad \le C(h^s + \|\boldsymbol{E}_0 - \boldsymbol{E}_h(0)\| + \|\boldsymbol{B}_0 - \boldsymbol{B}_h(0)\|), \quad \textit{for all } t \in [0, T].\end{aligned}$$

Proof. Let $\mathbf{Q}_h^{\boldsymbol{Z}}$ be the $\boldsymbol{L}^2_{\mu^{-1}}$-projection from $\boldsymbol{H}_0(\mathrm{div};\Omega)$ to $\mathbf{curl}\,\boldsymbol{U}_h$. Since $\mathrm{div}\,\boldsymbol{B} = 0$, by the commuting diagram II.1, we know that $\mathrm{div}\,\boldsymbol{r}_h\boldsymbol{B} = 0$ and thus $\boldsymbol{r}_h\boldsymbol{B} \in \mathbf{curl}\,\boldsymbol{U}_h$. Therefore, by (2.2) and the regularity of the solution given in Theorem III.2, we have that

$$\|\mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B} - \boldsymbol{B}\| \le \|\boldsymbol{r}_h\boldsymbol{B} - \boldsymbol{B}\| \le Ch^s. \tag{3.27}$$

We follow the strategy in [45] and split $\boldsymbol{B}_h(t) = \boldsymbol{B}_h^+(t) + \boldsymbol{B}_h^\perp(t)$ where $\boldsymbol{B}_h^+(t)$ belongs to $\mathbf{curl}\,\boldsymbol{U}_h$ and $\boldsymbol{B}_h^\perp(t)$ is in the $\boldsymbol{L}^2_{\mu^{-1}}$-orthogonal complement of $\mathbf{curl}\,\boldsymbol{U}_h$ in $\boldsymbol{V}_h$.

Due to (2.21), $(\boldsymbol{E}_h, \boldsymbol{B}_h^+) \in \boldsymbol{U}_h \times \mathbf{curl}\,\boldsymbol{U}_h$ satisfy

$$\begin{aligned}(\varepsilon\boldsymbol{E}_{h,t}, \boldsymbol{u}_h) - (\mu^{-1}\boldsymbol{B}_h^+, \mathbf{curl}\,\boldsymbol{u}_h) &= -(\boldsymbol{J}, \boldsymbol{u}_h), && \text{for all } \boldsymbol{u}_h \in \boldsymbol{U}_h,\\ (\mu^{-1}\boldsymbol{B}_{h,t}^+, \boldsymbol{z}_h) + (\mathbf{curl}\,\boldsymbol{E}_h, \mu^{-1}\boldsymbol{z}_h) &= 0, && \text{for all } \boldsymbol{z}_h \in \mathbf{curl}\,\boldsymbol{U}_h.\end{aligned} \tag{3.28}$$

In the same way, we can see that $\boldsymbol{B}_{h,t}^{\perp}(t) = 0$ for any t.

From (2.20) and definitions of $\mathbf{Q}_h^{\boldsymbol{Z}}$ and $\boldsymbol{\pi}_h$ it follows that, for any $\boldsymbol{u}_h \in \boldsymbol{U}_h$,

$$
\begin{aligned}
(\mu^{-1}\mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}_t, \mathbf{curl}\,\boldsymbol{u}_h) &= (\mu^{-1}\boldsymbol{B}_t, \mathbf{curl}\,\boldsymbol{u}_h) \\
&= -(\mathbf{curl}\,\boldsymbol{E}, \mu^{-1}\mathbf{curl}\,\boldsymbol{u}_h) = -(\mathbf{curl}\,\boldsymbol{\pi}_h\boldsymbol{E}, \mu^{-1}\mathbf{curl}\,\boldsymbol{u}_h).
\end{aligned}
$$

Note that both $\mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}_t$ and $\mathbf{curl}\,\boldsymbol{\pi}_h\boldsymbol{E}$ belong to $\mathbf{curl}\,\boldsymbol{U}_h$. This implies that

$$
\mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}_t + \mathbf{curl}\,\boldsymbol{\pi}_h\boldsymbol{E} = 0.
$$

By the second equation of (3.28),

$$
(\mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}_t - \boldsymbol{B}_{h,t}^{+}, \mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B} - \boldsymbol{B}_h^{+})_{\mu^{-1}} = -(\mathbf{curl}\,(\boldsymbol{\pi}_h\boldsymbol{E} - \boldsymbol{E}_h), \mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B} - \boldsymbol{B}_h^{+})_{\mu^{-1}}. \tag{3.29}
$$

Moreover, by the definition of $\mathbf{Q}_h^{\boldsymbol{Z}}$ and the first equations of (3.24) and (3.28), we have

$$
\begin{aligned}
(\boldsymbol{\pi}_h\boldsymbol{E}_t - \boldsymbol{E}_{h,t}, \boldsymbol{u}_h)_{\varepsilon} &+ (\boldsymbol{B}_h^{+} - \mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}, \mathbf{curl}\,\boldsymbol{u}_h)_{\mu^{-1}} \\
&= (\boldsymbol{\pi}_h\boldsymbol{E}_t - \boldsymbol{E}_{h,t}, \boldsymbol{u}_h)_{\varepsilon} + (\boldsymbol{B}_h^{+} - \boldsymbol{B}, \mathbf{curl}\,\boldsymbol{u}_h)_{\mu^{-1}} \\
&= (\boldsymbol{\pi}_h\boldsymbol{E}_t - \boldsymbol{E}_{h,t}, \boldsymbol{u}_h)_{\varepsilon} - (\boldsymbol{E}_t - \boldsymbol{E}_{h,t}, \boldsymbol{u}_h)_{\varepsilon} \\
&= (\boldsymbol{\pi}_h\boldsymbol{E}_t - \boldsymbol{E}_t, \boldsymbol{u}_h)_{\varepsilon},
\end{aligned}
$$

for any $\boldsymbol{u}_h \in \boldsymbol{U}_h$. In particular, choosing $\boldsymbol{u}_h = \boldsymbol{\pi}_h\boldsymbol{E} - \boldsymbol{E}_h$, we get that

$$
\begin{aligned}
(\boldsymbol{\pi}_h\boldsymbol{E}_t - \boldsymbol{E}_{h,t}, \boldsymbol{\pi}_h\boldsymbol{E} - \boldsymbol{E}_h)_{\varepsilon} &+ (\boldsymbol{B}_h^{+} - \mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}, \mathbf{curl}\,(\boldsymbol{\pi}_h\boldsymbol{E} - \boldsymbol{E}_h))_{\mu^{-1}} \\
&= (\boldsymbol{\pi}_h\boldsymbol{E}_t - \boldsymbol{E}_t, \boldsymbol{\pi}_h\boldsymbol{E} - \boldsymbol{E}_h)_{\varepsilon}.
\end{aligned} \tag{3.30}
$$

Adding (3.29) and (3.30) together yields

$$\frac{d}{dt}\Big[(\boldsymbol{\pi}_h\boldsymbol{E}-\boldsymbol{E}_h,\boldsymbol{\pi}_h\boldsymbol{E}-\boldsymbol{E}_h)_\varepsilon+(\mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}-\boldsymbol{B}_h^+,\mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}-\boldsymbol{B}_h^+)_{\mu^{-1}}\Big]$$
$$=2(\boldsymbol{\pi}_h\boldsymbol{E}_t-\boldsymbol{E}_t,\boldsymbol{\pi}_h\boldsymbol{E}-\boldsymbol{E}_h)_\varepsilon.$$

Using $\boldsymbol{B}_{h,t}^\perp=0$ and the orthogonal property between $\boldsymbol{B}_h^\perp$ and $\mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}-\boldsymbol{B}_h^+$, we have that

$$\frac{d}{dt}(\mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}-\boldsymbol{B}_h^+,\mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}-\boldsymbol{B}_h^+)_{\mu^{-1}}=\frac{d}{dt}(\mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}-\boldsymbol{B}_h,\mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}-\boldsymbol{B}_h)_{\mu^{-1}},$$

and thus

$$\frac{d}{dt}\Big[(\boldsymbol{\pi}_h\boldsymbol{E}-\boldsymbol{E}_h,\boldsymbol{\pi}_h\boldsymbol{E}-\boldsymbol{E}_h)_\varepsilon+(\mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}-\boldsymbol{B}_h,\mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}-\boldsymbol{B}_h)_{\mu^{-1}}\Big]$$
$$=2(\boldsymbol{\pi}_h\boldsymbol{E}_t-\boldsymbol{E}_t,\boldsymbol{\pi}_h\boldsymbol{E}-\boldsymbol{E}_h)_\varepsilon. \tag{3.31}$$

Integrating both sides of (3.31) over $[0,t]$ yields

$$\begin{aligned}(\boldsymbol{\pi}_h\boldsymbol{E}-\boldsymbol{E}_h,\boldsymbol{\pi}_h\boldsymbol{E}-\boldsymbol{E}_h)_\varepsilon&+(\mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}-\boldsymbol{B}_h,\mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}-\boldsymbol{B}_h)_{\mu^{-1}}\\
&=(\boldsymbol{\pi}_h\boldsymbol{E}_0-\boldsymbol{E}_h(0),\boldsymbol{\pi}_h\boldsymbol{E}_0-\boldsymbol{E}_h(0))_\varepsilon\\
&+(\mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}_0-\boldsymbol{B}_h(0),\mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}_0-\boldsymbol{B}_h(0))_{\mu^{-1}}\\
&+2\int_0^t(\boldsymbol{\pi}_h\boldsymbol{E}_t-\boldsymbol{E}_t,\boldsymbol{\pi}_h\boldsymbol{E}-\boldsymbol{E}_h)_\varepsilon\,d\tau\\
&\le C(h^{2s}+\|\boldsymbol{E}_0-\boldsymbol{E}_h(0)\|^2+\|\boldsymbol{B}_0-\boldsymbol{B}_h(0)\|^2)\\
&+Ch^s\int_0^t\|\boldsymbol{\pi}_h\boldsymbol{E}-\boldsymbol{E}_h\|\,d\tau.\end{aligned}$$

For the last inequality, we have used (3.10), (3.27) and the triangle inequality. It follows from Gronwall's inequality that

$$\|\boldsymbol{\pi}_h\boldsymbol{E}-\boldsymbol{E}_h\|+\|\mathbf{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}-\boldsymbol{B}_h\|\le C(h^s+\|\boldsymbol{E}_0-\boldsymbol{E}_h(0)\|+\|\boldsymbol{B}_0-\boldsymbol{B}_h(0)\|). \tag{3.32}$$

The desired estimate then follows from (3.32), (3.27) and the triangle inequality. □

Remark III.3. *If the initial approximation* $\boldsymbol{B}_h(0)$ *is divergence free, the splitting of* $\boldsymbol{B}_h(t)$ *into* $\boldsymbol{B}_h^+(t) + \boldsymbol{B}_h^\perp(t)$ *in the proof is not necessary because of* $\boldsymbol{B}_h^\perp(t) \equiv 0$. *But the above theorem shows that the initial approximation* $\boldsymbol{B}_h(0)$ *does not need to be divergence free for the semidiscrete scheme* (2.21) *to result in good approximations of* $\boldsymbol{E}$ *and* $\boldsymbol{B}$.

CHAPTER IV

OVERLAPPING SCHWARZ METHODS

In this chapter, we will analyze overlapping Schwarz methods for the problem (2.24) in Chapter II. We only consider the case when ε and μ are constants. To be more specific, we will study overlapping Schwarz preconditioners for the problem: Find $\boldsymbol{u}_h \in \boldsymbol{U}_h$ such that

$$A(\boldsymbol{u}_h, \boldsymbol{v}_h) \equiv \alpha(\boldsymbol{u}_h, \boldsymbol{v}_h) + (\mathbf{curl}\, \boldsymbol{u}_h, \mathbf{curl}\, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \quad \text{for all } \boldsymbol{v} \in \boldsymbol{U}_h, \tag{4.1}$$

where α is a positive number. The above equation can be written as

$$\mathbf{A}_h^{\boldsymbol{U}} \boldsymbol{u}_h = \boldsymbol{f}_h \equiv \mathbf{Q}_h^{\boldsymbol{U}} \boldsymbol{f}, \tag{4.2}$$

where $\mathbf{Q}_h^{\boldsymbol{U}}$ is the $\boldsymbol{L}^2$-projection onto $\boldsymbol{U}_h$, and $\mathbf{A}_h^{\boldsymbol{U}} : \boldsymbol{U} \to \boldsymbol{U}_h$ is defined by $(\mathbf{A}_h^{\boldsymbol{U}} \boldsymbol{u}, \boldsymbol{v}_h) = \mathbf{A}(\boldsymbol{u}, \boldsymbol{v}_h)$, for all $\boldsymbol{v}_h \in \boldsymbol{U}_h$. When ε and μ are not constants, the discrete system corresponding to (2.24) can be preconditioned by preconditioners for (4.2) provided jumps of ε or μ cross interfaces are not too large.

We begin our analysis with introducing a regular decomposition of vector fields in $\boldsymbol{H}_0(\mathbf{curl}; \Omega)$ in Section A. After giving the construction of both additive and multiplicative preconditioners in Section B, we provide a stable decomposition in Section C which is critical in the estimate of condition number of the preconditioned system. Our results hold uniformly for $0 < \alpha < \infty$ under standard conditions on the overlapping subdomains. In Section D, we give the results of numerical experiments to illustrate the theory.

A. Decompositions of $\boldsymbol{H}_0(\mathbf{curl};\Omega)$

Due to the different behavior of $A(\cdot,\cdot)$ on solenoidal and irrotational vector fields, the Helmholtz decomposition is an important tool in the analysis. For any $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, we have the continuous Helmholtz decomposition

$$\boldsymbol{u} = \boldsymbol{z} + \nabla\varphi, \tag{4.3}$$

where $\boldsymbol{z} \in \boldsymbol{H}_0(\mathbf{curl};\Omega), \operatorname{div}\boldsymbol{z} = 0$ and $\varphi \in H_0^1(\Omega)$. Unfortunately, the vector field $\boldsymbol{z}$ in (4.3) does not, in general, belong to $\boldsymbol{H}^1(\Omega)$ when the domain Ω is not convex. Our analysis is based on a decomposition of $\boldsymbol{z}$. The following two lemmas provide the construction and estimates.

Lemma IV.1. *For any $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, there exists $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$ such that*

$$\mathbf{curl}\,\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{u} \quad \textit{and} \quad \operatorname{div}\boldsymbol{w} = 0 \textit{ in } \Omega,$$

and the following estimates hold:

$$\|\boldsymbol{w}\| \leq \|\boldsymbol{u}\| \quad \textit{and} \quad |\boldsymbol{w}|_1 \leq \sqrt{2}\|\mathbf{curl}\,\boldsymbol{u}\|.$$

Proof. The proof follows the argument of Theorem 3.4, chapter I in [32]. Denote by $\widetilde{\boldsymbol{u}}$ the extension by zero of $\boldsymbol{u}$. Then $\widetilde{\boldsymbol{u}}$ is in $\boldsymbol{H}(\mathbf{curl}\,,\mathbb{R}^3)$. Let $\boldsymbol{v} = \mathbf{curl}\,\widetilde{\boldsymbol{u}}$. Note that $\boldsymbol{v}$ has compact support.

Let $\widehat{\boldsymbol{u}}$ and $\widehat{\boldsymbol{v}}$ be the Fourier transforms of $\widetilde{\boldsymbol{u}}$ and $\boldsymbol{v}$ respectively. Since $\operatorname{div}\boldsymbol{v} = 0$ and $\boldsymbol{v} = \mathbf{curl}\,\widetilde{\boldsymbol{u}}$, we have

$$\xi\cdot\widehat{v} = 0 \quad \text{and} \quad \widehat{\boldsymbol{v}} = i\xi \times \widehat{\boldsymbol{u}},$$

where $i = \sqrt{-1}$ and $\xi = (\xi_1, \xi_2, \xi_3)^T$ stands for the dual variable of $\boldsymbol{x} = (x_1, x_2, x_3)^T$.

Define $\widehat{\boldsymbol{w}} \equiv (\boldsymbol{I} - \frac{1}{|\xi|^2}\xi\xi^T)\widehat{\boldsymbol{u}}$ where $\boldsymbol{I}$ is the identity matrix. It is not hard to see that the matrix $\boldsymbol{I} - \frac{1}{|\xi|^2}\xi\xi^T$ has the eigenvalue 0 corresponding to the eigenvector ξ, and the eigenvalue 1 of multiplicity two corresponding to two linearly independent eigenvectors orthogonal to ξ. This shows that $\|\widehat{w}\| \leq \|\widehat{\boldsymbol{u}}\|$ and thus the inverse Fourier transform $\boldsymbol{w}$ of $\widehat{\boldsymbol{w}}$ satisfies

$$\|\boldsymbol{w}\|_{0,\mathbb{R}^3} = \|\widehat{\boldsymbol{w}}\|_{0,\mathbb{R}^3} \leq \|\widehat{\boldsymbol{u}}\|_{0,\mathbb{R}^3} = \|\widetilde{\boldsymbol{u}}\|_{0,\mathbb{R}^3} = \|\boldsymbol{u}\|.$$

By the construction of $\widehat{\boldsymbol{w}}$, we also have

$$\xi \cdot \widehat{\boldsymbol{w}} = \xi \cdot \widehat{\boldsymbol{u}} - \frac{1}{|\xi|^2}\xi \cdot \xi(\xi^T\widehat{\boldsymbol{u}}) = 0$$

and

$$i\xi \times \widehat{\boldsymbol{w}} = i\xi \times \widehat{\boldsymbol{u}} - \frac{i}{|\xi|^2}(\xi^T\widehat{\boldsymbol{u}})\xi \times \xi = i\xi \times \widehat{\boldsymbol{u}}.$$

Thus,

$$\operatorname{div} \boldsymbol{w} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{w} = \mathbf{curl}\, \widetilde{\boldsymbol{u}}.$$

Since $\widehat{\boldsymbol{v}} = i\xi \times \widehat{\boldsymbol{u}}$,

$$\begin{aligned}\xi \times \widehat{\boldsymbol{v}} &= i\xi \times (\xi \times \widehat{\boldsymbol{u}}) = i[(\xi^T\widehat{\boldsymbol{u}})\xi - |\xi|^2\widehat{\boldsymbol{u}}] \\ &= -i|\xi|^2(\widehat{\boldsymbol{u}} - \frac{1}{|\xi|^2}\xi\xi^T\widehat{\boldsymbol{u}}) = -i|\xi|^2\widehat{\boldsymbol{w}}.\end{aligned}$$

It immediately follows that $|\boldsymbol{w}|_{1,\mathbb{R}^3} \leq \sqrt{2}\|\boldsymbol{v}\|_{0,\mathbb{R}^3} \leq \sqrt{2}\|\mathbf{curl}\, \boldsymbol{u}\|$.

The restriction $\boldsymbol{w}$ to Ω is the desired potential. This completes the proof of the lemma. □

The following lemma is an improvement of Proposition 5.1 in [25]. It provides the additional stability estimate $\|\boldsymbol{w}\| \leq C\|\boldsymbol{z}\|$. The proof mainly follows the argument given there. Note that some modification has to be done for the case that $\partial\Omega$ has multiple components.

Lemma IV.2. *For any* $\boldsymbol{z} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ *with* $\operatorname{div}\boldsymbol{z} = 0$ *in* Ω*, there exist* $\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$ *and* $\psi \in H^1(\Omega)$ *with* ψ *being constant on each connected component of* $\partial\Omega$ *such that,*

$$\boldsymbol{z} = \boldsymbol{w} + \nabla\psi,$$

and the following estimates hold:

$$\|\boldsymbol{w}\| + \|\psi\|_1 \leq C\|\boldsymbol{z}\| \quad \textit{and} \quad \|\boldsymbol{w}\|_1 \leq C\|\mathbf{curl}\,\boldsymbol{z}\|.$$

Proof. Let Γ_i, $1 \leq i \leq I$, be the internal connected components of $\partial\Omega$ and Γ_0 the boundary of the only unbounded connected component of $\mathbb{R}^3\backslash\overline{\Omega}$.

Define q_i to be the unique solution in $H^1(\Omega)$ of the problem [3]

$$\begin{cases} -\triangle q_i = 0 \quad \text{in} \quad \Omega, \\ q_i\big|_{\Gamma_0} = 0, \quad q_i\big|_{\Gamma_k} = C_{ik}, 1 \leq k \leq I, \end{cases}$$

where C_{ik} are constants on Γ_k. These constants are uniquely determined by the following conditions

$$\left\langle \frac{\partial q_i}{\partial \boldsymbol{n}}, 1 \right\rangle_{\Gamma_0} = -1, \quad \left\langle \frac{\partial q_i}{\partial \boldsymbol{n}}, 1 \right\rangle_{\Gamma_k} = \delta_{ik}, \ 1 \leq k \leq I.$$

For $\boldsymbol{z}$ given above, we define $\mathring{\boldsymbol{z}}$ by

$$\mathring{\boldsymbol{z}} = \boldsymbol{z} - \sum_{i=1}^{I} \langle \boldsymbol{z}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} \nabla q_i.$$

Then $\mathring{\boldsymbol{z}} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ satisfies that

$$\mathbf{curl}\,\mathring{\boldsymbol{z}} = \mathbf{curl}\,\boldsymbol{z}, \quad \operatorname{div}\mathring{\boldsymbol{z}} = 0,$$

$$\langle \mathring{\boldsymbol{z}}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_k} = \langle \boldsymbol{z}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_k} - \sum_{i=1}^{I} \langle \boldsymbol{z}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} \left\langle \frac{\partial q_i}{\partial \boldsymbol{n}}, 1 \right\rangle_{\Gamma_k} = 0, \quad 1 \leq k \leq I,$$

and

$$\|\mathring{\boldsymbol{z}}\| \leq \|\boldsymbol{z}\| + \sum_{i=1}^{I} |\langle \boldsymbol{z}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i}| \cdot \|\nabla q_i\|$$
$$\leq \|\boldsymbol{z}\| + C\|\boldsymbol{z}\|_{\boldsymbol{H}(\mathrm{div};\Omega)} \leq C\|\boldsymbol{z}\|.$$

It follows from Corollary 3.19 of [3] that

$$\|\mathring{\boldsymbol{z}}\| \leq C\|\mathbf{curl}\,\mathring{\boldsymbol{z}}\|. \tag{4.4}$$

Denote by $\widetilde{\boldsymbol{z}}$ the extension by zero of $\mathring{\boldsymbol{z}}$ to an open ball $B(0;r)$ which contains $\overline{\Omega}$. Let $\Omega^c \equiv B(0;r)\backslash\overline{\Omega}$. By Lemma IV.1, there is a $\widetilde{\boldsymbol{w}} \in \boldsymbol{H}^1(B(0;r))$ such that

$$\mathbf{curl}\,\widetilde{\boldsymbol{w}} = \mathbf{curl}\,\widetilde{\boldsymbol{z}} \quad \text{and} \quad \mathrm{div}\,\widetilde{\boldsymbol{w}} = 0.$$

Moreover, $\|\widetilde{\boldsymbol{w}}\|_{0,B(0;r)} \leq \|\mathring{\boldsymbol{z}}\|$ and $\|\widetilde{\boldsymbol{w}}\|_{1,B(0;r)} \leq \sqrt{2}\|\mathring{\boldsymbol{z}}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \leq C\|\mathbf{curl}\,\mathring{\boldsymbol{z}}\|$. In the last inequality, we used 4.4.

Since $\mathbf{curl}\,(\widetilde{\boldsymbol{w}} - \widetilde{\boldsymbol{z}}) = 0$, there is a $\widetilde{\varphi} \in H^1(B(0;r))/\mathbb{R}$ such that $\widetilde{\boldsymbol{w}} - \widetilde{\boldsymbol{z}} = \nabla\widetilde{\varphi}$ and $\|\widetilde{\varphi}\|_{1,B(0;r)} \leq C\|\widetilde{\boldsymbol{w}} - \widetilde{\boldsymbol{z}}\|_{0,B(0;r)}$ (cf. Theorem 2.9, Chapter I in [32]). Note that in Ω^c, $\nabla\widetilde{\varphi} = \widetilde{\boldsymbol{w}} \in \boldsymbol{H}^1(\Omega^c)$ since $\widetilde{\boldsymbol{z}} = 0$ and thus $\widetilde{\varphi} \in H^2(\Omega^c)$. Using Theorem 5 in [61], we can extend this $\widetilde{\varphi}$ in $H^2(\Omega^c)$ to φ defined on $B(0;r)$ satisfying

$$\|\varphi\|_{1,B(0;r)} \leq C\|\widetilde{\varphi}\|_{1,\Omega^c} \leq C\|\widetilde{\boldsymbol{w}} - \widetilde{\boldsymbol{z}}\|_{0,B(0;r)} \tag{4.5}$$

and

$$\|\varphi\|_{2,B(0;r)} \leq C\|\widetilde{\varphi}\|_{2,\Omega^c} \leq C(\|\widetilde{\boldsymbol{w}}\|_{1,\Omega^c} + \|\widetilde{\boldsymbol{z}}\|). \tag{4.6}$$

Now, we have

$$\widetilde{\boldsymbol{z}} = \widetilde{\boldsymbol{w}} - \nabla\widetilde{\varphi} = (\widetilde{\boldsymbol{w}} - \nabla\varphi) + \nabla(\varphi - \widetilde{\varphi}).$$

Note that $\widetilde{\boldsymbol{w}} - \nabla\varphi$ is in $\boldsymbol{H}^1(B(0;r))$ and its trace to $\partial\Omega$ from Ω^c vanishes. Thus,

$\widetilde{\boldsymbol{w}} - \nabla\varphi$ is in $\boldsymbol{H}_0^1(\Omega)$ and satisfies

$$\|\widetilde{\boldsymbol{w}} - \nabla\varphi\|_{0,B(0;r)} \le C\|\widetilde{\boldsymbol{w}}\|_{0,B(0;r)} + C\|\widetilde{\boldsymbol{w}} - \widetilde{\boldsymbol{z}}\|_{0,B(0;r)} \le C\|\mathring{\boldsymbol{z}}\| \le C\|\boldsymbol{z}\|$$

and

$$\|\widetilde{\boldsymbol{w}} - \nabla\varphi\|_{1,B(0;r)} \le C(\|\widetilde{\boldsymbol{w}}\|_{1,B(0;r)} + \|\widetilde{\boldsymbol{z}}\|) \le C\|\mathbf{curl}\,\mathring{\boldsymbol{z}}\| = C\|\mathbf{curl}\,\boldsymbol{z}\|.$$

We complete the proof by setting $\boldsymbol{w}$ to be the restriction to Ω of $\widetilde{\boldsymbol{w}} - \nabla\varphi$ and ψ to be the sum of $\sum_{i=1}^{I}\langle \boldsymbol{z}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} q_i$ and the restriction to Ω of $\widetilde{\varphi} - \varphi$. □

B. Overlapping Schwarz preconditioners

In this section, we give two overlapping Schwarz preconditioners for the discrete system corresponding to (4.2). The overlapping Schwarz algorithms as described in [28, 34, 64] are based on two levels of partitioning of Ω. The first is a coarse partitioning into (non-overlapping) tetrahedra $\{\Omega_i : \ i = 1, \ldots, N_0\}$. This forms a mesh $\mathcal{T}_H$ of mesh size H. Next, each Ω_i is further partitioned into finer tetrahedra $\{\tau_i^j : \ j = 1, 2, \ldots, N_i\}$. The fine partitioning gives the fine mesh $\mathcal{T}_h$ of mesh size h. Both $\mathcal{T}_H$ and $\mathcal{T}_h$ are assumed to be regular.

Along with this partitioning, we assume that we are given another sequence of (overlapping) subdomains Ω_j' $j = 1, \ldots, N$ in such a way that $\partial\Omega_j'$ aligns with the h-level mesh. Then each subdomain Ω_j' is also partitioned by tetrahedra in $\mathcal{T}_h$ and the space

$$\boldsymbol{U}_h^j = \boldsymbol{U}_h \cap \boldsymbol{H}_0(\mathbf{curl}\,; \Omega_j'), \quad j = 1, \ldots, N,$$

is also a Nedelec finite element space. In the above definition, we regard $\boldsymbol{H}_0(\mathbf{curl}; \Omega_j')$ as a subset of $\boldsymbol{H}_0(\mathbf{curl}; \Omega)$ by identifying functions in $\boldsymbol{H}_0(\mathbf{curl}\,; \Omega_j')$ with their extension by zero. It is convenient to set $\Omega_0' = \Omega$ and $\boldsymbol{U}_h^0 = \boldsymbol{U}_H$. Similarly, we define the Lagrange finite element space $S_h^j, j = 0, 1, \ldots, N$ by replacing $\boldsymbol{H}_0(\mathbf{curl}\,; \Omega_j')$ with $H_0^1(\Omega_j')$.

We assume that subdomains $\{\Omega_j'\}$ are such that there is a partition of unity $\{\theta_j\}_{j=1}^N$ where the partition functions are piecewise linear with respect to the fine mesh and satisfy

$$\|\nabla \theta_j\|_\infty \le C H^{-1}, \text{ for } j = 1, \ldots, N. \tag{4.7}$$

We finally assume that the subdomains $\{\Omega_j'\}$ satisfy a limited overlap property, i.e., each point of Ω is contained in at most n_0 subdomains where n_0 is independent of H and h.

One can, for example, define the overlapping subdomains to be regions associated with vertices of the coarse mesh, i.e., Ω_j' is the interior of the union of the closures of the coarse grid tetrahedra which share the j'th vertex. In this case, the partition of unity functions can be taken to be the nodal finite element basis functions associated with the conforming piecewise linear coarse grid approximation to $H^1(\Omega)$. Alternatively, one can use the classical approach of defining the overlapping subdomains by extending the original coarse grid subdomains $\{\Omega_j\}$ so that

$$\text{dist}(\partial\Omega_j' \cap \Omega, \partial\Omega_j \cap \Omega) \ge \delta H \quad \text{for all } j = 1, \ldots, N. \tag{4.8}$$

Here δ is some constant independent of h and H.

A key property to establish the effectiveness of the overlapping Schwarz preconditioners is the following stability result. Its proof will be given in the next section.

Lemma IV.3. *Suppose that the overlapping subdomains and partition of unity satisfy the conditions above. Then there is a constant C_{stab} such that for all $\boldsymbol{u} \in \boldsymbol{U}_h$, we have* *a decomposition* $\boldsymbol{u} = \sum_{j=0}^{N} \boldsymbol{u}_j$ *with* $\boldsymbol{u}_j \in \boldsymbol{U}_h^j$ *satisfying*

$$\sum_{j=0}^{N} \mathbf{A}(\boldsymbol{u}_j, \boldsymbol{u}_j) \leq C_{stab} \mathbf{A}(\boldsymbol{u}, \boldsymbol{u}).$$

The overlapping Schwarz methods uses the solvers on the overlapping subregions $\{\Omega_j'\}$. For $j = 0, 1, \ldots, N$, we define $\mathbf{A}_j : \boldsymbol{U}_h^j \to \boldsymbol{U}_h^j$ by

$$(\mathbf{A}_j \boldsymbol{u}, \boldsymbol{w}) = \mathbf{A}(\boldsymbol{v}, \boldsymbol{w}), \quad \text{for all } \boldsymbol{w} \in \boldsymbol{U}_h^j,$$

and set $\mathbf{Q}_j : \boldsymbol{U}_h \to \boldsymbol{U}_h^j$ to be the $\boldsymbol{L}^2(\Omega)$-projection.

The additive Schwarz preconditioner $\boldsymbol{B}_a : \boldsymbol{U}_h \to \boldsymbol{U}_h$ is defined by

$$\boldsymbol{B}_a = \sum_{j=0}^{N} \mathbf{A}_j^{-1} \mathbf{Q}_j. \tag{4.9}$$

The symmetric multiplicative Schwarz preconditioner $\boldsymbol{B}_m : \boldsymbol{U}_h \to \boldsymbol{U}_h$ is defined as follows. For a given $\boldsymbol{g} \in \boldsymbol{U}_h$, we let $\boldsymbol{B}_m \boldsymbol{g} = \boldsymbol{u}^N \in \boldsymbol{U}_h$, where the $\boldsymbol{u}^N$ is defined by the iteration $\boldsymbol{u}^{-N-1} = 0$, and

$$\boldsymbol{u}^j = \boldsymbol{u}^{j-1} - \mathbf{A}_{|j|}^{-1} \mathbf{Q}_{|j|} (\boldsymbol{g} - \mathbf{A}_h \boldsymbol{u}^{j-1}), \ j = -N, -N+1, \ldots, N. \tag{4.10}$$

In practice, one can replace $\mathbf{A}_j^{-1}$ by preconditioner for $\mathbf{A}_j$ in either algorithm and still get robust preconditioners for the operator $\mathbf{A}_h^{\boldsymbol{U}}$. The results for the termwise preconditioned algorithm easily follow [16] from those for (4.9) and (4.10) which we give below.

The following theorem provides the upper bound for the conditioner number of the additive and multiplicative Schwarz preconditioners. Its proof is well known (cf. [16, 60]) and follows from the assumptions on the overlapping subdomains and Lemma IV.3.

Theorem IV.1. *Under the assumption of Lemma IV.3, for any* $\boldsymbol{u} \in \boldsymbol{U}_h$*, we have*

$$C_{stab}^{-1}\,\mathbf{A}(\boldsymbol{u},\boldsymbol{u}) \le \mathbf{A}(\boldsymbol{B}_a\,\mathbf{A}_h^{\boldsymbol{U}}\boldsymbol{u},\boldsymbol{u}) \le n_0\mathbf{A}(\boldsymbol{u},\boldsymbol{u}),$$

and

$$(C_{stab}\,n_0^2)^{-1}\,\mathbf{A}(\boldsymbol{u},\boldsymbol{u}) \le \mathbf{A}(\boldsymbol{B}_m\,\mathbf{A}_h^{\boldsymbol{U}}\boldsymbol{u},\boldsymbol{u}) \le \mathbf{A}(\boldsymbol{u},\boldsymbol{u}).$$

Remark IV.1. *The above theorem guarantees that the condition number for the preconditioned system remains bounded independently of* h *and* H*. This means that, for example, a preconditioned conjugate gradient iteration using these preconditioners is guaranteed to converge at a rate which can be bounded independently of* h *and* H*.*

Remark IV.2. *The theorem suggests that the additive method has a smaller condition number than the multiplicative. In practice this is not the case. In numerical experiments, it is observed that the multiplicative method has a smaller condition number.*

C. Analysis of overlapping Schwarz methods

In our analysis, we will use the interpolation operator $\mathbf{\Pi}_h$ and the $\boldsymbol{L}^2$-projection $\mathbf{Q}_h^{\boldsymbol{U}}$ onto $\boldsymbol{U}_h$. From Lemma III.5, we have

$$\|\boldsymbol{u} - \mathbf{\Pi}_h\boldsymbol{u}\| \le Ch|\boldsymbol{u}|_1, \quad \text{for all } \boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \text{ such that } \mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{V}_h. \tag{4.11}$$

For $\mathbf{Q}_h^{\boldsymbol{U}}$, we have the following stability and error estimates suggested in [34]:

$$\|\boldsymbol{u} - \mathbf{Q}_h^{\boldsymbol{U}}\boldsymbol{u}\| + h\|\mathbf{curl}\,\mathbf{Q}_h^{\boldsymbol{U}}\boldsymbol{u}\| \le Ch|\boldsymbol{u}|_1, \quad \text{for all } \boldsymbol{u} \in H^1(\Omega) \tag{4.12}$$

although its proof was not given there. In a private communication, Hiptmair suggested a proof of (4.12) using the operator $\mathfrak{P}_h$ introduced in [7]. The projector $\mathfrak{P}_h$ was defined locally and replaced integration on the edges with integration on the faces.

This produces an interpolation which is well defined on vector fields in $\boldsymbol{H}^1$. By applying a Bramble-Hilbert argument, Lemma 5 of [7] shows that $\|\boldsymbol{u}-\mathfrak{P}_h\boldsymbol{u}\| \le Ch|\boldsymbol{u}|_1$, and $\|\mathbf{curl}\,\mathfrak{P}_h\boldsymbol{u}\| \le C|\boldsymbol{u}|_1$, for all $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$. The first estimate of (4.12) follows from the best approximation property of $\mathbf{Q}_h^{\boldsymbol{U}}$, and the second follows from

$$\|\mathbf{curl}\,\mathbf{Q}_h^{\boldsymbol{U}}\boldsymbol{u}\| \le Ch^{-1}\|(\mathbf{Q}_h^{\boldsymbol{U}} - \mathfrak{P}_h)\boldsymbol{u}\| + C\|\mathbf{curl}\,\mathfrak{P}_h\boldsymbol{u}\|.$$

We now give a proof of Lemma IV.3.

Proof of Lemma IV.3. Pick an arbitrary $\boldsymbol{u} \in \boldsymbol{U}_h$ and let $\boldsymbol{u} = \boldsymbol{z} + \nabla\varphi$ be its continuous Helmholtz decomposition. Splitting $\boldsymbol{z} = \boldsymbol{w} + \nabla\psi$ as in Lemma IV.2 gives

$$\boldsymbol{u} = \boldsymbol{w} + \nabla p, \tag{4.13}$$

where $\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$ and $p = \varphi + \psi \in H^1(\Omega)$ with p being constant on each connected component of $\partial\Omega$ satisfy

$$\|\boldsymbol{w}\| + \|p\|_1 \le C\|\boldsymbol{u}\| \quad \text{and} \quad |\boldsymbol{w}|_1 \le C\|\mathbf{curl}\,\boldsymbol{u}\|. \tag{4.14}$$

Since $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$ and $\mathbf{curl}\,\boldsymbol{w} \in \boldsymbol{V}_h$, we can apply $\boldsymbol{\Pi}_h$ to both sides of (4.13) to get

$$\boldsymbol{u} = \boldsymbol{\Pi}_h\boldsymbol{w} + \nabla p^h, \tag{4.15}$$

where $p^h \in \overline{S}_h$ is constant on each connected component of $\partial\Omega$ (see the proof of Lemma 5.10, Chapter III of [32]). We will decompose $\boldsymbol{\Pi}_h\boldsymbol{w}$ and p^h separately.

For the decomposition of p^h, we define the piecewise linear function p_0 in $\overline{S}_h$ by

$$p_0 = \begin{cases} Q_H\, p^h, & \text{at nodes of } \mathcal{T}_H \text{ in } \Omega, \\ p^h, & \text{at nodes on } \partial\Omega, \end{cases} \tag{4.16}$$

where Q_H is the L^2-projection onto $\overline{S}_H$. Using partition of unity $\{\theta_j\}_{j=1}^N$ introduced in the previous section, we define the decomposition of p^h by

$$p^h = p_0 + \sum_{j=1}^{N} I_h(\theta_j(p^h - p_0)) \equiv p_0 + \sum_{j=1}^{N} p_j, \tag{4.17}$$

where I_h is the interpolation operator on S_h. Note that ∇p_j, $j = 0, \ldots, N$, belongs to U_j because p_0 is constant on each component of $\partial\Omega$ and $p^h - p_0$ vanishes on $\partial\Omega$.

To show the stability of the decomposition (4.17), we first note that

$$\|p_0 - p^h\| \leq CH\|\nabla p^h\| \quad \text{and} \quad \|\nabla(p_0 - p^h)\| \leq C\|\nabla p^h\|.$$

For details, we refer to Section 4 in [17]. Therefore, using (4.7) and the finite overlapping assumption, we have that

$$\begin{aligned}
\|\nabla p_0\|^2 + \sum_{j=1}^{N} \|\nabla p_j\|^2 &\leq C\|\nabla p^h\|^2 + C\sum_{j=1}^{N} \|\nabla\theta_j(p^h - p_0)\|^2 \\
&\leq C\|\nabla p^h\|^2 + C\sum_{j=1}^{N}\Big\{H^{-2}\|p^h - p_0\|^2_{\boldsymbol{L}^2(\Omega_j')} + \|\nabla(p^h - p_0)\|^2_{\boldsymbol{L}^2(\Omega_j')}\Big\} \\
&\leq C\|\nabla p^h\|^2,
\end{aligned}$$

and thus

$$\sum_{j=0}^{N} \mathbf{A}(\nabla p_j, \nabla p_j) = \alpha\sum_{j=0}^{N} \|\nabla p_j\|^2 \leq C\mathbf{A}(\nabla p^h, \nabla p^h) \leq C\mathbf{A}(\boldsymbol{u}, \boldsymbol{u}). \tag{4.18}$$

To deal with $\boldsymbol{\Pi}_h\boldsymbol{w}$ in (4.15), we first eliminate the low frequency components by subtracting $\mathbf{Q}_H^{\boldsymbol{U}}\boldsymbol{w}$ from $\boldsymbol{w}$, and get

$$\boldsymbol{\Pi}_h\boldsymbol{w} = (\boldsymbol{\Pi}_h\boldsymbol{w} - \mathbf{Q}_H^{\boldsymbol{U}}\boldsymbol{w}) + \mathbf{Q}_H^{\boldsymbol{U}}\boldsymbol{w} \equiv \boldsymbol{w}^h + \boldsymbol{w}_0, \tag{4.19}$$

By (4.11), (4.12) and (4.14), $\boldsymbol{w}_0$ and $\boldsymbol{w}^h$ satisfy,

$$\mathbf{A}(\boldsymbol{w}_0, \boldsymbol{w}_0) \leq \alpha\|\boldsymbol{w}\|^2 + C|\boldsymbol{w}|_1^2 \leq C\mathbf{A}(\boldsymbol{u}, \boldsymbol{u}), \tag{4.20}$$

$$\|\boldsymbol{w}^h\| \le \|\boldsymbol{\Pi}_h\boldsymbol{w} - \boldsymbol{w}\| + \|\boldsymbol{w} - \mathbf{Q}_H^{\boldsymbol{U}}\boldsymbol{w}\| \le CH|\boldsymbol{w}|_1 \le CH\|\mathbf{curl}\,\boldsymbol{u}\|. \tag{4.21}$$

Alternatively, we have the bound

$$\begin{aligned}\|\boldsymbol{w}^h\| &\le \|\boldsymbol{\Pi}_h\boldsymbol{w} - \boldsymbol{w}\| + \|\boldsymbol{w} - \mathbf{Q}_H^{\boldsymbol{U}}\boldsymbol{w}\| \\ &\le C(h\|\mathbf{curl}\,\boldsymbol{u}\| + \|\boldsymbol{w}\|) \le C\|\boldsymbol{u}\|. \end{aligned} \tag{4.22}$$

Finally, by (4.15) and (4.12),

$$\begin{aligned}\|\mathbf{curl}\,\boldsymbol{w}^h\| &\le \|\mathbf{curl}\,\boldsymbol{\Pi}_h\boldsymbol{w}\| + \|\mathbf{curl}\,\mathbf{Q}_H^{\boldsymbol{U}}\boldsymbol{w}\| \\ &\le \|\mathbf{curl}\,\boldsymbol{u}\| + C|\boldsymbol{w}|_1 \le C\|\mathbf{curl}\,\boldsymbol{u}\|. \end{aligned} \tag{4.23}$$

The remainder $\boldsymbol{w}^h$ is decomposed in a classical way. We use the partition of unity $\{\theta_j\}_{j=1}^N$ introduced earlier and define $\boldsymbol{w}_j = \boldsymbol{\Pi}_h(\theta_j\boldsymbol{w}^h)$, for $j = 1, \ldots, N$. Using the fact that the partition functions $\{\theta_j\}$ are piecewise linear with respect to the fine grid mesh, it can be shown (cf. Lemma 4.5 in [64]) that

$$\|\boldsymbol{\Pi}_h(\theta_j\boldsymbol{w}^h)\| \le C\|\theta_j\boldsymbol{w}^h\| \qquad \text{and}$$
$$\|\mathbf{curl}\,\boldsymbol{\Pi}_h(\theta_j\boldsymbol{w}^h) \le C\|\mathbf{curl}\,\theta_j\boldsymbol{w}^h\|.$$

The argument given there uses the property that $\theta_j\boldsymbol{w}^h$ is a piecewise polynomial of fixed order. Thus, we have

$$\|\boldsymbol{w}_j\| \le C\|\theta_j\boldsymbol{w}^h\| \le C\|\boldsymbol{w}^h\|_{\boldsymbol{L}^2(\Omega_j')}$$

and

$$\begin{aligned}\|\mathbf{curl}\,\boldsymbol{w}_j\| &\le C\|\mathbf{curl}\,\theta_j\boldsymbol{w}^h\| \\ &\le C(\|\nabla\theta_j\|_{L^\infty}\|\boldsymbol{w}^h\|_{\boldsymbol{L}^2(\Omega)(\Omega_j')} + \|\mathbf{curl}\,\boldsymbol{w}^h\|_{\boldsymbol{L}^2(\Omega)(\Omega_j')}) \\ &\le C(H^{-1}\|\boldsymbol{w}^h\|_{\boldsymbol{L}^2(\Omega)(\Omega_j')} + \|\mathbf{curl}\,\boldsymbol{w}^h\|_{\boldsymbol{L}^2(\Omega)(\Omega_j')}). \end{aligned}$$

The above inequalities and the limited overlap property of the subdomains imply that

$$\begin{aligned}\sum_{j=1}^{N} \mathbf{A}(\boldsymbol{w}_j, \boldsymbol{w}_j) &\leq C((\alpha + H^{-2})\|\boldsymbol{w}^h\|^2 + \|\mathbf{curl}\,\boldsymbol{w}^h\|^2) \\ &\leq C(\alpha\|\boldsymbol{u}\|^2 + \|\mathbf{curl}\,\boldsymbol{u}\|^2) = CA(\boldsymbol{u}, \boldsymbol{u}).\end{aligned} \tag{4.24}$$

The last inequality above followed from applying (4.21) and (4.22).

Finally, setting $\boldsymbol{u}_j = \boldsymbol{w}_j + \nabla p_j$ gives the desired decomposition of $\boldsymbol{u}$. Indeed, combining (4.18), (4.20), and (4.24) shows that

$$\begin{aligned}\sum_{j=0}^{N} \mathbf{A}(\boldsymbol{u}_j, \boldsymbol{u}_j) &\leq 2\mathbf{A}(\boldsymbol{w}_0, \boldsymbol{w}_0) + 2\sum_{j=1}^{N} \mathbf{A}(\boldsymbol{w}_j, \boldsymbol{w}_j) + 2\sum_{j=0}^{N} \mathbf{A}(\nabla p_j, \nabla p_j) \\ &\leq C\mathbf{A}(\boldsymbol{u}, \boldsymbol{u}).\end{aligned}$$

This completes the proof of Lemma IV.3. □

D. Numerical results

In this section we report the results of numerical experiments confirming and illustrating the theory of previous sections. All of the computations to be described use lowest order Nedelec elements on cubes.

The domain Ω is defined to be the three-dimensional domain $(0,1)^3/[0,1/2]^3$. On this domain, the solenoidal component of the Helmholtz decomposition is generally not in $\boldsymbol{H}^1(\Omega)$.

We take the coarse grid to be the 7 cubes of size $[0,1/2]^3$, whose union is the closure of Ω. Ω is meshed uniformly by cubic elements of size h. Overlapping subdomains are constructed by adjoining just enough fine elements to the coarse elements so that (4.8) holds.

Equation (4.1) with various α was solved using the preconditioned Conjugate

Table I. Condition numbers of $\boldsymbol{B}_a \mathbf{A}_h^{\boldsymbol{U}}$ with $\delta = 0.1$

α	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10	10^2	10^3	10^4
$h = 1/4$	7.18	7.18	7.18	7.18	7.18	7.20	7.24	7.74	7.95
$h = 1/8$	7.78	7.78	7.77	7.77	7.71	7.20	7.01	7.05	7.07
$h = 1/16$	13.17	13.17	13.17	13.16	13.11	12.38	7.00	7.00	7.00
$h = 1/32$	13.24	13.24	13.24	13.23	13.18	12.43	7.01	7.00	7.00
$h = 1/64$	13.26	13.26	13.26	13.24	13.19	12.44	7.01	7.00	7.00

Table II. Condition numbers of $\boldsymbol{B}_a \mathbf{A}_h^{\boldsymbol{U}}$ with $\delta = 0.2$

α	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10	10^2	10^3	10^4
$h = 1/4$	7.18	7.18	7.18	7.18	7.18	7.20	7.24	7.74	7.95
$h = 1/8$	7.78	7.78	7.77	7.77	7.71	7.20	7.01	7.05	7.07
$h = 1/16$	7.95	7.95	7.95	7.95	7.90	7.27	6.97	7.00	7.00
$h = 1/32$	7.91	7.91	7.91	7.91	7.86	7.26	6.98	7.00	7.00
$h = 1/64$	8.80	8.80	8.80	8.80	8.76	7.94	6.98	7.00	7.00

Gradient method. For the additive and multiplicative preconditioners, the Conjugate Gradient method without preconditioning was used to solve the discrete problems on the coarse mesh and on the subdomains. The condition numbers of the preconditioned system as a function of h were obtained by using a Lanczos technique [33].

In Table I and Table II, we report the condition numbers of the preconditioned system as a function of h for various values of α using the additive Schwarz preconditioner (4.9) with $\delta = 0.1$ and $\delta = 0.2$, respectively. The results are uniform with respect to α and h. Note that larger values of δ yield better preconditioners.

The condition numbers of the preconditioned system using multiplicative preconditioner (4.10) with $\delta = 0.1$ are given in Table III. The multiplicative preconditioner

Table III. Condition numbers of $\boldsymbol{B}_m \mathbf{A}_h^{\boldsymbol{U}}$ with $\delta = 0.1$

α	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10	10^2	10^3	10^4
$h = 1/4$	1.02	1.02	1.02	1.02	1.02	1.004	1.00025	1.005	1.008
$h = 1/8$	1.08	1.08	1.08	1.08	1.07	1.05	1.001	1.0007	1.005
$h = 1/16$	1.34	1.34	1.34	1.34	1.33	1.25	1.06	1.0002	1.002
$h = 1/32$	1.35	1.35	1.35	1.35	1.34	1.26	1.06	1.0002	1.
$h = 1/64$	1.35	1.35	1.35	1.35	1.34	1.26	1.09	1.00032	1.

performs better than the additive preconditioner in terms of the condition numbers. Indeed the condition numbers for large α end up being very close to one.

CHAPTER V

MULTIGRID METHODS

In this chapter, we will analyze multigrid methods for the problem (4.1): Find $\boldsymbol{u}_h \in \boldsymbol{U}_h$ such that

$$\alpha(\boldsymbol{u}_h, \boldsymbol{v}_h) + (\mathbf{curl}\, \boldsymbol{u}_h, \mathbf{curl}\, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \quad \text{for all } \boldsymbol{v} \in \boldsymbol{U}_h. \tag{5.1}$$

Multigrid methods are natural extensions of domain decomposition methods given in Chapter IV.

We take the abstract theory in [14, 15] as the basis of our analysis. In Section A, we introduce two new innerproducts in the finite element space $\boldsymbol{U}_h \subseteq \boldsymbol{H}_0(\mathbf{curl}; \Omega)$ by means of continuous and discrete Helmholtz decompositions. These innerproducts will serve as the base innerproduct in the abstract theory of multigrid analysis. In Section B, we describe BPX-type and V-cycle multigrid methods based on smoothers proposed by Hiptmair [35]. We present the analysis of both methods in Section C. Essentially we construct the multilevel stable decomposition and show that the strengthen Cauchy-Schwarz inequality holds.

We will only consider the case when $\alpha = 1$ and $\partial\Omega$ has one connected component.

A. New innerproducts in $\boldsymbol{H}_0(\mathbf{curl}; \Omega)$

Let $\boldsymbol{u}$ be in the Nedelec finite element space $\boldsymbol{U}_h$ and $\boldsymbol{u} = \boldsymbol{z}_h + \nabla\phi_h$, the discrete Helmholtz decomposition (2.3). Recall that $\boldsymbol{z}_h \in \boldsymbol{U}_h$ is $\boldsymbol{L}^2(\Omega)$-orthogonal to ∇S_h, and ϕ_h belongs to S_h. We define the norm $\|\cdot\|_{*,h}$ on $\boldsymbol{U}_h$ by

$$\|\boldsymbol{u}\|_{*,h}^2 = \|\boldsymbol{z}_h\|^2 + \|\phi_h\|^2. \tag{5.2}$$

This norm naturally induces an innerproduct $(\cdot,\cdot)_{*,h}$ in $\boldsymbol{U}_h$ by the parallelogram identity:

$$(\boldsymbol{u},\boldsymbol{v})_{*,h} = \frac{1}{4}\big[\|\boldsymbol{u}+\boldsymbol{v}\|_{*,h}^2 - \|\boldsymbol{u}-\boldsymbol{v}\|_{*,h}^2\big], \quad \text{for all } \boldsymbol{u},\boldsymbol{v}\in\boldsymbol{U}_h. \tag{5.3}$$

On the other hand, $\boldsymbol{u}\in\boldsymbol{U}_h$ is a function in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, and thus has the continuous Helmholtz decomposition $\boldsymbol{u}=\boldsymbol{z}+\nabla\phi$, where $\boldsymbol{z}\in\boldsymbol{H}_0(\mathbf{curl};\Omega)$ has zero divergence and ϕ belongs to $H_0^1(\Omega)$. We define the norm $\|\cdot\|_*$ on $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ by

$$\|\boldsymbol{u}\|_*^2 = \|\boldsymbol{z}\|^2 + \|\varphi\|^2. \tag{5.4}$$

Similar to (5.3), this norm induces an innerproduct $(\cdot,\cdot)_*$ on the space $\boldsymbol{H}_0(\mathbf{curl};\Omega)$.

The following lemma shows that both norms have a certain minimum property.

Lemma V.1. *Let $\boldsymbol{u}=\boldsymbol{w}_h+\nabla\psi_h$ be any decomposition of $\boldsymbol{u}\in\boldsymbol{U}_h$ with $\boldsymbol{w}_h\in\boldsymbol{U}_h$ and $\psi_h\in S_h$. Then,*

$$\|\boldsymbol{u}\|_{*,h} \le C(\|\boldsymbol{w}_h\| + \|\psi_h\|). \tag{5.5}$$

similarly, let $\boldsymbol{u}=\boldsymbol{w}+\nabla\psi$ be any decomposition of $\boldsymbol{u}\in\boldsymbol{H}_0(\mathbf{curl};\Omega)$ with $\boldsymbol{w}\in\boldsymbol{H}_0(\mathbf{curl};\Omega)$ and $\psi\in H_0^1(\Omega)$. Then,

$$\|\boldsymbol{u}\|_* \le C(\|\boldsymbol{w}\| + \|\psi\|). \tag{5.6}$$

Proof. For (5.5), let $\boldsymbol{u}=\boldsymbol{w}_h+\nabla\psi_h$ be as above, and $\boldsymbol{u}=\boldsymbol{z}_h+\nabla\varphi_h$ be the discrete Helmholtz decomposition of $\boldsymbol{u}$. We first have

$$\|\boldsymbol{z}_h\|^2 = (\boldsymbol{z}_h, \boldsymbol{w}_h+\nabla\psi_h-\nabla\varphi_h) = (\boldsymbol{z}_h,\boldsymbol{w}_h) \le \|\boldsymbol{z}_h\|\|\boldsymbol{w}_h\|,$$

and thus

$$\|\boldsymbol{z}_h\| \le \|\boldsymbol{w}_h\|. \tag{5.7}$$

Secondly, let $\mu_h \in S_h$ solve

$$(\nabla \mu_h, \nabla \chi) = (\varphi_h, \chi), \quad \text{for all } \chi \in S_h. \tag{5.8}$$

Then we have $\|\nabla \mu_h\| \leq C\|\varphi_h\|$ and

$$\begin{aligned}
\|\varphi_h\|^2 &= (\nabla \mu_h, \nabla \varphi_h) = (\nabla \mu_h, \nabla \psi_h + \boldsymbol{w}_h - \boldsymbol{z}_h) \\
&= (\nabla \mu_h, \nabla \psi_h + \boldsymbol{w}_h) = (\varphi_h, \psi_h) + (\nabla \mu_h, \boldsymbol{w}_h) \\
&\leq \|\varphi_h\|\|\psi_h\| + \|\nabla \mu_h\|\|\boldsymbol{w}_h\| \leq C\|\varphi_h\|(\|\psi_h\| + \|\boldsymbol{w}_h\|),
\end{aligned}$$

which implies $\|\varphi_h\| \leq C(\|\psi_h\| + \|\boldsymbol{w}_h\|)$. Combining the above gives the desired estimate.

The proof of (5.6) is similar except we replace μ_h satisfying (5.8) by $\mu \in H_0^1(\Omega)$ satisfying

$$(\nabla \mu, \nabla \chi) = (\varphi, \chi), \quad \text{for all } \chi \in H_0^1(\Omega), \tag{5.9}$$

where φ is the solenoidal part of Helmholtz decomposition of $\boldsymbol{u}$. □

Although $\|\cdot\|_{*,h}$ is convenient in dealing with elements in $\boldsymbol{U}_h$, we need the level-independent norm $\|\cdot\|_*$ for the multilevel analysis. It turns out that these two norms are equivalent on $\boldsymbol{U}_h$. To show this equivalence, we need the following inverse inequalities for both components in Helmholtz decomposition of a discrete function in $\boldsymbol{U}_h$.

Lemma V.2. *Let $\boldsymbol{u} = \boldsymbol{z} + \nabla\varphi$ be the Helmholtz decomposition of $\boldsymbol{u} \in \boldsymbol{U}_h$. Then, we have the following inverse inequalities.*

$$\|\mathbf{curl}\, \boldsymbol{z}\| \leq Ch^{-1}\|\boldsymbol{z}\|, \tag{5.10}$$

and

$$\|\nabla \varphi\| \leq Ch^{-1}\|\boldsymbol{u}\|_*. \tag{5.11}$$

Proof. The proof of (5.10) uses a scaling argument. Let $\widehat{\tau}$ be the reference tetrahedron of unit size. Each tetrahedron $\tau \in \mathcal{T}_h$ can then be obtained from $\widehat{\tau}$ by the affine map

$$F_\tau(\widehat{\boldsymbol{x}}) = B_\tau \widehat{\boldsymbol{x}} + \boldsymbol{b}_\tau. \tag{5.12}$$

Under the map

$$\widehat{\boldsymbol{u}} = B_\tau^T \boldsymbol{u} \circ F_\tau, \tag{5.13}$$

we have

$$\mathbf{curl}\, \boldsymbol{u}(\boldsymbol{x}) = B_\tau^{-T} \mathbf{curl}\, \widehat{\boldsymbol{u}}(\widehat{\boldsymbol{x}}) B_\tau^{-1},$$

and thus $\mathbf{curl}\, \widehat{\boldsymbol{u}}(\widehat{\boldsymbol{x}})$ is a polynomial if $\mathbf{curl}\, \boldsymbol{u}(\boldsymbol{x})$ is a polynomial on τ. A standard scaling argument [49] shows that

$$C\|\widehat{\boldsymbol{u}}\|_{0,\widehat{\tau}}^2 \leq h_\tau^{-1} \|\boldsymbol{u}\|_{0,\tau}^2 \leq C^{-1} \|\widehat{\boldsymbol{u}}\|_{0,\widehat{\tau}}^2, \tag{5.14}$$

and

$$C\|\mathbf{curl}\, \boldsymbol{u}\|_{0,\tau}^2 \leq h_\tau^{-1} \|\mathbf{curl}\, \widehat{\boldsymbol{u}}\|_{0,\widehat{\tau}}^2 \leq C^{-1} \|\mathbf{curl}\, \boldsymbol{u}\|_{0,\tau}^2. \tag{5.15}$$

Then, we have

$$\begin{aligned} \|\mathbf{curl}\, \boldsymbol{z}\|^2 &= \sum_\tau \|\mathbf{curl}\, \boldsymbol{z}\|_{0,\tau}^2 \leq C \sum_\tau h_\tau^{-1} \|\mathbf{curl}\, \widehat{\boldsymbol{z}}\|_{0,\widehat{\tau}}^2 \\ &\leq C \sum_\tau h_\tau^{-1} \|\mathbf{curl}\, \widehat{\boldsymbol{z}}\|_{-1,\widehat{\tau}}^2, \end{aligned}$$

since $\mathbf{curl}\, \boldsymbol{z} = \mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{V}_h$ and all norms are equivalent on a finite dimensional space. By the definition of the norm $\|\cdot\|_{-1}$ and integration by parts, we further have

$$\begin{aligned} \|\mathbf{curl}\, \boldsymbol{z}\|^2 &\leq C \sum_\tau h_\tau^{-1} \Big(\sup_{\boldsymbol{v} \in \boldsymbol{H}_0^1(\widehat{\tau})} \frac{(\mathbf{curl}\, \widehat{\boldsymbol{z}}, \boldsymbol{v})}{\|\boldsymbol{v}\|_1} \Big)^2 \leq C \sum_\tau h_\tau^{-1} \|\widehat{\boldsymbol{z}}\|_{0,\widehat{\tau}}^2 \\ &\leq C \sum_\tau h_\tau^{-2} \|\boldsymbol{z}\|_{0,\tau}^2 \leq C h^{-2} \|\boldsymbol{z}\|^2. \end{aligned}$$

For (5.11), let $\boldsymbol{u} = \boldsymbol{z}_h + \nabla \varphi_h$ be the discrete decomposition of $\boldsymbol{u}$ and $\mu \in H_0^1(\Omega)$

solve

$$(\nabla\mu, \nabla\chi) = (\varphi_h, \chi), \quad \text{for all } \chi \in H_0^1(\Omega).$$

Then, we have

$$\begin{aligned}\|\varphi_h\|^2 &= (\nabla\mu, \nabla\varphi_h) = (\nabla\mu, \nabla\varphi - \boldsymbol{z}_h) \\ &= (\varphi_h, \varphi) - (\nabla\mu, \boldsymbol{z}_h) \le C\|\varphi_h\|(\|\varphi\| + \|\boldsymbol{z}_h\|),\end{aligned} \tag{5.16}$$

which implies that $\|\varphi_h\| \le C(\|\varphi\| + \|\boldsymbol{z}_h\|)$. Therefore, by the inverse inequality, we have

$$\|\nabla\varphi_h\| \le Ch^{-1}\|\varphi_h\| \le Ch^{-1}(\|\varphi\| + \|\boldsymbol{z}_h\|). \tag{5.17}$$

Split $\boldsymbol{z} = \boldsymbol{w} + \nabla\psi$ as in Lemma IV.2. Note that $\boldsymbol{w}$ belongs to $\boldsymbol{H}^1(\Omega)$, and $\mathbf{curl}\,\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{u}$ belongs to $\boldsymbol{V}_h$. Thus, $\Pi_h\boldsymbol{w}$ makes senses by Lemma III.5. Since $\boldsymbol{u} \in \boldsymbol{U}_h$, we have, by Lemma 5.10, Chapter III in [32],

$$\boldsymbol{u} = \Pi_h\boldsymbol{w} + \Pi_h\nabla(\varphi + \psi) = \Pi_h\boldsymbol{w} + \nabla\mu_h$$

for some $\mu_h \in S_h$. By (5.7), we get

$$\|\boldsymbol{z}_h\| \le \|\Pi_h\boldsymbol{w}\| \le \|\boldsymbol{w} - \Pi_h\boldsymbol{w}\| + \|\boldsymbol{w}\| \le Ch\|\boldsymbol{w}\|_1 + \|\boldsymbol{w}\|,$$

where we used Lemma III.5 for the last inequality. Thus, using the stability in Lemma IV.2 and the inverse inequality (5.10), we have

$$\|\boldsymbol{z}_h\| \le Ch\|\boldsymbol{z}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + C\|\boldsymbol{z}\| \le C\|\boldsymbol{z}\|. \tag{5.18}$$

Finally combining (5.17) and (5.18) yields

$$\begin{aligned}\|\nabla\varphi\| &\le \|\nabla\varphi - \nabla\varphi_h\| + \|\nabla\varphi_h\| \\ &\le \|\boldsymbol{z} - \boldsymbol{z}_h\| + Ch^{-1}(\|\varphi\| + \|\boldsymbol{z}_h\|) \\ &\le Ch^{-1}(\|\boldsymbol{z}\| + \|\varphi\|) = Ch^{-1}\|\boldsymbol{u}\|_*.\end{aligned}$$

□

Theorem V.1. *There is a constant C independent of h such that*

$$C^{-1}\|\boldsymbol{u}\|_* \le \|\boldsymbol{u}\|_{*,h} \le C\|\boldsymbol{u}\|_*, \quad \textit{for all } \boldsymbol{u} \in \boldsymbol{U}_h.$$

Proof. Lemma V.1 gives lower bound for $\|\boldsymbol{u}\|_{*,h}$. For the upper bound, let $\boldsymbol{u} = \boldsymbol{z}+\nabla\varphi$ and $\boldsymbol{u} = \boldsymbol{z}_h + \nabla\varphi_h$ be the Helmholtz decomposition and the discrete Helmholtz decomposition of $\boldsymbol{u}$ respectively. By the second inequality of (5.17) and (5.18), we have

$$\|\boldsymbol{u}\|_{*,h} = \|\boldsymbol{z}_h\| + \|\varphi_h\| \le C(\|\varphi\| + \|\boldsymbol{z}_h\|) \le C\|\boldsymbol{u}\|_*.$$

□

B. Multigrid methods and smoothers

We consider a nested sequence of triangulations of Ω, $\{\mathcal{T}_j,\ 1 \le j \le J\}$. Assume that $h_j \approx \gamma^{-j}$ for some positive constant γ. A typical γ is around 2. In the following, we will use h instead of h_J when convenient.

These nested triangulations give rise to the nested spaces $S_j \equiv S_{h_j}, \boldsymbol{U}_j \equiv \boldsymbol{U}_{h_j}$ and $\boldsymbol{V}_j \equiv \boldsymbol{V}_{h_j}$. For example, we have

$$\boldsymbol{U}_1 \subseteq \boldsymbol{U}_2 \subseteq \cdots \subseteq \boldsymbol{U}_J.$$

For each j, the $(\cdot,\cdot)_*$-projection $\mathbf{Q}_j : \boldsymbol{H}_0(\mathbf{curl};\Omega) \to \boldsymbol{U}_j$ is defined by

$$(\mathbf{Q}_j \boldsymbol{u}, \boldsymbol{v})_* = (\boldsymbol{u}, \boldsymbol{v})_*, \quad \text{for all } \boldsymbol{v} \in \boldsymbol{U}_j, \tag{5.19}$$

and the operator $\mathbf{A}_j : \boldsymbol{H}_0(\mathbf{curl};\Omega) \to \boldsymbol{U}_j$ is defined by

$$(\mathbf{A}_j \boldsymbol{u}, \boldsymbol{v}_j)_* = A(\boldsymbol{u}, \boldsymbol{v}_j), \quad \text{for all } \boldsymbol{v}_j \in \boldsymbol{U}_j. \tag{5.20}$$

Let λ_j be maximum eigenvalue of $\mathbf{A}_j$. The following lemma gives a relation between λ_j and h_j.

Lemma V.3. *There exits a constant C such that*

$$C^{-1}\lambda_j \leq h_j^{-2} \leq C\lambda_j, \quad \textit{for all } 1 \leq j \leq J.$$

Proof. Let $\boldsymbol{u} = \boldsymbol{z} + \nabla\varphi$ be the discrete Helmholtz decomposition of $\boldsymbol{u} \in \boldsymbol{U}_j$. Then, by Lemma V.1 and V.2 we have

$$\begin{aligned} A(\boldsymbol{u}, \boldsymbol{u}) &= A(\boldsymbol{z}, \boldsymbol{z}) + \|\nabla\varphi\|^2 \\ &\leq \|\boldsymbol{z}\|^2 + Ch_j^{-2}\|\boldsymbol{z}\|^2 + Ch_j^{-2}\|\boldsymbol{u}\|_*^2 \leq Ch_j^{-2}\|\boldsymbol{u}\|_*^2, \end{aligned}$$

and thus $\lambda_j \leq Ch_j^{-2}$.

On the other hand, taking $\varphi \in S_j$ such that $\|\nabla\varphi\| \geq Ch_j^{-1}\|\varphi\|$ (see e.g. [19]), we have

$$A(\nabla\varphi, \nabla\varphi) = \|\nabla\varphi\|^2 \geq Ch_j^{-2}\|\varphi\|^2 = Ch_j^{-2}\|\nabla\varphi\|_{*,h_j}^2 \geq Ch_j^{-2}\|\nabla\varphi\|_*^2,$$

and thus $\lambda_j \geq Ch_j^{-2}$. □

The discretization of problem (4.1) can be rewritten: Find $\boldsymbol{u} \in \boldsymbol{U}_J$ such that

$$A(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}_*, \boldsymbol{v})_*, \quad \text{for all } \boldsymbol{v} \in \boldsymbol{U}_J,$$

where $\boldsymbol{f}_* \in \boldsymbol{U}_J$ satisfies

$$(\boldsymbol{f}_*, \boldsymbol{v})_* = (\boldsymbol{f}, \boldsymbol{v}), \quad \text{for all } \boldsymbol{v} \in \boldsymbol{U}_J.$$

Using $\mathbf{A}_J$, the above problem can be written as

$$\mathbf{A}_J \boldsymbol{u} = \boldsymbol{f}_*.$$

Given $(\cdot,\cdot)_*$-symmetric and positive definite smoothers $\mathbf{R}_j : \boldsymbol{U}_j \to \boldsymbol{U}_j$, we can define the BPX-type preconditioner $\boldsymbol{G}_a$ by

$$\boldsymbol{G}_a = \sum_{j=1}^{J} \mathbf{R}_j \mathbf{Q}_j.$$

Remark V.1. *If the action of $\mathbf{R}_j f_*$ can be computed using only $(f_*, \phi_{j,i})_*$ data where $\{\phi_{j,i}\}_i$ is the finite element basis of $\boldsymbol{U}_j$, then the action $\boldsymbol{G}_a f_*$ can be implemented without the solution of the Gram matrix problem for $\mathbf{Q}_j$ or the explicit computation of $\boldsymbol{f}_*$. This observation remains valid also for the multiplicative version $\boldsymbol{G}_m$ which we define below. For details, we refer to [19]. We will give an example of such a smoother at the end of this section.*

The V-cycle multigrid preconditioner $\boldsymbol{G}_m$ is defined recursively. Let $\boldsymbol{G}_m^1 = \mathbf{A}_1^{-1}$ and, for $j > 1$, we define $\boldsymbol{G}_m^j \boldsymbol{b} = \boldsymbol{y}_1$ where

$$\begin{aligned}
\boldsymbol{y}_0 &= 0, \\
\boldsymbol{y}_{\frac{1}{3}} &= \boldsymbol{y}_0 + \mathbf{R}_j \mathbf{Q}_j (\boldsymbol{f}_* - \mathbf{A}\boldsymbol{y}_0), \\
\boldsymbol{y}_{\frac{2}{3}} &= \boldsymbol{y}_{\frac{1}{3}} + \boldsymbol{G}_m^{j-1} \mathbf{Q}_{j-1} (\boldsymbol{f}_* - \mathbf{A}\boldsymbol{y}_{\frac{1}{3}}), \\
\boldsymbol{y}_1 &= \boldsymbol{y}_{\frac{2}{3}} + \mathbf{R}_j \mathbf{Q}_j (\boldsymbol{f}_* - \mathbf{A}\boldsymbol{y}_{\frac{2}{3}}).
\end{aligned}$$

We write $\boldsymbol{G}_m = \boldsymbol{G}_m^J$.

The following conditions are used for estimating the multilevel preconditioners.

Condition V.1. *There is a constant* ω_1 *not depending on* $1 \le j \le J$ *such that the* $(\cdot,\cdot)_*$*-symmetric and positive definite smoothers,* $\mathbf{R}_j$*, satisfy*

$$\omega_1(\mathbf{R}_j^{-1}\boldsymbol{v},\boldsymbol{v})_* \le \lambda_j^{-1}(\boldsymbol{v},\boldsymbol{v})_*, \quad \textit{for all } \boldsymbol{v} \in \boldsymbol{U}_j,\ j \ge 1.$$

Condition V.2. *There is a constant* $\omega_2 \in (0,2)$ *not depending on* $1 \le j \le J$ *such that the* $(\cdot,\cdot)_*$*-symmetric and positive definite smoothers,* $\mathbf{R}_j$*, satisfy*

$$A(\boldsymbol{v},\boldsymbol{v}) \le \omega_2\,(\mathbf{R}_j^{-1}\boldsymbol{v},\boldsymbol{v})_*, \quad \textit{for all } \boldsymbol{v} \in \boldsymbol{U}_j,\ j \ge 1.$$

Condition V.3. *For any* $\boldsymbol{u} \in \boldsymbol{U}_J$*, there is a multilevel decomposition* $\boldsymbol{u} = \sum_{j=1}^{J} \boldsymbol{u}_j$ *with* $\boldsymbol{u}_j \in \boldsymbol{U}_j$ *and a constant* C_{smd} *not depending on* J *such that*

$$\sum_{j=1}^{J} \lambda_j \|\boldsymbol{u}_j\|_*^2 \le C_{smd}\, A(\boldsymbol{u},\boldsymbol{u}).$$

Condition V.4. *There is a constant* C_{scs} *and a number* $\varepsilon \in (0,1)$ *not depending on* J *such that for all* $1 \le i \le j \le J$

$$A(\boldsymbol{u}_i,\boldsymbol{u}_j) \le C_{scs}\,\varepsilon^{j-i}\lambda_j^{1/2}\|\boldsymbol{u}_i\|_{\mathbf{A}}\|\boldsymbol{u}_j\|_*, \quad \textit{for all } \boldsymbol{u}_i \in \boldsymbol{U}_i, \boldsymbol{u}_j \in \boldsymbol{U}_j,$$

where $\|\cdot\|_{\mathbf{A}}^2 \equiv A(\cdot,\cdot)$.

The following theorem (cf. [19, 21]) shows that provided that the appropriate conditions are satisfied, the condition number $K(\boldsymbol{G}_a\mathbf{A}_J)$ is bounded by some constant not depending on J.

Theorem V.2. *Assume that the smoothers* $\mathbf{R}_j, 1 \le j \le J$*, satisfy Condition V.1. Assume, in addition, that Conditions V.3 and V.4 hold. Then,* $\boldsymbol{G}_a$ *satisfies*

$$\frac{\omega_1}{C_{smd}}(\boldsymbol{G}_a^{-1}\boldsymbol{u},\boldsymbol{u})_* \le A(\boldsymbol{u},\boldsymbol{u}) \le \frac{2C_{scs}}{1-\varepsilon}(\boldsymbol{G}_a^{-1}\boldsymbol{u},\boldsymbol{u})_*, \quad \textit{for all } \boldsymbol{v} \in \boldsymbol{U}_J.$$

For the V-cycle algorithm, we have (cf. [19, 21])

Theorem V.3. *Assume that the smoothers* $\mathbf{R}_j, 1 \le j \le J$, *satisfy Conditions V.1 and V.2. Assume, in addition, that Conditions V.3 and V.4 hold. Then,* $\boldsymbol{G}_m$ *satisfies*

$$0 \le A([I - \boldsymbol{G}_m \mathbf{A}_J]\boldsymbol{v}, \boldsymbol{v}) \le (1-\delta)A(\boldsymbol{v}, \boldsymbol{v}), \quad \textit{for all } \boldsymbol{v} \in \boldsymbol{U}_J,$$

where

$$\delta = \frac{(2-\omega 2)\omega_1}{C_{smd}(1 + 2C_{scs}/(1-\varepsilon))^2}.$$

The proof of the above two theorems can also be derived from Lemma 4.6, Theorem 4.1 and 4.4 in [67] with simple manipulations.

The construction of smoothers satisfying the above conditions is based on the decomposition of $\boldsymbol{U}_h$ and S_h as sums of spaces supported in small patches. Let Ω_h^v be the union of the tetrahedra having the vertex v. We denote by S_h^v the set of functions in S_h whose support is contained in $\overline{\Omega}_h^v$. Similarly, we define $\boldsymbol{U}_h^e \subset \boldsymbol{U}_h$ corresponding to an edge e. Functions in S_h^v and $\boldsymbol{U}_h^e$ will be called patch functions. Let $\mathcal{V}_h$ and $\mathcal{E}_h$ be the sets of all vertices and edges respectively in the triangulation $\mathcal{T}_h$. It is possible [4] to decompose $p \in S_h$ as $p = \sum_{v \in \mathcal{V}_h} \varphi^v$ and $\boldsymbol{u} \in \boldsymbol{U}_h$ as $\boldsymbol{u} = \sum_{e \in \mathcal{E}_h} \boldsymbol{\Phi}^e$ where φ^v and $\boldsymbol{\Phi}^e$ belong to S_h^v and $\boldsymbol{U}_h^e$ respectively.

Both decompositions follow the same lines. For example, to obtain the edge-based decomposition of $\boldsymbol{U}_h$, we note that the degrees of freedom of the space $\boldsymbol{U}_h$ give rise to a decomposition of $\boldsymbol{u} \in \boldsymbol{U}_h$ as $\boldsymbol{u} = \sum \boldsymbol{u}^\kappa$ where the sum runs over all degrees of freedom of $\boldsymbol{U}_h$ and $\boldsymbol{u}^\kappa$ is the element of $\boldsymbol{U}_h$ with all degrees of freedom other than κ set equal to zero. It is clear that $\|\boldsymbol{u}^\kappa\| \le C\|\boldsymbol{u}\|_{L^2(\text{supp}\, \boldsymbol{u}^\kappa)}$. Now to each degree of freedom κ, we assign an edge e such that $\boldsymbol{u}^\kappa \in \boldsymbol{U}_h^e$. When $\boldsymbol{U}_h$ is a higher order Nedelec element space, such a choice is not unique. $\boldsymbol{\Phi}^e$ in the desired edge-base decomposition is then the sum of all $\boldsymbol{u}^\kappa$ corresponding to e. By the limited overlapping property of basis functions, we have that

$$\sum_{v\in\mathcal{V}_h} \|\varphi^v\|^2 \le C\|p\|^2 \quad \text{and} \quad \sum_{e\in\mathcal{E}_h} \|\mathbf{\Phi}^e\|^2 \le C\|\boldsymbol{u}\|^2. \tag{5.21}$$

To end this section, we give the construction of the additive smoother by Hiptmair [35]. The multiplicative version can be constructed based on the same space decomposition. For details of such construction, we refer to [19, 67].

In analogue to (5.19) and (5.20), we can define $\mathbf{Q}_j^e$ and $\mathbf{A}_j^e$ onto $\boldsymbol{U}_h^e$. Similarly, we can define $\mathbf{Q}_j^v$ and $\mathbf{A}_j^v$ into ∇S_h^v. Then, the additive smoother $\mathbf{R}_j^a$ is given by

$$\mathbf{R}_j^a = \sum_{e\in\mathcal{E}_h} (\mathbf{A}_j^e)^{-1}\mathbf{Q}_j^e + \sum_{v\in\mathcal{V}_h} (\mathbf{A}_j^v)^{-1}\mathbf{Q}_j^v, \quad \text{for all } 1 \le j \le J. \tag{5.22}$$

The evaluation of $\mathbf{R}_j^a$ is local and only depends on the $(\cdot,\cdot)_*$-innerproduct data.

Theorem V.4. *Let $\mathbf{R}_j^a$ be as in* (5.22). *Then $\mathbf{R}_j^a, j = 1,\dots,J$, satisfy Condition V.1. A properly scaled smoother $\gamma\mathbf{R}_j^a$ will satisfy Conditions V.1 and V.2.*

Proof. Clearly $\mathbf{R}_j^a$ are symmetric with respect to $(\cdot,\cdot)_*$. It is well known that (see e.g. [21]) for any $\boldsymbol{u} \in \boldsymbol{U}_j$,

$$\Big((\mathbf{R}_j^a)^{-1}\boldsymbol{u}, \boldsymbol{u}\Big)_* = \inf_{\{\boldsymbol{z}^e,\varphi^v\}} \sum_{e\in\mathcal{E}_h} \mathbf{A}(\boldsymbol{z}^e, \boldsymbol{z}^e) + \sum_{v\in\mathcal{V}_h} \mathbf{A}(\nabla\varphi^v, \nabla\varphi^v), \tag{5.23}$$

where the inf is taken over all $\boldsymbol{z}^e \in \boldsymbol{U}_j^e$ and $\varphi^v \in S_j^v$ such that $\boldsymbol{u} = \sum_e \boldsymbol{z}^e + \sum_v \nabla\varphi^v$.

Let $\boldsymbol{u} = \boldsymbol{z}_j + \nabla\varphi_j$ be the discrete Helmholtz decomposition of $\boldsymbol{u}$. Decompose

$$\boldsymbol{z}_j = \sum_e \boldsymbol{z}_j^e \quad \text{and} \quad \varphi_j = \sum_v \varphi_j^v,$$

as we specified in (5.21). Then, we have

$$\begin{aligned}\Big((\mathbf{R}_j^a)^{-1}\boldsymbol{u},\boldsymbol{u}\Big)_* &\le \sum_e \mathbf{A}(\boldsymbol{z}_j^e,\boldsymbol{z}_j^e)+\sum_v \|\nabla\varphi_j^v\|^2\\ &\le \sum_e (1+Ch_j^{-2})\|\boldsymbol{z}_j^e\|^2 + Ch_j^{-2}\sum_v \|\varphi_j^v\|^2\\ &\le Ch_j^{-2}\big(\|\boldsymbol{z}_j\|^2+\|\varphi_j\|^2\big) = Ch_j^{-2}\|\boldsymbol{u}\|_*^2.\end{aligned}$$

In the last inequality we used (5.21). This shows that $\mathbf{R}_j^a$ satisfies Condition V.1.

We now show the second part. For any decomposition of $\boldsymbol{u}$ to patch functions $\boldsymbol{u}=\sum_e \boldsymbol{z}^e+\sum_v \nabla\varphi^v$ where $\boldsymbol{z}^e\in\boldsymbol{U}_j^e$ and $\varphi^v\in S_j^v$, by the limited overlapping of the support of patch functions, we have

$$A(\boldsymbol{u},\boldsymbol{u}) \le C_0\sum_e A(\boldsymbol{z}^e,\boldsymbol{z}^e)+C\sum_v A(\nabla\varphi^v,\nabla\varphi^v),$$

and thus, using (5.23),

$$A(\boldsymbol{u},\boldsymbol{u}) \le C_0\Big((\mathbf{R}_j^a)^{-1}\boldsymbol{u},\boldsymbol{u}\Big)_*.$$

Taking γ such that $\omega_2\equiv C_0/\gamma\in(0,2)$, we complete the proof.

□

C. Analysis of multigrid methods

In this section, we will give the proof of Conditions V.3 and V.4, which are key parts of the multilevel analysis. The equivalence of the norms $\|\cdot\|_*$ and $\|\cdot\|_{*,h}$ on $\boldsymbol{U}_h$ plays an important role in this section.

It is required in [35] that the solenoidal component of Helmholtz decomposition of any function $\boldsymbol{u}$ in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ be $\boldsymbol{H}^1$-regular, which fails when the domain Ω is not convex. For an eddy-current problem on the non-convex domain, Hiptmair [36] analyzed multilevel methods using approximate Helmholtz-decompositions of the function space $\boldsymbol{H}(\mathbf{curl};\Omega)$ into an $\boldsymbol{H}^1$-regular subspace and gradients. The approach used here is to decompose the solenoidal component further as in Lemma IV.2.

The following lemma was essentially shown in [35] with $\boldsymbol{w}$ being the solenoidal component of the Helmholtz decomposition. However, not like the solenoidal component only satisfying $\boldsymbol{w} \times \boldsymbol{n} = 0$ on $\partial\Omega$, the regular component of the decomposition in Lemma IV.2 is in $\boldsymbol{H}_0^1(\Omega)$. This allows us to use the $\boldsymbol{L}^2$-projection $\mathbf{Q}_h^S$ onto $(S_h^1)^3$ instead of $(\overline{S}_h^1)^3$, and thus simplifies the construction for high order Nedelec elements. For the lowest order case, we still use Hiptmair's construction in [35].

Lemma V.4. *Let* $\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$ *be such that* $\mathbf{curl}\,\boldsymbol{w} \in \boldsymbol{V}_h$. *Then, there are* $\boldsymbol{w}_j \in \boldsymbol{U}_j$ *such that* $\boldsymbol{\Pi}_h \boldsymbol{w} = \sum_{j=1}^J \boldsymbol{w}_j$ *and*

$$\sum_{j=1}^J h_j^{-2} \|\boldsymbol{w}_j\|^2 \le C |\boldsymbol{w}|_1^2. \tag{5.24}$$

Proof. If k, the order of the Nedelec element, is greater than 1, we construct the multilevel decomposition as follows. Let $\mathbf{Q}_h^S$ be the $\boldsymbol{L}^2$-projection onto $(S_h^1)^3$, the space of vector-valued piecewise-linear Lagrange finite elements. It is well known (see e.g. [54, 67, 70]) that we have $\boldsymbol{w}_j \in (S_h^1)^3 \subset \boldsymbol{U}_h$ such that $\mathbf{Q}_h^S \boldsymbol{w} = \sum_{j=1}^J \boldsymbol{w}_j$ and

$$\sum_{j=1}^J h_j^{-2} \|\boldsymbol{w}_j\|^2 \le C |\mathbf{Q}_h^S \boldsymbol{w}|_1^2 \le C |\boldsymbol{w}|_1^2.$$

Note that $\boldsymbol{\Pi}_h \boldsymbol{w} - \mathbf{Q}_h^S \boldsymbol{w}$ belongs to $\boldsymbol{U}_h$ and satisfies

$$\|\boldsymbol{\Pi}_h \boldsymbol{w} - \mathbf{Q}_h^S \boldsymbol{w}\| \le C \|\boldsymbol{\Pi}_h \boldsymbol{w} - \boldsymbol{w}\| + \|\mathbf{Q}_h^S \boldsymbol{w} - \boldsymbol{w}\| \le Ch |\boldsymbol{w}|_1.$$

Setting $\boldsymbol{w}_j$ to be $\boldsymbol{w}_j + \boldsymbol{\Pi}_h \boldsymbol{w} - \mathbf{Q}_h^S \boldsymbol{w}$, we get the desired decomposition satisfying (5.24).

If k is equal to 1, the construction is done in Section 5 of [35]. In this case, Lemma 5.2 of [35] is critical. □

Theorem V.5. *For any* $\boldsymbol{u} \in \boldsymbol{U}_J$*, there is a stable multilevel decomposition* $\boldsymbol{u} = \sum_{j=1}^{J} \boldsymbol{u}_j$ *and a constant* C *not depending on* J*, such that*

$$\sum_{j=1}^{J} \lambda_j \|\boldsymbol{u}_j\|_*^2 \le C \|\boldsymbol{u}\|_{\mathbf{A}}^2.$$

Proof. Let $\boldsymbol{u} = \boldsymbol{z} + \nabla\varphi$ be the Helmholtz decomposition of $\boldsymbol{u} \in \boldsymbol{U}_h$. Split $\boldsymbol{z} = \boldsymbol{w} + \nabla\psi$ as in Lemma IV.2. Thus, we have

$$\boldsymbol{u} = \boldsymbol{w} + \nabla\phi,$$

where $\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$ and $\phi = \varphi + \psi \in H_0^1(\Omega)$ satisfy

$$\|\boldsymbol{w}\|_1 + \|\phi\|_1 \le C \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}.$$

Furthermore, similar to the proof of Theorem V.1, we have a discrete decomposition

$$\boldsymbol{u} = \Pi_h \boldsymbol{u} = \Pi_h \boldsymbol{w} + \nabla\phi_h$$

for some $\phi_h \in S_h$. Note that

$$\begin{aligned} \|\nabla\phi_h\| &\le \|\nabla\phi\| + \|\boldsymbol{w} - \Pi_h \boldsymbol{w}\| \le \|\boldsymbol{u}\| + Ch|\boldsymbol{w}|_1 \\ &\le \|\boldsymbol{u}\| + Ch\|\mathbf{curl}\,\boldsymbol{u}\| \le C\|\boldsymbol{u}\|. \end{aligned}$$

We will split $\Pi_h \boldsymbol{w}$ and $\nabla\phi_h$ separately. For $\Pi_h \boldsymbol{w}$, by Lemma V.4, we have $\boldsymbol{w}_j \in \boldsymbol{U}_j, j = 1, 2, \ldots, J$ such that $\boldsymbol{\Pi}_h \boldsymbol{w} = \sum_{j=1}^{J} \boldsymbol{w}_j$ and

$$\sum_{j=1}^{J} h_j^{-2} \|\boldsymbol{w}_j\|^2 \le C |\boldsymbol{w}|_1^2. \tag{5.25}$$

For $\nabla\phi_h$, by well known results (see e.g. [54, 67, 70]), we have $\phi_j \in S_j, j = 1, 2, \ldots, J$

such that $\phi_h = \sum_{j=1}^{J} \phi_j$ and

$$\sum_{j=1}^{J} h_j^{-2} \|\phi_j\|^2 \leq C\|\phi\|_1^2 \leq C\|\boldsymbol{u}\|^2. \tag{5.26}$$

Finally, letting $\boldsymbol{u}_j = \boldsymbol{w}_j + \nabla\phi_j$ in $\boldsymbol{U}_j$, we have $\boldsymbol{u} = \sum_{j=1}^{J} \boldsymbol{u}_j$ and, by (5.25), (5.26), and Lemma V.1,

$$\sum_{j=1}^{J} h_j^{-2} \|\boldsymbol{u}_j\|_*^2 \leq C \sum_{j=1}^{J} h_j^{-2} (\|\boldsymbol{w}_j\|^2 + \|\phi_j\|^2) \leq C\|\boldsymbol{u}\|_{\mathbf{A}}^2.$$

□

The following lemma [20] says that the discontinuous piecewise polynomials have certain regularity. The proof uses the real method of interpolation of Lions and Peetre [43].

Lemma V.5. *Assume that we are given a quasi-uniform triangulation $\mathcal{T}_h$ of size h on Ω and consider any function $\boldsymbol{v}$ of discontinuous piecewise polynomials up to degree k on this triangulation. Then, $\boldsymbol{v}$ is in $\boldsymbol{H}^s(\Omega)$ for any $s \in [0, 1/2)$, and satisfies that*

$$\|\boldsymbol{v}\|_s \leq C h^{-s} \|\boldsymbol{v}\|.$$

To prove Condition V.4, with $\lambda_j \approx h_j^{-2}$ and $h_j \approx \gamma^{-j}$, we only need to show the following theorem. Our proof is valid for Nedelec spaces of any order.

Theorem V.6. *Let $1 \leq i \leq j \leq J$. Then, for any $s \in [0, \frac{1}{2})$,*

$$A(\boldsymbol{u}_i, \boldsymbol{u}_j) \leq C \Big(\frac{h_j}{h_i}\Big)^s h_j^{-1} \|\boldsymbol{u}_i\|_{\mathbf{A}} \|\boldsymbol{u}_j\|_*, \quad \textit{for all } \boldsymbol{u}_i \in \boldsymbol{U}_i, \boldsymbol{u}_j \in \boldsymbol{U}_j.$$

Proof. Let $\boldsymbol{u}_i = \boldsymbol{z} + \nabla\varphi$ and $\boldsymbol{u}_j = \boldsymbol{w} + \nabla\psi$ be the corresponding Helmholtz decompositions. Then,

$$\begin{aligned} A(\boldsymbol{u}_i, \boldsymbol{u}_j) &= A(\boldsymbol{u}_i, \boldsymbol{w}) + (\boldsymbol{u}_i, \nabla\psi) \\ &= (\boldsymbol{u}_i, \boldsymbol{w}) + (\mathbf{curl}\, \boldsymbol{u}_i, \mathbf{curl}\, \boldsymbol{w}) + (\boldsymbol{u}_i, \nabla\psi). \end{aligned}$$

We estimate the above three terms separately. For the first term, we have $(\boldsymbol{u}_i, \boldsymbol{w}) \le \|\boldsymbol{u}_i\|\|\boldsymbol{w}\|$. For the second term, we have

$$\begin{aligned} (\mathbf{curl}\, \boldsymbol{u}_i, \mathbf{curl}\, \boldsymbol{w}) &\le C\|\mathbf{curl}\, \boldsymbol{u}_i\|_s \|\mathbf{curl}\, \boldsymbol{w}\|_{-s} \\ &\le C h_i^{-s} \|\mathbf{curl}\, \boldsymbol{u}_i\| \cdot h_j^{s-1} \|\boldsymbol{w}\|, \end{aligned}$$

where the last inequality follows from $\|\mathbf{curl}\, \boldsymbol{w}\|_{-1} \le C\|\boldsymbol{w}\|$, Lemma V.2, and the convexity. For the third term, we use the same technique to get

$$(\boldsymbol{u}_i, \nabla\psi) \le C\|\boldsymbol{u}_i\|_s \|\nabla\psi\|_{-s} \le C h_i^{-s} \|\boldsymbol{u}_i\| \cdot h_j^{s-1} \|\boldsymbol{u}_j\|_*.$$

Combining the above yields the desired inequality. □

CHAPTER VI

CONCLUSIONS

Our studies provide a rigorous theoretical analysis of some existing numerical techniques applied to the time dependent Maxwell's problem. We build up a solid basis for the study of regularity of solutions to time dependent Maxwell's interface problem, give optimal error estimates of the finite element method for the time dependent Maxwell's problem, and analyzed some preconditioning techniques for linear system arising in solving the discretized time dependent Maxwell's problem.

In Chapter II and III, we reduce the regularity of solutions to Maxwell's equations to the regularity of solutions of certain Laplacian interface problems via a lemma in [24] and thus one can use results that are available on this subject (see e.g. [42, 53]). This gives us a clear idea where singularities of solutions come from and how much regularity we can use in the error analysis of numerical approximations to Maxwell's equations.

In Chapter III, based on regularity results, we give optimal error estimates for the semidiscrete finite element scheme for the time dependent Maxwell's interface problem using Nedelec and Raviart-Thomas elements. This generalizes the result in [45]. The error estimates of the Fortin operator $\boldsymbol{\pi}_h$ obtained in Section B may also be used to study finite element methods to time-harmonic Maxwell's equations and eigenvalue problems [9].

In Chapter IV, we extend the convergence analysis in [34, 64] for overlapping Schwarz methods to the case of a simply-connected computational domain. The results are uniform with respect to mesh sizes and the time step. An important tool in our analysis is Lemma IV.2, by which we are able to decompose the weakly solenoidal component of Helmholtz decomposition into a regular $\boldsymbol{H}^1$-field and a gradient. This technique is also crucial in our analysis of multigrid methods.

In Chapter V we present a new convergence analysis of multigrid methods using the frame work in [19, 21]. This is made possible by introducing new (base) inner-products in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$. Then the estimate of condition numbers of both additive and multiplicative multilevel preconditioners follows from the abstract theory. Our analysis is valid for rather general computational domains.

There are still a few topics not discussed in this dissertation. The first is that the model Maxwell's equations studied here do not cover many interesting physical problems [11]. For example, the material occupying Ω is assumed to be a dielectric medium, and homogeneous boundary conditions are set on the whole boundary. Secondly, although implicit schemes are theoretically better than explicit schemes, we have not done any numerical experiments to support our theory. The third is the construction of robust preconditioners with respect to coefficients ε and μ. It is believed that substructuring methods provide robust preconditioners for second order elliptic problems with strongly discontinuous coefficients [40, 58] and there is already pioneering work on Maxwell's equations [2, 66] for specific situations. Being short of time, I have to leave these interesting problems to future study.

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APPENDIX

APPLICATION OF SEMIGROUP ON PARTIAL DIFFERENTIAL EQUATIONS

In this appendix, we state some definitions and properties of semigroup of class (C_0) [69] and its application [56] to the following abstract Cauchy problem:

$$\begin{cases} \frac{d}{dt}u(t) + Au(t) = f(t) \quad 0 \leq t \leq T, \\ u(0) = u_0, \end{cases} \tag{1}$$

where A is a linear operator from the Hilbert space $(X, (\cdot, \cdot))$ to itself, $f(t) : [0, T] \to X$ is an X-valued function, and u_0 is the initial condition.

Definition 1. *If $\{T_t; t \geq 0\} \subseteq L(X, X)$ satisfy the conditions*

$$T_t T_s = T_{t+s}, \quad \textit{for all } t, s \geq 0, \tag{2}$$

$$T_0 = I, \tag{3}$$

$$\lim_{t \to t_0} \|T_t b - T_{t_0} b\| = 0, \quad \textit{for all } b \in X, \tag{4}$$

then $\{T_t; t \geq 0\}$ is called a semigroup of class (C_0).

Lemma 1. *Let $\{T_t; t \geq 0\}$ be a semigroup of class (C_0). There exist constants $M > 0$ and $\beta < \infty$ such that*

$$\|T_t\| \leq M e^{\beta t}, \quad \textit{for all } t \geq 0. \tag{5}$$

The operator A is said to be the infinitesimal generator of the semigroup $\{T_t; t \geq 0\}$ of class (C_0) if

$$A\, b = \lim_{h \to 0+} \frac{T_h b - b}{h}. \tag{6}$$

Theorem 1 (M. Stone). *If the densely defined operator A is skew symmetric and satisfies $D(A) = D(A^*)$, A is the infinitesimal generator of some semigroup U_t of class (C_0).*

Theorem 2. *Let X be a Hilbert space, and $-A$ the infinitesimal generator of the semigroup T_t of class (C_0) on B. If $f \in C^1([0,T],X)$ and $u_0 \in D(A)$, the function $u(t)$ given by*

$$u(t) = T_t u_0 + \int_0^t T_{t-s} f(s)\, ds. \tag{7}$$

belongs to $C^1([0,T],X)$ and solves the problem (1).

Moreover, for each $t \in [0,T]$, $u(t) \in D(A)$ satisfies

$$\|u(t)\| + \|u_t(t)\| \le M e^{\beta t}\Big\{\|u_0\| + \|Au_0\| + \|f(0)\| + \int_0^t (\|f(s)\| + \|f'(s)\|)\, ds\Big\}, \tag{8}$$

where M and β are in Lemma 1.

Proof. We will only show (8). Let $u^H(t) = T_t u_0$ and $u^I(t) = \int_0^t T_{t-s} f(s)\, ds$. It is easy to see that

$$\frac{u^H(t+h) - u^H(t)}{h} = T_t \frac{T_h u_0 - u_0}{h} \to -T_t A u_0, \quad \text{as} \quad h \to 0,$$

and thus

$$\|u^H(t)\| + \|u^H_t(t)\| = \|T_t u_0\| + \|T_t A u_0\| \le M e^{\beta t}(\|u_0\| + \|Au_0\|). \tag{9}$$

For $u^I(t)$ we have

$$\begin{aligned}\frac{u^I(t+h) - u^I(t)}{h} &= \frac{1}{h}\int_0^{t+h} T_{t+h-s} f(s)\, ds - \frac{1}{h}\int_0^t T_{t-s} f(s)\, ds \\ &= \frac{1}{h}\int_{-h}^t T_{t-s} f(s+h)\, ds - \frac{1}{h}\int_0^t T_{t-s} f(s)\, ds \\ &= \int_0^t T_{t-s} \frac{f(s+h) - f(s)}{h}\, ds + \frac{1}{h}\int_t^{t+h} T_s f(t+h-s)\, ds.\end{aligned}$$

Note that

$$\|(f(s+h)-f(s))/h\| = \|h^{-1}\int_s^{s+h} f'(\tau)d\tau\| \le \max_{0\le\tau\le T}\|f'(\tau)\|.$$

By Lebesgue control theorem, we have that

$$\begin{aligned}
&\|\int_0^t T_{t-s}\frac{f(s+h)-f(s)}{h}\,ds - \int_0^t T_{t-s}f'(s)\,ds\| \\
&\le \int_0^t \|T_{t-s}\|\,\|\frac{f(s+h)-f(s)}{h} - f'(s)\|\,ds \\
&\le M\int_0^t e^{\beta(t-s)}\|\frac{f(s+h)-f(s)}{h} - f'(s)\|\,ds \\
&\to 0, \quad \text{as} \quad h\to 0,
\end{aligned}$$

where we have used Lemma 1. Similarly, we can show that

$$\frac{1}{h}\int_t^{t+h} T_s f(t+h-s)\,ds \to T_t f(0), \quad \text{as} \quad h\to 0.$$

Therefore, $u_t^I(t) = \int_0^t T_{t-s}f'(s)\,ds + T_t f(0)$ and thus

$$\begin{aligned}
&\|u^I(t)\| + \|u_t^I(t)\| \\
&= \|\int_0^t T_{t-s}f(s)\,ds\| + \|\int_0^t T_{t-s}f'(s)\,ds + T_t f(0)\| \\
&\le M\int_0^t e^{\beta(t-s)}(\|f(s)\| + \|f'(s)\|)\,ds + Me^{\beta t}\|f(0)\|. \qquad (10)
\end{aligned}$$

The desired estimate (8) follows from (9), (10) and triangle inequality. □

From the proof we can see that the derivative of the solution u is

$$u'(t) = T_t(-Au_0 + f_0) + \int_0^t T_{t-s}f'(s)\,ds.$$

Comparing the above with (7), we know that $u'(t)$ satisfies that

$$\begin{cases} \frac{d}{dt}v(t) + Av(t) = f'(t) \quad 0 < t \leq T, \\ v(0) = -Au_0 + f_0, \end{cases} \tag{11}$$

Repeating the argument m times, we can get the following corollary.

Corollary 1. *Under assumptions in Theorem 2, if* $f \in C^m([0,T], X)$*, and* $(-A)^{k-1}u_0 + \sum_{i=0}^{k-2}(-A)^i f^{(k-i-2)}(0) \in D(A)$ *for all* $k = 1, 2, \ldots, m$*, the solution* $u(t)$ *given by* (7) *belongs to* $C^m([0,T], X)$*.*

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