L^∞-ERROR ESTIMATES AND SUPERCONVERGENCE IN MAXIMUM NORM OF MIXED FINITE ELEMENT METHODS FOR NONFICKIAN FLOWS IN POROUS MEDIA

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ABSTRACT. On the basis of the estimates for the regularized Green's functions with memory terms, optimal L^∞-error estimates are established for the nonFickian flow of fluid in porous media by means of a mixed Ritz-Volterra projection. Moreover, local L^∞-superconvergence estimates for the velocity along the Gauss lines and for the pressure at the Gauss points are derived for the mixed finite element method, and global L^∞-superconvergence estimates for the velocity and the pressure are also investigated by virtue of an interpolation post-processing technique. Meanwhile, some useful a-posteriori error estimators are presented for this mixed finite element method.

1. INTRODUCTION

The nonFickian flow of fluid in porous media can be modeled by an integro-differential equation: Find $u = u(x,t)$ such that

$$\begin{aligned}
u_t &= \nabla \cdot \sigma + cu + f && \text{in } \Omega \times J, \\
\sigma &= A(t) \cdot \nabla u - \int_0^t B(t,s) \cdot \nabla u(s) ds && \text{in } \Omega \times J, \\
u &= g && \text{on } \partial\Omega \times J, \\
u &= u_0(x) && x \in \Omega,\ t = 0,
\end{aligned} \tag{1.1}$$

where $\Omega \subset R^d$ $(d = 2, 3)$ is an open bounded domain with smooth boundary $\partial\Omega$, $J = (0, T)$ with $T > 0$, $A(t) = A(x,t)$ and $B(t,s) = B(x,t,s)$ are two 2×2 or 3×3 matrices, and A is positive definite, $c \leq 0$, f, g and u_0 are known smooth functions. This kind of flow is complicated by the history effect characterizing various mixing length growth of the flow, which has been investigated, for example, in [9, 10].

The numerical approximations of the problem (1.1) are available in extensive literature. See, for instance, [2, 3, 11, 13, 14, 18-21].

In the present paper, the approximate solutions of (1.1) are studied by mixed finite element methods. Optimal L^∞-error estimates are obtained by employing a mixed Ritz-Volterra projection introduced in [11]. In addition, local L^∞-superconvergence estimates for the velocity along the Gauss lines and for the pressure at the Gauss points are derived, and with the aid of an interpolation post-processing method global L^∞-superconvergence

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estimates are also considered for the velocity and the pressure. As a result of the global superconvergence, a-posteriori error indicators of the mixed finite element method are presented in the paper.

The paper is organized in the following manner. In Section 2, we give the approximate sub-space and the approximate problem. Two regularized Green's functions and a Ritz-Volterra projection with memory terms for the mixed form for the problem (1.1) are introduced in Section 3. Also, in Section 3 the L^1-error estimates for the mixed finite element approximations of the regularized Green's functions are stated, and the L^∞-error estimates for the mixed Ritz-Volterra projection are established. In Section 4, optimal error estimates in maximum norm are given for the mixed finite element approximations. Section 5 is devoted to the local and global L^∞-superconvergence analysis of the mixed finite element method, by which some a-posteriori error estimators are obtained for the mixed finite element method. Finally, the L^1-error estimates for the mixed finite element approximations of the regularized Green's functions are proved in Section 6.

2. The mixed finite element method

In this section, we give the mixed finite element approximate scheme for the parabolic integro-differential equation (1.1). For simplicity, the method will be presented on plane domains.

Let $W := L^2(\Omega)$ be the standard L^2 space on Ω with norm $\|\cdot\|_0$. Denote by

$$\mathbf{V} := H(\mathrm{div}, \Omega) = \left\{\sigma \in (L^2(\Omega))^2 \mid \nabla\cdot\sigma \in L^2(\Omega)\right\}$$

the Hilbert space equipped with the following norm:

$$\|\sigma\|_{\mathbf{V}} := \left(\|\sigma\|_0^2 + \|\nabla\cdot\sigma\|_0^2\right)^{\frac{1}{2}}.$$

There are several ways to discretize the problem (1.1) based on the variables σ and u; each method corresponds to a particular variational form of (1.1) [13, 20, 21].

Let T_h be a finite element partition of Ω into triangles or quadrilaterals which is quasi-uniform. Let $\mathbf{V}_h \times W_h$ denote a pair of finite element spaces satisfying the Brezzi-Babuska condition. Although there are now several choices for $\mathbf{V}_h$ and W_h, here we only consider the Raviart-Thomas elements of order $k \geq 0$ [23]. The extension to other stable elements can be made without any difficulty.

Recall from [13] that the week mixed formulation of (1.1) is given by finding $(u, \sigma) \in W \times \mathbf{V}$ such that

$$\begin{aligned}
&(u_t, w) - (\nabla\cdot\sigma, w) - (cu, w) = (f, w), && w \in W,\\
&(\alpha\sigma, \mathbf{v}) + \int_0^t (M(t,s)\sigma(s), \mathbf{v})ds + (\nabla\cdot\mathbf{v}, u) = \langle g, \mathbf{v}\cdot\mathbf{n}\rangle, && \mathbf{v} \in \mathbf{V},\\
&u(0,x) = u_0(x) \quad \text{in } L^2(\Omega),
\end{aligned} \tag{2.1}$$

where $\alpha = A^{-1}(t)$, $M(t,s) = R(t,s)A^{-1}(s)$ and $R(t,s)$ is the resolvent of the matrix $A^{-1}(t)B(t,s)$ and is given by

$$R(t,s) = A^{-1}(t)B(t,s) + \int_s^t A^{-1}(t)B(t,\tau)\, R(\tau,s)ds, \quad t > s \geq 0.$$

Here $\langle\cdot,\cdot\rangle$ indicates the L^2-inner product on $\partial\Omega$.

The corresponding semi-discrete version is to seek a pair $(u_h, \sigma_h) \in W_h \times \mathbf{V}_h$ such that

$$\begin{aligned}
&(u_{h,t}, w_h) - (\nabla\cdot\sigma_h, w_h) - (cu_h, w_h) = (f, w_h), && w_h \in W_h,\\
&(\alpha\sigma_h, \mathbf{v}_h) + \int_0^t (M(t,s)\sigma_h(s), \mathbf{v}_h)ds + (u_h, \nabla\cdot\mathbf{v}_h) = \langle g, \mathbf{n}\cdot\mathbf{v}_h\rangle, && \mathbf{v}_h \in \mathbf{V}_h.
\end{aligned} \tag{2.2}$$

The discrete initial condition $u_h(0,x) = u_{0,h}$, where $u_{0,h} \in W_h$ is some appropriately chosen approximation of the initial data $u_0(x)$, should be added to (2.2) for starting. The pair (u_h, σ_h) is a semi-discrete approximation of the true solution of (1.1) in the finite element space $W_h \times \mathbf{V}_h$ [1, 6, 11, 13, 14], where $\sigma_h(0,x)$ is chosen to satisfy the equation (2.2) with $t = 0$; namely, it is related to $u_{0,h}$ as follows:

$$(\alpha\sigma_h(0), \mathbf{v}_h) + (u_{0,h}, \nabla\cdot\mathbf{v}_h) = \langle g_0, \mathbf{n}\cdot\mathbf{v}_h\rangle, \tag{2.3}$$

where $g_0 = g(0,x)$ is the initial value of the boundary data.

From (2.1) and (2.2) we derive the following mixed finite element error equation:

$$\begin{aligned}
&(u_t - u_{h,t}, w_h) - (\nabla\cdot(\sigma-\sigma_h), w_h) - (c(u-u_h), w_h) = 0, && w_h \in W_h,\\
&(\alpha(\sigma-\sigma_h), \mathbf{v}_h) + \int_0^t (M(t,s)(\sigma-\sigma_h)(s), \mathbf{v}_h)ds + (u-u_h, \nabla\cdot\mathbf{v}_h) = 0, && \mathbf{v}_h \in \mathbf{V}_h.
\end{aligned} \tag{2.4}$$

Throughout the paper, we often need the following Raviart-Thomas projection [7, 23].

$$\Pi_h^k \times P_h^k : \mathbf{V}\times W \to \mathbf{V}_h \times W_h,$$

which has the properties:

(i) P_h^k is the $L^2(\Omega)$ projection;

(ii) Π_h^k and P_h^k satisfy

$$(\nabla\cdot(\sigma - \mathbf{\Pi}_h^k\sigma), w_h) = 0,\ w_h \in W_h \quad \text{and} \quad (\nabla\cdot\mathbf{v}_h, u - P_h^k u) = 0,\ \mathbf{v}_h \in \mathbf{V}_h. \tag{2.5}$$

(iii) the following approximation properties hold

$$\begin{array}{lll}
\|\sigma - \mathbf{\Pi}_h^k\sigma\|_{0,p} \le Ch^r\|\sigma\|_{r,p}, & 1 \le r \le k+1, & 1 \le p \le \infty,\\
\|\nabla\cdot(\sigma - \Pi_h^k\sigma)\|_{-s,p} \le Ch^{r+s}\|\nabla\cdot\sigma\|_{r,p}, & 0 \le r,\ s \le k+1, & 1 \le p \le \infty,\\
\|u - P_h^k u\|_{-s,p} \le Ch^{r+s}\|u\|_{r,p}, & 0 \le r,\ s \le k+1, & 1 \le p \le \infty.
\end{array} \tag{2.6}$$

3. The mixed Ritz-Volterra projection and its L^∞-error estimates

In this section, we consider optimal and superconvergent error estimates in L^∞-norm for the mixed Ritz-Volterra projection. It is well-known that the regularized Green's function plays an essential role in the analysis of maximum norm and superconvergence for finite element methods and mixed finite element methods of elliptic equations [8, 12, 17, 25-27] and parabolic equations [17]. For the finite element method of parabolic integro-differential equations, maximum norm and superconvergence have been obtained in [18, 19] using the modified regularized Green's function with memory term. Here we consider the mixed finite element approximations for parabolic equations with memory, and it is expected that certain modification form of the standard regularized Green's function with memory should be introduced, analysed and used in our analysis.

Let us define the following two linear operators $M*$ and $M**$ for any smooth function $f(t)$ defined on $(0,T)$ by

$$(M*f)(t) := \int_0^t M(t,s)f(s)ds \quad \text{and} \quad (M**f)(t) := \int_t^T M(s,t)f(s)ds.$$

Then, we have

Lemma 3.1. *There holds*

$$< M*f, g >_T := \int_0^T M*f(t)\ g(t)dt = \int_0^T f(t)\ M**g(t)dt := < f, M**g >_T .$$

Proof. The result follows from exchanging the order of integration. □

Lemma 3.2. *Assume that* $f(t),\ g(t) \in L^1(0,T^*)$ *and there exists* $C > 0$ *such that for any non-negative* $\phi(t) \in C^\infty(0,T)$,

$$\left|\int_0^T f(t)\phi(t)dt\right| \le C\left|\int_0^T g(t)(1+\phi(t))dt\right|, \quad 0 \le T \le T^*.$$

Then, we have

$$|f(t)| \le C\left|g(t) + \int_0^t g(s)ds\right|, \quad \forall t \in (0,T),\ a.e.$$

Especially,

$$|f(t)| \le C|g(t)|, \quad \forall t \in (0,T),\ a.e. \quad if$$

$$\left|\int_0^T f(t)\phi(t)dt\right| \le C\left|\int_0^T g(t)\phi(t)dt\right|.$$

Proof. Take $\mu > 0$ and let

$$\phi_\mu(t,t_0) = \begin{cases} (C_\mu)^{-1}\exp\left(-\frac{\mu^2}{\mu^2-|t-t_0|^2}\right), & |t-t_0| < \mu, \\ 0, & |t-t_0| \ge \mu, \end{cases}$$

where t_0 is any fixed point in $(0,T)$ and $C_\mu := \mu\int_{|t|<1}\exp\left(-\frac{1}{1-t^2}\right)dt$. We see easily that for almost all $t_0 \in (0,T)$,

$$f(t_0) = \lim_{\mu\to 0}\int_0^T f(t)\phi_\mu(t,t_0)dt, \quad f \in C^\infty(0,T).$$

Thus, if we take $f_n(t) \in C^\infty(0,T)$ such that $f_n(t) \to f(t)$ as $n \to \infty$ in $L^1(0,T)$, then the result is true for all $f_n(t)$. Therefore, it is true for $f(t)$ via a limiting procedure. □

For an arbitrary point $z_0 \in \bar{\Omega}$, let

$$\beta(z,z_0) := (|z-z_0|^2 + \theta^2)^{1/2}$$

be the weight function used in [24, 25, 27], where $z = (x,y) \in R^2$, $\theta = \gamma h$ and γ is a positive number chosen appropriately. Moreover, as usual, for any $\alpha \in R$ we define the weighted norms by

$$||u||^2_{\beta^\alpha,Q} := \int_Q \beta^\alpha u^2 d\Omega,$$

and $||\cdot||_{\beta^\alpha}$ is the weighted norm for $Q = \Omega$. Then, we have [25, 27]

$$\int_\Omega \beta^{-2}d\Omega \le C\log\frac{1}{h}. \tag{3.1}$$

Next we shall define two regularized Green's functions with memory terms for the problem (1.1) in mixed form in the fasion analogous to that employed earlier for Galerkin methods [27]. Our main results concerning the regularized Green's functions and their mixed finite element approximations are L^1-error estimates which are useful for establishing L^∞-error estimates and superconvergence in maximum norm for the mixed finite element solution of (1.1).

For simplicity, we assume that $c = 0$. Thus, for an arbitrary point $z_0 \in \bar{\Omega}$ the first pair modified regularized Green's function $(\mathbf{G}_1,\lambda_1) = (\mathbf{G}_1(z,z_0),\lambda_1(z,z_0))$ with memory is defined as the solution of the following system:

$$\begin{aligned} \alpha\mathbf{G}_1 + M**\mathbf{G}_1 - \nabla\lambda_1 &= 0, && \text{in } \Omega\times(0,T), \\ \text{div}\mathbf{G}_1 &= \delta_1^h\phi_1(t), && \text{in } \Omega\times(0,T), \\ \lambda_1 &= 0, && \text{on } \partial\Omega\times(0,T), \end{aligned} \tag{3.2}$$

where $\phi_1(t) \in C^\infty(0,T)$, and $\delta_1^h = \delta_1^h(z,z_0) \in W_h$ is the regularized Dirac δ-function at any fixed point $z_0 \in \bar{\Omega}$ such that ([8, 12, 25, 26])

$$||w_h||_\infty \leq C|(w_h, \delta_1^h)|, \quad w_h \in W_h. \tag{3.3}$$

We also introduce the second pair regularized Green's function $(\mathbf{G}_2, \lambda_2) = (\mathbf{G}_2(z,z_0), \lambda_2(z,z_0))$ such that

$$\begin{array}{rcll} \alpha \mathbf{G}_2 + M ** \mathbf{G}_2 - \nabla \lambda_2 &=& \delta_2^h \phi_2(t), & \text{in } \Omega \times (0,T), \\ \mathrm{div} \mathbf{G}_2 &=& 0, & \text{in } \Omega \times (0,T), \\ \lambda_2 &=& 0, & \text{on } \partial\Omega \times (0,T), \end{array} \tag{3.4}$$

where $\phi(t) \in C^\infty(0,T)$ and δ_2^h is either $(\delta_2^h, 0)$ or $(0, \delta_2^h)$ with δ_2^h being a regularized Dirac δ-function at z_0, which depends upon the needs of our analysis, such that an analogue of (3.3) is also valid for δ_2^h. In addition, δ_2^h, $\phi_1(t)$ and $\phi_2(t)$ are required to satisfy

$$\delta_2^h \geq 0, \ \int_\Omega \delta_2^h d\Omega = 1; \ \phi_i(t) \geq 0, \ \int_0^T \phi_i(t) dt \leq 1, \ i = 1, 2. \tag{3.5}$$

Now and in what follows in this paper, the domain Ω is assume to be H^2-regular [7]. Therefore, it is not difficult to show (see, for example, $(3.6a) - (3.6d)$ in [25]) that the following Theorem 3.1 is true.

Theorem 3.1. *There exists a positice constant $C > 0$, independent of $h, t, \phi_1(t)$, such that*

$$\begin{aligned} &||\nabla \lambda_1||_0 \leq C \left(\log \frac{1}{h} \right)^{1/2} (1 + \phi_1(t)), \\ &||\nabla^2 \lambda_1||_0 \leq C h^{-1} (1 + \phi_1(t)), \\ &||\nabla^2 \lambda_1||_{\beta^2} \leq C \left(\log \frac{1}{h} \right)^{1/2} (1 + \phi_1(t)), \\ &||\nabla^2 \lambda_1||_{L^1(\Omega)} \leq C \log \frac{1}{h} (1 + \phi_1(t)). \end{aligned}$$

Theorem 3.2. *Assume that $(\mathbf{G}_1, \lambda_1)$ and $(\mathbf{G}_1^h, \lambda_1^h)$ are the exact solution and the mixed finite element approximation of (3.2), respectively. Then, there exists a positive constant $C > 0$, independent of h, t, ϕ_1, such that*

$$\begin{aligned} &||\mathbf{G}_1^h - \mathbf{G}_1||_0 \leq C(1 + \phi_1(t)), \\ &||\mathbf{G}_1^h - \mathbf{G}_1||_{L^1(\Omega)} \leq C h \log \frac{1}{h} (1 + \phi_1(t)), \\ &||\lambda_1^h - \lambda_1||_0 \leq C h \left(\log \frac{1}{h} \right)^{1/2} (1 + \phi_1(t)). \end{aligned}$$

Theorem 3.3. *Assume that $(\mathbf{G}_2, \lambda_2)$ and $(\mathbf{G}_2^h, \lambda_2^h)$ are the exact solution and the mixed finite element approximation of (3.4), respectively. Then, there exists a positive constant $C > 0$, independent of h, t, ϕ_2, such that*

$$\begin{aligned} &||\mathbf{G}_2^h - \mathbf{G}_2||_0 \leq C h^{-1} (1 + \phi_2(t)), \\ &||\mathbf{G}_2^h - \mathbf{G}_2||_{L^1(\Omega)} \leq C \left(\log \frac{1}{h} \right)^{1/2} (1 + \phi_2(t)), \\ &||\lambda_2^h - \lambda_2||_0 \leq C(1 + \phi_2(t)), \end{aligned}$$

$$||\lambda_2||_0 \le C\left(1+|\log h|^{1/2}\right)(1+\phi_2(t)),$$
$$||\nabla\lambda_2||_0 \le Ch^{-1}(1+\phi_2(t)),$$
$$||\nabla\lambda_2||_{L^1(\Omega)} \le C\log\frac{1}{h}(1+\phi_2(t)),$$
$$||\nabla^2\lambda_2||_{L^1(\Omega)} \le Ch^{-1}\left(\log\frac{1}{h}\right)^{1/2}(1+\phi_2(t)).$$

We would like to point out the estimate

$$||\nabla^2\lambda_2||_{L^1(\Omega)} \le Ch^{-1}\left(\log\frac{1}{h}\right)^{1/2}(1+\phi_2(t))$$

is not sharp, since it can be improved to

$$||\nabla^2\lambda_2||_{L^1(\Omega)} \le Ch^{-1}(1+\phi_2(t)) \tag{3.6}$$

if the domain is smooth enough. A proof of (3.6) can be found in [24].

Remark 3.1 The proofs of Theorems 3.2 and 3.3 will be postponed to Section 6 where the weighted norm estimates are used.

Following the procedure for Theorems 3.3 and 3.4 in [25] together with the application of Gronwall's lemma, we can also obtain the following Theorems 3.4 and 3.5.

Theorem 3.4. *Assume that Ω is a plane rectangular domain and $q \in [1,\infty]$. Then, we have*

$$\begin{aligned} ||\mathbf{G}_1^h||_q &\le Ch^{min\{0,\frac{2}{q}-1\}}|\log h|^{1/2}(1+\phi_1(t)) \\ ||\mathbf{G}_1-\mathbf{G}_1^h||_q &\le (C(q)+C|\log h|)\,h^{1-\frac{2}{p}}(1+\phi_1(t)),\ 1<q<\infty, \end{aligned}$$

where $p=\frac{q}{q-1}$ is the conjugate of q.

Theorem 3.5. *For $q \in [1,\infty]$, there hold*

$$\begin{aligned} ||\mathbf{G}_2^h||_q &\le \begin{cases} Ch^{-\frac{2}{p}}|\log h|(1+\phi_2(t)), & 1\le q<2,\\ Ch^{-\frac{2}{p}}(1+\phi_2(t)), & q\ge 2, \end{cases} \\ ||\mathbf{G}_2-\mathbf{G}_2^h||_q &\le \left(C(q)+C|\log h|^{1/2}\right)h^{-\frac{2}{p}}(1+\phi_2(t)),\ 1<q<\infty, \end{aligned}$$

where $p=\frac{q}{q-1}$.

In the following we shall show the error estimates in maximum norms for the mixed finite element approximation of (1.1). To this end, we first introduce the mixed Ritz-Volterra projection [11].

Definition 3.1 For $(u,\sigma)\in W\times\mathbf{V}$ we define a pair $(\bar{u}_h,\bar{\sigma}_h):[0,T]\to\mathbf{W}_h\times\mathbf{V}_h$ such that

$$\begin{aligned} (\alpha(\sigma-\bar{\sigma}_h)+M*(\sigma-\bar{\sigma}_h),\mathbf{v}_h)+(u-\bar{u}_h,\mathrm{div}\mathbf{v}_h) &= 0, \quad \mathbf{v}_h\in\mathbf{V}_h \\ (\mathrm{div}(\sigma-\bar{\sigma}_h),w_h) &= 0, \quad w_h\in W_h, \end{aligned} \tag{3.7}$$

where $\alpha=A^{-1}$. The pair $(\bar{u}_h,\bar{\sigma})$ is called the mixed Ritz-Volterra projection of (u,σ). It has been proved in [11] that the solution of (3.7) exists uniquely for a given pair (u,σ).

Lemma 3.3. *Assume that $(\bar{u}_h,\bar{\sigma}_h)$ is the mixed Ritz-Volterra projection of $(u,\sigma)\in W\times\mathbf{V}$. Then we have*

$$\int_0^T(\bar{u}_h-P_h^k u,\delta_1^h)\phi_1(t)dt = \int_0^T\left(\alpha(\sigma-\Pi_h^k\sigma)+M*(\sigma-\Pi_h^k\sigma),\ \mathbf{G}_1^h\right)dt,$$
$$\int_0^T(\bar{\sigma}_h-\Pi_h^k\sigma,\delta_2^h)\phi_2(t)dt = \int_0^T\left(\alpha(\sigma-\Pi_h^k\sigma)+M*(\sigma-\Pi_h^k\sigma),\ \mathbf{G}_2^h\right)dt.$$

Proof. It follows from (3.2) and its corresponding mixed finite element error equation to (2.4) that

$$(\bar{u}_h - P_h^k u, \delta_1^h \phi_1(t)) = (\bar{u}_h - P_h^k u, \mathrm{div}\mathbf{G}_1) = (\bar{u}_h - P_h^k u, \mathrm{div}\mathbf{G}_1^h).$$

Note that P_h^k is a local L^2-projection operator. Thus, we know from (2.5) that

$$(\bar{u}_h - P_h^k u, \delta_1^h \phi_1(t)) = (\bar{u}_h - u, \mathrm{div}\mathbf{G}_1^h)$$

which, together with (3.7), leads to

$$\begin{aligned}(\bar{u}_h - P_h^k u, \delta_1^h \phi_1(t)) &= (\alpha(\sigma - \bar{\sigma}_h) + M * (\sigma - \bar{\sigma}_h), \mathbf{G}_1^h) \\ &= (\alpha(\sigma - \Pi_h^k \sigma) + M * (\sigma - \Pi_h^k \sigma),\ \mathbf{G}_1^h) \\ &\quad + (\alpha(\Pi_h^k \sigma - \bar{\sigma}_h) + M * (\Pi_h^k \sigma - \bar{\sigma}_h),\ \mathbf{G}_1^h).\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^T (\bar{u}_h - P_h^k u, \delta_1^h)\phi_1(t)dt &= \int_0^T (\alpha(\sigma - \Pi_h^k \sigma) + M * (\sigma - \Pi_h^k \sigma),\ \mathbf{G}_1^h)dt \\ &\quad + \int_0^T (\alpha(\Pi_h^k \sigma - \bar{\sigma}_h) + M * (\Pi_h^k \sigma - \bar{\sigma}_h),\ \mathbf{G}_1^h)dt \\ &:= K_1 + K_2.\end{aligned} \tag{3.8}$$

However, it follows from Lemma 3.1 and the mixed finite element approximation of (3.2) as well as Green's formula that

$$\begin{aligned}K_2 &:= \int_0^T (\alpha\mathbf{G}_1^h + M * *\mathbf{G}_1^h,\ \Pi_h^k \sigma - \bar{\sigma}_h)dt \\ &= \int_0^T (\nabla\lambda_1^h,\ \Pi_h^k \sigma - \bar{\sigma}_h)dt \\ &= -\int_0^T (\lambda_1^h, \mathrm{div}(\Pi_h^k \sigma - \bar{\sigma}_h))dt,\end{aligned}$$

which, together with (2.5) and (3.7), yields

$$K_2 = -\int_0^T (\lambda_1^h, \mathrm{div}(\Pi_h^k \sigma - \sigma))dt - \int_0^T (\lambda_1^h, \mathrm{div}(\sigma - \bar{\sigma}_h))dt = 0.$$

Thus, from (3.8) we know that the first identity in Lemma 3.3 is true.

To prove the second identity, we use (3.4) and its corresponding mixed finite element error equation to (2.4) to see that

$$\begin{aligned}(\bar{\sigma}_h - \Pi_h^k \sigma, \delta_2^h)\phi_2(t) &= (\alpha\mathbf{G}_2 + M * *\mathbf{G}_2, \bar{\sigma}_h - \Pi_h^k \sigma) - (\nabla\lambda_2, \bar{\sigma}_h - \Pi_h^k \sigma) \\ &= (\alpha\mathbf{G}_2^h + M * *\mathbf{G}_2^h, \bar{\sigma}_h - \Pi_h^k \sigma) + (\lambda_2^h, \mathrm{div}(\bar{\sigma}_h - \Pi_h^k \sigma)).\end{aligned}$$

Thus, by means of Lemma 3.1, (2.5) and (3.7) we have

$$\begin{aligned}
\int_0^T (\bar\sigma_h - \Pi_h^k\sigma, \delta_2^h)\phi_2(t)dt &= \int_0^T (\alpha\mathbf{G}_2^h + M**\mathbf{G}_2^h, \bar\sigma_h - \Pi_h^k\sigma)dt \\
&\quad + \int_0^T (\lambda_2^h, \mathrm{div}(\bar\sigma_h - \Pi_h^k\sigma))dt \\
&= \int_0^T (\alpha(\bar\sigma_h - \Pi_h^k\sigma) + M*(\bar\sigma_h - \Pi_h^k\sigma),\ \mathbf{G}_2^h)dt \\
&+ \int_0^T (\lambda_2^h, \mathrm{div}(\bar\sigma_h - \sigma))dt + \int_0^T (\lambda_2^h, \mathrm{div}(\sigma - \Pi_h^k\sigma))dt \\
&= \int_0^T (\alpha(\bar\sigma_h - \Pi_h^k\sigma) + M*(\bar\sigma_h - \Pi_h^k\sigma),\ \mathbf{G}_2^h)dt \\
&= \int_0^T (\alpha(\sigma - \Pi_h^k\sigma) + M*(\sigma - \Pi_h^k\sigma), \mathbf{G}_2^h)dt \\
&+ \int_0^T (u - \bar u_h, \mathrm{div}\mathbf{G}_2^h)dt \\
&= \int_0^T (\alpha(\sigma - \Pi_h^k\sigma) + M*(\sigma - \Pi_h^k\sigma),\ \mathbf{G}_2^h)dt,
\end{aligned}$$

where $\mathrm{div}\mathbf{G}_2^h = 0$ has been used. This completes the proof. □

We are now ready to show the maximum norm error estimate for the mixed Ritz-Volterra projection. First, we consider it for $\bar u_h - P_h^k u$.

Theorem 3.6. *Let $(\bar u_h, \bar\sigma_h)$ be the Ritz-Volterra projection of (u,σ). Then, there exists a constant $C > 0$, independent of h and t, such that*

$$||\bar u_h - P_h^k u||_\infty \le \begin{cases} Ch\log\frac{1}{h}\left(|||\sigma - \Pi_h^0\sigma|||_\infty + (\log\frac{1}{h})^{-1/2}|||(I - P_h^0)\nabla\cdot\sigma|||_0\right), \\ C\left(|||\sigma - \Pi_h^0\sigma|||_0 + h\left(\log\frac{1}{h}\right)^{1/2}|||(I - P_h^0)\nabla\cdot\sigma|||_0\right), \\ Ch\log\frac{1}{h}\left(|||\sigma - \Pi_h^k\sigma|||_\infty + h|||(I - P_h^k)\nabla\cdot\sigma|||_\infty\right), & k \ge 1, \end{cases}$$

where $|||u|||_{r,p} := ||u(t)||_{r,p} + \int_0^t ||u(s)||_{r,p}ds$, $-\infty \le r \le \infty$, $1 \le p \le \infty$, $t > 0$. As usual, $|||u|||_{r,p}$ is simply denoted by $|||u|||_r$ when $p = 2$.

Proof. For any point $z_0 \in \bar\Omega$, let δ_1^h be the regularized Dirac δ-function associated with this point z_0, and then we find from Lemma 3.3 that

$$\begin{aligned}
\int_0^T (\bar u_h - P_h^k u, \delta_1^h)\phi_1(t)dt &= \int_0^T \left(\alpha(\sigma - \Pi_h^k\sigma) + M*(\sigma - \Pi_h^k\sigma),\ \mathbf{G}_1^h - \mathbf{G}_1\right)dt \\
&\quad + \int_0^T \left(\alpha(\sigma - \Pi_h^k\sigma) + M*(\sigma - \Pi_h^k\sigma),\ \mathbf{G}_1\right)dt \\
&:= K_{11} + K_{22}.
\end{aligned}$$

It is easy to see from lemma 3.1, (2.5) and (3.2) that

$$
\begin{aligned}
K_{22} &= \int_0^T \left(\alpha \mathbf{G}_1 + M ** \mathbf{G}_1,\ \sigma - \Pi_h^k \sigma\right) dt \\
&= \int_0^T \left(\nabla \lambda_1,\ \sigma - \Pi_h^k \sigma\right) dt \\
&= -\int_0^T \left(\lambda_1, \mathrm{div}(\sigma - \Pi_h^k \sigma)\right) dt \\
&= -\int_0^T \left(\lambda_1 - P_h^k \lambda_1, \mathrm{div}(\sigma - \Pi_h^k \sigma)\right) dt \\
&= -\int_0^T \left(\lambda_1 - P_h^k \lambda_1, \mathrm{div}\sigma\right) dt \\
&= -\int_0^T (\lambda_1 - P_h^k \lambda_1, (I - P_h^k)\mathrm{div}\sigma) dt.
\end{aligned}
\tag{3.9}
$$

Thus, we have for $k = 0$ that

$$
\begin{aligned}
&\left| \int_0^T (\bar{u}_h - P_h^0 u, \delta_1^h) \phi_1(t) dt \right| \\
&\leq \begin{cases}
C \int_0^T \left(|||\sigma - \Pi_h^0 \sigma|||_\infty ||\mathbf{G}_1^h - \mathbf{G}_1||_{L^1(\Omega)} + ||\lambda_1 - P_h^0 \lambda_1||_0 ||(I - P_h^0)\mathrm{div}\sigma||_0 \right) dt, \\
\text{or} \\
C \int_0^T \left(|||\sigma - \Pi_h^0 \sigma|||_0 ||\mathbf{G}_1^h - \mathbf{G}_1||_0 + ||(I - P_h^0)\lambda_1||_0 ||(I - P_h^0)\mathrm{div}\sigma||_0 \right) dt.
\end{cases}
\end{aligned}
$$

Noticing that for $k = 0$ by Theorem 3.1,

$$
||\lambda_1 - P_h^0 \lambda_1||_0 \leq Ch ||\nabla \lambda_1||_0 \leq Ch \left(\log \frac{1}{h} \right)^{1/2} (1 + \phi_1(t)),
$$

it follows from the above inequality and Theorem 3.2 that for $k = 0$

$$
\begin{aligned}
&\left| \int_0^T (\bar{u}_h - P_h^0 u, \delta_1^h) \phi_1(t) dt \right| \\
&\leq \begin{cases}
Ch \log \frac{1}{h} \int_0^T \left(|||\sigma - \Pi_h^0 \sigma|||_\infty + \left(\log \frac{1}{h} \right)^{-1/2} ||(I - P_h^0)\mathrm{div}\sigma||_0 \right) (1 + \phi_1(t)) dt, \\
\text{or} \\
C \int_0^T \left(|||\sigma - \Pi_h^0 \sigma|||_0 + h \left(\log \frac{1}{h} \right)^{1/2} ||(I - P_h^0)\mathrm{div}\sigma||_0 \right) (1 + \phi_1(t)) dt.
\end{cases}
\end{aligned}
$$

We now see from Lemma 3.2 and the arbitrariness of $\phi_1(t)$ that

$$
\left| (\bar{u}_h - P_h^0 u, \delta_1^h) \right| \leq \begin{cases}
Ch \log \frac{1}{h} \left(|||\sigma - \Pi_h^0 \sigma|||_\infty + \left(\log \frac{1}{h} \right)^{-1/2} |||(I - P_h^0)\mathrm{div}\sigma|||_0 \right) \\
\text{or} \\
C \left(|||\sigma - \Pi_h^0 \sigma|||_0 + h \left(\log \frac{1}{h} \right)^{1/2} |||(I - P_h^0)\mathrm{div}\sigma|||_0 \right),
\end{cases}
$$

from which and (3.3) we derive that for $k=0$

$$\|\bar{u}_h - P_h^0 u\|_\infty \le \begin{cases} Ch\log\frac{1}{h}\left(|||\sigma - \Pi_h^0\sigma|||_\infty + \left(\log\frac{1}{h}\right)^{-1/2} |||(I-P_h^0)\text{div}\sigma|||_0\right), \\ C\left(|||\sigma - \Pi_h^0\sigma|||_0 + h\left(\log\frac{1}{h}\right)^{1/2} |||(I-P_h^0)\text{div}\sigma|||_0\right). \end{cases}$$

Therefore, Theorem 3.6 is true for $k=0$.

For $k \ge 1$, we have

$$\begin{aligned} &\left|\int_0^T (\bar{u}_h - P_h^k u, \delta_1^h)\phi_1(t)dt\right| \\ &\quad \le C\int_0^T \left(|||\sigma - \Pi_h^k\sigma|||_\infty \|\mathbf{G}_1^h - \mathbf{G}_1\|_{L^1(\Omega)} + \|(I-P_h^k)\lambda_1\|_{L^1(\Omega)}\|(I-P_h^k)\text{div}\sigma\|_\infty\right) dt. \end{aligned}$$

It follows from Theorem 3.1 that

$$\|(I-P_h^k)\lambda_1\|_{L^1(\Omega)} \le Ch^2\|\nabla^2\lambda_1\|_{L^1(\Omega)} \le Ch^2\log\frac{1}{h}(1+\phi_1(t)).$$

Hence, we find from Theorem 3.2 that

$$\begin{aligned} &\left|\int_0^T (\bar{u}_h - P_h^k u, \delta_1^h)\phi(t)dt\right| \\ &\quad \le Ch\log\frac{1}{h}\int_0^T \left(|||\sigma - \Pi_h^k\sigma|||_\infty + h\|(I-P_h^k)\text{div}\sigma\|_\infty\right)(1+\phi_1(t))dt, \end{aligned}$$

which, together with Lemma 3.2 and (3.3), yields that for $k \ge 1$

$$\|\bar{u}_h - P_h^k u\|_\infty \le Ch\log\frac{1}{h}\left(|||\sigma - \Pi_h^k\sigma|||_\infty + h|||(I-P_h^k)\text{div}\sigma|||_\infty\right).$$

Therefore, the proof of Theorem 3.6 is complete. □

Theorem 3.7. *Under the conditions as for Theorem 3.6, there exists a constant $C>0$, independent of h and t, such that*

$$\|\sigma - \bar{\sigma}_h\|_\infty \le C\left(\log\frac{1}{h}\right)^{1/2}\left(|||\sigma - \Pi_h^k\sigma|||_\infty + h\left(\log\frac{1}{h}\right)^{\delta_{k0}/2} |||(I-P_h^k)\,div\sigma|||_\infty\right),$$

where δ_{kj} is the usual Kronecker symbol.

Proof. It suffices to bound $\bar{\sigma}_h - \Pi_h^k\sigma$ in L^∞-norm. By Lemma 3.3 we have that

$$\begin{aligned} \int_0^T (\bar{\sigma}_h - \Pi_h^k\sigma, \delta_2^h)\phi_2(t)dt &= \int_0^T \left(\alpha(\sigma - \Pi_h^k\sigma) + M*(\sigma - \Pi_h^k\sigma), \mathbf{G}_2^h - \mathbf{G}_2\right)dt \\ &\quad + \int_0^T \left(\alpha(\sigma - \Pi_h^k\sigma) + M*(\sigma - \Pi_h^k\sigma), \mathbf{G}_2\right)dt \\ &:= M_1 + M_2. \end{aligned}$$

Similar to (3.9), it follows from Lemma 3.1, (2.5) and (3.4) that

$$\begin{aligned} M_2 &= \int_0^T (\alpha\mathbf{G}_2 + M**\mathbf{G}_2, \sigma - \Pi_h^k\sigma)dt \\ &= \int_0^T (\nabla\lambda_2 + \delta_2^h\phi_2(t), \sigma - \Pi_h^k\sigma)dt \\ &= -\int_0^T (\lambda_2, \text{div}(\sigma - \Pi_h^k\sigma))dt + \int_0^T (\delta_2^h, \sigma - \Pi_h^k\sigma)\phi_2(t)dt \\ &= \int_0^T (P_h^k\lambda_2 - \lambda_2, (I-P_h^k)\text{div}\sigma)dt + \int_0^T (\delta_2^h, \sigma - \Pi_h^k\sigma)\phi_2(t)dt. \end{aligned}$$

And then, we find that

$$\left|\int_0^T (\bar\sigma_h - \Pi_h^k\sigma, \delta_2^h)\phi_2(t)dt\right| \le C\int_0^T |||\sigma - \Pi_h^k\sigma|||_\infty \left(||\mathbf{G}_2^h - \mathbf{G}_2||_{L^1(\Omega)} + ||\delta_2^h||_{L^1(\Omega)}\phi_2(t)\right)dt + \int_0^T ||\lambda_2 - P_h^k\lambda_2||_{L^1(\Omega)}||(I-P_h^k)\text{div}\sigma||_\infty dt,$$

which in turn implies by (3.5), Theorem 3.3,

$$||\mathbf{G}_2^h - \mathbf{G}_2||_{L^1(\Omega)} \le C\left(\log\frac{1}{h}\right)^{1/2}(1+\phi_2(t))$$

$$||\lambda_2 - P_h^k\lambda_2||_{L^1(\Omega)} \le \begin{cases} Ch\log\frac{1}{h}(1+\phi_2(t)), & k=0,\\ Ch\left(\log\frac{1}{h}\right)^{1/2}(1+\phi_2(t)), & k\ge 1,\end{cases}$$

that

$$\left|\int_0^T (\bar\sigma_h - \Pi_h^k\sigma, \delta_2^h)\phi_2(t)dt\right| \le C\left(\log\frac{1}{h}\right)^{1/2}\int_0^T |||\sigma - \Pi_h^k\sigma|||_\infty(1+\phi_2(t))dt + Ch\left(\log\frac{1}{h}\right)^{\frac{1+\delta_{k0}}{2}}\int_0^T ||(I-P_h^k)\text{div}\sigma||_\infty(1+\phi_2(t))dt.$$

Thus, it follows from Lemma 3.2 that

$$\left|(\bar\sigma_h - \Pi_h^k\sigma, \delta_2^h)\right| \le C\left(\log\frac{1}{h}\right)^{1/2}\left\{|||\sigma - \Pi_h^k\sigma|||_\infty + h\left(\log\frac{1}{h}\right)^{\delta_{k0}/2}|||(I-P_h^k)\text{div}\sigma|||_\infty\right\},$$

and by virtue of the analogue of (3.3) for δ_2^h that

$$||\bar\sigma_h - \Pi_h^k\sigma||_\infty \le C\left(\log\frac{1}{h}\right)^{1/2}\left\{|||\sigma - \Pi_h^k\sigma|||_\infty + h\left(\log\frac{1}{h}\right)^{\delta_{k0}/2}|||(I-P_h^k)\text{div}\sigma|||_\infty\right\}$$

which, together with the standard triangle inequality, implies Theorem 3.7. □

Remark 3.2 By (3.6) we have

$$||\lambda_2 - P_h^k\lambda_2||_{L^1(\Omega)} \le Ch(1+\phi_2(t)), \quad k\ge 1, \tag{3.10}$$

for sufficiently regular $\partial\Omega$. Thus, Theorem 3.7 can be improved to become

$$||\sigma - \bar\sigma_h||_\infty \le C\left\{|\log h|^{1/2}|||\sigma - \Pi_h^k\sigma|||_\infty + h|||(I-P_h^k)\text{div}\sigma|||_\infty\right\}. \tag{3.11}$$

for $k\ge 1$ if $\partial\Omega$ is sufficiently smooth.

Corollary 3.1. *Under the assumptions of Theorem 3.6, we have*

$$||\bar u_h - P_h^k u||_\infty \le \begin{cases} Ch^2|\log h|\left(|||\sigma|||_{1,\infty} + |\log h|^{-1/2}|||\sigma|||_2\right), & k=0,\\ Ch^{k+2}|\log h||||\sigma|||_{k+1,\infty}, & k\ge 1.\end{cases}$$

Proof. By (2.6) we have for the interpolation operators Π_h^k and P_h^k that

$$||\mathbf{f} - \Pi_h^k\mathbf{f}||_{0,p} \le Ch^{k+1}||\mathbf{f}||_{k+1,p}, \qquad 1\le p\le\infty,$$
$$||g - P_h^k g||_{0,p} \le Ch^{k+1}||g||_{k+1,p}, \qquad 1\le p\le\infty.$$

Then, we find from Theorem 3.6 that for $k=0$

$$\begin{aligned}||\bar u_h - P_h^0 u||_\infty &\le Ch\log\frac{1}{h}\left(|||\sigma - \Pi_h^0\sigma|||_\infty + |\log h|^{-1/2}|||(I-P_h^0)\text{div}\sigma|||_0\right)\\ &\le Ch^2|\log h|\left(|||\sigma|||_{1,\infty} + |\log h|^{-1/2}|||\sigma|||_2\right).\end{aligned}$$

Also, we can obtain the result for $k \geq 1$. □

Similarly, from Theorem 3.7 we can derive

Corollary 3.2. *We have under the assumptions of Theorem 3.6 that*

$$||\sigma - \bar{\sigma}_h||_\infty \leq Ch^{k+1}|\log h|^{(\delta_{k0}+1)/2}|||\sigma|||_{k+1,\infty}, \quad k \geq 0.$$

4. Optimal L^∞-error estimates for mixed finite element solutions

In this section we consider error estimates in maximum norms for the mixed finite element approximation of (1.1) by means of the L^∞-error estimates for the mixed Ritz-Volterra projections and the estimates for the regularized Green's functions given in the last section. First, the following error estimate of $||u_t - u_{h,t}||$ is demonstrated for the future needs. To this purpose, we recall from [11] the following two lemmas.

Lemma 4.1. *Assume that the matrix $A(t)$ is positive define. Then, the norms $||\sigma||_0^2 := (\sigma, \sigma)$ and $||\sigma||_{A^{-1}}^2 := (A^{-1}\sigma, \sigma)$ are equivalent.*

Lemma 4.2. *Let $(\bar{u}_h, \bar{\sigma}_h)$ be the mixed Ritz-Volterra projection of $(u, \sigma) \in W \times \mathbf{V}$ defined by (3.7). Then, there is a positive constant $C > 0$, independent of $h > 0$ small enough, such that the error $(u - \bar{u}_h, \sigma - \bar{\sigma}_h)$ can be estimated for any positive integer m by*

$$\begin{aligned} ||D_t^m(u - \bar{u}_h)||_0 &\leq C \begin{cases} h|||u(t)|||_{2,2,m}, & k = 0, \\ h^r|||u(t)|||_{r,2,m}, & k \geq 1 \text{ and } 2 \leq r \leq k+1, \end{cases} \\ ||D_t^m(\sigma - \bar{\sigma}_h)||_0 &\leq Ch^r|||u(t)|||_{r+1,2,m}, \qquad 1 \leq r \leq k+1, \end{aligned}$$

where $|||u(t)|||_{r,p,m} := \sum_{j=0}^{m} ||D_t^j u(t)||_{r,p} + \int_0^t \sum_{j=0}^{m} ||D_t^j u(s)||_{r,p} ds$, $-\infty \leq r \leq \infty$, $1 \leq p \leq \infty$, $t \geq 0$.

Theorem 4.1. *Assume that (u, σ) and (u_h, σ_h) are the solutions of (2.1) and (2.2), respectively, and $(u_h(0), \sigma_h(0))$ are chosen as follows:*

$$\begin{aligned} &(\alpha(0)(\sigma_h(0) - \sigma(0)), \mathbf{v}_h) + (div\mathbf{v}_h, u_h(0) - u_0) = 0, && \mathbf{v}_h \in \mathbf{V}_h, \\ &(div(\sigma_h(0) - \sigma(0)), w_h) = 0, && w_h \in W_h. \end{aligned} \tag{4.1}$$

Then we have for $k = 0$ that

$$||u_t - u_{h,t}||_0 \leq Ch\left\{||u||_2 + ||u_t||_2 + \left[\int_0^t (||u||_2^2 + ||u_t||_2^2 + ||u_{tt}||_2^2) ds\right]^{1/2}\right\}$$

and for $k \geq 1$ that

$$||u_t - u_{h,t}||_0 \leq Ch^{k+1}\left\{||u||_{k+1} + ||u_t||_{k+1} + \left[\int_0^t (||u||_{k+1}^2 + ||u_t||_{k+1}^2 + ||u_{tt}||_{k+1}^2) ds\right]^{1/2}\right\}.$$

Proof. Let

$$\begin{aligned} u - u_h &= (u - \bar{u}_h) + (\bar{u}_h - u_h) := \rho + \rho_h, \\ \sigma - \sigma_h &= (\sigma - \bar{\sigma}_h) + (\bar{\sigma}_h - \sigma_h) := \theta + \theta_h, \end{aligned}$$

where $(\bar{u}_h, \bar{\sigma}_h)$ is the Ritz-Volterra projection of (u, σ). Then, by Lemma 4.2 we have

$$\begin{aligned} ||\rho_t||_0 &\leq \begin{cases} Ch|||u|||_{2,2,1}, & k = 0, \\ Ch^{k+1}|||u|||_{k+1,2,1}, & k \geq 1, \end{cases} \\ ||\rho_{tt}||_0 &\leq \begin{cases} Ch|||u|||_{2,2,2}, & k = 0, \\ Ch^{k+1}|||u|||_{k+1,2,2}, & k \geq 1. \end{cases} \end{aligned} \tag{4.2}$$

Thus, only $||\rho_{h,t}||_0$ needs to be estimated in order to get the estimate for $||u_t - u_{h,t}||_0$.

From (3.7) and (4.1) we derive that

$$\begin{aligned}
&(\alpha(0)\theta_h(0), \mathbf{v}_h) + (\mathrm{div}\mathbf{v}_h, \rho_h(0)) = 0, && \mathbf{v}_h \in \mathbf{V}_h,\\
&(\mathrm{div}\theta_h(0), w_h) = 0, && w_h \in W_h,
\end{aligned}$$

which, together with the uniqueness of the solution to (3.7), implies

$$\theta_h(0) = \rho_h(0) = 0. \tag{4.3}$$

It follows from (3.7) and the mixed finite element error equation (2.4) that (ρ_h, θ_h) satisfies

$$\begin{aligned}
&(\alpha\theta_h + M * \theta_h, \mathbf{v}_h) + (\mathrm{div}\mathbf{v}_h, \rho_h) = 0, && \mathbf{v}_h \in \mathbf{V}_h,\\
&(\rho_{h,t}, w_h) - (\mathrm{div}\theta_h, w_h) = -(\rho_t, w_h), && w_h \in W_h.
\end{aligned} \tag{4.4}$$

Therefore, setting $w_h = \rho_h$ and $\mathbf{v}_h = \theta_h$ in (4.4) we obtain from their sum that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||\rho_h||_0^2 + ||\theta_h||_{A^{-1}}^2 &= -(M * \theta_h, \theta_h) - (\rho_t, \rho_h)\\
&\le C\int_0^t ||\theta_h(s)||_0 ds||\theta_h||_0 + ||\rho_t||_0||\rho_h||_0,
\end{aligned}$$

and by means of Lemma 4.1 that

$$\frac{1}{2}\frac{d}{dt}||\rho_h||_0^2 + ||\theta_h||_{A^{-1}}^2 \le C\left(||\rho_h||_0^2 + \int_0^t ||\theta_h||_{A^{-1}}^2 ds\right) + \frac{1}{2}\left(||\theta_h||_{A^{-1}}^2 + ||\rho_t||_0^2\right).$$

Noticing (4.3) we have by integrating from 0 to t that

$$||\rho_h||_0^2 + \int_0^t ||\theta_h||_{A^{-1}}^2 ds \le \int_0^t \left[||\rho_h||_0^2 + \int_0^s ||\theta_h(\tau)||_{A^{-1}}^2 d\tau\right] ds + \int_0^t ||\rho_t||_0^2 ds,$$

which, together with Gronwall's lemma, implies

$$\int_0^t ||\theta_h(s)||_{A^{-1}}^2 ds + ||\rho_h||_0^2 \le C\int_0^t ||\rho_t||_0^2 ds,$$

and then

$$||\rho_h||_0^2 \le C\int_0^t ||\rho_t||_0^2 ds. \tag{4.5}$$

In order to get the estimate for $\theta_h(t)$, we first differentiate (4.4) to obtain

$$(\alpha_t\theta_h + \alpha\theta_{h,t} + M(t,t)\theta_h + M_t * \theta_h, \mathbf{v}_h) + (\mathrm{div}\mathbf{v}_h, \rho_{h,t}) = 0, \quad \mathbf{v}_h \in \mathbf{V}_h,$$

and then we have by setting $\mathbf{v}_h = \theta_h$ in the above equation and $w_h = \rho_{h,t}$ in (4.4) that

$$||\rho_{h,t}||_0^2 + (\alpha\theta_{h,t}, \theta_h) + (\alpha_t\theta_h, \theta_h) = -(M\theta_h + M_t * \theta_h, \theta_h) - (\rho_t, \rho_{h,t}). \tag{4.6}$$

Since

$$\alpha(\theta_h^2)_t = (\alpha\theta_h^2)_t - \alpha_t\theta_h^2,$$

then

$$\begin{aligned}
(\alpha\theta_{h,t}, \theta_h) &= \int_\Omega \alpha\theta_{h,t}\theta_h = \frac{1}{2}\int_\Omega \alpha\frac{d}{dt}(\theta_h^2)\\
&= \frac{1}{2}\int_\Omega \frac{d}{dt}(\alpha\theta_h^2) - \frac{1}{2}\int_\Omega \alpha_t\theta_h^2\\
&= \frac{1}{2}\frac{d}{dt}||\theta_h||_{A^{-1}}^2 - \frac{1}{2}(\alpha_t\theta_h, \theta_h).
\end{aligned}$$

Hence, (4.6) can be rewritten as

$$||\rho_{h,t}||_0^2 + \frac{1}{2}\frac{d}{dt}||\theta_h||_{A^{-1}}^2 + \frac{1}{2}(\alpha_t\theta_h, \theta_h) = -(M\theta_h + M_t * \theta_h, \theta_h) - (\rho_t, \rho_{h,t}).$$

Thus, from the ϵ-inequality we derive that

$$||\rho_{h,t}||_0^2 + \frac{d}{dt}||\theta_h||_{A^{-1}}^2 \le C\left\{||\theta_h||_0^2 + \int_0^t ||\theta_h||_0^2 ds + ||\rho_t||_0^2\right\},$$

and then via integrating from 0 to t, Lemma 4.1, (4.3) and Gronwall's lemma that

$$||\theta_h||_0^2 \le C\int_0^t ||\rho_t||_0^2 ds. \tag{4.7}$$

Differentiate (4.4) to obtain that

$$\begin{aligned} &(\alpha_t\theta_h + \alpha\theta_{h,t} + M(t,t)\theta_h + M_t * \theta_h, \mathbf{v}_h) + (\mathrm{div}\mathbf{v}_h, \rho_{h,t}) = 0, && \mathbf{v}_h \in \mathbf{V}_h, \\ &(\rho_{h,tt}, w_h) - (\mathrm{div}\theta_{h,t}, w_h) = -(\rho_{tt}, w_h), && w_h \in W_h. \end{aligned} \tag{4.8}$$

And hence, setting $\mathbf{v}_h = \theta_{h,t}$ and $w_h = \rho_{h,t}$ in (4.8) we have from their sum that

$$\frac{1}{2}\frac{d}{dt}||\rho_{h,t}||_0^2 + (\alpha\theta_{h,t}, \theta_{h,t}) \le C\left(||\theta_h||_0 + \int_0^t ||\theta_h||_0 ds\right)^2 + ||\rho_{tt}||_0^2 + ||\rho_{h,t}||_0^2.$$

Integrating the above inequality from 0 to t leads to

$$||\rho_{h,t}||_0^2 \le ||\rho_{h,t}(0)||_0^2 + C\int_0^t ||\theta_h||_0^2 ds + \int_0^t ||\rho_{tt}||_0^2 ds + \int_0^t ||\rho_{h,t}||_0^2 ds,$$

and Gronwall's lemma and (4.7) imply

$$||\rho_{h,t}||_0^2 \le C\left\{||\rho_{h,t}(0)||_0^2 + \int_0^t ||\rho_t||_0^2 ds + \int_0^t ||\rho_{tt}||_0^2 ds\right\}. \tag{4.9}$$

Let $t = 0$ and $w_h = \rho_{h,t}(0)$ in (4.4) to obtain by (4.3) that

$$||\rho_{h,t}(0)||_0^2 = -(\rho_t(0), \rho_{h,t}(0)) \le ||\rho_t(0)||_0 ||\rho_{h,t}(0)||_0$$

or

$$||\rho_{h,t}(0)||_0 \le ||\rho_t(0)||_0$$

which, together with (4.9) and (4.2), leads to

$$\begin{aligned} &||\rho_{h,t}||_0^2 \le C\left\{||\rho_t(0)||_0^2 + \int_0^t (||\rho_t||_0^2 + ||\rho_{tt}||_0^2) ds\right\} \\ &\le \begin{cases} Ch^2\left[||u(0)||_2^2 + ||u_t(0)||_2^2 + \int_0^t (||u||_2^2 + ||u_t||_2^2 + ||u_{tt}||_2^2) ds\right], & k = 0, \\ Ch^{2k+2}\left[||u(0)||_{k+1}^2 + ||u_t(0)||_{k+1}^2 + \int_0^t (||u||_{k+1}^2 + ||u_t||_{k+1}^2 + ||u_{tt}||_{k+1}^2) ds\right], & k \ge 1. \end{cases} \end{aligned}$$

Therefore, we have for $k = 0$ that

$$||u_t - u_{h,t}||_0 \le Ch\left\{||u||_2 + ||u_t||_2 + \left[\int_0^t (||u||_2^2 + ||u_t||_2^2 + ||u_{tt}||_2^2) ds\right]^{1/2}\right\}$$

and for $k \ge 1$ that

$$||u_t - u_{h,t}||_0 \le Ch^{k+1}\left\{||u||_{k+1} + ||u_t||_{k+1} + \left[\int_0^t (||u||_{k+1}^2 + ||u_t||_{k+1}^2 + ||u_{tt}||_{k+1}^2) ds\right]^{1/2}\right\}.$$

□

Theorem 4.2. *We have under the assumptions of Theorem 4.1 that for* $k=0$

$$\begin{aligned}||u-u_h||_\infty &\le Ch\left[||u||_{1,\infty}+|\log h|^{1/2}(||u||_2+||u_t||_2)\right]\\ &\quad +Ch|\log h|^{1/2}\left[\int_0^t(||u||_2^2+||u_t||_2^2+||u_{tt}||_2^2)ds\right]^{1/2}\end{aligned}$$

and

$$\begin{aligned}||\sigma-\sigma_h||_\infty &\le Ch|\log h|^{1/2}\left(|\log h|^{1/2}|||u|||_{2,\infty}+||u||_2+||u_t||_2\right)\\ &\quad +Ch|\log h|^{1/2}\left[\int_0^t(||u||_2^2+||u_t||_2^2+||u_{tt}||_2^2)ds\right]^{1/2};\end{aligned}$$

for $k\ge 1$

$$\begin{aligned}||u-u_h||_\infty &\le Ch^{k+1}|\log h|^{1/2}\left(|\log h|^{1/2}|||u|||_{k+1,\infty}+||u||_{k+1}+||u_t||_{k+1}\right)\\ &\quad +Ch^{k+1}|\log h|^{1/2}\left[\int_0^t(||u||_{k+1}^2+||u_t||_{k+1}^2+||u_{tt}||_{k+1}^2)ds\right]^{1/2}\end{aligned}$$

and

$$\begin{aligned}||\sigma-\sigma_h||_\infty &\le Ch^{k+1}|\log h|^{1/2}\left(|||u|||_{k+2,\infty}+||u||_{k+1}+||u_t||_{k+1}\right)\\ &\quad +Ch^{k+1}|\log h|^{1/2}\left[\int_0^t(||u||_{k+1}^2+||u_t||_{k+1}^2+||u_{tt}||_{k+1}^2)ds\right]^{1/2}.\end{aligned}$$

Proof. Also, we decompose the errors as

$$\begin{aligned}u-u_h &= (u-\bar{u}_h)+(\bar{u}_h-u_h) := \rho+\rho_h,\\ \sigma-\sigma_h &= (\sigma-\bar{\sigma}_h)+(\bar{\sigma}_h-\sigma_h) := \theta+\theta_h.\end{aligned}$$

Then, from Theorems 3.6 and 3.7 we know

$$\begin{aligned}||\rho||_\infty &\le ||u-P_h^k u||_\infty+||P_h^k u-\bar{u}_h||_\infty\\ &\le \begin{cases} Ch\left(||u||_{1,\infty}+|\log h|^{1/2}|||u|||_2\right), & k=0,\\ Ch^{k+1}|\log h||||u|||_{k+1,\infty}, & k\ge 1,\end{cases}\\ ||\theta||_\infty &\le Ch^{k+1}|\log h|^{(\delta_{k0}+1)/2}|||\sigma|||_{k+1,\infty}.\end{aligned}\tag{4.10}$$

Therefore, only $||\rho_h||_\infty$ and $||\theta_h||_\infty$ are left to be estimated.

Set $\mathbf{v}_h=\mathbf{G}_1^h$ in (4.4) to obtain from the mixed finite element approximation of (3.2) that

$$(\delta_1^h\phi_1(t),\rho_h)=(\mathrm{div}\mathbf{G}_1^h,\rho_h)=-(\alpha\theta_h+M*\theta_h,\mathbf{G}_1^h),$$

so that it follows from the integration, Lemma 3.1 and the mixed finite element solution of (3.2) that

$$\begin{aligned}\int_0^T(\delta_1^h,\rho_h)\phi_1(t)dt &= -\int_0^T(\alpha\theta_h+M*\theta_h,\mathbf{G}_1^h)dt\\ &= -\int_0^T(\alpha\mathbf{G}_1^h+M**\mathbf{G}_1^h,\theta_h)dt\\ &= \int_0^T(\lambda_1^h,\mathrm{div}\theta_h)dt\\ &= \int_0^T(\lambda_1^h-\lambda_1,\mathrm{div}\theta_h)dt+\int_0^T(\lambda_1,\mathrm{div}\theta_h)dt.\end{aligned}\tag{4.11}$$

Since $\lambda_1|_{\partial\Omega} = 0$, it follows from Theorems 3.1 and 3.2 that

$$||\lambda_1||_0 \le C||\nabla\lambda_1||_0 \le C|\log h|^{1/2}(1+\phi_1(t)),$$
$$||\lambda_1 - \lambda_1^h||_0 \le Ch|\log h|^{1/2}(1+\phi_1(t)).$$

Hence, from (4.11) we find that

$$\begin{aligned}\int_0^T (\delta_1^h, \rho_h)\phi_1(t)dt &\le Ch|\log h|^{1/2}\int_0^T ||\mathrm{div}\theta_h||_0(1+\phi_1(t))dt \\ &+C|\log h|^{1/2}\int_0^T ||\mathrm{div}\theta_h||_0(1+\phi_1(t))dt,\end{aligned}$$

and by Lemma 3.2 and (3.3) that

$$||\rho_h||_\infty \le C|\log h|^{1/2}(h+1)|||\mathrm{div}\theta_h|||_0. \tag{4.12}$$

We know from (3.7) and the mixed finite element error equation (2.4) that

$$\begin{aligned}&(\mathrm{div}(\sigma - \bar{\sigma}_h), w_h) = 0, && w_h \in W_h,\\ &(\mathrm{div}(\sigma - \sigma_h), w_h) = (u_t - u_{h,t}, w_h), && w_h \in W_h.\end{aligned}$$

This implies

$$(\mathrm{div}\theta_h, w_h) = (\mathrm{div}(\bar{\sigma}_h - \sigma_h), w_h) = (\mathrm{div}(\sigma - \sigma_h), w_h) = (u_t - u_{h,t}, w_h),\ \ w_h \in W_h,$$

from which we have by means of the arbitrariness of $w_h \in W_h$ that

$$||\mathrm{div}\theta_h||_0 \le ||u_t - u_{h,t}||_0. \tag{4.13}$$

Combining (4.12) with (4.13) and Theorem 4.1 leads to

$$||\rho_h||_\infty \le \begin{cases} Ch|\log h|^{1/2}\left[||u_0||_2 + ||u_t(0)||_2 + ||u||_2 + ||u_t||_2\right] \\ +Ch|\log h|^{1/2}\left[\int_0^t (||u||_2^2 + ||u_t||_2^2 + ||u_{tt}||_2^2)ds\right]^{1/2}, & k=0,\\ Ch^{k+1}|\log h|^{1/2}\left[||u_0||_{k+1} + ||u_t(0)||_{k+1} + ||u||_{k+1} + ||u_t||_{k+1}\right] \\ +Ch^{k+1}|\log h|^{1/2}\left[\int_0^t (||u||_{k+1}^2 + ||u_t||_{k+1}^2 + ||u_{tt}||_{k+1}^2)ds\right]^{1/2}, & k \ge 1.\end{cases} \tag{4.14}$$

Thus we obtain according to (4.10) and (4.14) that for $k = 0$

$$\begin{aligned}||u - u_h||_\infty &\le Ch\left[||u||_{1,\infty} + |\log h|^{1/2}(||u||_2 + ||u_t||_2)\right] \\ &+Ch|\log h|^{1/2}\left[\int_0^t (||u||_2^2 + ||u_t||_2^2 + ||u_{tt}||_2^2)ds\right]^{1/2},\end{aligned}$$

and for $k \ge 1$,

$$\begin{aligned}||u - u_h||_\infty &\le Ch^{k+1}|\log h|^{1/2}\left(|\log h|^{1/2}|||u|||_{k+1,\infty} + ||u||_{k+1} + ||u_t||_{k+1}\right) \\ &+Ch^{k+1}|\log h|^{1/2}\left[\int_0^t (||u||_{k+1}^2 + ||u_t||_{k+1}^2 + ||u_{tt}||_{k+1}^2)ds\right]^{1/2}.\end{aligned}$$

Next we shall give the proof of the estimate for $||\sigma - \sigma_h||_\infty$. For this purpose, set $\mathbf{v}_h = \mathbf{G}_2^h$ in (4.4) to get according to (3.4) and its corresponding mixed finite element error equation that

$$\begin{aligned}(\alpha\theta_h + M * \theta_h, \mathbf{G}_2^h) &= -(\mathrm{div}\mathbf{G}_2^h, \rho_h) \\ &= -(\mathrm{div}(\mathbf{G}_2^h - \mathbf{G}_2), \rho_h) - (\mathrm{div}\mathbf{G}_2, \rho_h) \\ &= 0.\end{aligned}$$

Thus, we see from Lemma 3.1, the mixed finite element approximation of (3.4) and Green's formula that

$$0 = \int_0^T (\alpha\theta_h + M * \theta_h, \mathbf{G}_2^h)dt \quad = \int_0^T (\alpha\mathbf{G}_2^h + M * *\mathbf{G}_2^h, \theta_h)dt$$
$$= \int_0^T (\delta_2^h, \theta_h)\phi_2(t)dt - \int_0^T (\lambda_2^h, \mathrm{div}\theta_h)dt,$$

which, together with Theorem 3.3, implies that

$$\left|\int_0^T (\delta_2^h, \theta_h)\phi_2(t)dt\right| \quad = \left|\int_0^T (\lambda_2^h - \lambda_2, \mathrm{div}\theta_h)dt + \int_0^T (\lambda_2, \mathrm{div}\theta_h)dt\right|$$
$$\leq \int_0^T C||\mathrm{div}\theta_h||_0(1+\phi_2(t))dt$$
$$+ \int_0^T C(1+|\log h|^{1/2})||\mathrm{div}\theta_h||_0(1+\phi_2(t))dt.$$

Thus, from Lemma 3.2 and (4.13) we derive that

$$\left|(\delta_2^h, \theta_h)\right| \leq C|\log h|^{1/2}|||\mathrm{div}\theta_h|||_0 \leq C|\log h|^{1/2}|||u_t - u_{h,t}|||_0$$

which, together with Theorem 4.1 and an analogue of (3.3) for δ_2^h, demonstrates that

$$||\theta_h||_\infty \leq \begin{cases} Ch|\log h|^{1/2}(||u||_2 + ||u_t||_2) \\ +Ch|\log h|^{1/2}\left[\int_0^t (||u||_2^2 + ||u_t||_2^2 + ||u_{tt}||_2^2)ds\right]^{1/2}, & k=0, \\ Ch^{k+1}|\log h|^{1/2}(||u||_{k+1} + ||u_t||_{k+1}) \\ +Ch^{k+1}|\log h|^{1/2}\left[\int_0^t (||u||_{k+1}^2 + ||u_t||_{k+1}^2 + ||u_{tt}||_{k+1}^2)ds\right]^{1/2}, & k \geq 1, \end{cases}$$

and according to (4.10) that

$$||\sigma - \sigma_h||_\infty \leq \begin{cases} Ch|\log h||||u|||_{2,\infty} + Ch|\log h|^{1/2}(||u||_2 + ||u_t||_2) \\ +Ch|\log h|^{1/2}\left[\int_0^t (||u||_2^2 + ||u_t||_2^2 + ||u_{tt}||_2^2)ds\right]^{1/2}, & k=0, \\ Ch^{k+1}|\log h|^{1/2}(|||u|||_{k+2,\infty} + ||u||_{k+1} + ||u_t||_{k+1}) \\ +Ch^{k+1}|\log h|^{1/2}\left[\int_0^t (||u||_{k+1}^2 + ||u_t||_{k+1}^2 + ||u_{tt}||_{k+1}^2)ds\right]^{1/2}, & k \geq 1. \end{cases}$$

□

5. Superconvergence estimates in L^∞-norm and a-posteriori error estimates

In the past years, the superconvergence of mixed finite element methods has received considerable attention. See, for example, [8, 12, 17] for elliptic equations, [4, 5, 17] for parabolic equations and [11] for partial integro-differential equations.

The aim of this section is to give local and global maximum norm superconvergence error estimates and a-posteriori error estimators for the mixed finite element approximation of (2.1). First of all, we consider the local superconvergence. To this end, let us define some seminorms as follows.

Following [12] we assume that $\Omega \subset R^2$ is a rectangle and $e = [a,b] \times [c,d] \in T_h$ is an arbitrary element of the partition T_h. We denote by $(g_1, g_2, \cdots, g_{k+1})$ the Gauss points in $[a,b]$ and $(\hat{g}_1, \hat{g}_2, \cdots, \hat{g}_{k+1})$ the Gauss points in $[c,d]$, and define

$$\begin{aligned} |||w|||_{*,\infty} &:= \max_{e\in T_h} \max_{1\le i,j\le k+1} |w(g_i, \hat{g}_j)|, \\ |||\mathbf{v}|||_{*,\infty} &:= |||v_1|||^*_{\infty,1} + |||v_2|||^*_{\infty,2}, \end{aligned}$$

where

$$\begin{aligned} |||v_1|||^*_{\infty,1} &:= \max_{e\in T_h} \max_{1\le j\le k+1} \max_{(x,\hat{g}_j)\in e} |v_1(x, \hat{g}_j)|, \\ |||v_2|||^*_{\infty,2} &:= \max_{e\in T_h} \max_{1\le i\le k+1} \max_{(g_i,y)\in e} |v_2(g_i, y)|. \end{aligned}$$

From [8, 17] and [11] we recall the following Lemma 5.1 and Lemma 5.2, respectively.

Lemma 5.1. *Let σ be a sufficiently smooth vector-valued function, $B = (b_{ij})$ be a 2×2 matrix with $b_{ij} \in W^{1,\infty}(\Omega)$ and Ω be a rectangular domain which is partitioned into rectangular elements. Then, we have*

$$|(B\cdot(\sigma - \Pi_h^k\sigma), \mathbf{v}_h)| \le Ch^{k+2}|\sigma|_{k+2,p}||\mathbf{v}_h||_{0,p'}, \quad \forall \mathbf{v}_h \in \mathbf{V}_h,$$

where $|f|_{m,q} := \left(\sum_{|i|=m} ||D^i f||^q_{0,q,\Omega}\right)^{1/q}$, $1 \le q < \infty$, $|f|_{m,\infty} := \max_{|i|=m}\{ess\sup_{\Omega}|D^i f|\}$ *and* $p' = \frac{p}{p-1}$ *is the conjugate of* $p \ge 1$.

Lemma 5.2. *Let $(\bar{u}_h, \bar{\sigma}_h)$ be the mixed Ritz-Volterra projection of (u,σ). Then, we have*

$$||\bar{u}_h - P_h^k u||_W + ||\bar{\sigma}_h - \Pi_h^k\sigma||_{\mathbf{V}} \le Ch^{k+2}(||u||_{k+1} + |||\sigma|||_{k+2}),$$

where $||u||_W := ||u||_0$ *and* $||\sigma||_{\mathbf{V}} := \left(||\sigma||_0^2 + ||\nabla\cdot\sigma||_0^2\right)^{1/2}$.

Theorem 5.1. *Assume that (u,σ) and (u_h, σ_h) are the solutions of (2.1) and (2.2), respectively, and $(u_h(0), \sigma_h(0))$ are chosen to satisfy (4.1). If the exact solution u and σ satisfies σ, $\sigma_t \in (H^{k+2}(\Omega))^2$, then we have*

$$\begin{aligned} ||u_h - P_h^k u||_0 &+ ||(u_h - P_h^k u)_t||_0 + ||\sigma_h - \Pi_h^k\sigma||_0 \\ &\le Ch^{k+2}\left\{||u_0||_{k+2} + ||\sigma(0)||_{k+2} + \left[\int_0^t (||\sigma||^2_{k+2} + ||\sigma_t||^2_{k+2})ds\right]^{1/2}\right\}. \end{aligned}$$

Proof. Let $\rho_h^* := u_h - P_h^k u$ and $\theta_h^* := \sigma_h - \Pi_h^k\sigma$. Then, it follows from the mixed finite element error equation (2.4) and (2.5) that

$$\begin{aligned} &(\alpha\theta_h^* + M * \theta_h^*, \mathbf{v}_h) + (\rho_h^*, \nabla\cdot\mathbf{v}_h) = (\alpha(\sigma - \Pi_h^k\sigma) + M*(\sigma - \Pi_h^k\sigma), \mathbf{v}_h), \quad && \mathbf{v}_h \in \mathbf{V}_h, \\ &(\rho^*_{h,t}, w_h) - (\nabla\cdot\theta_h^*, w_h) = 0, && w_h \in W_h. \end{aligned} \tag{5.1}$$

Thus, letting $w_h = \rho_h^*$ and $\mathbf{v}_h = \theta_h^*$ in (5.1) we obtain from Lemmas 4.1, 5.1 and the ϵ-type inequality that

$$\frac{d}{dt}||\rho_h^*||_0^2 + ||\theta_h^*||_0^2 \le C\left\{\int_0^t ||\theta_h^*||_0^2 ds + h^{k+2}||\sigma||_{k+2}||\theta_h^*||_0\right\},$$

or

$$\frac{d}{dt}||\rho_h^*||_0^2 + ||\theta_h^*||_0^2 \le C\left\{\int_0^t ||\theta_h^*||_0^2 ds + h^{2(k+2)}||\sigma||^2_{k+2}\right\}.$$

Hence, integrating from 0 to t and using Gronwall's lemma yield

$$||\rho_h^*||_0^2 + \int_0^t ||\theta_h^*||_0^2 ds \le C\left\{||\rho_h^*(0)||_0^2 + h^{2(k+2)}\int_0^t ||\sigma||^2_{k+2}ds\right\}. \tag{5.2}$$

From (4.3) we know

$$\bar{u}_h(0) - u_h(0) = \bar{\sigma}_h(0) - \sigma_h(0) = 0. \tag{5.3}$$

Therefore, from Lemma 5.2 we know

$$||\rho_h^*(0)||_0 = ||\bar{u}_h(0) - P_h^k u_0||_0 \le Ch^{k+2}(||u_0||_{k+2} + ||\sigma(0)||_{k+2}),$$

and then from (5.2) we further obtain

$$||\rho_h^*||_0 \le Ch^{k+2}\left\{||u_0||_{k+2} + ||\sigma(0)||_{k+2} + \left(\int_0^t ||\sigma||_{k+2}^2 ds\right)^{1/2}\right\}. \tag{5.4}$$

Again, we have according to (5.3) and Lemma 5.2 that

$$||\theta_h^*(0)||_0 = ||\bar{\sigma}_h(0) - \Pi_h^k\sigma(0)||_0 \le Ch^{k+2}(||u_0||_{k+2} + ||\sigma(0)||_{k+2}). \tag{5.5}$$

The second equation in (5.1) implies

$$\rho_{h,t}^* = \nabla\cdot\theta_h^* \tag{5.6}$$

which, together with (5.3) and Lemma 5.2, demonstrates

$$\begin{aligned} ||\rho_{h,t}^*(0)||_0 &= ||\nabla\cdot\theta_h^*(0)||_0 = ||\nabla\cdot(\bar{\sigma}_h - \Pi_h^k\sigma)(0)||_0 \\ &\le Ch^{k+2}(||u_0||_{k+2} + ||\sigma(0)||_{k+2}). \end{aligned} \tag{5.7}$$

Following the steps for θ_h and $\rho_{h,t}$ in Theorem 4.1 and using the initial approximations (5.5) and (5.7) we can also obtain

$$||\theta_h^*||_0 + ||\rho_{h,t}^*||_0 \le Ch^{k+2}\left\{||u_0||_{k+2} + ||\sigma(0)||_{k+2} + \left[\int_0^t (||\sigma||_{k+2}^2 + ||\sigma_t||_{k+2}^2)ds\right]^{1/2}\right\}. \tag{5.8}$$

(5.4) and (5.8) complete the proof of Theorem 5.1. □

Theorem 5.2. *If there is, besides the conditions of Theorem 5.1, $\sigma \in \left(W^{k+2,\infty}(\Omega)\right)^2$, then we have*

$$\begin{aligned} &|\log h|^{1/2}||u_h - P_h^k u||_\infty + ||\sigma_h - \Pi_h^k\sigma||_\infty \\ &\le Ch^{k+2}|\log h|\left\{||u_0||_{k+2} + ||\sigma(0)||_{k+2} + |||\sigma|||_{k+2,\infty} + \left[\int_0^t (||\sigma||_{k+2}^2 + ||\sigma_t||_{k+2}^2)ds\right]^{1/2}\right\}. \end{aligned}$$

Proof. Set $\mathbf{v}_h = \mathbf{G}_1^h$ in (5.1) to obtain that

$$(\alpha\theta_h^* + M * \theta_h^*, \mathbf{G}_1^h) + (\rho_h^*, \nabla\cdot\mathbf{G}_1^h) = (\alpha(\sigma - \Pi_h^k\sigma) + M * (\alpha - \Pi_h^k\sigma), \mathbf{G}_1^h),$$

and by means of the mixed finite element approximation of (3.2), Theorems 3.4, 5.1 and Lemmas 5.1, 3.1 that

$$\begin{aligned}\left|\int_0^T (\rho_h^*, \delta_1^h)\phi_1(t)dt\right| &\leq \left|\int_0^T (\alpha\theta_h^* + M * \theta_h^*, \mathbf{G}_1^h)dt\right| \\ &\quad + Ch^{k+2}\int_0^T |||\sigma|||_{k+2}|\log h|^{1/2}(1+\phi_1(t))dt \\ &= \left|\int_0^T (\alpha\mathbf{G}_1^h + M * *\mathbf{G}_1^h, \theta_h^*)dt\right| \\ &\quad + Ch^{k+2}|\log h|^{1/2}\int_0^T |||\sigma|||_{k+2}(1+\phi_1(t))dt \\ &\leq Ch^{k+2}|\log h|^{1/2}\int_0^T (1+\phi_1(t))(||u_0||_{k+2} + ||\sigma(0)||_{k+2} + ||\sigma||_{k+2})dt \\ &\quad + Ch^{k+2}|\log h|^{1/2}\int_0^T (1+\phi_1(t))\left[\int_0^t (||\sigma||_{k+2}^2 + ||\sigma_t||_{k+2}^2)ds\right]^{1/2} dt.\end{aligned}$$

Thus, Lemma 3.2 and (3.3) imply

$$\begin{aligned}||\rho_h^*||_\infty &\leq Ch^{k+2}|\log h|^{1/2}[||u_0||_{k+2} + ||\sigma(0)||_{k+2} + ||\sigma||_{k+2}] \\ &\quad + Ch^{k+2}|\log h|^{1/2}\left[\int_0^t (||\sigma||_{k+2}^2 + ||\sigma_t||_{k+2}^2)ds\right]^{1/2}.\end{aligned}$$

Letting $\mathbf{v}_h = \mathbf{G}_2^h$ in (5.1) we have

$$(\alpha\theta_h^* + M * \theta_h^*, \mathbf{G}_2^h) + (\rho_h^*, \nabla\cdot\mathbf{G}_2^h) = (\alpha(\sigma - \Pi_h^k\sigma) + M * (\sigma - \Pi_h^k\sigma), \mathbf{G}_2^h).$$

Since $\nabla\cdot\mathbf{G}_2^h = 0$ by the mixed finite element approximation of (3.4), it follows from Lemmas 3.1, 5.1 and the mixed finite element approximation of (3.4) that

$$\int_0^T (\alpha\mathbf{G}_2^h + M * *\mathbf{G}_2^h, \theta_h^*)dt \leq Ch^{k+2}\int_0^T |||\sigma|||_{k+2,\infty}||\mathbf{G}_2^h||_{L^1(\Omega)}dt.$$

Therefore, we obtain by means of the mixed finite element approximation of (3.4) and Theorems 3.5, 3.3, (5.6) and (5.8) that

$$\begin{aligned}\left|\int_0^T (\delta_2^h, \theta_h^*)dt\right|\phi_2(t)dt &\leq \int_0^T |(\lambda_2^h, \nabla\cdot\theta_h^*)|dt + Ch^{k+2}\int_0^T |||\sigma|||_{k+2,\infty}|\log h|(1+\phi_2(t))dt \\ &\leq \int_0^T ||\lambda_2^h||_0||\rho_{h,t}^*||_0 dt + Ch^{k+2}|\log h|\int_0^T |||\sigma|||_{k+2,\infty}(1+\phi_2(t))dt \\ &\leq Ch^{k+2}|\log h|\int_0^T (||u_0||_{k+2} + ||\sigma(0)||_{k+2} + |||\sigma|||_{k+2,\infty})(1+\phi_2(t))dt \\ &\quad + Ch^{k+2}|\log h|\int_0^T \left[\int_0^t (||\sigma||_{k+2}^2 + ||\sigma_t||_{k+2}^2)ds\right]^{1/2}(1+\phi_2(t))dt,\end{aligned}$$

and Lemma 3.2 yields

$$\begin{aligned}||\theta_h^*||_\infty &\leq Ch^{k+2}|\log h|(||u_0||_{k+2} + ||\sigma(0)||_{k+2} + |||\sigma|||_{k+2,\infty}) \\ &\quad + Ch^{k+2}|\log h|\left[\int_0^t (||\sigma||_{k+2}^2 + ||\sigma_t||_{k+2}^2)ds\right]^{1/2}.\end{aligned}$$

□

Remark 5.1 From Lemmas 3.3 and 5.1 we can also obtain the following L^∞-norm superconvergence for the mixed Ritz-Volterra projection of (u,σ):

$$|\log h|^{1/2}||\bar{u}_h - P_h^k u||_\infty + ||\bar{\sigma}_h - \Pi_h^k \sigma||_\infty \le Ch^{k+2}|\log h|||\sigma|||_{k+2,\infty}.$$

Hence, there holds the L^∞-superconvergence estimate under the conditions of Theorem 5.2,

$$\begin{aligned} |\log h|^{1/2}||\bar{u}_h - u_h||_\infty + ||\bar{\sigma}_h - \sigma_h||_\infty &\le Ch^{k+2}|\log h|(||u_0||_{k+2} + ||\sigma(0)||_{k+2} + |||\sigma|||_{k+2,\infty}) \\ &+ Ch^{k+2}|\log h|\left[\int_0^t (||\sigma||_{k+2}^2 + ||\sigma_t||_{k+2}^2) ds\right]^{1/2}. \end{aligned}$$

In order to obtain the local superconvergence for the mixed finite element solution (u_h, σ_h), we need the following lemmas which come from [12] and [8], respectively.

Lemma 5.3. *Assume that* $u \in W^{k+2,\infty}(\Omega)$. *Then,*

$$|||u - P_h^k u|||_{*,\infty} \le Ch^{k+2}||u||_{k+2,\infty}.$$

Lemma 5.4. *If* $\sigma \in \left(W^{k+2,\infty}(\Omega)\right)^2$, *then we have*

$$|||\sigma - \Pi_h^k \sigma|||_{*,\infty} \le Ch^{k+2}||\sigma||_{k+2,\infty}.$$

We are now in the position to get our local superconvergence on the Gauss points for the approximation of the pressure field and along the Gauss lines for the approximation of the velocity field, respectively.

Theorem 5.3. *If there holds, besides the conditions of Theorem 5.2,* $u \in W^{k+2,\infty}(\Omega)$, *then we have*

$$\begin{aligned} & |\log h|^{1/2}|||u - u_h|||_{*,\infty} + |||\sigma - \sigma_h|||_{*,\infty} \\ & \le Ch^{k+2}|\log h|[||u_0||_{k+2} + ||\sigma(0)||_{k+2} + ||u||_{k+2,\infty}] \\ & + Ch^{k+2}|\log h|\left\{|||\sigma|||_{k+2,\infty} + \left[\int_0^t (||\sigma||_{k+2}^2 + ||\sigma_t||_{k+2}^2) ds\right]^{1/2}\right\}. \end{aligned}$$

Proof. From Theorem 5.2 and Lemma 5.3 we know

$$\begin{aligned} |||u - u_h|||_{*,\infty} &\le |||u - P_h^k u|||_{*,\infty} + |||P_h^k u - u_h|||_{*,\infty} \\ &\le Ch^{k+2}||u||_{k+2,\infty} + Ch^{k+2}|\log h|^{1/2}[||u_0||_{k+2} + ||\sigma(0)||_{k+2}] \\ &+ Ch^{k+2}|\log h|^{1/2}\left\{||\sigma||_{k+2} + \left[\int_0^t (||\sigma||_{k+2}^2 + ||\sigma_t||_{k+2}^2) ds\right]^{1/2}\right\} \\ &\le Ch^{k+2}|\log h|^{1/2}\left[|\log h|^{-1/2}||u||_{k+2,\infty} + ||u_0||_{k+2} + ||\sigma(0)||_{k+2}\right] \\ &+ Ch^{k+2}|\log h|^{1/2}\left\{||\sigma||_{k+2} + \left[\int_0^t (||\sigma||_{k+2}^2 + ||\sigma_t||_{k+2}^2) ds\right]^{1/2}\right\}. \end{aligned}$$

Similarly, we can also obtain by means of Theorem 5.2 and Lemma 5.4 that

$$\begin{aligned} |||\sigma - \Pi_h^k \sigma|||_{*,\infty} &\le Ch^{k+2}|\log h|\,[||u_0||_{k+2} + ||\sigma(0)||_{k+2} + |||\sigma|||_{k+2,\infty}] \\ &+ Ch^{k+2}|\log h|\left[\int_0^t (||\sigma||_{k+2}^2 + ||\sigma_t||_{k+2}^2) ds\right]^{1/2}. \end{aligned}$$

□

Next we shall consider the global superconvergence for the pressure and the velocity fields by virtue of interpolation post-processing methods. Analogous to [11] we need to construct two post-processing interpolation operators Π_{2h}^{k+1} and P_{2h}^{k+1} to satisfy

$$
\begin{aligned}
&\Pi_{2h}^{k+1}\Pi_h^k = \Pi_{2h}^{k+1}, && \\
&||\Pi_{2h}^{k+1}\mathbf{v}_h||_{0,p} \le C||\mathbf{v}_h||_{0,p}, && \forall \mathbf{v}_h \in \mathbf{V}_h, \\
&||\Pi_{2h}^{k+1}\sigma - \sigma||_{0,p} \le Ch^{k+2}||\sigma||_{k+2,p}, && \forall \sigma \in (W^{k+2,p}(\Omega))^2, \\
&P_{2h}^{k+1}P_h^k = P_{2h}^{k+1}, && \\
&||P_{2h}^{k+1}w_h||_{0,p} \le C||w_h||_{0,p}, && \forall w_h \in W_h, \\
&||P_{2h}^{k+1}u - u||_{0,p} \le Ch^{k+2}||u||_{k+2,p}, && \forall u \in W^{k+2,p}(\Omega),
\end{aligned}
\tag{5.9}
$$

where $1 \le p \le \infty$ and $||\cdot||_{0,\infty} = ||\cdot||_\infty$. Here we take for example $k = 3$ to demonstrate the construction of the projection interpolation operators Π_{2h}^{k+1} and P_{2h}^{k+1} satisfying (5.9). To this purpose, we assume that the rectangular partition T_h has been obtained from $T_{2h} = \{\tau\}$ with mesh size $2h$ by subdividing each element of T_{2h} into four small congruent rectangles. Let $\tau := \bigcup_{i=1}^{4} e_i$ with $e_i \in T_h$. Thus, we can define two projection operators Π_{2h}^4 and P_{2h}^4 associated with T_{2h} of degree at most 4 in x and y on τ, respectively, according to the following conditions:

$$
\begin{aligned}
&\Pi_{2h}^4\sigma|_\tau \in (Q_{4,4}(\tau))^2, && P_{2h}^4 u|_\tau \in Q_{4,4}(\tau), \\
&\int_{l_i} (\sigma - \Pi_{2h}^4\sigma)\cdot \mathbf{n} q ds = 0, && \forall q \in P_2(l_i),\ i = 1,2,\cdots,12, \\
&\int_{e_i} (\sigma - \Pi_{2h}^4\sigma) = 0, && i = 1,2,3,4, \\
&\int_{\tau} (\sigma - \Pi_{2h}^4\sigma)\cdot\phi = 0, && \forall \phi \in (Q_{1,1}(\tau)\backslash Q_{0,0}(\tau))^2,\ \text{and} \\
&\int_{e_i} (u - P_{2h}^4 u)\psi = 0, && \forall \psi \in Q_{2,1}(e_i),\ i = 1,2,3,4, \\
&\int_{\tau} (u - P_{2h}^4 u)\psi = 0, && \forall \psi \in Q_{3,0}(\tau)\backslash Q_{2,0}(\tau),\ \text{respectively,}
\end{aligned}
$$

where l_i $(i = 1,2,\cdots,12)$ is one of the twelve sides of the four small elements e_i $(i = 1,2,3,4)$.

Similarly, we can also define Π_{2h}^{k+1} and P_{2h}^{k+1} for the case of $k \ne 3$ such that (5.9) is satisfied.

By the two projection interpolation operators Π_{2h}^{k+1} and P_{2h}^{k+1} we can immediately gain the following global superconvergence theorem.

Theorem 5.4. *Assume that (u,σ) and (u_h,σ_h) are the solutions of (2.1) and (2.2), respectively. Then, we have under the conditions of Theorem 5.3 that*

$$
\begin{aligned}
&|\log h|^{1/2}||P_{2h}^{k+1}u_h - u||_\infty + ||\Pi_{2h}^{k+1}\sigma_h - \sigma||_\infty \\
&\quad \le Ch^{k+2}|\log h|[||u_0||_{k+2} + ||\sigma(0)||_{k+2} + |\log h|^{-1/2}||u||_{k+2,\infty}] \\
&\quad + Ch^{k+2}|\log h|\left\{|||\sigma|||_{k+2,\infty} + \left[\int_0^t (||\sigma||_{k+2}^2 + ||\sigma_t||_{k+2}^2) ds\right]^{1/2}\right\}.
\end{aligned}
$$

Proof. We see from one of the properties of the operator P_{2h}^{k+1} described in (5.9) that

$$P_{2h}^{k+1}u_h - u = P_{2h}^{k+1}(u_h - P_h^k u) + (P_{2h}^{k+1}u - u).$$

Therefore, it follows from Theorem 5.2 and (5.9) that

$$\begin{aligned} ||P_{2h}^{k+1}u - u||_\infty &\le C||u_h - P_h^k u||_\infty + Ch^{k+2}||u||_{k+2,\infty} \\ &\le Ch^{k+2}|\log h|^{1/2}\left(|\log h|^{-1/2}||u||_{k+2,\infty} + ||u_0||_{k+2} + ||\sigma(0)||_{k+2}\right) \\ &+ Ch^{k+2}|\log h|^{1/2}\left\{||\sigma||_{k+2} + \left[\int_0^t (||\sigma||_{k+2}^2 + ||\sigma_t||_{k+2}^2)ds\right]^{1/2}\right\}. \end{aligned}$$

Analogously, we can obtain

$$\begin{aligned} ||\Pi_{2h}^{k+1}\sigma - \sigma||_\infty &\le Ch^{k+2}|\log h|[||u_0||_{k+2} + ||\sigma(0)||_{k+2} + |||\sigma|||_{k+2,\infty}] \\ &+ Ch^{k+2}|\log h|\left[\int_0^t (||\sigma||_{k+2}^2 + ||\sigma_t||_{k+2}^2)ds\right]^{1/2}. \end{aligned}$$

□

Remark 5.2 From the superconvergence estimates of $||\bar{u}_h - P_h^k u||_\infty$ and $||\bar{\sigma}_h - \Pi_h^k\sigma||_\infty$ indicated in Remark 5.1 we can also obtain the following global superconvergence under the conditions of Theorem 5.3 by the interpolation post-processing method:

$$|\log h|^{1/2}||P_{2h}^{k+1}\bar{u}_h - u||_\infty + ||\Pi_{2h}^{k+1}\bar{\sigma}_h - \sigma||_\infty \le Ch^{k+2}|\log h|(||u||_{k+2,\infty} + |||\sigma|||_{k+2,\infty}).$$

As a by-product, Theorem 5.4 can be employed to construct a-posteriori error estimators to assess the accuracy of the mixed finite element solution in applications. In fact, we have

Theorem 5.5. *We have under the conditions of Theorem 5.3 that*

$$||u - u_h||_\infty = ||P_{2h}^{k+1}u_h - u_h||_\infty + O\left(h^{k+2}|\log h|^{1/2}\right), \tag{5.10}$$

$$||\sigma - \sigma_h||_\infty = ||\Pi_{2h}^{k+1}\sigma_h - \sigma_h||_\infty + O\left(h^{k+2}|\log h|\right). \tag{5.11}$$

In addition, if there exist positive constants C_1, C_2 and small ϵ_1, $\epsilon_2 \in (0,1)$ such that

$$||u - u_h||_\infty \ge C_1 h^{k+2-\epsilon_1}, \tag{5.12}$$

$$||\sigma - \sigma_h||_\infty \ge C_2 h^{k+2-\epsilon_2}, \tag{5.13}$$

then there hold

$$\lim_{h\to 0} \frac{||u - u_h||_\infty}{||P_{2h}^{k+1}u_h - u_h||_\infty} = 1, \tag{5.14}$$

$$\lim_{h\to 0} \frac{||\sigma - \sigma_h||_\infty}{||\Pi_{2h}^{k+1}\sigma_h - \sigma_h||_\infty} = 1, \tag{5.15}$$

Proof. Following the procedure for Theorem 5.3 in [11] we can immediately obtain the desired results. □

We see from (5.10) that the computable error quantity $||P_{2h}^{k+1}u_h - u_h||_\infty$ is the principal part of the mixed finite element error $||u - u_h||_\infty$. Moreover, by (5.14) it can be used as a reliable a-posteriori error indicator to assess the accuracy of the mixed finite element solution under the condition (5.12). Meanwhile, (5.12) seems to be a reasonable assumption since $O(h^{k+1})$ is the optimal convergence rate of the mixed finite element solution in L^∞-norm subject to the conditions of Theorem 5.3. The same comments are also valid for (5.11), (5.13) and (5.15).

In the previous sections, we have seen that the regularized Green's functions play an important role in the analysis of convergence and superconvergence estimates in maximum norms for the mixed finite element method of (1.1). We present the proofs of Theorems 3.2 and 3.3 in this section. The proofs are based on a series of lemmas. First, we prove the following result.

Lemma 6.1. *We have under the assumptions of Theorem 3.2 that*

$$||\mathbf{G}_1^h - \mathbf{G}_1||_0 \leq C(1 + \phi_1(t)).$$

Proof. From (3.2) we know

$$||\mathbf{G}_1||_0 \leq C\left(||\nabla\lambda_1||_0 + \int_t^T ||\mathbf{G}_1||_0 ds\right),$$

and by means of Gronwall's lemma and Theorem 3.1,

$$||\mathbf{G}_1||_0 \leq C||\nabla\lambda_1||_0 \leq C|\log h|^{1/2}(1 + \phi_1(t)),$$

which yields via following the similar arguments to those above and using the estimate for $||\nabla^2\lambda_1||_0$ in Theorem 3.1 that

$$||\mathrm{div}\mathbf{G}_1||_0 \leq Ch^{-1}(1 + \phi_1(t)) + C|\log h|^{1/2}(1 + \phi_1(t)) \leq Ch^{-1}(1 + \phi_1(t)).$$

Decompose the error $\mathbf{G}_1 - \mathbf{G}_1^h$ as follows:

$$\mathbf{G}_1 - \mathbf{G}_1^h = (\mathbf{G}_1 - \Pi_h^k\mathbf{G}_1) + (\Pi_h^k\mathbf{G}_1 - \mathbf{G}_1^h) := \theta^{**} + \theta_h^{**}.$$

Then, θ_h^{**} satisfies the following equation according to (2.5) and the corresponding mixed finite element error equation of (3.2) to (2.4) that

$$\begin{aligned}(\alpha\theta_h^{**} + M * *\theta_h^{**}, \mathbf{v}_h) &= -(\alpha\theta^{**} + M * *\theta^{**}, \mathbf{v}_h) + (\nabla(\lambda_1 - \lambda_1^h), \mathbf{v}_h)\\ &= -(\alpha\theta^{**} + M * *\theta^{**}, \mathbf{v}_h) - (\lambda_1 - \lambda_1^h, \nabla\cdot\mathbf{v}_h)\\ &= -(\alpha\theta^{**} + M * *\theta^{**}, \mathbf{v}_h) - (P_h^k\lambda_1 - \lambda_1^h, \nabla\cdot\mathbf{v}_h), \quad \mathbf{v}_h \in \mathbf{V}_h.\end{aligned}$$

Since

$$(P_h^k\lambda_1 - \lambda_1^h, \nabla\cdot\theta_h^{**}) = 0$$

by (2.5) and the mixed finite element error equation of (3.2), taking $\mathbf{v}_h = \theta_h^{**}$ in the above equation leads to

$$(\alpha\theta_h^{**} + M * *\theta_h^{**}, \theta_h^{**}) = -(\alpha\theta^{**} + M * *\theta^{**}, \theta_h^{**}).$$

Thus, we have by Lemma 4.1 that

$$||\theta_h^{**}||_0^2 \leq C\left(\int_t^T ||\theta_h^{**}||_0 ds\right)^2 + \epsilon||\theta_h^{**}||_0^2 + C\left(||\theta^{**}||_0 + \int_t^T ||\theta^{**}||_0 ds\right)^2,$$

or

$$||\theta_h^{**}||_0 \leq C\left(\int_t^T ||\theta_h^{**}||_0 ds + ||\theta^{**}||_0 + \int_t^T ||\theta^{**}||_0 ds\right)$$

which, together with Gronwall's lemma, implies

$$||\theta_h^{**}||_0 \leq C\left(||\theta^{**}||_0 + \int_t^T ||\theta^{**}||_0 ds\right).$$

Hence, we obtain by virtue of the above estimate for $\mathrm{div}\mathbf{G}_1$ in L^2-norm and (3.5) that

$$\begin{aligned}||\mathbf{G}_1-\mathbf{G}_1^h||_0 &\le C\left(||\mathbf{G}_1-\Pi_h^k\mathbf{G}_1||_0+\int_t^T||\mathbf{G}_1-\Pi_h^k\mathbf{G}_1||_0ds\right)\\ &\le Ch\left(||\mathrm{div}\mathbf{G}_1||_0+\int_t^T||\mathrm{div}\mathbf{G}_1||_0ds\right)\\ &\le C(1+\phi_1(t)).\end{aligned}$$

Thus, the proof of Lemma 6.1 is complete. □.

Lemma 6.2. *Under the assumptions of Theorem 3.2,*

$$\begin{aligned}||\lambda_1^h-P_h^k\lambda_1||_0 &\le Ch(1+\phi_1(t)),\\ ||\lambda_1^h-\lambda_1||_0 &\le Ch|\log h|^{\delta_{k0}/2}(1+\phi_1(t)).\end{aligned}$$

Proof. Let $(\mathbf{w},\lambda)\in\mathbf{V}\times L^2(\Omega)$ be defined such that

$$\begin{aligned}\alpha\mathbf{w}+M*\mathbf{w}-\nabla\lambda &=0, && \text{in } \Omega\times(0,T),\\ \mathrm{div}\mathbf{w} &=(\lambda_1^h-P_h^k\lambda_1)\phi(t), && \text{in } \Omega\times(0,T),\\ \lambda &=0, && \text{on } \partial\Omega\times(0,T),\end{aligned}\tag{6.1}$$

where $\phi(t)\ge 0$ and $\int_0^T\phi(t)dt\le 1$. Clearly, $(\mathbf{w},\lambda)$ is well defined and satisfies

$$||\nabla^2\lambda||_0\le C\left(||\lambda_1^h-P_h^k\lambda_1||_0\phi(t)+\int_0^t||\lambda_1^h-P_h^k\lambda_1||_0\phi(s)ds\right)$$

by the regularity assumption on Ω. Now, it follows from (2.5), the corresponding mixed finite element error equation of (3.2) to (2.4) and Lemma 3.1 that

$$\begin{aligned}\int_0^T||\lambda_1^h-P_h^k\lambda_1||_0^2\phi(t)dt &=\int_0^T(\lambda_1^h-P_h^k\lambda_1,\mathrm{div}\mathbf{w})dt\\ &=\int_0^T(\lambda_1^h-P_h^k\lambda_1,\mathrm{div}\Pi_h^k\mathbf{w})dt\\ &=\int_0^T(\lambda_1^h-\lambda_1,\mathrm{div}\Pi_h^k\mathbf{w})dt\\ &=\int_0^T(\alpha(\mathbf{G}_1-\mathbf{G}_1^h)+M**(\mathbf{G}_1-\mathbf{G}_1^h),\Pi_h^k\mathbf{w})dt\\ &=\int_0^T(\alpha(\mathbf{G}_1-\mathbf{G}_1^h)+M**(\mathbf{G}_1-\mathbf{G}_1^h),\Pi_h^k\mathbf{w}-\mathbf{w})dt\\ &\quad+\int_0^T(\alpha\mathbf{w}+M*\mathbf{w},\mathbf{G}_1-\mathbf{G}_1^h)dt:=N_1+N_2.\end{aligned}\tag{6.2}$$

Obviously, we have via using (3.2) and its mixed finite element approximation as well as (2.5) that

$$\begin{aligned} N_2 &:= \int_0^T (\nabla\lambda, \mathbf{G}_1 - \mathbf{G}_1^h)dt = \int_0^T (\lambda, \mathrm{div}(\mathbf{G}_1^h - \mathbf{G}_1))dt \\ &= \int_0^T (\lambda, \mathrm{div}\mathbf{G}_1^h)dt - \int_0^T (\lambda, \delta_1^h)\phi_1(t)dt \\ &= \int_0^T (P_h^k\lambda, \mathrm{div}\mathbf{G}_1^h)dt - \int_0^T (\lambda, \delta_1^h)\phi_1(t)dt \\ &= \int_0^T (P_h^k\lambda, P_h^k\delta_1^h)\phi_1(t)dt - \int_0^T (\lambda, \delta_1^h)\phi_1(t)dt \\ &= \int_0^T (P_h^k\lambda, \delta_1^h)\phi_1(t)dt - \int_0^T (\lambda, \delta_1^h)\phi_1(t)dt \\ &= \int_0^T (P_h^k\lambda - \lambda, \delta_1^h)\phi_1(t)dt. \end{aligned}$$

And then, N_2 can be estimated by Lemma 3.1 and the estimate for $||\nabla^2\lambda||_0$ as for $k \geq 1$

$$\begin{aligned} |N_2| &\leq \int_0^T Ch^2||\nabla^2\lambda||_0||\delta_1^h||_0\phi_1(t)dt \\ &\leq Ch\int_0^T \left(||\lambda_1^h - P_h^k\lambda_1||_0\phi(t) + \int_0^t ||\lambda_1^h - P_h^k\lambda_1||_0\phi(s)ds\right)\phi_1(t)dt \\ &\leq Ch\int_0^T \left(\phi_1(t) + \int_t^T \phi_1(s)ds\right)||\lambda_1^h - P_h^k\lambda_1||_0\phi(t)dt \\ &\leq Ch\int_0^T (\phi_1(t)+1)||\lambda_1^h - P_h^k\lambda_1||_0\phi(t)dt. \end{aligned} \tag{6.3}$$

Similarly, we have for N_1 by virtue of Lemma 6.1 and (6.1) that

$$\begin{aligned} |N_1| &\leq \int_0^T C\left(||\mathbf{G}_1 - \mathbf{G}_1^h||_0 + \int_t^T ||\mathbf{G}_1 - \mathbf{G}_1^h||_0 ds\right)||\mathbf{w} - \Pi_h^k\mathbf{w}||_0 dt \\ &\leq Ch\int_0^T (1+\phi_1(t))||\mathrm{div}\mathbf{w}||_0 dt \\ &\leq Ch\int_0^T (1+\phi_1(t))||\lambda_1^h - P_h^k\lambda_1||_0\phi(t)dt. \end{aligned} \tag{6.4}$$

Combining (6.2) with (6.3) and (6.4) leads to

$$\int_0^T ||\lambda_1^h - P_h^k\lambda_1||_0^2\phi(t)dt \leq Ch\int_0^T (1+\phi_1(t))||\lambda_1^h - P_h^k\lambda_1||_0\phi(t)dt,$$

and by means of Lemma 3.2,

$$||\lambda_1^h - P_h^k\lambda_1||_0^2 \leq Ch(1+\phi_1(t))||\lambda_1^h - P_h^k\lambda_1||_0,$$

that is,

$$||\lambda_1^h - P_h^k\lambda_1||_0 \leq Ch(1+\phi_1(t)), \quad \text{for } k \geq 1.$$

It remains to treat N_2 for $k = 0$. It follows from

$$\begin{aligned} N_2 &= \int_0^T (P_h^0\lambda - \lambda, \delta_1^h)\phi_1(t)dt \\ &= \int_0^T (P_h^0\lambda - P_h^1\lambda, \delta_1^h)\phi_1(t)dt + \int_0^T (P_h^1\lambda - \lambda, \delta_1^h)\phi_1(t)dt \end{aligned}$$

and

$$\int_0^T (P_h^0\lambda - P_h^1\lambda, \delta_1^h)\phi_1(t)dt = 0 \quad \text{(see [25])}$$

that

$$N_2 = \int_0^T (P_h^1\lambda - \lambda, \delta_1^h)\phi_1(t)dt, \quad \text{for } k = 0,$$

which implies by the same arguments as those for (6.3) that

$$|N_2| \le Ch\int_0^T (1+\phi_1(t))||\lambda_1^h - P_h^0\lambda_1||_0\phi(t)dt.$$

This verifies

$$||\lambda_1^h - P_h^k\lambda_1||_0 \le Ch(1+\phi_1(t)).$$

Finally, the second inequality in Lemma 6.2 is an easy consequence of the first inequality in the same lemma and Theorem 3.1 together with the standard triangle inequality. □

Remark 6.1 Using the similar duality argument to that above we can easily obtain [25]

$$||\lambda_2^h - P_h^k\lambda_2||_0 + ||\lambda_2^h - \lambda_2||_0 \le C(1+\phi_2(t)).$$

Here we omit the details.

Lemma 6.3. *We have for Green's function $\mathbf{G}_1$ and its mixed finite element approximation $\mathbf{G}_1^h$ that*

$$||\mathbf{G}_1^h - \mathbf{G}_1||_{L^1(\Omega)} \le Ch|\log h|(1+\phi_1(t)).$$

Proof. By Schwartz inequality and (3.1) we have

$$\begin{aligned} ||\mathbf{G}_1^h - \mathbf{G}_1||_{L^1(\Omega)} &= \int_\Omega \beta^{-1}(\beta|\mathbf{G}_1^h - \mathbf{G}_1|)d\Omega \le \left(\int_\Omega \beta^{-2}d\Omega\right)^{1/2} ||\mathbf{G}_1^h - \mathbf{G}_1||_{\beta^2} \\ &\le C|\log h|^{1/2}||\mathbf{G}_1^h - \mathbf{G}_1||_{\beta^2}. \end{aligned} \tag{6.5}$$

Let

$$\mathbf{\Psi}_1 := \beta^2(\mathbf{G}_1 - \mathbf{G}_1^h).$$

Then, from Lemma 4.1 and the mixed finite element error equation of (3.2) we derive that

$$\begin{aligned} ||\mathbf{G}_1^h - \mathbf{G}_1||_{\beta^2}^2 &\le C_0(\alpha(\mathbf{G}_1 - \mathbf{G}_1^h), \mathbf{\Psi}_1) \\ &= C_0(\alpha(\mathbf{G}_1 - \mathbf{G}_1^h), \mathbf{\Psi}_1 - \Pi_h^k\mathbf{\Psi}_1) + C_0(\alpha(\mathbf{G}_1 - \mathbf{G}_1^h), \Pi_h^k\mathbf{\Psi}_1) \\ &= C_0(\alpha(\mathbf{G}_1 - \mathbf{G}_1^h), \mathbf{\Psi}_1 - \Pi_h^k\mathbf{\Psi}_1) \\ &\quad + C_0(\alpha(\mathbf{G}_1 - \mathbf{G}_1^h) + M**(\mathbf{G}_1 - \mathbf{G}_1^h), \Pi_h^k\mathbf{\Psi}_1) \\ &\quad - C_0(M**(\mathbf{G}_1 - \mathbf{G}_1^h), \Pi_h^k\mathbf{\Psi}_1) \\ &= C_0(\alpha(\mathbf{G}_1 - \mathbf{G}_1^h), \mathbf{\Psi}_1 - \Pi_h^k\mathbf{\Psi}_1) - C_0(\lambda_1 - \lambda_1^h, \text{div}\Pi_h^k\mathbf{\Psi}_1) \\ &\quad - C_0(M**(\mathbf{G}_1 - \mathbf{G}_1^h), \Pi_h^k\mathbf{\Psi}_1) \\ &:= M_1 + M_2 + M_3. \end{aligned} \tag{6.6}$$

Now we consider M_i's individually. First, it follows from Lemma 6.5 below that

$$\begin{aligned} |M_1| &\le C_0||\alpha(\mathbf{G}_1 - \mathbf{G}_1^h)||_{\beta^2} \cdot ||\mathbf{\Psi}_1 - \Pi_h^k\mathbf{\Psi}_1||_{\beta^{-2}} \\ &\le \epsilon||\mathbf{G}_1 - \mathbf{G}_1^h||_{\beta^2}^2 + Ch^2|\log h|(1+\phi_1(t))^2. \end{aligned} \tag{6.7}$$

We know from (2.5) that

$$(\text{div}\Pi_h^k\sigma, w_h) = (\text{div}\sigma, w_h), \quad \forall\, w_h \in W_h,$$

which, together with Lemma 6.2, implies

$$\begin{aligned}
|M_2| &= C_0|(\lambda_1-\lambda_1^h, \mathrm{div}\Pi_h^k \mathbf{\Psi}_1)| \\
&= C_0|(P_h^k\lambda_1-\lambda_1^h, \mathrm{div}\Pi_h^k \mathbf{\Psi}_1)| \\
&= C_0|(P_h^k\lambda_1-\lambda_1^h, \mathrm{div}\mathbf{\Psi}_1)| \\
&\le Ch(1+\phi_1(t))||\mathrm{div}\mathbf{\Psi}_1||_0.
\end{aligned} \tag{6.8}$$

Since there holds by (3.2)

$$\mathrm{div}\mathbf{\Psi}_1 = \nabla(\beta^2)\cdot(\mathbf{G}_1-\mathbf{G}_1^h) + \beta^2(\delta_1^h - P_h^k\delta_1^h)\phi_1(t),$$

we have

$$||\mathrm{div}\mathbf{\Psi}_1||_0 \le C||\mathbf{G}_1-\mathbf{G}_1^h||_{\beta^2} + Ch\phi_1(t).$$

Thus, we obtain from (6.8)

$$|M_2| \le Ch^2(1+\phi_1(t))^2 + \epsilon||\mathbf{G}_1-\mathbf{G}_1^h||_{\beta^2}^2. \tag{6.9}$$

It follows from Schwartz inequality, Lemma 6.5 and the equality

$$||\mathbf{\Psi}_1||_{\beta^{-2}} = ||\mathbf{G}_1-\mathbf{G}_1^h||_{\beta^2}$$

that

$$\begin{aligned}
|M_3| &= C_0\left|(M**(\mathbf{G}_1-\mathbf{G}_1^h), \Pi_h^k\mathbf{\Psi}_1-\mathbf{\Psi}_1) + (M**(\mathbf{G}_1-\mathbf{G}_1^h), \mathbf{\Psi}_1)\right| \\
&\le C\left(\int_t^T ||\mathbf{G}_1-\mathbf{G}_1^h||_{\beta^2}ds\right)||\Pi_h^k\mathbf{\Psi}_1-\mathbf{\Psi}_1||_{\beta^{-2}} \\
&\quad +C\left(\int_t^T ||\mathbf{G}_1-\mathbf{G}_1^h||_{\beta^2}ds\right)||\mathbf{\Psi}_1||_{\beta^{-2}} \\
&\le C\left(\int_t^T ||\mathbf{G}_1-\mathbf{G}_1^h||_{\beta^2}ds\right)h|\log h|^{1/2}(1+\phi_1(t)) \\
&\quad +C\left(\int_t^T ||\mathbf{G}_1-\mathbf{G}_1^h||_{\beta^2}ds\right)||\mathbf{G}_1-\mathbf{G}_1^h||_{\beta^2} \\
&\le \epsilon||\mathbf{G}_1-\mathbf{G}_1^h||_{\beta^2}^2 + C\left(\int_t^T ||\mathbf{G}_1-\mathbf{G}_1^h||_{\beta^2}ds\right)^2 \\
&\quad +Ch^2|\log h|(1+\phi_1(t))^2.
\end{aligned} \tag{6.10}$$

Combining (6.6) with (6.7), (6.9) and (6.10) gives for $\epsilon > 0$ small enough and fixed that

$$||\mathbf{G}_1-\mathbf{G}_1^h||_{\beta^2}^2 \le Ch^2|\log h|(1+\phi_1(t))^2 + C\left(\int_t^T ||\mathbf{G}_1-\mathbf{G}_1^h||_{\beta^2}ds\right)^2,$$

which in turn implies that

$$||\mathbf{G}_1-\mathbf{G}_1^h||_{\beta^2} \le Ch|\log h|^{1/2}(1+\phi_1(t)) + C\int_t^T ||\mathbf{G}_1-\mathbf{G}_1^h||_{\beta^2}ds,$$

so that Gronwall's lemma yields

$$||\mathbf{G}_1-\mathbf{G}_1^h||_{\beta^2} \le Ch|\log h|^{1/2}(1+\phi_1(t)). \tag{6.11}$$

Hence, Lemma 6.3 follows from (6.5) and (6.11). □.

Lemma 6.4. *Under the assumptions of Theorem 3.3,*

$$\begin{aligned}
&||\mathbf{G}_2-\mathbf{G}_2^h||_{L^1(\Omega)} \le C|\log h|^{1/2}(1+\phi_2(t)), \\
&||\mathbf{G}_2-\mathbf{G}_2^h||_0 \le Ch^{-1}(1+\phi_2(t)), \\
&||\nabla\lambda_2||_0 \le Ch^{-1}(1+\phi_2(t)).
\end{aligned}$$

Proof. We have by virtue of Schwartz inequality and (3.1) that

$$||\mathbf{G}_2 - \mathbf{G}_2^h||_{L^1(\Omega)} \le C|\log h|^{1/2}||\mathbf{G}_2 - \mathbf{G}_2^h||_{\beta^2}. \tag{6.12}$$

Let

$$\Psi_2 := \beta^2(\mathbf{G}_2 - \mathbf{G}_2^h).$$

Then, it follows from a similar argument to that for Lemma 6.3 that

$$\begin{aligned} ||\mathbf{G}_2 - \mathbf{G}_2^h||^2_{\beta^2} &\le C_0(\alpha(\mathbf{G}_2 - \mathbf{G}_2^h), \Psi_2) \\ &= C_0(\alpha(\mathbf{G}_2 - \mathbf{G}_2^h), \Psi_2 - \Pi_h^k\Psi_2) - C_0(\lambda_2 - \lambda_2^h, \mathrm{div}\Pi_h^k\Psi_2) \\ &\quad - C_0(M**(\mathbf{G}_2 - \mathbf{G}_2^h), \Pi_h^k\Psi_2) \\ &:= M_1' + M_2' + M_3'. \end{aligned} \tag{6.13}$$

Thus, we know from Lemma 6.5 below that

$$|M_1'| \le \epsilon||\mathbf{G}_2 - \mathbf{G}_2^h||^2_{\beta^2} + C(1 + \phi_2(t))^2. \tag{6.14}$$

Moreover, we see from Remark 6.1 and the same arguments as those for (6.8) that

$$|M_2'| \le C(1 + \phi_2(t))||\mathrm{div}\Psi_2||_0. \tag{6.15}$$

We derive from (3.4) that

$$\mathrm{div}\Psi_2 = \nabla(\beta^2) \cdot (G_2 - G_2^h),$$

which yields by (6.15) that

$$\begin{aligned} |M_2'| &\le C(1 + \phi_2(t))||\mathbf{G}_2 - \mathbf{G}_2^h||_{\beta^2} \\ &\le \epsilon||\mathbf{G}_2 - \mathbf{G}_2^h||^2_{\beta^2} + C(1 + \phi_2(t))^2. \end{aligned} \tag{6.16}$$

Also, we can obtain according to the similar steps for (6.10) that

$$|M_3'| \le \epsilon||\mathbf{G}_2 - \mathbf{G}_2^h||^2_{\beta^2} + C\left(\int_t^T ||\mathbf{G}_2 - \mathbf{G}_2^h||_{\beta^2} ds\right)^2 + C(1 + \phi_2(t))^2. \tag{6.17}$$

Combining (6.14), (6.16) and (6.17) with (6.13) leads to

$$||\mathbf{G}_2 - \mathbf{G}_2^h||_{\beta^2} \le C\left\{\int_t^T ||\mathbf{G}_2 - \mathbf{G}_2^h||_{\beta^2} ds + (1 + \phi_2(t))\right\},$$

and Gronwall's inequality implies

$$||\mathbf{G}_2 - \mathbf{G}_2^h||_{\beta^2} \le C(1 + \phi_2(t)).$$

Hence, from (6.12) we know

$$||\mathbf{G}_2 - \mathbf{G}_2^h||_{L^1(\Omega)} \le C|\log h|^{1/2}(1 + \phi_2(t)).$$

By the H^2-regularity assumption on the domain, there holds

$$||\nabla\lambda_2||_0 \le Ch^{-1}(1 + \phi_2(t)).$$

Thus, from [25] we know

$$||\mathbf{G}_2 - \mathbf{G}_2^h||_0 \le Ch^{-1}(1 + \phi_2(t)).$$

□

Lemma 6.5. *Let Ψ_i $(i = 1, 2)$ be the functions defined as before. Then, we have*

$$\begin{aligned} ||\Psi_1 - \Pi_h^k\Psi_1||_{\beta^{-2}} &\le Ch|\log h|^{1/2}(1 + \phi_1(t)), \\ ||\Psi_2 - \Pi_h^k\Psi_2||_{\beta^{-2}} &\le C(1 + \phi_2(t)). \end{aligned}$$

Proof. Recall

$$\Psi_i = \beta^2(\mathbf{G}_i - \mathbf{G}_i^h), \quad i = 1, 2,$$

and rewrite them as

$$\begin{aligned}\Psi_i &= \beta^2(\mathbf{G}_i - \Pi_h^k \mathbf{G}_i) + \beta^2(\Pi_h^k \mathbf{G}_i - \mathbf{G}_i^h) \\ &:= \Psi_{i1} + \Psi_{i2}.\end{aligned}$$

Thus,

$$||\Psi_i - \Pi_h^k \Psi_i||_{\beta^{-2}} \leq ||\Psi_{i1} - \Pi_h^k \Psi_{i1}||_{\beta^{-2}} + ||\Psi_{i2} - \Pi_h^k \Psi_{i2}||_{\beta^{-2}}. \tag{6.18}$$

Since Π_h^k is a local projection operator, it follows from [25] that

$$||\Psi_{i1} - \Pi_h^k \Psi_{i1}||_{\beta^{-2}} \leq C||\Psi_{i1}||_{\beta^{-2}} \leq C||\mathbf{G}_i - \Pi_h^k \mathbf{G}_i||_{\beta^2} \leq Ch||\nabla^2 \lambda_i||_{\beta^2}.$$

Then, Theorem 3.1 and (6.28) below lead to

$$||\Psi_{i1} - \Pi_h^k \Psi_{i1}||_{\beta^{-2}} \leq \begin{cases} Ch|\log h|^{1/2}(1 + \phi_1(t)), & \text{for } i = 1, \\ C(1 + \phi_2(t)), & \text{for } i = 2. \end{cases} \tag{6.19}$$

Following [25] we obtain from Lemma 6.1 and the estimate for $||\nabla \lambda_2||_0$ in Lemma 6.4 that

$$||\Psi_{i2} - \Pi_h^k \Psi_{i2}||_{\beta^{-2}} \leq \begin{cases} Ch(1 + \phi_1(t)), & \text{for } i = 1, \\ C(1 + \phi_2(t)), & \text{for } i = 2. \end{cases} \tag{6.20}$$

Now, (6.19) and (6.20) lead (6.18) to

$$||\Psi_i - \Pi_h^k \Psi_i||_{\beta^{-2}} \leq \begin{cases} Ch|\log h|^{1/2}(1 + \phi_1(t)), & \text{for } i = 1, \\ C(1 + \phi_2(t)), & \text{for } i = 2, \end{cases}$$

which verifies the conclusions of Lemma 6.5. □

Lemma 6.6. *Under the assumptions of Theorem 3.3 there hold*

$$\begin{aligned}||\lambda_2||_0 &\leq C|\log h|^{1/2}(1 + \phi_2(t)), \\ ||\nabla \lambda_2||_{L^1(\Omega)} &\leq C|\log h|(1 + \phi_2(t)), \\ ||\nabla^2 \lambda_2||_{L^1(\Omega)} &\leq Ch^{-1}|\log h|^{1/2}(1 + \phi_2(t)).\end{aligned}$$

Proof. For the sake of simplicity of our analysis, here we assume that the matrices A and B in (1.1) are independent of the spatial variable x.

From Schwarz's inequality and (3.1) we know

$$||\nabla \lambda_2||_{L^1(\Omega)} \leq C|\log h|^{1/2}||\nabla \lambda_2||_{\beta^2}. \tag{6.21}$$

Furthermore, it follows from (3.4) and Green's formula that

$$\begin{aligned}||\nabla \lambda_2||_{\beta^2}^2 &= (\nabla \lambda_2, \beta^2 \nabla \lambda_2) = -(\Delta \lambda_2, \beta^2 \lambda_2) + \frac{1}{2}(\lambda_2, \Delta(\beta^2)\lambda_2) \\ &\leq |(\text{div}\delta_2^h \phi_2(t), \beta^2 \lambda_2)| + C||\lambda_2||_0^2 \\ &\leq C(\phi_2^2(t) + ||\lambda_2||_0^2).\end{aligned} \tag{6.22}$$

Now, let us consider the following auxiliary Dirichlet problem to bound $||\lambda_2||_0$:

$$\begin{aligned}-\Delta r &= \lambda_2 \qquad \text{in } \Omega, \\ r &= 0 \qquad \text{on } \partial\Omega.\end{aligned}$$

Then, we know from the regularity assumption on the domain Ω that

$$||\nabla^2 r||_0 \leq C||\lambda_2||_0. \tag{6.23}$$

In addition, it follows from (3.4) and Green's formula that

$$\begin{aligned}||\lambda_2||_0^2 &= (\nabla \lambda_2, \nabla r) = -(\nabla^2 \lambda_2, r) \\ &= (\text{div}\delta_2^h, r)\phi_2(t) = -(\delta_2^h, \nabla r)\phi_2(t) \\ &:= N_*.\end{aligned} \tag{6.24}$$

As done in [25], for N_* we further have according to (3.5), (6.23) and the standard inverse estimate,

$$||(\nabla r)^I||_\infty \leq C|\log h|^{1/2}||(\nabla r)^I||_1 \leq C|\log h|^{1/2}||\nabla^2 r||_0,$$

that

$$\begin{aligned} |N_*| &= \left|\left(\delta_2^h, \nabla r\right) \phi_2(t)\right| \\ &\leq \left|\left(\delta_2^h, \nabla r - (\nabla r)^I\right)\right| + \left|\left(\delta_2^h, (\nabla r)^I\right)\right| \phi_2(t) \\ &\leq C \left(||\nabla^2 r||_0 + ||\delta_2^h||_{L^1(\Omega)}||(\nabla r)^I||_\infty\right) \phi_2(t) \\ &\leq C \left(1 + |\log h|^{1/2}\right) ||\lambda_2||_0 \phi_2(t), \end{aligned} \tag{6.25}$$

where f^I stands for the standard locally regularized piecewise linear interpolation of f [see, for example, 25].

Combining (6.25) with (6.24) yields

$$||\lambda_2||_0 \leq C\left(1 + |\log h|^{1/2}\right)\phi_2(t). \tag{6.26}$$

Now, (6.26) and (6.22) lead (6.21) to

$$||\nabla \lambda_2||_{L^1(\Omega)} \leq C|\log h|(1 + \phi_2(t)).$$

Again, we use Schwarz's inequality and (3.1) to obtain

$$||\nabla^2 \lambda_2||_{L^1(\Omega)} \leq C|\log h|^{1/2}||\nabla^2 \lambda_2||_{\beta^2}. \tag{6.27}$$

Following [25] we further have

$$||\nabla^2 \lambda_2||_{\beta^2} \leq Ch^{-1}(1 + \phi_2(t)). \tag{6.28}$$

Thus,

$$||\nabla^2 \lambda_2||_{L^1(\Omega)} \leq Ch^{-1}|\log h|^{1/2}(1 + \phi_2(t)).$$

□

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