ON ITERATION AND APPROXIMATION METHODS FOR ANISOTROPIC PROBLEMS

C. Kim

ISC–01–03–MATH

ABSTRACT

On Iteration and Approximation
Methods for Anisotropic Problems. (December 2001)
Chisup Kim, B.S., Seoul National University, Korea
Chair of Advisory Committee: Dr. Joseph E. Pasciak

We study numerical methods and iterative solution techniques for a model second-order anisotropic elliptic partial differential equation on a unit square. In particular, we consider a mixed finite element approximation for this problem on uniform rectangular and triangular meshes and derive error estimates explicitly giving the behavior of the anisotropy parameter. To efficiently solve the resulting linear system from the mixed finite element problem, a two-level preconditioner is constructed and analyzed. Here, the fine and coarse level problems correspond to the mixed finite element and the standard finite element problems, respectively, on the same mesh. Utilizing the multigrid/multilevel preconditioners for the finite element problem, a multilevel preconditioner for the mixed system is obtained. To relate the mixed finite element problem with the standard finite element problem, we take the Schur complement of the mixed system in the rectangular case and an equivalent nonconforming problem in the triangular case. As smoothers in the multilevel preconditioners, the line Jacobi and line Gauss-Seidel smoothers are used. It is shown that this approach gives a preconditioner for the mixed system which is uniform both in the anisotropy parameter and the mesh size. Uniform multigrid preconditioners for the standard finite element method for the anisotropic problem are also discussed and numerical results for the two-level preconditioning are presented.

ACKNOWLEDGMENTS

First and foremost, I would like to thank Prof. Joseph Pasciak, my advisor. He was the true captain, not only when I served at times as a crew member of his boat but also in my studies. He set the directions, was precise in the navigation, and took care of me during the voyages. I deeply appreciate his patience and the full support he gave me in times of difficulty.

I am indebted to each and every one of the members of my advisory committee for their concern and guidance. The constant smile and encouragement of Prof. Raytcho Lazarov only made me work harder. Prof. Richard Ewing, probably the busiest man I have ever known, strangely enough was there whenever I needed him. Prof. David Dobson has always been helpful, especially in my beginning year here. The insightful comments and suggestions of Prof. Thomas Blasingame are appreciated as well.

It was a blessing for me to have additional mentors. I thank Prof. James Bramble for his wisdom and help. In his courses in finite elements and iterative methods, it seemed as if the methods themselves were speaking. Dr. Panayot Vassilevski of the Center for Applied Scientific Computing at the Lawrence Livermore National Laboratory has also been an essential part of my education.

I give my thanks to the fellow students who willingly shared with me their invaluable knowledge and experience which were earned through many sleepless nights of hard work. I learned a lot from them as well as the renowned professors.

My five years as a graduate student were both pleasant and fruitful. For this I am grateful to the Department of Mathematics, especially to Prof. Jay Walton, who has been the Graduate Advisor for most of the five years, and to Ms. Monique Stewart. In this respect, I must also thank Prof. Dongwoo Sheen of Seoul National University, my alma mater. He introduced me to the field of numerical analysis and had a decisive influence on my choice to come to study at Texas A&M.

Financial support was very important in my graduate study as well. I thank the Department of Mathematics and the Institute for Scientific Computation, through which I have been supported under grants from the National Science Foundation, the Texas Higher Education Coordinating Board, and the Environmental Protection Agency. Thanks to the Center for Applied Scientific Computing , I was also able to spend two wonderful and productive summers in Livermore. In addition, I would like to sincerely thank the organizers of the 10th Copper Mountain Conference on Multigrid Methods, who gave great encouragement to a struggling student by awarding a prize for a part of the current work.

There are still many names and faces that I hold only in my heart and prayers in order to keep the acknowledgments from becoming longer than the dissertation itself. Nevertheless, I must thank my family – my parents, my sister, and my brother – who have been and will always be my greatest source of comfort and support. Finally, I wish a healthy and peaceful late 70's and beyond for my maternal grandmother who taught me how to read, write, and count.

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CHAPTER I

INTRODUCTION

A number of physical phenomena and quantities can be described or modeled by partial differential equations and their solutions. For instance, elliptic boundary value problems arise in a large class of physical phenomena such as diffusive or steady state phenomena. Rigorous mathematical analysis gives the existence and uniqueness of the solution to a variety of such partial differential equations (see, e.g., [41], [42], and [56]). In many circumstances, however, such results assume a high degree of smoothness or based on arguments that are not constructive. Some solution methods such as the series methods often require the coefficients of the equation or the domain to satisfy smoothness conditions that are not satisfied in a large number of applications. Therefore, it is very difficult, if not impossible, to obtain solution functions in closed form to the partial differential equations in those applications. Fortunately, however, the numerical or approximate values of such solutions are sufficient for practical purposes. Thus, systematic and mathematically sound ways to obtain the numerical solutions are desired. Two issues naturally arise, approximation and efficiency. One would like to obtain a numerical solution of desired accuracy while consuming only the minimal amount of available computational resources.

In this thesis, we study the numerical solution of a boundary value problem. In particular, we consider a model second-order elliptic anisotropic partial differential equation on the unit square with zero Dirichlet boundary condition.

Anisotropic problems form a subclass of singularly perturbed problems. An operator $L = L(\varepsilon)$ depending on a parameter ε is called singularly perturbed if the

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This dissertation follows the style and format of Mathematics of Computation.

limiting operator $L(0) = \lim_{\varepsilon \to 0} L(\varepsilon)$ is of a different type than $L(\varepsilon)$ for $\varepsilon > 0$ (see, e.g. [48] or [57]). Our model anisotropic operator will be given by $L(\varepsilon) = \partial_x^2 + \varepsilon \partial_y^2$, where ε is a small positive constant. Numerical methods for convection diffusion problems with the second order term $\varepsilon(\partial_x^2 + \partial_y^2)$ can be found in, e.g., [57].

Applications of anisotropic problems are found, for example, in the modeling of porous media flows. These applications include oil reservoir or groundwater flow (see, e.g., [12], [33], or [38]), and electrical wave propagation in the human heart (see, e.g., [67] and the references therein). The porous media and porous formations encountered in nature are highly heterogeneous. That is, their properties such as hydraulic conductivity and porosity differ greatly in space. A simplified situation of heterogeneous media could be found in a stratified or layered medium. Our model anisotropic problem, which depicts the case where the flow is essentially forced to a fixed direction of the medium, could be used in the modeling of flow in a layer (when constant ε is used) or several such layers (with piecewise constant ε depending on one of the x- or y-directions). As simplified as the model is, it could serve as a building block for more complicated models and a benchmark problem for numerical simulations. The current work is an effort to provide a refined error estimate for the flow properties, for example the pressure and velocity, and efficient and robust numerical methods to obtain them.

For the anisotropic problem, the accuracy of the numerical solution and the efficiency of the solution methods will be addressed. In the study of finite element methods for elliptic equations, the ratio between the ellipticity constant and the continuity constant is an important quantity (see, e.g., [29] or [36]). This ratio for the anisotropic operator $L(\varepsilon)$, when ε approaches 0, can become so large that the standard methods lose robustness. Therefore, different techniques specifically tailored for the anisotropy are called for. We will study the mixed finite element approximation (see, e.g., [6], [7], [8], [30], [31], [40], [43], or [55]) for the anisotropic problem and derive some error estimates (Chapter III). To efficiently solve the linear system arising from the mixed finite element problem, we will apply multigrid/multilevel preconditioning techniques (see, e.g., [22], [32], [48], or [68]) and obtain uniform preconditioners (Chapter IV). In the analysis, special attention will be paid to how the constants in various estimates in the finite element and multigrid analysis depend on the anisotropy parameter ε. The methods will be robust if the constants are uniform with respect to, or independent of, ε. For a survey of robust multigrid methods for some parameter-dependent problems, see [26].

The mixed finite element methods for second-order elliptic partial differential equations in the uniformly elliptic case have been extensively studied (see, e.g., [39], [40], and [55]). An advantage of the mixed methods is that an approximation to the derivatives of the solution can be obtained directly rather than by differentiation of the approximated solution. In some applications, the derivatives of the solution have important physical meaning. For example, in porous media flow modeling and groundwater applications, the gradient of the solution of the differential equation represents the velocity field of the flow (see, e.g., [33] and [35]). When the Raviart-Thomas elements [55] are used in these applications, the normal component of the velocity or the flux is computed. This property can be used to construct locally conservative schemes for parabolic or time-dependent problems.

In our mixed finite element approximation, we will use the lowest order Raviart-Thomas elements. Some error estimates explicitly giving the behavior of the parameter ε will be given in Chapter III by applying the results of Falk and Osborn [40]. There is another, probably more widely applied, abstract framework for the study of the mixed finite element methods developed independently by Babuška [6], [7], [8] and Brezzi [30]. The purpose of the Falk and Osborn approach is to obtain error estimates without the Babuška-Brezzi condition also known as the discrete inf-sup condition. In contrast to the Babuška-Brezzi analysis, the Falk and Osborn approach [40] gives separate error estimates for the two variables in the mixed method, the original unknown for the partial differential equation and the Lagrange multiplier. It can be shown for some applications such as certain biharmonic problems that the discrete inf-sup condition is satisfied with mesh dependent norms and the results from Babuška-Brezzi and Falk-Osborn approaches are the same [9].

Multigrid methods provide arguably the most efficient preconditioners for elliptic problems. In these methods, attempts are made to make the iterative error smooth on the finer grid by using an operator called a smoother and to reduce the error on the coarser grid by solving a suitable problem on the coarser grid. The coarser grid problem contains fewer unknowns and hence is easier to solve than the finer problem. To design and analyze multigrid algorithms, several things need to be considered. They are the regularity of the solution of the partial differential equation, the approximation property of the finite element space, and the effectiveness of the smoother. It is also possible to achieve this goal without the regularity assumptions [18]. In the so called geometric multigrid, the grids and the corresponding finite element spaces are given *a priori.* There is another approach called algebraic multigrid which only assumes that the finest space is given and constructs coarser spaces by some coarsening algorithms (see, e.g., [32], [49], [58], [65], and [68]). Multigrid method can also be viewed as a subspace correction method (see, [17], [18], or [74]).

Multigrid methods for mixed finite element problems have been studied for second order selfadjoint uniformly elliptic equations (see, e.g., [2], [19], [28], [34], [35], [60], [69], [70], and [72]). In [19], the Schur complement of the mixed finite element equations is solved in connection with the standard finite element problem, whereas in [28], [34], and [35], the multigrid technique is applied to an equivalent nonconforming finite element problem. In [60], the multigrid technique is directly applied to the Schur complement problem in the framework of non-inherited forms [19]. Multigrid for saddle point problems in the context of mortar finite element method (see, e.g., [10] and [11]) is developed in [72].

We will use uniform rectangular and triangular meshes in the mixed finite element method that align with the anisotropy. It is known that for the grids that do not align with the anisotropy, the multigrid preconditioning technique applied to the finite element approximation of anisotropic problems loses robustness [62]. Li and Wheeler [51] use anisotropically refined conforming rectangular mesh and obtain different error estimate in the context of anisotropic reaction-diffusion equations. In [51], refined but still conforming meshes are used in the regions with boundary layers, which are typical in the solutions to anisotropic problems.

Our multilevel preconditioner will be based on the abstract two-level result by Bramble, Pasciak, and Zhang [21]. An advantage of this approach is that it does not require the approximation or regularity properties of the mixed anisotropic problem, which is the main cause of the difficulty in applying the multigrid technique to anisotropic problems. Other two-level results can be found, e.g., in [27] and [73].

In our two-level preconditioning, the "coarse" level problem will be the finite element problem, on the same mesh as the mixed problem, in both the rectangular and triangular cases. The "fine" level problem will be a symmetric positive definite problem obtained from the saddle point problem. In the rectangular case, the two-level approach taken here is identical to that in [19] in that the fine level problem is obtained from the Schur complement of the mixed problem. The analysis and implementation involve the use of a mesh dependent form. A similar form in the uniformly elliptic case was used in [59]. A similar mesh dependent form, however, cannot be defined in the triangular case due to the diagonal edges in the mesh. Thus, an algebraically equivalent nonconforming problem (see, e.g. [5], [28], and [34]) is taken as the fine level problem. We follow the approach of [34] since it does not involve any bubble functions and is simpler than the results such as [2], [5], and [28]. Multigrid methods for the mixed finite element problem in terms of such algebraically equivalent nonconforming problem are found in [28] and [34]. Nonconforming multigrid methods have been studied in [15] in the full elliptic regularity case and in [25] without full elliptic regularity.

For the above two-level formulations associated with the uniform rectangular and triangular meshes, the conditions in [21] will be verified with constants independent of ε. We then obtain a uniform multi-level or two-level preconditioner for the anisotropic mixed finite element problem by combining the work here with the existing multigrid preconditioners or solvers for anisotropic finite element equations (e.g., [23], [48], [54], [64], [62], and [63]). Uniform V-cycle convergence results for the standard finite element problem are given in [23], [54], [62], and [64] by establishing suitable approximation properties of the finite element spaces.

Smoothing and coarsening techniques are important in multigrid algorithms. To reduce the high frequency components of the iterative error, an effective smoother must be chosen. Then, to deal with the low frequency or the smooth components of the error, the coarse space must be identified in the right way. In isotropic or uniformly elliptic problems, full coarsening and point-wise smoothing give good results. For the anisotropic problems, either full coarsening combined with line smoothing or semi-coarsening with point-wise coarsening is used (see, e.g., [68]). In this work, we take the former approach. We will use line versions of smoothers such as Jacobi and Gauss-Seidel smoothers.

The rest of this thesis is organized as follows. In the next chapter, we introduce the anisotropic problem and formulate the corresponding mixed problem. Some regularity results that will be used later in the error estimates are established. In Chapter III, the mixed finite element method for the anisotropic problem is considered in the framework of Falk and Osborn [40] and some error estimates are derived. Two-level preconditioning is considered in Chapter IV. The conditions in Bramble, Pasciak, and Zhang [21] will be verified in both the rectangular and triangular cases. In Chapter V, multigrid techniques for the finite element approximation of the anisotropic problem will be reviewed. Numerical results will be given in Chapter VI. Finally, a summary of the current work and possible future research areas will be presented in Chapter VII.

CHAPTER II

MIXED FORMULATION OF THE ANISOTROPIC PROBLEM

In this chapter, we introduce the anisotropic problem and study its mixed formulation. We establish some regularity results which will be used in the next chapter in the error analysis of the mixed finite element approximation.

A. Anisotropic problem

We begin with a review of the definition of some Sobolev spaces. Let E be a domain, that is a bounded and connected set, in $\mathbb{R}$ or $\mathbb{R}^2$.

As usual, the space $L^2(E)$ denotes the set of square integrable functions on E with the inner product defined by

$$(\varphi, \phi)_E = \int_E \varphi \, \phi \, dx.$$

When E is a 1-dimensional domain, we will use the notation $\langle \cdot, \cdot \rangle_E$ to denote the $L^2(E)$ inner product. In both cases, the corresponding norm will be denoted by $\|\cdot\|_{0,E}$.

For a positive integer m, the space $H^m(E)$ consists of functions in $L^2(E)$ whose weak derivatives up to the m-th order are again contained in $L^2(E)$. Its norm $\|\cdot\|_m$ is defined by

$$\|\phi\|_{m,E} = \left(\|\phi\|_{0,E}^2 + \sum_{i=1}^{m} |\phi|_{m,E}^2 \right)^{1/2},$$

where $|\cdot|_{m,E}$ is the seminorm given by

$$|\phi|_{m,E} = \left(\sum_{|\delta|=m} \|D^\delta \phi\|_{0,E}^2 \right)^{1/2}.$$

Here, $\delta = (\delta_1, \delta_2)$ is an ordered pair of nonnegative integers with $|\delta| = \delta_1 + \delta_2$ and D^δ is the differential operator defined by

$$D^\delta = \left(\frac{\partial}{\partial x}\right)^{\delta_1} \left(\frac{\partial}{\partial y}\right)^{\delta_2}.$$

In this work, we will use subscripts to denote the derivatives as in the following examples:

$$\phi_x = \frac{\partial \phi}{\partial x} \quad \text{and} \quad \phi_{xy} = \frac{\partial^2 \phi}{\partial x \partial y}.$$

In addition, the space $H_0^1(E)$ will denote the subspace of $H^1(E)$ consisting of functions in $H^1(E)$ that vanish on the boundary ∂E. For vector valued functions $\mathbf{v} = (v_1, v_2)$, the Sobolev norms are defined by

$$\|\mathbf{v}\|_{m,E} = \left(\|v_1\|_{m,E}^2 + \|v_2\|_{m,E}^2 \right)^{1/2} \quad \text{for } m = 0, 1, 2, \cdots.$$

Let $\Omega = (0,1)^2$ be the unit square and $\partial\Omega$ be its boundary. When $E = \Omega$, we will not specify the domain in the norms: for example, $\|\cdot\|_0 = \|\cdot\|_{0,\Omega}$.

We consider the following model anisotropic elliptic problem of second order: *Given $f \in L^2(\Omega)$, find p satisfying*

$$\begin{aligned} \mathcal{L}p \equiv -(p_{xx} + \varepsilon p_{yy}) &= f \quad \text{in} \quad \Omega, \\ p &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.1}$$

where $0 < \varepsilon \leq 1$ is a constant. We are interested in the case where ε is small. Notice that we may as well take $\varepsilon \gg 1$ to be large. Extensions to more general boundary conditions are also possible.

B. Mixed formulation

We consider the weighted inner product defined by

$$(\mathbf{v},\mathbf{w})_\varepsilon = (v_1, w_1) + (\varepsilon^{-1} v_2, w_2).$$

The corresponding norm is denoted by $|||\cdot|||_0$.

The subspace $H(\mathrm{div};\Omega)$ of $(L^2(\Omega))^2$ is defined by

$$H(\mathrm{div};\Omega) = \{\mathbf{v} \in (L^2(\Omega))^2 \mid \nabla\cdot\mathbf{v} \in L^2(\Omega)\}$$

with the norm

$$|||\mathbf{v}|||_{\mathrm{div}} = (|||\mathbf{v}|||_0^2 + \|\nabla\cdot\mathbf{v}\|_0^2)^{1/2}.$$

The mixed formulation of the anisotropic problem (2.1) is then given as follows: *Given* $f \in L^2(\Omega)$, *find* $(\mathbf{u}, p) \in H(\mathrm{div};\Omega) \times L^2(\Omega)$ *satisfying*

$$\begin{aligned} (\mathbf{u},\mathbf{v})_\varepsilon \quad + (\nabla\cdot\mathbf{v}, p) &= 0 \qquad \text{for all } \mathbf{v} \in H(\mathrm{div};\Omega), \\ (\nabla\cdot\mathbf{u}, q) \qquad &= -(f,q) \quad \text{for all } q \in L^2(\Omega). \end{aligned} \tag{2.2}$$

For $f \in L^2(\Omega)$, the anisotropic problem (2.1) has a unique solution $p \in H^2(\Omega) \cap H_0^1(\Omega)$ (see, e.g., [47]). Now, take

$$\mathbf{u} = (p_x, \varepsilon p_y). \tag{2.3}$$

Clearly, $\nabla\cdot\mathbf{u} = f$ and hence $\mathbf{u} \in H(\mathrm{div};\Omega)$. Moreover, it is well known that the pair $(\mathbf{u}, p)$ solves the mixed problem (2.2) uniquely (see, e.g., [55]).

C. Regularity results

To prove a regularity result for the mixed problem (2.2), we need the following lemma. Recall that our domain $\Omega = (0,1)^2$ is the unit square and let

$$\Gamma_x = \{(x,y) \in \partial\Omega \,|\, y = 0 \text{ or } y = 1\}$$

and

$$\Gamma_y = \{(x,y) \in \partial\Omega \,|\, x = 0 \text{ or } x = 1\}.$$

Lemma II.1 *Let $w \in H^1(\Omega)$ and vanish on Γ_y. Then,*

$$\|w\|_0 \leq 2\|w_x\|_0.$$

Proof. By the chain rule,

$$(w^2(x,y))_x = 2w(x,y)w_x(x,y).$$

Let t be an arbitrary number in $[0,1]$. Then, since w vanishes on $\partial\Omega$, the fundamental theorem of calculus gives, for any fixed $y \in [0,1]$,

$$w^2(t,y) = \int_0^t 2w(x,y)w_x(x,y)\,dx \leq \int_0^1 2|w(x,y)|\,|w_x(x,y)|\,dx.$$

Now, integration over y and the Cauchy-Schwarz inequality yield

$$\int_0^1 w^2(t,y)\,dy \leq 2\|w\|_0\|w_x\|_0.$$

Finally, integrating both sides over t gives

$$\|w\|_0^2 \leq 2\|w\|_0\|w_x\|_0$$

and the result follows. □

Remark II.1 *Let $w \in H^1(\Omega)$ and vanish on Γ_x. Then, the same argument as above gives that*

$$\|w\|_0 \le 2\|w_y\|_0.$$

We now have a regularity result for the mixed problem.

Proposition II.1 *Let $(\mathbf{u}, p) \in H(\mathrm{div};\Omega) \times L^2(\Omega)$ be the unique solution of the mixed problem* (2.2)*. There exists a constant C, independent of ε, such that*

$$\|p\|_0 + |||\mathbf{u}|||_{\mathrm{div}} \le C\|f\|_0.$$

Proof. By the definition (2.3) of $\mathbf{u}$, we have

$$|||\mathbf{u}|||^2_{\mathrm{div}} = \|p_x\|_0^2 + \varepsilon\|p_y\|_0^2 + \|f\|_0^2. \tag{2.4}$$

Since $p \in H_0^1(\Omega)$, the previous lemma implies that $\|p\|_0$ is bounded by $|||\mathbf{u}|||_{\mathrm{div}}$. Thus, it suffices to bound the first two terms on the right hand side of (2.4).

Now, integration by parts and the zero Dirichlet boundary condition give

$$(f, p) = -(p_{xx} + \varepsilon p_{yy}, p) = \|p_x\|_0^2 + \varepsilon\|p_y\|_0^2.$$

Then, by the Cauchy-Schwarz inequality and Lemma II.1,

$$\begin{aligned}\|p_x\|_0^2 + \varepsilon\|p_y\|_0^2 &\le \|f\|_0\,\|p\|_0 \le 2\|f\|_0\,\|p_x\|_0 \\ &\le 2\|f\|_0^2 + \frac{1}{2}\|p_x\|_0^2\end{aligned}$$

and hence

$$\frac{1}{2}\left(\,\|p_x\|_0^2 + \varepsilon\|p_y\|_0^2\,\right) \le 2\|f\|_0^2 \tag{2.5}$$

and the proof is complete. □

The continuous inf-sup condition follows from this regularity result.

Proposition II.2 *There exists a positive constant C, independent of ε, such that*

$$\|q\|_0 \leq C \sup_{\mathbf{v}\in H(\mathrm{div};\Omega)} \frac{(\nabla\cdot\mathbf{v}, q)}{|||\mathbf{v}|||_{\mathrm{div}}} \quad \text{for all } q \in L^2(\Omega). \tag{2.6}$$

Proof. Let $q \in L^2(\Omega)$ and consider the problem: *Find $w \in H_0^1(\Omega)$ satisfying*

$$\begin{aligned} w_{xx} + \varepsilon w_{yy} &= q \quad \text{in} \quad \Omega, \\ w &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Take $\mathbf{v} = (w_x, \varepsilon w_y)$. Then, clearly $q = \nabla\cdot\mathbf{v}$. Therefore, by Proposition II.1,

$$\|q\|_0 = \frac{(\nabla\cdot\mathbf{v}, q)}{\|q\|_0} \leq C\frac{(\nabla\cdot\mathbf{v}, q)}{|||\mathbf{v}|||_{\mathrm{div}}} \leq C \sup_{\mathbf{v}\in H(\mathrm{div};\Omega)} \frac{(\nabla\cdot\mathbf{v}, q)}{|||\mathbf{v}|||_{\mathrm{div}}}$$

and the proof is complete. □

We now establish an estimate for the second derivatives of the solution $p \in H^2(\Omega)\cap H_0^1(\Omega)$ of (2.1). We begin with an integration by parts formula. We follow Grisvard [47, Section 4.3.1], where arbitrary polygonal domains and more general boundary conditions are considered. Define, for $s = 1$ and 2,

$$G^s(\Omega) = \left\{(v, w) \in (H^s(\Omega))^2 \,\middle|\, v = 0 \text{ on } \Gamma_x \text{ and } w = 0 \text{ on } \Gamma_y\right\}.$$

Lemma II.2 *For all $(v, w) \in G^1(\Omega)$,*

$$\int_\Omega v_x\, w_y\, dx\, dy = \int_\Omega v_y\, w_x\, dx\, dy.$$

Proof. It is easy to see that the identity holds for $(v, w) \in G^2(\Omega)$ by integration by parts and the boundary conditions. Moreover, it can be shown that $G^2(\Omega)$ is dense in $G^1(\Omega)$ [47, Lemma 4.3.1.3] and the lemma follows. □

Applying the above lemma to $(p_x, p_y) \in G^1(\Omega)$, we have the following integration by parts formula.

Lemma II.3 *Let $p \in H^2(\Omega) \cap H_0^1(\Omega)$. Then,*

$$\int_\Omega p_{xy}^2 \, dx \, dy = \int_\Omega p_{xx} \, p_{yy} \, dx \, dy.$$

Remark II.2 *In [47], it is shown that the above integration by parts formula holds for any polygonal domains with vanishing Dirichlet and Neumann boundary conditions.*

Remark II.3 *An alternative proof of the above lemma can be given using eigenfunction expansion [66, Lemma 3.1].*

By Lemma II.3, we have an estimate for the second derivatives of p:

Lemma II.4 *Let $p \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of* (2.1)*. Then,*

$$\int_\Omega (\varepsilon^{-1} p_{xx}^2 + 2p_{xy}^2 + \varepsilon p_{yy}^2) \, dx \, dy = \varepsilon^{-1} \int_\Omega |f|^2 \, dx \, dy.$$

Remark II.4 *A similar result for a certain case of variable ε is given in [23] (see Remark V.1 for details).*

We now have an estimate for $\|\mathbf{u}\|_1$.

Lemma II.5 *Let $p \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of* (2.1) *and $\mathbf{u} = (p_x, \varepsilon p_y)$. Then, there exists a constant C not depending on ε such that*

$$\|\mathbf{u}\|_1 \leq C\varepsilon^{-1/2} \|f\|_0.$$

Proof. By definition,

$$\begin{aligned}\|\mathbf{u}\|_1^2 &= \|p_x\|_1^2 + \|\varepsilon p_y\|_1^2 \\ &= \|p_x\|_0^2 + \|p_{xx}\|_0^2 + \|p_{xy}\|_0^2 + \varepsilon^2\big(\, \|p_y\|_0^2 + \|p_{yx}\|_0^2 + \|p_{yy}\|_0^2 \,\big).\end{aligned}$$

Now, by (2.5) and Lemma II.4, the result follows. □

CHAPTER III

MIXED FINITE ELEMENT APPROXIMATION

A mixed finite element approximation to the anisotropic problem (2.1) is studied in this chapter. Solvability of the discrete system as well as some error estimates will be obtained in the framework of Falk and Osborn [40].

Our mixed finite element spaces will be constructed on uniform rectangular and triangular meshes that align with the anisotropy. It is known that for non-aligning grids, multigrid preconditioning technique applied to the finite element approximation of anisotropic problems lose robustness [62].

A. Abstract framework

Let V, W, and H be three real Banach spaces with their respective norms $\|\cdot\|_V$, $\|\cdot\|_W$, and $\|\cdot\|_H$. We assume that V is continuously embedded in H, that is, there exists a constant C satisfying

$$\|v\|_H \leq C\|v\|_V \quad \text{for all } v \in V.$$

Let $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ be bounded bilinear forms on $H \times H$ and $V \times W$, respectively. In other words, there exist constants $\|a\|$ and $\|b\|$ satisfying

$$|a(v,w)| \leq \|a\|\, \|v\|_H \|w\|_H \quad \text{for all } v, w \in H, \tag{3.1}$$

and

$$|b(v,q)| \leq \|b\|\, \|v\|_V \|q\|_W \quad \text{for all } v \in V \text{ and } q \in W. \tag{3.2}$$

In addition, we assume that $a(\cdot,\cdot)$ is symmetric. Although this is not assumed in [40], it is sufficient for our purpose and slightly simplifies the presentation.

Let W' be the dual space of W. We consider the following problem: *Given $f \in W'$, find $(u,p) \in V \times W$ such that*

$$\begin{aligned} a(u,v) + b(v,p) &= 0 \qquad \text{for all } v \in V, \\ b(u,q) \qquad &= -\langle f,q \rangle \quad \text{for all } q \in W. \end{aligned} \tag{3.3}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between W' and W.

Let finite dimensional subspaces V_h of V and W_h of W be given. We also consider the following approximate problem to (3.3): *Given $f \in W'$, find $(u_h, p_h) \in V_h \times W_h$ such that*

$$\begin{aligned} a(u_h,v) + b(v,p_h) &= 0 \qquad \text{for all } v \in V_h, \\ b(u_h,q) \qquad &= -\langle f,q \rangle \quad \text{for all } q \in W_h. \end{aligned} \tag{3.4}$$

We now state a set of conditions from [40].

(H1) For each $f \in W'$, (3.3) has a unique solution.

(H2) Let $Z_h = \{v \in V_h \mid b(v,q) = 0 \text{ for all } q \in W_h\}$. There is a constant $\alpha > 0$, independent of h, such that

$$a(v,v) \geq \alpha \|v\|_H^2 \quad \text{for all } v \in Z_h.$$

(H3) Let (y_d, λ_d) be the solution of problem (3.3) with f in the right hand side replaced by d. Define $Y = \text{span}\{y_d \,|\, d \in W'\}$. There is an operator $\pi_h : Y \to V_h$ such that

$$b(y - \pi_h y, q) = 0 \quad \text{for all } y \in Y \text{ and } q \in W_h.$$

(H4) Let $Z = \{v \in V \mid b(v,q) = 0 \text{ for all } q \in W\}$. Then, $Z_h \subset Z$.

These conditions lead to the following theorems.

Theorem III.1 ([40, Theorem 1]) *Assume that hypotheses* (H1)–(H3) *hold. Then, the discrete problem* (3.4) *has a unique solution* $(u_h, p_h) \in V_h \times W_h$.

Theorem III.2 ([40, Theorems 2 and 3]) *Suppose that hypotheses* (H1)–(H4) *hold and let* (y_d, λ_d) *be given as in* (H3). *Then, with* π_h *satisfying* (H3), *the following error estimates hold:*

$$\|u - u_h\|_H \leq \left(1 + \frac{\|a\|}{\alpha}\right) \|u - \pi_h u\|_H, \tag{3.5}$$

and

$$\begin{aligned}\|p - p_h\|_W = \sup_{d \in W'} \frac{1}{\|d\|_{W'}} \{ & b(y_d - \pi_h y_d, p) + a(u_h - u, \pi_h y_d - y_d) \\ & + b(u - \pi_h u, \lambda_d)\}\end{aligned} \tag{3.6}$$

B. Mixed method for the anisotropic problem

We apply the abstract result in the previous section to our problem. Clearly, the mixed problem (2.2) is an example of (3.3) with spaces

$$V = H(\text{div}; \Omega), \quad W = L^2(\Omega), \quad H = (L^2(\Omega))^2,$$

with their respective norms

$$\|\cdot\|_V = |||\cdot|||_{\text{div}}, \quad \|\cdot\|_W = \|\cdot\|_0, \quad \|\cdot\|_H = |||\cdot|||_0,$$

and bilinear forms

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{v})_\varepsilon \quad \text{for all } \mathbf{u}, \mathbf{v} \in H,$$

$$b(\mathbf{v}, q) = (\nabla \cdot \mathbf{v}, q) \quad \text{for all } \mathbf{v} \in V \text{ and } q \in W.$$

One can immediately see that (3.1) and (3.2) are valid with $\|a\| = 1$ and $\|b\| = 1$. Moreover, condition (H1) holds as discussed in Section II.B and (H2) holds with $\alpha = 1$

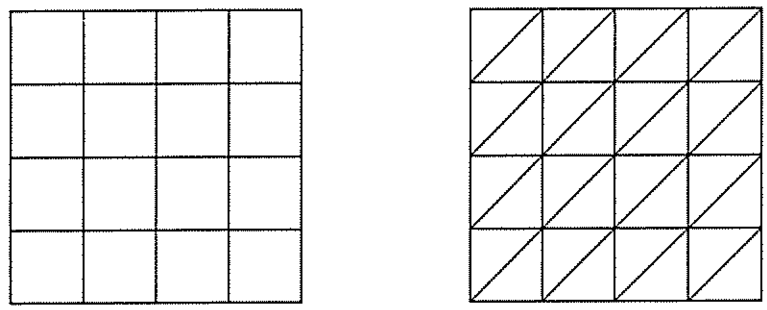

FIGURE 1. Rectangular and triangular meshes with $n = 4$.

for all $\mathbf{v} \in H$, in particular, for all $\mathbf{v} \in Z_h$.

We now describe our uniform rectangular or triangular mesh $\mathcal{T}$ as illustrated in Figure 1. The finite element spaces will be defined in terms of the mesh $\mathcal{T}$. To construct a uniform rectangular mesh, we divide the domain Ω into squares of the same size with vertices (ih, jh), $i, j = 0, 1, \cdots, n$, where $h = 1/n$ for some positive integer n. For a triangular mesh $\mathcal{T}$, we divide each square in the rectangular mesh into two right-angled triangles of the same size. Hypotenuses of a fixed orientation are used in this subdivision process. The mesh $\mathcal{T}$ will be regarded as the collection of thus constructed elements and $\tau \in \mathcal{T}$ will denote a rectangle or a triangle in the mesh.

We first consider the finite dimensional subspace W_h of $L^2(\Omega)$. In both the triangular and rectangular cases, W_h will be the space of piecewise constant functions with respect to the mesh $\mathcal{T}$ and can be written

$$M_0^{-1} = \left\{ q \in L^2(\Omega) \;\middle|\; q|_\tau \text{ is constant for all } \tau \in \mathcal{T} \right\}. \tag{3.7}$$

For V_h, we take the lowest order Raviart-Thomas space [55] with respect to the triangulation $\mathcal{T}$. To be specific, RT_0 is given in the triangular case by

$$RT_0 = \left\{ \mathbf{v} \in H(\text{div};\Omega) \;\middle|\; \mathbf{v}|_\tau = a_\tau \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_\tau \\ c_\tau \end{pmatrix} \text{ for all } \tau \in \mathcal{T} \right\} \tag{3.8}$$

and in the rectangular case by

$$RT_0 = \left\{ \mathbf{v} \in H(\text{div};\Omega) \;\middle|\; \mathbf{v}|_\tau = \begin{pmatrix} a_\tau x + b_\tau \\ c_\tau y + d_\tau \end{pmatrix} \text{ for all } \tau \in \mathcal{T} \right\}.$$

Here, a_τ, b_τ, c_τ, and d_τ denote constants that depend on the element τ. The condition $RT_0 \subset H(\text{div};\Omega)$ implies that each $\mathbf{v} \in RT_0$ has continuous normal component across the edges of the elements. To be specific, for each pair of elements τ_1 and τ_2 that share an edge e, we have

$$\mathbf{v}|_{\tau_1} \cdot \mathbf{n}_1 + \mathbf{v}|_{\tau_2} \cdot \mathbf{n}_2 = 0 \qquad \text{for all } \mathbf{v} \in RT_0,$$

where $\mathbf{n}_i$ is the outward normal vector on the edge e with respect to the element τ_i, $i = 1, 2$.

Let $\mathbf{n}$ be a normal vector on an edge e. We observe that, for any $\mathbf{v} \in RT_0$, the normal component $\mathbf{v} \cdot \mathbf{n}$ is constant on e. It is easy, then, to see from this fact and the continuity property of the normal component that $\mathbf{v} \in RT_0$ is completely determined by the values of its normal components on the edges.

Our mixed finite element problem corresponding to (3.4) is given by the following: *Given $f \in L^2(\Omega)$, find $(\mathbf{u}_h, p_h) \in RT_0 \times M_0^{-1}$ satisfying*

$$\begin{aligned} (\mathbf{u}_h, \mathbf{v})_\varepsilon \quad &+ (\nabla \cdot \mathbf{v}, p_h) = 0 && \text{for all } \mathbf{v} \in RT_0, \\ (\nabla \cdot \mathbf{u}_h, q) \quad &= -\,(f, q) && \text{for all } q \in M_0^{-1}. \end{aligned} \tag{3.9}$$

We now verify condition (H3). As was discussed in the previous chapter, $\lambda_d \in H^2(\Omega) \cap H_0^1(\Omega)$ and hence $Y \subset \left(H^1(\Omega)\right)^2$. Thus, it suffices to define π_h on $\left(H^1(\Omega)\right)^2$.

Let $\tau \in \mathcal{T}$ with edges $\{e_i\}_{i=1}^m$, where $m = 3$ in the triangular case and $m = 4$ in the rectangular case. Let $RT_0(\tau) = RT_0|_\tau$ be the lowest Raviart-Thomas space on τ and $R_0(\partial\tau)$ be defined by

$$R_0(\partial\tau) = \{\phi \in L^2(\partial\tau) \mid \phi|_{e_i} \text{ is constant for all } i = 1, \cdots, m\}.$$

Define an interpolation operator $\rho_\tau : (H^1(\tau))^2 \to RT(\tau)$ by

$$\int_{\partial\tau} (\rho_\tau \mathbf{v} - \mathbf{v}) \cdot \mathbf{n}\, \phi \, ds = 0 \quad \text{for all } \phi \in R_0(\partial\tau). \tag{3.10}$$

Remark III.1 *Alternatively, ρ_τ can be defined on the space*

$$(L^q(\tau))^2 \cap H(\mathrm{div};\tau) = \{\mathbf{v} \in (L^q(\tau))^2 \mid \nabla \cdot \mathbf{v} \in L^2(\tau)\}$$

for some fixed $q > 2$ (see [31]).

The global interpolation operator $\pi_h : (H^1(\Omega))^2 \to RT_0$ is now defined by

$$(\pi_h \mathbf{v})|_\tau = \rho_\tau(\mathbf{v}|_\tau).$$

Since (3.10) holds for all $\tau \in \mathcal{T}$, we have

$$b(\mathbf{v} - \pi_h \mathbf{v}, q) = 0 \quad \text{for all } v \in (H^1(\Omega))^2 \text{ and } q \in M_0^{-1}$$

and (H3) is satisfied.

We also note that this operator satisfies, for $\mathbf{v} \in (H^1(\Omega))^2$,

$$\|\mathbf{v} - \pi_h \mathbf{v}\|_0 \le Ch\|\mathbf{v}\|_1 \tag{3.11}$$

and

$$\|\nabla \cdot (\mathbf{v} - \pi_h \mathbf{v})\|_0 \le Ch^s |\nabla \cdot \mathbf{v}|_s \quad \text{for } s = 0, 1 \tag{3.12}$$

(see, e.g., [31] and [40]). Here and in the rest of this work, C will be used to denote a constant that depends neither on the parameter ε nor on the mesh size h.

Now that conditions (H1)–(H3) are verified, Theorem III.1 gives the following existence and uniqueness result for the mixed finite element problem (3.9).

Theorem III.3 *The mixed finite element problem* (3.9) *for the anisotropic equation* (2.1) *has a unique solution* $(\mathbf{u}_h, p_h) \in RT_0 \times M_0^{-1}$.

Next, we check condition (H4). For $\mathbf{v} \in Z_h$, we have $b(\mathbf{v}, \varphi) = (\nabla \cdot \mathbf{v}, \varphi) = 0$ for all $\varphi \in M_0^{-1}$. Since $\nabla \cdot \mathbf{v} \in M_0^{-1}$, this implies that $\nabla \cdot \mathbf{v} = 0$ and hence $\mathbf{v} \in Z$. Therefore, $Z_h \subset Z$ and (H4) holds.

We will need a couple of lemmas for the error estimates.

Lemma III.1 *Let* $p \in H^2(\Omega) \cap H_0^1(\Omega)$ *and* $\mathbf{u} = (p_x, \varepsilon p_y)$. *Then,*

$$|||\mathbf{u} - \pi_h \mathbf{u}|||_0 \le C\varepsilon^{-1} h \|\mathcal{L}p\|_0.$$

Proof. From the definition of the weighted norm $|||\cdot|||_0$ and (3.11), we have

$$|||\mathbf{u} - \pi_h \mathbf{u}|||_0^2 \le \varepsilon^{-1} \|\mathbf{u} - \pi_h \mathbf{u}\|_0^2 \le C\varepsilon^{-1} h^2 \|\mathbf{u}\|_1^2.$$

Now, Lemma II.5 gives the result. □

Lemma III.2 *Let* $p \in H^2(\Omega) \cap H_0^1(\Omega)$. *Then,*

$$\inf_{\varphi \in M_0^{-1}} \|p - \varphi\|_0 \le C\varepsilon^{-1/2} h \|\mathcal{L}p\|_0.$$

Proof. From the approximation property of M_0^{-1} (see, e.g., [29]), we have

$$\inf_{\varphi \in M_0^{-1}} \|p - \varphi\|_0 \le Ch\|p\|_1.$$

Now, by (2.5)

$$\|p\|_1 \leq C\varepsilon^{-1/2}\|\mathcal{L}p\|_0$$

and the proof is complete. □

Applying Theorem III.2, we obtain the following error estimates for our mixed approximation.

Theorem III.4 *Let* $(\mathbf{u}, p) \in V \times W$ *and* $(\mathbf{u}_h, p_h) \in RT_0 \times M_0^{-1}$ *be the solutions of the continuous problem* (2.2) *and the mixed finite element problem* (3.9), *respectively. Then,*

$$|||\mathbf{u} - \mathbf{u}_h|||_0 \leq C\varepsilon^{-1}h\|f\|_0. \tag{3.13}$$

$$\|p - p_h\|_0 \leq C(h^2\varepsilon^{-2} + h\varepsilon^{-1/2})\|f\|_0. \tag{3.14}$$

Proof. From (3.5) in Theorem III.2, we have

$$|||\mathbf{u} - \mathbf{u}_h|||_0 \leq 2|||\mathbf{u} - \pi_h\mathbf{u}|||_0 \tag{3.15}$$

since $\|a\| = 1$ and $\alpha = 1$. Equation (3.13) now follows from Lemma III.1.

We now prove (3.14). Let $(\mathbf{y}_d, \lambda_d)$ be as defined in (H3). Applying (3.6) of Theorem III.2 gives

$$\begin{aligned}\|p - p_h\|_0 = \sup_{d \in L^2(\Omega)} \frac{1}{\|d\|_0}\{&(\nabla \cdot (\mathbf{y}_d - \pi_h\mathbf{y}_d), p - \varphi) + (\mathbf{u}_h - \mathbf{u}, \pi_h\mathbf{y}_d - \mathbf{y}_d)_\varepsilon \\ &+ (\nabla \cdot (\mathbf{u} - \pi_h\mathbf{u}), \lambda_d - \eta)\}\end{aligned} \tag{3.16}$$

for all $\varphi, \eta \in M_0^{-1}$. Cauchy-Schwarz inequality gives

$$\inf_{\varphi \in M_0^{-1}} |(\nabla \cdot (\mathbf{y}_d - \pi_h\mathbf{y}_d), p - \varphi)| \leq \|\nabla \cdot (\pi_h\mathbf{y}_d - \mathbf{y}_d)\|_0 \inf_{\varphi \in M_0^{-1}} \|p - \varphi\|_0.$$

From (3.12) and Proposition II.1

$$\|\nabla \cdot (\pi_h \mathbf{y}_d - \mathbf{y}_d)\|_0 \le C\|\nabla \cdot \mathbf{y}_d\|_0 \le C\|d\|_0.$$

Thus, we have

$$\inf_{\varphi \in M_0^{-1}} |(\nabla \cdot (\mathbf{y}_d - \pi_h \mathbf{y}_d), p - \varphi)| \le Ch\varepsilon^{-1/2}\|d\|_0 \|f\|_0. \tag{3.17}$$

A similar argument gives

$$\inf_{\eta \in M_0^{-1}} |(\nabla \cdot (\mathbf{u} - \pi_h \mathbf{u}), \lambda_d - \eta)| \le Ch\varepsilon^{-1/2}\|d\|_0 \|f\|_0. \tag{3.18}$$

Furthermore, Lemma III.1 and (3.13) give

$$\begin{aligned} |(\mathbf{u} - \mathbf{u}_h, \mathbf{y}_d - \pi_h \mathbf{y}_d)_\varepsilon| &\le \|\mathbf{u} - \mathbf{u}_h\|_H \|\mathbf{y}_d - \pi_h \mathbf{y}_d\|_H \\ &\le C\varepsilon^{-2}h^2\|f\|_0 \|d\|_0. \end{aligned} \tag{3.19}$$

Combining (3.16)–(3.19) gives

$$\|p - p_h\|_0 \le C(h^2\varepsilon^{-2} + h\varepsilon^{-1/2})\|f\|_0$$

and the proof is complete. □

CHAPTER IV

TWO-LEVEL PRECONDITIONING

In this chapter, we study a two-level preconditioner for the linear system arising from the mixed finite element approximation (3.9) for the second order anisotropic problem (2.1). In our two-level approach, the "fine" level problem will be derived from the mixed finite element problem discussed in the previous section. The "coarse" level problem, on the other hand, will be the corresponding standard finite element approximation of the anisotropic problem (2.1) on the same uniform mesh.

We consider both the rectangular and triangular meshes constructed in Chapter II. These two cases will be treated separately. For the rectangular mesh, we exploit the fact that our mesh aligns with the anisotropy of the problem and construct a mesh dependent norm which enables us to perform the analysis in terms of the nodal values. For the triangular case, we use the equivalence between Raviart-Thomas mixed methods and certain nonconforming methods (see, e.g., [5], [28], or [34]).

We begin with the two-level theory by Bramble, Pasciak, and Zhang [21, Section 2]. In this framework, the regularity or the approximation properties of the mixed problem, which are the main causes of the difficulties in the anisotropic problem, are not required.

A. A two-level preconditioning result

Let Q_h and Q_H be two finite dimensional spaces, each with two inner products. We denote the inner products on Q_h by $(\cdot,\cdot)_h$ and $A_h(\cdot,\cdot)$, while those on Q_H by $(\cdot,\cdot)_H$ and $A_H(\cdot,\cdot)$. In addition, we assume that the two spaces Q_h and Q_H are related by a connection operator $\mathcal{I}_h : Q_H \to Q_h$. The adjoint $\mathcal{I}_h^t : Q_h \to Q_H$ of $\mathcal{I}_h$ is defined by

$$(\mathcal{I}_h^t \varphi, \phi)_H = (\varphi, \mathcal{I}_h \phi)_h \quad \text{for all } \varphi \in Q_h \text{ and } \phi \in Q_H. \tag{4.1}$$

We consider the following pair of variational problems: *Find* $u_h \in Q_h$ *and* $u_H \in Q_H$ *such that*

$$A_h(u_h, \chi) = (f, \chi)_h \quad \text{for all } \chi \in Q_h,$$

$$A_H(u_H, \chi) = (f, \chi)_H \quad \text{for all } \chi \in Q_H.$$

Discrete operators A_h and A_H on Q_h and Q_H are defined, respectively, by

$$(A_h v, \chi)_h = A_h(v, \chi) \quad \text{for all } v, \chi \in Q_h,$$

$$(A_H v, \chi)_H = A_H(v, \chi) \quad \text{for all } v, \chi \in Q_H.$$

We are interested in the construction of preconditioners for A_h, the "fine" grid operator, by using A_H^{-1}, namely a "coarse" grid solve, or a good preconditioner for A_H. To this end, let $\mathcal{J}_h : Q_h \to Q_h$ be a symmetric positive definite operator and consider the preconditioner for A_h given by

$$B_h^{-1} = \mathcal{I}_h A_H^{-1} \mathcal{I}_h^t + \mathcal{J}_h. \tag{4.2}$$

Note that this is exactly the two-level additive preconditioner with $\mathcal{J}_h$ as the smoother on the fine level.

Remark IV.1 *In the definition of the preconditioner,* A_H^{-1} *could be replaced by a preconditioner* B_H^{-1} *for* A_H *and* B_h^{-1} *is given by*

$$B_h^{-1} = \mathcal{I}_h B_H^{-1} \mathcal{I}_h^t + \mathcal{J}_h. \tag{4.3}$$

Now, assume that the spaces Q_h and Q_H belong to a common, possibly infinite dimensional, normed linear space Λ with norm $\|\cdot\|_\Lambda$. We introduce a set of conditions from [21]. One involves the smoother and the remaining two the approximation properties between the spaces Q_h and Q_H.

(M1) The operator $\mathcal{J}_h$ behaves like a smoother; i.e. there exist positive constants c_1 and c_2 such that

$$c_1^{-1} A_h(q,q) \le (\mathcal{J}_h^{-1} q, q)_h \le c_2 \{ \|q\|_\Lambda^2 + A_h(q,q) \} \quad \text{for all } q \in Q_h.$$

(M2) The space Q_H approximates Q_h in the sense that there exists a positive constant c_3 such that

$$\inf_{\chi \in Q_H} \{ \|q - \chi\|_\Lambda^2 + A_H(\chi,\chi) \} \le c_3 A_h(q,q) \quad \text{for all } q \in Q_h.$$

(M3) The operator $\mathcal{I}_h$ provides a stable approximation to Q_H in the sense that there exists a positive constant c_4 such that

$$\|\mathcal{I}_h q - q\|_\Lambda^2 + A_h(\mathcal{I}_h q, \mathcal{I}_h q) \le c_4 A_H(q,q) \quad \text{for all } q \in Q_H.$$

Then the following theorem holds.

Theorem IV.1 ([21, Theorem 2.2]) *Assume that the conditions* (M1)–(M3) *hold for $\mathcal{I}_h$, $\mathcal{J}_h$, Q_h, and Q_H. Then*

$$C_1 A_h(q,q) \le A_h(B_h^{-1} A_h q, q) \le C_2 A_h(q,q) \quad \textit{for all } q \in Q_h$$

or, equivalently,

$$C_1 (B_h q, q)_h \le (A_h q, q)_h \le C_2 (B_h q, q)_h \quad \textit{for all } q \in Q_h.$$

Here, $C_1 = \big(c_3 + 2c_2\{1 + c_3(1 + c_4)\}\big)^{-1}$ and $C_2 = c_4 + c_1$.

B. Rectangular mesh

We apply the two-level result in the previous section to the anisotropic mixed problem (3.9). We will describe our choice of the spaces and their inner products. Next, we will study the smoothing properties of line Jacobi and line Gauss-Seidel smoothers and verify condition (M1) for these smoothers. Finally, the approximation conditions (M2) and (M3) will be established. Therefore, by Theorem IV.1, the two-level preconditioners (4.2) and (4.3) will provide uniform preconditioners.

1. Spaces and inner products

We take the space $Q_h = M_0^{-1}$ defined in (3.7). The other space Q_H is taken to be the space of continuous piecewise bilinear functions with respect to the mesh $\mathcal{T}$ that vanish on the boundary $\partial\Omega$. To be specific,

$$Q_H = \{q \in H_0^1(\Omega) \mid q|_\tau = (a_\tau + b_\tau x)(c_\tau + d_\tau y) \text{ for all } \tau \in \mathcal{T}\}. \tag{4.4}$$

This is the standard finite element space for the anisotropic problem on rectangular mesh. The inner product $A_H(\cdot,\cdot)$ is defined accordingly by

$$A_H(\varphi, \phi) = \int_\Omega (\varphi_x \phi_x + \varepsilon \varphi_y \phi_y)\, dx\, dy \quad \text{for all } \varphi, \phi \in Q_H. \tag{4.5}$$

In addition, we take $(\cdot,\cdot)_h = (\cdot,\cdot)_H = (\cdot,\cdot)$ and $\Lambda = L^2(\Omega)$ equipped with the weighted norm defined by

$$\|q\|_\Lambda^2 = \varepsilon h^{-2} \|q\|_0.$$

Note that, for the piecewise constant functions $q \in Q_h$, this norm can be written

$$\|q\|_\Lambda^2 = \varepsilon \sum_{\tau \in \mathcal{T}} q_\tau^2, \tag{4.6}$$

where $q_\tau = q|_\tau$. Clearly, the finite element spaces Q_h and Q_H are contained in Λ.

To define the inner product $A_h(\cdot,\cdot)$ on Q_h, we define discrete operators $A : RT_0 \to RT_0$, $B : RT_0 \to Q_h$, and $B^t : Q_h \to RT_0$ by

$$(A\mathbf{v}, \mathbf{w}) = (\mathbf{v}, \mathbf{w})_\varepsilon \qquad \text{for all } \mathbf{v}, \mathbf{w} \in RT_0,$$
$$(B\mathbf{v}, q) = (\nabla \cdot \mathbf{v}, q) \qquad \text{for all } \mathbf{v} \in RT_0, q \in Q_h,$$

and

$$(B^t q, \mathbf{v}) = (\nabla \cdot \mathbf{v}, q) \qquad \text{for all } q \in Q_h, \mathbf{v} \in RT_0.$$

Then, (3.9) can be rewritten in matrix form by

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_h \\ p_h \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathcal{Q}_0 f \end{pmatrix}. \tag{4.7}$$

Here, $\mathcal{Q}_0$ denotes the $L^2(\Omega)$ orthogonal projection onto Q_h defined by

$$(\mathcal{Q}_0\varphi, \phi) = (\varphi, \phi) \qquad \text{for all } \varphi \in L^2(\Omega) \text{ and } \phi \in Q_h.$$

By block Gaussian elimination, we see that the solution p_h of (3.9) satisfies

$$BA^{-1}B^t p_h = \mathcal{Q}_0 f. \tag{4.8}$$

The form $A_h(\cdot,\cdot)$ is then defined by

$$A_h(\varphi, \phi) = (BA^{-1}B^t\varphi, \phi) \qquad \text{for all } \varphi, \phi \in Q_h. \tag{4.9}$$

It is easy to check that $A_h(\cdot,\cdot)$ is an inner product, i.e. a symmetric positive definite bilinear form.

Remark IV.2 *In order to solve equation* (4.8) *by an iterative method, the action of the discrete operator* $BA^{-1}B^t$ *need to be evaluated. Following the definition, the*

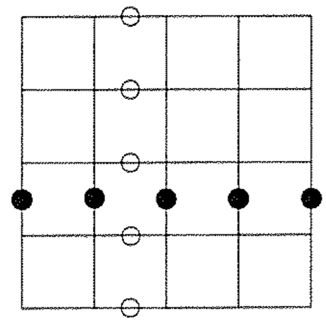

FIGURE 2. Edges associated with the tridiagonal systems of order $n+1$ in the x and y directions.

action of the operators B and B^t can easily be computed. Moreover, the operator B does not involve the parameter ε and is exactly the same as the corresponding term in the mixed finite element method for uniformly bounded second order problems (see, e.g., [31] or [55]). Thus, existing subroutines can be used to evaluate B. The action of B^t is given by the transpose of the matrix that computes the action of B. The evaluation of the action of A^{-1} clearly involves an inversion of a linear system. Yet, from the definition of the lowest order Raviart-Thomas space RT_0 and the inner product $(\cdot,\cdot)_\varepsilon$, it is easy to see that this linear system is block tridiagonal. In fact, on an $n \times n$ rectangular mesh, the evaluation of A^{-1} only involves the solution of $2n$ symmetric positive definite, n systems resulting from each of the x and y directions, tridiagonal systems of order $n+1$ (see Figure 2). These tridiagonal systems can be solved efficiently by direct solvers such as the routines in LAPACK [1]. We also note that the solution of the $2n$ tridiagonal systems can be done in parallel.

2. Line Jacobi smoother

To define the line Jacobi smoother, we introduce the following horizontal strip decomposition of Ω. For $j = 1, ..., n$, define

$$\Omega_j = \{(x, y) \in \Omega \mid (j-1)h < y < jh\}.$$

Then,

$$\overline{\Omega} = \bigcup_{j=1}^{n} \overline{\Omega_j}$$

and the finite element space Q_h is partitioned accordingly as

$$Q_h = \sum_{j=1}^{n} Q_{h,j},$$

where

$$Q_{h,j} = \{q \in Q_h \mid q = 0 \text{ in } \Omega \setminus \Omega_j\}.$$

Now a line Jacobi smoother $\widetilde{\mathcal{J}}_h : Q_h \to Q_h$ can be defined by

$$\widetilde{\mathcal{J}}_h = \sum_{j=1}^{n} A_{h,j}^{-1} \mathcal{Q}_{h,j}. \tag{4.10}$$

Here, $\mathcal{Q}_{h,j}$ is the L^2 projection of Q_h onto $Q_{h,j}$ and $A_{h,j}$ is defined by

$$(A_{h,j}\theta, \chi) = A_h(\theta, \chi) \quad \text{for all } \theta, \chi \in Q_{h,j}.$$

Note that the computation of $A_{h,j}^{-1}$ involves the inversion of the stiffness matrix corresponding to the form $A_h(\cdot,\cdot)$. This matrix is full and it is not practical to compute $A_{h,j}^{-1}$. To overcome this computational difficulty, we use a mesh dependent form defined below. We will also show that this mesh dependent form gives an inner product which is equivalent to the bilinear form $A_h(\cdot,\cdot)$ in the sense that will be made

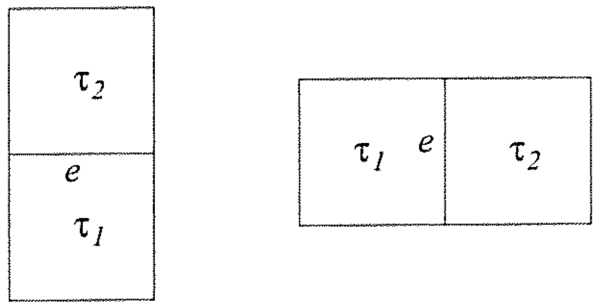

FIGURE 3. Labels for the neighboring elements of an edge e.

clear later.

We begin the definition of the mesh dependent form by setting up some notation. Let $\mathcal{E}$ be the collection of all edges in the mesh $\mathcal{T}$. The collection of edges that are parallel to the x- and y-axes are denoted respectively by $\mathcal{E}_x$ and $\mathcal{E}_y$. In addition, let $\mathcal{E}_0$ be the set of internal edges and $\mathcal{E}_\Gamma$ of boundary edges. Note that $e \in \mathcal{E}_\Gamma$ if and only if $e \subset \partial\Omega$. For an edge e, we define the jump $\left[q\right]_e$ of a function q as follows. Given an internal edge e, let τ_1 and τ_2 be the two rectangular elements that share e as a common edge. The element τ_1 is chosen so that the outward unit normal vector $\mathbf{n}$ on e with respect to τ_1 is given by $\mathbf{n} = (1, 0)$ if $e \in \mathcal{E}_y$ and by $\mathbf{n} = (0, 1)$ if $e \in \mathcal{E}_x$ (see Figure 3). If $e \in \mathcal{E}_\Gamma$, we may assume that one of τ_1 and τ_2 is the reflection with respect to e of the other which is in $\mathcal{T}$ and that q vanishes outside Ω. The jump $[q]_e$ of $q \in Q_h$ across an edge e is then defined by

$$\left[q\right]_e = q|_{\tau_1} - q|_{\tau_2}.$$

We define the mesh dependent form $\widetilde{A}_h(\cdot, \cdot)$ by

$$\widetilde{A}_h(\varphi, \phi) = \varepsilon \sum_{e \in \mathcal{E}_x} \left[\varphi\right]_e \left[\phi\right]_e + \sum_{e \in \mathcal{E}_y} \left[\varphi\right]_e \left[\phi\right]_e. \tag{4.11}$$

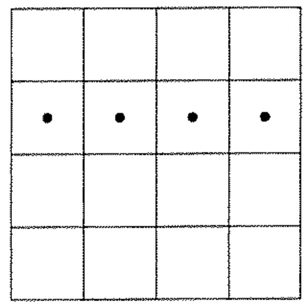

FIGURE 4. Degrees of freedom in $Q_{h,j}$ in a rectangular mesh.

Our line Jacobi smoother $\mathcal{J}_h : Q_h \to Q_h$ is now given by

$$\mathcal{J}_h = \sum_{j=1}^{n} \widetilde{A}_{h,j}^{-1} \mathcal{Q}_{h,j}, \tag{4.12}$$

where $\widetilde{A}_{h,j} : Q_{h,j} \to Q_{h,j}$ is an operator defined by

$$(\widetilde{A}_{h,j}\theta, \chi) = \widetilde{A}_h(\theta, \chi) \qquad \text{for all } \theta, \chi \in Q_{h,j}. \tag{4.13}$$

From the definition of $\widetilde{A}_h(\cdot,\cdot)$, it is easy to see that the computation of $\widetilde{A}_{h,j}^{-1}$ involves the inversion of a tridiagonal matrix (see Figure 4).

Next, we show that the two forms $A_h(\cdot,\cdot)$ and $\widetilde{A}_h(\cdot,\cdot)$ are equivalent. A similar result in the case of $\varepsilon \simeq 1$ was given in [59]. Here and in the following, the notations $A \simeq B$, $A \lesssim B$, and $A \gtrsim B$ will be used to mean $cA \leq B \leq CA$, $A \leq CB$, and $A \geq cB$, respectively, with constants c and C. These constants will be independent of the anisotropy parameter ε and the mesh size h. We use the following lemma, whose obvious proof is omitted.

Lemma IV.1 *Let $\tau \in \mathcal{T}$ be a rectangle with edges $e_1, e_2 \in \mathcal{E}_x$ and $e_3, e_4 \in \mathcal{E}_y$. Then,* *for all* $\mathbf{v} = (v_1, v_2) \in RT_0(\tau)$, *the following relations hold:*

$$\|v_1\|_{0,\tau}^2 \simeq h\big(\|v_1\|_{0,e_3}^2 + \|v_1\|_{0,e_4}^2\big),$$
$$\|v_2\|_{0,\tau}^2 \simeq h\big(\|v_2\|_{0,e_1}^2 + \|v_2\|_{0,e_2}^2\big).$$

Here, h is the length of each edge.

We now have the following equivalence result.

Proposition IV.1 *There exist positive constants c and C, independent of h and ε, such that*

$$cA_h(q,q) \leq \widetilde{A}_h(q,q) \leq CA_h(q,q) \quad \textit{for all } q \in Q_h. \tag{4.14}$$

Proof. For $q \in Q_h$, it is easy to see that

$$A_h(q,q) = \sup_{\mathbf{v} \in RT_0} \frac{(\nabla \cdot \mathbf{v}, q)^2}{(\mathbf{v}, \mathbf{v})_\varepsilon}. \tag{4.15}$$

We begin with the left inequality of (4.14). Let $q \in Q_h$ be given. Then, for each $\mathbf{v} \in RT_0$, since $\nabla \cdot \mathbf{v} \in L^2(\Omega)$, we have

$$\begin{aligned}(\nabla \cdot \mathbf{v}, q) &= \sum_{\tau \in \mathcal{T}} (\nabla \cdot \mathbf{v}, q)_\tau \\ &= \sum_{\tau \in \mathcal{T}} \big\{ -(\mathbf{v}, \nabla q)_\tau + \langle \mathbf{v} \cdot \mathbf{n}, q \rangle_{\partial\tau} \big\}.\end{aligned}$$

The last equality was obtained by integration by parts on each τ. Here, $\mathbf{n}$ denotes the outward normal vector on $\partial\tau$. As q is constant on each τ, we are left only with the boundary integrals on $\partial\tau$. Then, since the normal component $\mathbf{v} \cdot \mathbf{n}$ of $\mathbf{v} \in RT_0$ is continuous across each edge e, we have

$$(\nabla \cdot \mathbf{v}, q) = \sum_{e \in \mathcal{E}_x} \langle v_2, \lceil q \rceil \rangle_e + \sum_{e \in \mathcal{E}_y} \langle v_1, \lceil q \rceil \rangle_e. \tag{4.16}$$

Applying the Cauchy-Schwarz inequality to the first sum on the right hand side of the above identity gives

$$\begin{aligned}\sum_{e\in\mathcal{E}_x}\langle v_2, [q]\rangle_e &\le \sum_{e\in\mathcal{E}_x}\|v_2\|_{0,e}\,\|[q]\|_{0,e}\\ &\le \left(\varepsilon^{-1}h\sum_{e\in\mathcal{E}_x}\|v_2\|_{0,e}^2\right)^{1/2}\left(\varepsilon h^{-1}\sum_{e\in\mathcal{E}_x}\|[q]\|_{0,e}^2\right)^{1/2}.\end{aligned} \tag{4.17}$$

From Lemma IV.1, for each edge $e \in \mathcal{E}_x$ and any element τ containing e as an edge, we have

$$h\,\|v_2\|_{0,e}^2 \le C\,\|v_2\|_{0,\tau}^2.$$

Thus,

$$\varepsilon^{-1}h\sum_{e\in\mathcal{E}_x}\|v_2\|_{0,e}^2 \le C\varepsilon^{-1}\sum_{\tau\in\mathcal{T}}\|v_2\|_{0,\tau}^2 = C\varepsilon^{-1}\|v_2\|_0^2. \tag{4.18}$$

The second sum in the right hand side of (4.16) can be treated similarly. Then, combining (4.16)–(4.18) gives

$$(\nabla\cdot\mathbf{v}, q) \le C\left\{\varepsilon^{-1/2}\|v_2\|_0\left(\varepsilon\sum_{e\in\mathcal{E}_x}[q]_e^2\right)^{1/2} + \|v_1\|_0\left(\sum_{e\in\mathcal{E}_y}[q]_e^2\right)^{1/2}\right\}.$$

Applying Cauchy-Schwarz inequality, we obtain

$$(\nabla\cdot\mathbf{v}, q) \le C(\mathbf{v},\mathbf{v})_\varepsilon^{1/2}\widetilde{A}_h(q,q)^{1/2}.$$

Since $\mathbf{v} \in RT_0$ was arbitrary, the left inequality of the lemma follows from (4.15).

To prove the upper inequality, it suffices to show that, for each $q \in Q_h$, there exists a $\mathbf{v} \in RT_0$ such that

$$\widetilde{A}_h(q,q)^{1/2} \le C\frac{(\nabla\cdot\mathbf{v}, q)}{(\mathbf{v},\mathbf{v})_\varepsilon^{1/2}}.$$

For this, it is sufficient to find a $\mathbf{v} \in RT_0$ satisfying

$$\widetilde{A}_h(q,q) = (\nabla \cdot \mathbf{v}, q) \tag{4.19}$$

and

$$(\mathbf{v},\mathbf{v})_\varepsilon^{1/2} \le C\widetilde{A}_h(q,q)^{1/2}. \tag{4.20}$$

Recall that $\mathbf{v} \in RT_0$ is completely determined by the values of its normal components $\mathbf{v}\cdot\mathbf{n}$ on each edge $e \in \mathcal{E}$. Here, we fix $\mathbf{n} = (1,0)$ on $e \in \mathcal{E}_y$ and $\mathbf{n} = (0,1)$ on $e \in \mathcal{E}_x$. We now take $\mathbf{v} \in RT_0$ such that

$$\mathbf{v}\cdot\mathbf{n} = \begin{cases} v_2 = \varepsilon\, h^{-1}\lceil q \rceil & \text{if } e \in \mathcal{E}_x, \\ v_1 = h^{-1}\lceil q \rceil & \text{if } e \in \mathcal{E}_y. \end{cases} \tag{4.21}$$

Then, since q is constant on each $\tau \in \mathcal{T}$,

$$\begin{aligned}(\nabla\cdot\mathbf{v}, q) &= \sum_{\tau\in\mathcal{T}} (\nabla\cdot\mathbf{v}, q)_\tau = \sum_{\tau\in\mathcal{T}} \langle \mathbf{v}\cdot\mathbf{n}, q\rangle_{\partial\tau} \\ &= \sum_{e\in\mathcal{E}_x} \langle \varepsilon h^{-1}\lceil q\rceil, \lceil q\rceil\rangle_e + \sum_{e\in\mathcal{E}_y} \langle h^{-1}\lceil q\rceil, \lceil q\rceil\rangle_e \\ &= \widetilde{A}_h(q,q).\end{aligned}$$

We conclude the proof by showing (4.20). By Lemma IV.1 and the construction (4.21) of $\mathbf{v}$, we have

$$\begin{aligned}(\mathbf{v},\mathbf{v})_\varepsilon &= \sum_{\tau\in\mathcal{T}} \left(\|v_1\|_{0,\tau}^2 + \varepsilon^{-1}\|v_2\|_{0,\tau}^2\right) \\ &\le C\left(h\sum_{e\in\mathcal{E}_y} \|\mathbf{v}\cdot\mathbf{n}\|_{0,e}^2 + \varepsilon^{-1} h \sum_{e\in\mathcal{E}_x} \|\mathbf{v}\cdot\mathbf{n}\|_{0,e}^2 \right) \\ &= C\widetilde{A}_h(q,q)\end{aligned}$$

and we have the desired result. □

Remark IV.3 *A suitable mesh-dependent form can be defined in the case of piecewise constant ε. Let an edge $e \in \mathcal{E}$. We denote by $\tilde{\varepsilon}_e$ the harmonic average of ε on an edge e. In other words,*

$$\tilde{\varepsilon}_e = \begin{cases} \dfrac{2}{(\varepsilon|_{\tau_1})^{-1} + (\varepsilon|_{\tau_2})^{-1}} & \text{if } e \in \mathcal{E}_0, \\ \varepsilon|_\tau & \text{if } e \in \mathcal{E}_\Gamma. \end{cases}$$

Here, τ_1, τ_2, and τ denote the neighboring elements of e. Define the mesh dependent inner product $\widetilde{A}_h(\cdot,\cdot)$ by

$$\widetilde{A}_h(\varphi,\phi) = \sum_{e\in\mathcal{E}_x} \tilde{\varepsilon} \lceil\varphi\rceil_e \lceil\phi\rceil_e + \sum_{e\in\mathcal{E}_y} \lceil\varphi\rceil_e \lceil\phi\rceil_e.$$

It can be shown that the corresponding norm $|||\cdot|||_{\widetilde{A}_h}$ satisfies Proposition IV.1.

To establish the smoothing property of our line Jacobi smoother, we will need the following standard lemma for additive multigrid analysis, which can be found, for example, in [22, Lemma 4.1].

Lemma IV.2 *Let M be a Hilbert space with inner product $(\cdot,\cdot)$. Let M_k, $k = 1, ..., J$, be subspaces of M and $M = \sum_{k=1}^{J} M_k$. Denote by $\mathcal{Q}_k$ the orthogonal projectors onto M_k. Let $R_k : M_k \to M_k$ be symmetric and positive definite and $B_a = \sum_{k=1}^{J} R_k \mathcal{Q}_k$. Then, B_a^{-1} exists and is characterized by*

$$(B_a^{-1}v, v) = \min_{\{v_k\}} \sum_{k=1}^{J} (R_k^{-1}v_k, v_k),$$

where the minimum is taken over all $v_k \in M_k$ such that $v = \sum v_k$.

We now show condition (M1). Let $q \in Q_h$ and consider the decomposition $q = \sum_{j=1}^{n} q_j$, with $q_j \in Q_{h,j}$. From the construction, it is clear that $q_j = q|_{\Omega_j}$ and this decomposition is unique. Thus, applying the above lemma to our line Jacobi smoother $\mathcal{J}_h$ defined in (4.12) gives

$$(\mathcal{J}_h^{-1}q, q) = \sum_{j=1}^{n}(\widetilde{A}_{h,j}q_j, q_j) = \sum_{j=1}^{n}\widetilde{A}_h(q_j, q_j).$$

Note that, for each $e \in \mathcal{E}_x$,

$$[q_j]_e = \begin{cases} \pm q_\tau & \text{if } e \subset \overline{\tau} \text{ for some } \tau \subset \Omega_j, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have

$$(\mathcal{J}_h^{-1}q, q) \simeq \left(\varepsilon \sum_{\tau \in \mathcal{T}} q_\tau^2 + \sum_{e \in \mathcal{E}_y} [q]_e^2 \right). \tag{4.22}$$

Again, by Proposition IV.1, for each $q \in Q_h$,

$$A_h(q, q) \simeq \widetilde{A}_h(q, q) \lesssim \varepsilon \sum_{\tau \in \mathcal{T}} q_\tau^2 + \sum_{e \in \mathcal{E}_y} [q]_e^2. \tag{4.23}$$

Combining (4.22) and (4.23) gives

$$A_h(q, q) \lesssim (\mathcal{J}_h^{-1}q, q).$$

A similar argument, combined with (4.6), gives

$$(\mathcal{J}_h^{-1}q, q) \lesssim \|q\|_\Lambda^2 + A_h(q, q).$$

3. Line Gauss-Seidel smoother

We consider a line Gauss-Seidel smoother. We will need the projection $\widetilde{P}_{h,j} : Q_h \to Q_{h,j}$ with respect to the $\widetilde{A}_h(\cdot, \cdot)$ inner product defined by

$$\widetilde{A}_h(\widetilde{P}_{h,j}q, \chi) = \widetilde{A}_h(q, \chi) \quad \text{for all } q \in Q_h \text{ and } \chi \in Q_{h,j}.$$

Clearly, the relation

$$\widetilde{P}_{h,j} = \widetilde{A}_{h,j}^{-1} Q_{h,j} \widetilde{A}_h \tag{4.24}$$

holds. Here, $\widetilde{A}_h : Q_h \to Q_h$ is defined by

$$(\widetilde{A}_h q, \chi) = \widetilde{A}_h(q, \chi) \quad \text{for all } q \in Q_h \text{ and } \chi \in Q_h.$$

Our line Gauss-Seidel smoother is defined by

$$\mathcal{G}_h = \{I - (I - \widetilde{P}_{h,n})(I - \widetilde{P}_{h,n-1}) \cdots (I - \widetilde{P}_{h,1})\} \widetilde{A}_h^{-1} \tag{4.25}$$

and can be computed by the following algorithm.

Algorithm IV.1 (Line Gauss-Seidel Smoother) *Given $g \in Q_h$, compute $\mathcal{G}_h g \in Q_h$ by:*

1. *Set $q_0 = 0$.*
2. *For $j = 1, 2, \cdots, n$, define*

$$q_j = q_{j-1} + \widetilde{A}_{h,j}^{-1} Q_{h,j} (g - \widetilde{A}_h q_{j-1}).$$

3. *Set $\mathcal{G}_h g = q_n$.*

Remark IV.4 *Notice that $\widetilde{A}_h$, not A_h, is used in step 2 of the above algorithm. This is both for the computation and the analysis.*

Computationally, the current algorithm involves the action of the stiffness matrix corresponding to the operator $\widetilde{A}_h$, which has at most 5 nonzero entries in each row. On the other hand, the operator A_h leads to a stiffness matrix which is full.

Using A_h poses a difficulty in the analysis also. As will be seen later, we will use in the proof of Lemma IV.3, the fact that $\widetilde{P}_{h,j}$ is a projector. If A_h were used in the *above algorithm,* $\widetilde{P}_{h,j}$ *would be defined by*

$$\widetilde{A}_h(\widetilde{P}_{h,j}q, \chi) = A_h(q, \chi) \quad \text{for all } q \in Q_h \text{ and } \chi \in Q_{h,j}.$$

Clearly, $\widetilde{P}_{h,j}$ *is no longer a projector.*

Recall that a symmetric smoother is required in the two level preconditioner B_h^{-1}. Since $\mathcal{G}_h$ is not symmetric, we will consider the symmetric line Gauss-Seidel smoother given by

$$\overline{\mathcal{G}}_h = \mathcal{G}_h + \mathcal{G}_h^t - \mathcal{G}_h^t \widetilde{A}_h \mathcal{G}_h. \tag{4.26}$$

This corresponds to sweeping in one direction as in the above algorithm followed by a sweep in the opposite direction. Here, $\mathcal{G}_h^t$ is the adjoint of $\mathcal{G}_h$ with respect to the inner product $(\cdot,\cdot)$. We will show that $\overline{\mathcal{G}}_h$ satisfies the smoothing condition (M1). We begin with a lemma which gives an estimate for $\overline{\mathcal{G}}_h$ in terms of the line Jacobi smoother $\mathcal{J}_h$. The proof follows well-known argument found, for example, in [16].

Lemma IV.3 *Let* $\mathcal{J}_h$ *be the line Jacobi smoother defined in* (4.12) *and* $\overline{\mathcal{G}}_h$ *the symmetric line Gauss-Seidel smoother. Then, there exists a constant* C *such that*

$$\widetilde{A}_h(\mathcal{J}_h\widetilde{A}_h q, q) \le C\widetilde{A}_h(\overline{\mathcal{G}}_h\widetilde{A}_h q, q) \quad \text{for all } q \in Q_h.$$

Proof. From the definition of $\mathcal{J}_h$ and (4.24), we have

$$\mathcal{J}_h\widetilde{A}_h = \sum_{j=1}^{n} \widetilde{A}_{h,j}^{-1} \mathcal{Q}_{h,j}\widetilde{A}_h = \sum_{j=1}^{n} \widetilde{P}_{h,j}.$$

This implies that

$$\widetilde{A}_h(\mathcal{J}_h\widetilde{A}_h q, q) = \sum_{j=1}^{n} \widetilde{A}_h(\widetilde{P}_{h,j}q, \widetilde{P}_{h,j}q). \tag{4.27}$$

Define $E_0 = I$ and, for $i = 1, 2, \cdots, n$,

$$E_i = (I - \widetilde{P}_{h,i})E_{i-1} = (I - \widetilde{P}_{h,i}) \cdots (I - \widetilde{P}_{h,1}).$$

Then, from the definition (4.25) of $\mathcal{G}_h$,

$$I - \mathcal{G}_h \widetilde{A}_h = E_n$$

and hence

$$\widetilde{A}_h(\overline{\mathcal{G}}_h \widetilde{A}_h q, q) = \widetilde{A}_h(q, q) - \widetilde{A}_h(E_n q, E_n q).$$

Moreover, using the fact that $\widetilde{P}_{h,j}$ is a projector with respect to the inner product $\widetilde{A}_h(\cdot, \cdot)$, we have

$$\begin{aligned}
\widetilde{A}_h(E_{j-1}q, E_{j-1}q) - \widetilde{A}_h(E_j q, E_j q) \\
&= \widetilde{A}_h(E_{j-1}q, E_{j-1}q) - \widetilde{A}_h((I - \widetilde{P}_{h,j})E_{j-1}q, E_{j-1}q) \\
&= \widetilde{A}_h(\widetilde{P}_{h,j}E_{j-1}q, E_{j-1}q) \\
&= \widetilde{A}_h(\widetilde{P}_{h,j}E_{j-1}q, \widetilde{P}_{h,j}E_{j-1}q).
\end{aligned}$$

Thus, summing over j gives

$$\widetilde{A}_h(\overline{\mathcal{G}}_h \widetilde{A}_h q, q) = \sum_{j=1}^{n} \widetilde{A}_h(\widetilde{P}_{h,j}E_{j-1}q, \widetilde{P}_{h,j}E_{j-1}q). \tag{4.28}$$

Now, from the definition of E_j, we clearly have

$$E_{i-1} - E_i = \widetilde{P}_{h,i}E_{i-1}$$

and

$$I - E_j = \sum_{i=j-1}^{j} \widetilde{P}_{h,i}E_{i-1}.$$

Since $\widetilde{P}_{h,j}$ is a projection, we have

$$\widetilde{P}_{h,j}E_j = \widetilde{P}_{h,j}(I - \widetilde{P}_{h,j})E_{j-1} = 0.$$

It is clear that

$$\widetilde{P}_{h,j}\widetilde{P}_{h,i} = 0 \quad \text{for } i < j-1. \tag{4.29}$$

Thus,

$$\widetilde{P}_{h,j}q = \widetilde{P}_{h,j}(I - E_j)q = \sum_{i=j-1}^{j} \widetilde{P}_{h,j}\widetilde{P}_{h,i}E_{i-1}q. \tag{4.30}$$

Combining (4.27)–(4.30) and applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\widetilde{A}_h(\mathcal{J}_h\widetilde{A}_hq, q) &= \sum_{j=1}^{n} \widetilde{A}_h(\sum_{i=j-1}^{j} \widetilde{P}_{h,j}\widetilde{P}_{h,i}E_{i-1}q, \widetilde{P}_{h,j}q) \\ &\lesssim \widetilde{A}_h(\mathcal{J}_h\widetilde{A}_hq, q)^{1/2}\widetilde{A}_h(\overline{\mathcal{G}}_h\widetilde{A}_hq, q)^{1/2}\end{aligned}$$

and the lemma follows. □

Using this lemma, we can establish the smoothing property of the symmetric line Gauss-Seidel smoother $\overline{\mathcal{G}}_h$.

Lemma IV.4 *The symmetric line Gauss-Seidel smoother $\overline{\mathcal{G}}_h$ satisfies condition (M1). That is,*

$$A_h(q,q) \lesssim (\overline{\mathcal{G}}_h^{-1}q, q) \lesssim \{\|q\|_\Lambda^2 + A_h(q,q)\} \quad \textit{for all } q \in Q_h.$$

Proof. From the definition of $\overline{\mathcal{G}}_h$, we have

$$0 \le \widetilde{A}_h((I - \mathcal{G}_h\widetilde{A}_h)q, (I - \mathcal{G}_h\widetilde{A}_h)q) = \widetilde{A}_h(q,q) - \widetilde{A}_h(\overline{\mathcal{G}}_h\widetilde{A}_hq, q).$$

This and Proposition IV.1 give the lower estimate.

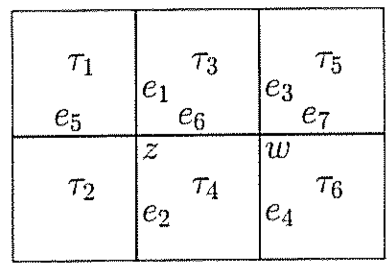

FIGURE 5. A portion of the rectangular mesh $\mathcal{T}$ for the verification of (M2).

Now, from Lemma IV.3, we have

$$(\overline{\mathcal{G}}_h^{-1} q, q) \lesssim (\mathcal{J}_h q, q) \quad \text{for all } q \in Q_h.$$

Since the line Jacobi smoother $\mathcal{J}_h$ satisfies condition (M1), this implies the upper estimate. □

4. Approximation conditions

We begin with the verification of (M2). Given $q \in Q_h$, we choose χ to be the function in Q_H whose value at a node or vertex z in the mesh $\mathcal{T}$ is defined by

$$\chi(z) = \begin{cases} \dfrac{1}{4} \sum_i q_i & \text{if } z \in \Omega, \\ 0 & \text{if } z \in \partial\Omega, \end{cases}$$

where q_i is the value of the piecewise constant function q on τ_i and the sum is taken over all i such that τ_i has z as a vertex (see, e.g. Figure 5). It is easy to see that, for a bilinear function φ on a rectangle τ with vertices $\{z_i\}_{i=1}^4$,

$$\|\varphi\|_{0,\tau}^2 \approx |\tau| \sum_{i=1}^4 \varphi(z_i)^2.$$

Applying this fact to $q - \chi$, we have

$$\|q - \chi\|_\Lambda^2 \simeq \varepsilon \sum_{\tau \in \mathcal{T}} \sum_{z \in Z(\tau)} \big(q(z) - \chi(z)\big)^2, \tag{4.31}$$

where $Z(\tau)$ denotes the set of all vertices of τ. Let z be an interior vertex. For example, take z and $\tau = \tau_3$ be as in Figure 5. Then, on τ,

$$\begin{aligned}\big(q(z) - \chi(z)\big)^2 &= \Big(q_3 - \frac{1}{4}(q_1 + q_2 + q_3 + q_4)\Big)^2 \\ &= \frac{1}{16}\Big(2(q_3 - q_1) + (q_1 - q_2) + (q_3 - q_4)\Big)^2 \\ &\lesssim [q]_{e_1}^2 + [q]_{e_5}^2 + [q]_{e_6}^2.\end{aligned}$$

If $z \in \overline{\tau}$ is a boundary node, then $\chi(z) = 0$ and

$$\big(q(z) - \chi(z)\big)^2 = [q]_e^2,$$

where e is any boundary edge of τ. Thus,

$$\|q - \chi\|_\Lambda^2 \leq C\Big(\varepsilon \sum_{e \in \mathcal{E}} [q]_e^2\Big) \leq C\widetilde{A}_h(q, q). \tag{4.32}$$

Now, we check the second term on the left hand side of (M2), namely

$$A_H(\chi, \chi) = \|\chi_x\|_0^2 + \varepsilon\|\chi_y\|_0^2.$$

We clearly have, for all $\chi \in Q_H$,

$$A_H(\chi, \chi) \simeq \sum_{e \in \mathcal{E}_x} \big(\chi(z_e) - \chi(w_e)\big)^2 + \varepsilon \sum_{e \in \mathcal{E}_y} \big(\chi(z_e) - \chi(w_e)\big)^2. \tag{4.33}$$

Here, z_e and w_e denote the end points of an edge e. Let $e \in \mathcal{E}_x$. Suppose that e has both end points in Ω. Take, for example, $e = e_6$ in Figure 5. Then,

$$\begin{aligned} 4\big(\chi(z) - \chi(w)\big) &= (q_1 + q_2 + q_3 + q_4) - (q_3 + q_4 + q_5 + q_6) \\ &= (q_1 - q_3) + (q_2 - q_4) + (q_3 - q_5) + (q_4 - q_6) \\ &= \lceil q \rceil_{e_1} + \lceil q \rceil_{e_2} + \lceil q \rceil_{e_3} + \lceil q \rceil_{e_4}. \end{aligned}$$

The remaining cases are similar. Thus,

$$\sum_{e \in \mathcal{E}_x} \big(\chi(z_e) - \chi(w_e)\big)^2 \le C \sum_{e \in \mathcal{E}_y} \lceil q \rceil_e^2 \tag{4.34}$$

The same argument gives

$$\sum_{e \in \mathcal{E}_y} \big(\chi(z_e) - \chi(w_e)\big)^2 \le C \sum_{e \in \mathcal{E}_x} \lceil q \rceil_e^2. \tag{4.35}$$

Combining (4.33), (4.34) and (4.35), we obtain

$$A_H(\chi, \chi) \le C\widetilde{A}_h(q, q). \tag{4.36}$$

Now, this, together with (4.32) and Proposition IV.1, gives

$$\|q - \chi\|_\Lambda^2 + A_H(\chi, \chi) \le CA_h(q, q)$$

and (M2) follows.

Finally, we verify (M3). We begin with the definition of the connection operator $\mathcal{I}_h$ which takes Q_H into Q_h. Let τ be a rectangle with vertices $\{z_i\}_{i=1}^4$ and $q \in Q_H$ be given. Then, $\mathcal{I}_h q \in Q_h$ on τ is defined by

$$(\mathcal{I}_h q)|_\tau = \frac{1}{|\tau|} \int_\tau q(x)\, dx = \frac{1}{4} \sum_{i=1}^4 q(z_i).$$

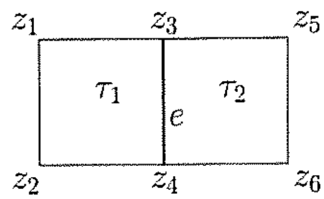

FIGURE 6. A portion of the rectangular mesh $\mathcal{T}$ for the verification of (M3).

Let $q \in Q_H$ be given. Then, we have, as in (4.31),

$$\|\mathcal{I}_h q - q\|_\Lambda^2 \simeq \varepsilon \sum_{\tau \in \mathcal{T}} \sum (\mathcal{I}_h q(z) - q(z))^2, \tag{4.37}$$

with the inner sum taken over the vertices of τ. Moreover, Proposition (IV.1) gives

$$A_h(\mathcal{I}_h q, \mathcal{I}_h q) \simeq \varepsilon \sum_{e \in \mathcal{E}_x} [\mathcal{I}_h q]_e^2 + \sum_{e \in \mathcal{E}_y} [\mathcal{I}_h q]_e^2 \tag{4.38}$$

and (4.33) gives

$$A_H(q,q) \simeq \sum_{e\in\mathcal{E}_x} \big(q(z_e) - q(w_e)\big)^2 + \varepsilon \sum_{e\in\mathcal{E}_y} \big(q(z_e) - q(w_e)\big)^2. \tag{4.39}$$

We study the typical terms in the sums (4.37) and (4.38). For instance, let $\tau = \tau_1$, $z = z_1$ and e be as in Figure 6. Then, at z in τ,

$$\begin{aligned} 4\big(\mathcal{I}_h q(z) - q(z)\big) &= \big(q(z_1) + q(z_2) + q(z_3) + q(z_4)\big) - 4q(z_1) \\ &= \big(q(z_2) - q(z_1)\big) + 2\big(q(z_3) - q(z_1)\big) + \big(q(z_4) - q(z_3)\big) \end{aligned}$$

and on e,

$$\begin{aligned} [\mathcal{I}_h q]_e^2 &= \left(\frac{1}{4}\sum_{i=1}^{4} q(z_i) - \frac{1}{4}\sum_{i=3}^{6} q(z_i)\right)^2 \\ &= \frac{1}{16}\left(\sum_{i=1}^{4}\big(q(z_i) - q(z_{i+2})\big)\right)^2. \end{aligned}$$

Now, the equivalence relations (4.37)–(4.39) give (M3).

C. Triangular mesh

In the rectangular case, the mesh dependent form $\widetilde{A}_h(\cdot,\cdot)$ defined in (4.11) played an important role in the construction and the analysis of the smoothers. Recall that this form was defined in terms of properly weighted jumps of functions across the edges. We have established in Proposition IV.1 that this form is equivalent to the form $A_h(\cdot,\cdot)$ induced by the Schur complement with constants independent of the mesh size h and the anisotropy parameter ε.

To construct such a mesh dependent form in the triangular case, appropriate weights need to be determined for the edges. The normal component of $\mathbf{v} = (v_1, v_2) \in RT_0$ across diagonal edges involve both v_1 and v_2. This poses a difficulty in choosing a suitable weight corresponding to the jumps across such edges. Moreover, the computation of A^{-1} in the bilinear form $A_h(\cdot,\cdot)$ induced by the Schur complement of the mixed finite element system is somewhat more complicated than in the rectangular case. In the current case, A leads to a banded system. Such a system still is relatively easy to invert, but does not possess the nice computational features of the block tridiagonal system generated in the rectangular case (cf. Remark IV.2).

For the above reasons, we take a different approach here. We will use a certain nonconforming finite element approximation to the anisotropic problem from which the solution to the mixed finite element problem can easily be obtained (see, e.g., [5] or [34]).

1. Equivalent nonconforming problem

Let $\mathcal{E}_0$ be the set of all internal edges in the uniform triangular mesh $\mathcal{T}$. Define the space L_0 by

$$L_0 = \left\{ \mu \in L^2\left(\bigcup_{e\in\mathcal{E}_0} e \right) \;\middle|\; \mu|_e \text{ is constant for all } e \in \mathcal{E}_0 \right\}.$$

Recall the lowest order Raviart-Thomas space RT_0 defined in (3.8). Each $\mathbf{v} \in RT_0$ has continuous normal component across the edges of the elements. In the triangular case, we will use the space RT_0^{-1} defined by

$$RT_0^{-1} = \left\{ \mathbf{v} \in (L^2(\Omega))^2 \;\middle|\; \mathbf{v}|_\tau = a_\tau \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_\tau \\ c_\tau \end{pmatrix} \text{ for all } \tau \in \mathcal{T} \right\}.$$

Note that, unlike RT_0, there is no continuity condition associated with the space RT_0^{-1}. We will also use the space of piecewise constant functions M_0^{-1} defined in (3.7).

Consider the problem: *Find* $(\widetilde{\mathbf{u}}_h, \widetilde{p}_h, \lambda_h) \in RT_0^{-1} \times M_0^{-1} \times L_0$ *such that*

$$(\widetilde{\mathbf{u}}_h, \mathbf{v})_\varepsilon + \sum_{\tau\in\mathcal{T}} \left\{ (\nabla\cdot\mathbf{v}, \widetilde{p}_h)_\tau - \langle \mathbf{v}\cdot\mathbf{n}_\tau, \lambda_h\rangle_{\partial\tau\backslash\partial\Omega} \right\} = 0 \qquad \text{for all } \mathbf{v} \in RT_0^{-1}, \tag{4.40a}$$

$$\sum_{\tau\in\mathcal{T}} (\nabla\cdot\widetilde{\mathbf{u}}_h, q)_\tau = -(f, q) \qquad \text{for all } q \in M_0^{-1}, \tag{4.40b}$$

$$-\sum_{\tau\in\mathcal{T}} \langle \widetilde{\mathbf{u}}_h\cdot\mathbf{n}_\tau, \mu\rangle_{\partial\tau\backslash\partial\Omega} = 0 \qquad \text{for all } \mu \in L_0. \tag{4.40c}$$

In this problem, the variable λ_h can be regarded as a Lagrange multiplier corresponding to the continuity condition imposed on the space RT_0.

Remark IV.5 *It can be shown that* $\widetilde{\mathbf{u}}_h = \mathbf{u}_h$ *and* $\widetilde{p_h} = p_h$*, where* $(\mathbf{u}_h, p_h)$ *is the unique solution pair of the mixed finite element problem* (3.9) *(see, e.g., [5], [28], or [34]).*

Let $\lambda = \{\lambda_i\}$ be the vector corresponding to the coefficients of the piecewise constant function $\lambda_h \in L_0$. This means that $\lambda_i = \lambda_h|_{e_i}$ for all $e_i \in \mathcal{E}_0$. Suppose that λ is known. Then, $\widetilde{\mathbf{u}}_h$ and $\widetilde{p_h}$ can be computed from the system (4.40a)–(4.40c) by substituting λ. Moreover, we notice that none of the spaces RT_0^{-1}, M_0^{-1}, and L_0 has any continuity conditions and are defined completely element wise. Therefore, computation of $\widetilde{\mathbf{u}}_h$ and $\widetilde{p_h}$ from λ involves simple element-by-element computation (see, e.g., [5]). Then, as stated in Remark IV.5, we have the solution $(\mathbf{u}, p) \in RT_0 \times M_0^{-1}$ of the mixed finite element problem (3.9).

As a means to obtain λ, we now describe a certain finite element problem which has λ as the solution. Let Q_h be the space of non-conforming piecewise linear finite elements with respect to the triangulation $\mathcal{T}$ called the Crouzeix-Raviart elements [37]. That is,

$$Q_h = \left\{ q \in L^2(\Omega) \;\middle|\; \begin{array}{l} q|_\tau \text{ is linear on each } \tau \in \mathcal{T}, \\ \text{and } q \text{ is ``continuous'' at the midpoint of each } e \in \mathcal{E}_0 \end{array} \right\}. \tag{4.41}$$

For each $q \in Q_h$ and $\tau \in \mathcal{T}$, the linear function $q|_\tau$ is determined by the values of q at the midpoints of the edges (see Figure 7). Let $e \in \mathcal{E}_0$ be an interior edge whose midpoint is denoted by m_e. Let τ_1 and τ_2 be the triangles in $\mathcal{T}$ that share e as a common edge. By the "continuity" of $q \in Q_h$ at the midpoint m_e in the definition

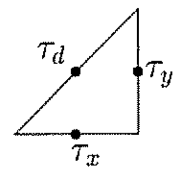

FIGURE 7. A typical triangle τ in the mesh $\mathcal{T}$ with labels for midpoints of the edges.

(4.41), we mean that

$$q|_{\tau_1}(m_e) = q|_{\tau_2}(m_e).$$

The standard nodal basis of the nonconforming space Q_h is defined as follows. Given $e \in \mathcal{E}_0$, the corresponding nodal basis function $\phi_e \in Q_h$ has the nodal values

$$\phi_e(m_\alpha) = \begin{cases} 1 & \text{if } \alpha = e, \\ 0 & \text{otherwise,} \end{cases}$$

for all $\alpha \in \mathcal{E}_0$.

As a special case of [34], we have the next theorem.

Theorem IV.2 *Consider the nonconforming finite element problem:* Find $\lambda \in Q_h$ satisfying

$$A_h(\lambda, q) = (\mathcal{Q}_0 f, q) \quad \textit{for all } q \in Q_h, \tag{4.42}$$

where

$$A_h(p, q) = \sum_{\tau \in \mathcal{T}} \int_\tau (p_x q_x + \varepsilon p_y q_y)\, dx\, dy \quad \textit{for all } p, q \in Q_h \tag{4.43}$$

and $\mathcal{Q}_0$ is the L^2 projection of $L^2(\Omega)$ onto the piecewise constant space M_0^{-1}. Let the linear system arising from this nonconforming problem using the standard nodal basis *for Q_h be given by*

$$M\widetilde{\lambda} = F. \tag{4.44}$$

Then, $\widetilde{\lambda} = \lambda$.

In addition to the above choice of Q_h and $A_h(\cdot,\cdot)$, we take $A_H(\cdot,\cdot)$ by (4.5) and Q_H to be the standard finite element space for (2.1) on triangular mesh, that is

$$Q_H = \{q \in H_0^1(\Omega) \mid q|_\tau \text{ is linear for all } \tau \in \mathcal{T}\}. \tag{4.45}$$

Also, as in the previous section, we take $(\cdot,\cdot)_h = (\cdot,\cdot)_H = (\cdot,\cdot)$.

The larger space Λ and its norm $\|\cdot\|_\Lambda$ are also taken as in the rectangular case. Consider the following quadrature rule on a triangle τ:

$$(\phi,\psi)_\tau \approx \frac{|\tau|}{3}\sum_{i=1}^{3}\phi(m_i)\psi(m_i),$$

where the m_i's are the midpoints of the edges. This quadrature rule is exact for linear functions ϕ and ψ. Thus, for each $q \in Q_h$, we have

$$\|q\|_\Lambda^2 = \varepsilon h^{-2}\sum_{\tau\in\mathcal{T}}\|q\|_{0,\tau} = \frac{\varepsilon}{3}\sum_{e\in\mathcal{E}_0} q(m_e)^2. \tag{4.46}$$

Here, we used the fact that every midpoint of an interior edge is associated with exactly two triangles in $\mathcal{T}$. Moreover, for any $q \in Q_h$, we have

$$A_h(q,q) = 2\sum_{\tau\in\mathcal{T}}\left(\big(q(\tau_d) - q(\tau_y)\big)^2 + \varepsilon\big(q(\tau_d) - q(\tau_x)\big)^2\right), \tag{4.47}$$

where τ_d, τ_x, and τ_y are the midpoints of edges labeled in Figure 7. Note also that the equivalence relation (4.33) for $A_H(\cdot,\cdot)$ also holds in the triangular case.

2. Line Jacobi and Gauss-Seidel smoothers

To define the line smoothers, we consider a decomposition of the space Q_h into subspaces. Let $\mathcal{V}$ be the collection of the midpoints of all edges in $\mathcal{E}_0$. For $j = 1, 2, \cdots, 2n-1$, we define $\mathcal{V}_j$ and $Q_{h,j}$ by

$$\mathcal{V}_j = \{(x, y) \in \mathcal{V} \,|\, y = jh/2\},$$

and

$$Q_{h,j} = \{q \in Q_h \,|\, q(z) = 0 \text{ for all } z \in \mathcal{V} \setminus \mathcal{V}_j\}.$$

Clearly,

$$Q_h = \sum_{j=1}^{2n-1} Q_{h,j}.$$

The line Jacobi smoother $\mathcal{J}_h$ is given as in (4.10) by

$$\mathcal{J}_h = \sum_{j=1}^{2n-1} A_{h,j}^{-1} \mathcal{Q}_{h,j}.$$

Then, by Lemma IV.2, $\mathcal{J}_h$ is invertible and its inverse is given by

$$(\mathcal{J}_h^{-1} q, q) = \sum_{j=1}^{2n-1} A_h(q_j, q_j).$$

Here, $q = \sum_{j=1}^{2n-1} q_j$ is the unique decomposition of $q \in Q_h$ as the sum of $q_j \in Q_{h,j}$. Now, (4.46) and (4.47) immediately imply (M1).

Next, we consider the line Gauss-Seidel smoother $\mathcal{G}_h$. Define the projection $P_{h,j} : Q_h \to Q_{h,j}$ with respect to the $A_h(\cdot, \cdot)$ inner product given by

$$A_h(P_{h,j} q, \chi) = A_h(q, \chi) \quad \text{for all } q \in Q_h \text{ and } \chi \in Q_{h,j}.$$

Then, the line Gauss-Seidel smoother $\mathcal{G}_h$ is given as in (4.25) by

$$\mathcal{G}_h = \{I - (I - P_{h,2n-1})(I - P_{h,2n}) \cdots (I - P_{h,1})\} A_h^{-1}.$$

The line Gauss-Seidel smoother $\mathcal{G}_h$ can be implemented by the following algorithm.

Algorithm IV.2 (Line Gauss-Seidel Smoother) *Given $g \in Q_h$, compute $\mathcal{G}_h g \in Q_h$ by the following:*

1. *Set $q_0 = 0$.*
2. *For $j = 1, 2, \cdots, 2n-1$, define*

$$q_j = q_{j-1} + A_{h,j}^{-1} Q_{h,j}(g - A_h q_{j-1}).$$

3. *Set $\mathcal{G}_h g = q_{2n-1}$.*

Notice that, unlike in the rectangular case, we use A_h and $P_{h,j}$ in the Gauss-Seidel smoother. Since

$$P_{h,j} P_{h,i} = 0 \quad \text{for } i < j-1,$$

it can be shown, as in Lemma IV.3, that

$$A_h(\mathcal{J}_h A_h q, q) \leq C A_h(\overline{\mathcal{G}}_h A_h q, q) \qquad \text{for all } q \in Q_h,$$

where $\overline{\mathcal{G}}_h$ is the line Gauss-Seidel smoother given in (4.26). Condition (M1) for $\overline{\mathcal{G}}_h$ now follows from the fact that the line Jacobi smoother $\mathcal{J}_h$ satisfies (M1).

We remark that it is easy to invert $A_{h,j}$ in the application of the smoothers $\mathcal{J}_h$ and $\mathcal{G}_h$. In fact, the operator $A_{h,j}^{-1}$ involves the solution of a diagonal system when j is even and a tridiagonal system when j is odd (see Figure 8).

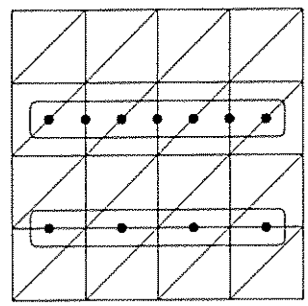

FIGURE 8. Degrees of freedom in $Q_{h,j}$ in a triangular mesh.

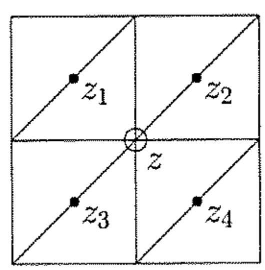

FIGURE 9. A portion of the triangular mesh $\mathcal{T}$.

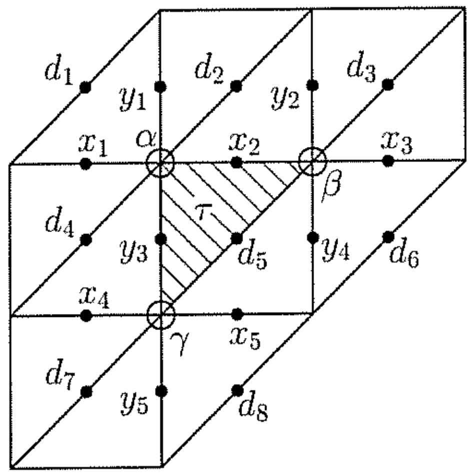

FIGURE 10. A portion of the triangular mesh $\mathcal{T}$ for the verification of (M2).

3. Approximation conditions

We begin with condition (M2). Given $q \in Q_h$, we take $\chi \in Q_H$ with the value at a node z defined by

$$\chi(z) = \begin{cases} \dfrac{1}{4}\displaystyle\sum_{i=1}^{4} q(z_i) & \text{if } z \in \Omega, \\ 0 & \text{if } z \in \partial\Omega, \end{cases}$$

where the z_i's are the 4 nodes around z as shown in Figure 9.

Let τ be a triangle in $\mathcal{T}$. We only demonstrate the case where all three vertices of τ are contained in Ω. For other cases, a slight modification of the arguments in the rectangular case is sufficient. For $\tau \in \mathcal{T}$, define $N(\tau)$ by

$$N(\tau) = \{\sigma \in \mathcal{T} \mid \overline{\sigma} \cap \overline{\tau} \neq \emptyset\}.$$

For example, if we take τ as in Figure 10, then $N(\tau)$ consists precisely of all the triangles in that figure. Define, for the nodes in $N(\tau)$,

$$I_y(\tau) = \{(i,j) \mid y_i, d_j \in \overline{\sigma} \text{ for some } \sigma \in N(\tau)\}$$

and $I_x(\tau)$ in a similar way. Then, as in the rectangular case,

$$(\chi(\alpha) - \chi(\beta))^2 \le C \sum_{(i,j)\in\mathcal{I}_y(\tau)} \left(q(y_i) - q(d_j)\right)^2$$

and

$$(\chi(\alpha) - \chi(\gamma))^2 \le C \sum_{(i,j)\in\mathcal{I}_x(\tau)} \left(q(x_i) - q(d_j)\right)^2.$$

Summing over all $\tau \in \mathcal{T}$ and applying the equivalence relations (4.33) and (4.47), we obtain

$$A_H(\chi, \chi) \le C A_h(q, q). \tag{4.48}$$

Let z be the midpoint of an edge $e \in \mathcal{E}$. Then, by adding and subtracting values of q at several midpoints of the edges, we have

$$\left(q(z) - \chi(z)\right)^2 \le C \left(\sum_{(i,j)\in I_y(\tau)} \left(q(y_i) - q(d_j)\right)^2 + \sum_{(i,j)\in I_x(\tau)} \left(q(x_i) - q(d_j)\right)^2 \right).$$

Summing this over all edges $e \in \mathcal{E}$, we obtain

$$\|q - \chi\|_{\Lambda}^2 \le C A_h(q, q)$$

from (4.46) and (4.47). This and (4.48) give (M2).

Finally, since $Q_H \subset Q_h$, we may take $\mathcal{I}_h$ to be the natural injection of Q_H into Q_h. Then,

$$\mathcal{I}_h q = q \quad \text{and} \quad A_H(q, q) = A_h(q, q)$$

for all $q \in Q_H$. This implies (M3) with $c_4 = 1$.

Remark IV.6 *Conditions (M1)–(M3) can also be verified when Q_H is taken as in the rectangular case by* (4.4)*. The current choice of Q_H as in* (4.45) *simplifies the verification of the conditions, that of (M3) in particular.*

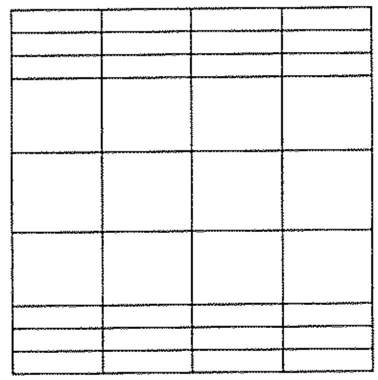

FIGURE 11. An example of the anisotropically refined rectangular mesh.

D. Anisotropically refined rectangular mesh

Here, we consider an anisotropically refined rectangular mesh used by Li and Wheeler [51] shown in Figure 11, On this mesh, Li and Wheeler [51] obtain an uniform error estimate for the reaction-diffusion equation: *Given* $f \in L^2(\Omega)$, *find* p *satisfying*

$$\begin{aligned} -(p_{xx} + \varepsilon p_{yy}) + b(x,y)p = f \quad &\text{in} \quad \Omega, \\ p = 0 \quad &\text{on} \quad \partial\Omega, \end{aligned} \tag{4.49}$$

where $b(x, y) > \beta^2 > 0$.

The mesh is defined as follows. Let $\sigma_y = \beta^{-1}\varepsilon|\ln \varepsilon|$ and split Ω into three subdomains $\Omega_1 = (0,1)\times(0,\sigma_x)$, $\Omega_2 = (0,1)\times(\sigma_x, 1-\sigma_x)$, and $\Omega_3 = (0,1)\times(1-\sigma_x, 1)$. Now, in the x-direction, the domain Ω is quasiuniformly divided into N subintervals. In the y-direction, on the other hand, each subdomain Ω_i is quasiuniformly divided into N_i subintervals, where each $N_i \simeq N$.

For the reaction-diffusion problem (4.49), the continuous and discrete mixed problems are formulated as in (2.2) and (3.9) respectively, with an extra term which corresponds to the reaction term. The following error estimate holds [51, Theorem

3.12]:

$$|||\mathbf{u}-\mathbf{u}_h|||_0 + \|p-p_h\|_0 \le CN^{-1}.$$

Our analysis for the two-level preconditioner in the uniform rectangular mesh carries over to this anisotropically refined mesh with slight modification. For example, the norm $\|\cdot\|_\Lambda$ in the space $\Lambda = L^2(\Omega)$ is defined in this case by

$$\|q\|_\Lambda^2 = \varepsilon \sum_{\tau\in\mathcal{T}} \frac{1}{|\tau|}\|q\|_{0,\tau}^2,$$

where $|\tau|$ is the area of the rectangle τ. The rest of the components of the two-level preconditioner are taken similarly as in the uniform rectangular mesh case. Then, since the verification of conditions (M1)–(M3) is based on local element by element or stripwise arguments, the two-level preconditioner with line Jacobi smoother gives a uniform preconditioner for our mixed finite element system for the anisotropic problem on the anisotropically refined rectangular mesh.

CHAPTER V

MULTIGRID METHODS FOR THE FINITE ELEMENT APPROXIMATION OF THE ANISOTROPIC PROBLEM

In this chapter, we review multigrid techniques for the standard finite element approximation of the anisotropic problem. In particular, we will present the uniform result by Bramble and Zhang [23] in detail since this algorithm is used in our numerical experiments in the next chapter. Other uniform results have been developed by Neuss [54] and Stevenson [62], [63]. Combined with such a method, our two-level preconditioner studied in the last chapter gives a uniform preconditioner for the linear system resulting from the mixed finite element approximation for the anisotropic problem.

Two main components of the multigrid method are smoothing and coarsening. Smoothing operators reduce the high frequency components of the iterative error whereas the coarse grid solve reduces the low frequency or the smooth components of the error. When applied to anisotropic problems, the standard pointwise smoothing and full coarsening strategy does not work well. The remedy is to use either line smoothing or semicoarsening or both (see the numerical examples presented in [32] and [68]). The coarse grid defined by semicoarsening is given by doubling the mesh size in one direction without changing the mesh size in the other.

We begin with the semicoarsening/pointwise smoothing case for the anisotropic problem. An analysis of the smoother in this case applying the local Fourier analysis by Brandt [24] is found in [68]. A robust multilevel BPX-type [20] additive preconditioner based on tensor product type subspace splitting with semi-coarsening was developed by Griebel and Oswald [46].

In the case of full coarsening/line smoothing applied to the anisotropic problem, a uniform V-cycle result is given by Bramble and Zhang [23]. Moreover, it deals with a certain case of variable coefficients as discussed in Remark V.1. A generalization of the results of Hackbusch [48] and Mandel, McCormick, and Bank [52] is given by Stevenson which guarantees the robustness of the V-cycle multigrid algorithm [62], [63]. The robustness of V-cycle with one line Jacobi post-smoothing step (and no pre-smoothing) is proved by Neuss in [54] by generalizing the results of Xu [74] and Yserentant [75]. There is also a robust hierarchical basis type preconditioner by Stevenson [61]. This preconditioner is shown to be robust when the direction of anisotropy aligns with the three directions in a triangular mesh.

In the case of variable ε, there are several computational strategies. One is alternating line relaxation, where the x- and y- line smoothing steps are taken alternately. Another is multiple semicoarsening. In this approach, semicoarsening both in the x- or y- direction take place. For the multiple semicoarsening algorithm, see [53]. In addition, there is flexible multiple semicoarsening algorithm in which only a subset of the coarse grids are used (see, e.g., [71]). Compared with the full coarsening/line smoothing algorithm in [23], the above computational approaches can handle a larger class of variable ε (see Remark V.1 for the assumptions on ε in [23]). However, no analysis as thorough as [23] is available as yet for the strategies discussed in this paragraph.

We now present the uniform V-cycle multigrid method analyzed by Bramble and Zhang [23]. We begin with the following construction of a sequence of nested uniform rectangular or triangular meshes $\mathcal{T}_1, \cdots, \mathcal{T}_J \equiv \mathcal{T}$. Let $\mathcal{T}_1$ be the uniform rectangular or triangular mesh with $n = 2$ (see Chapter III and Figure 1). The mesh $\mathcal{T}_j$ for $2 \le j \le J$ is constructed from $\mathcal{T}_{j-1}$ by subdividing each rectangular or triangular element $\tau \in \mathcal{T}_{j-1}$ into 4 elements in $\mathcal{T}_j$ by connecting the centers of the edges (see Figure 12). In other words, $\mathcal{T}_j$ is the uniform mesh with $n = 2^j$ and mesh size

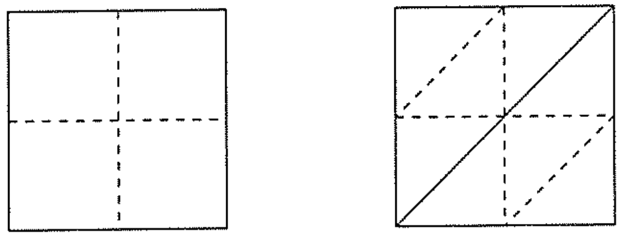

FIGURE 12. Element(s) of meshes $\mathcal{T}_j$ and $\mathcal{T}_{j-1}$.

$h_j = 1/n$.

Associated with the above sequence of nested meshes, we consider the sequence of nested finite element spaces

$$M_1 \subset M_2 \subset \cdots \subset M_J = Q_H,$$

where the space M_j is the standard finite element space given in the rectangular case by

$$M_j = \{\phi \in H_0^1(\Omega) \mid \phi \text{ is bilinear on each } \tau \in \mathcal{T}_j\}$$

and in the triangular case by

$$M_j = \{\phi \in H_0^1(\Omega) \mid \phi \text{ is linear on each } \tau \in \mathcal{T}_j\}.$$

Recall that $A_H(\cdot,\cdot)$ was given in (4.5) by

$$A_H(\varphi, \phi) = \int_\Omega (\varphi_x \phi_x + \varepsilon \varphi_y \phi_y)\, dx\, dy.$$

The corresponding norm and the corresponding operator norm will be denoted by $\|\cdot\|_{A_H}$. We define $A_j : M_j \to M_j$ by

$$(A_j \phi, \chi) = A_H(\phi, \chi) \quad \text{for all } \chi \in M_j.$$

As usual, $\mathcal{Q}_j : L^2(\Omega) \to M_j$ will denote the L^2 orthogonal projection onto M_j. In addition, we let $\mathcal{P}_j : H_0^1(\Omega) \to M_j$ be the Galerkin projection defined by

$$A_H(\mathcal{P}_j q, \chi) = A_H(q, \chi) \quad \text{for all } \chi \in M_j.$$

The Galerkin projection satisfies the following approximation property in both the rectangular and triangular cases [23, Lemma 4.3]: for all $q \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\|(I - \mathcal{P}_j)q\|_0 \leq C\varepsilon^{-1} h_j^2 \|(I - \mathcal{P}_j)q\|_{A_H}^2. \tag{5.1}$$

In the multigrid algorithm, smoothing operators are also needed. Let $R_j : M_j \to M_j$ denote a smother and R_j^t its adjoint with respect to the $(\cdot, \cdot)$ inner product. Specific smoothers and their properties will be discussed later.

Given an initial guess $q_0 \in M_j$, a multigrid algorithm produces a sequence of approximations to $q_j = A_j^{-1} f_j$ by

$$q^{m+1} = \mathrm{Mg}_j(q^m, f_j), \quad m = 0, 1, 2, \cdots,$$

where $\mathrm{Mg}_j : M_j \times M_j \to M_j$ is defined recursively as follows.

Algorithm V.1 *For $j = 1$ and $q, g \in M_1$, set $\mathrm{Mg}_1(q, g) = A_1^{-1} g$. For $j > 1$ and $q, g \in M_j$, $\mathrm{Mg}_j(q, g)$ is defined as follows:*

1. *Pre-smoothing:*

$$q' = q + R_k^t(g - A_j q).$$

2. *Coarse grid correction:*

$$q'' = q' + \mathrm{Mg}_{j-1}(0, \mathcal{Q}_{j-1}(g - A_j q'))$$

3. Post-smoothing:

$$q''' = q'' + R_k(g - A_j q'').$$

4. Set $\mathrm{Mg}_j(q, g) = q'''$.

Clearly, the above algorithm is linear, i.e.,

$$\mathrm{Mg}_j(\alpha(p, f) + (q, g)) = \alpha \mathrm{Mg}_j(p, f) + \mathrm{Mg}_j(q, g)$$

for all $(p, f), (q, g) \in M_j \times M_j$ and $\alpha \in \mathbb{R}$. Moreover, the multigrid algorithm is consistent, i.e.,

$$\mathrm{Mg}_j(q, A_j q) = q$$

for all $q \in M_j$. Consider the linear operator $B_j : M_j \to M_j$ given by

$$B_j g = \mathrm{Mg}_j(0, g) \quad \text{for all } g \in M_j.$$

It is easy to see, then, from the linearity and the consistency of the multigrid algorithm, that

$$\mathrm{Mg}_j(q, g) = q + B_j(g - A_j q).$$

In addition, we note that step 3 of Algorithm V.1 makes B_j symmetric with respect to the $(\cdot, \cdot)$ inner product and hence B_j can be used as a preconditioner for A_j. In particular, we take $B_H^{-1} = B_J$ in our two-level preconditioner (4.3) for the mixed finite element problem.

In [23], the smoothing condition and the regularity and approximation condition of the theory of Braess and Hackbusch [14] are combined as follows.

(SA1) The operator R_j is symmetric and there is a constant $\theta < 1$ such that the spectrum $\sigma(I - R_jA_j) \subset [-1, \theta)$ for all j. Moreover, there is a constant C_M independent of j and ε such that

$$(R_j^{-1}(I - \mathcal{P}_{j-1})q, (I - \mathcal{P}_{j-1})q) \leq C_M A_H((I - \mathcal{P}_{j-1})q, q) \quad \text{for all } q \in M_j.$$

(SA2) For the operator R_j, $\|I - R_jA_j\|_{A_H} < 1$. Moreover, there is a constant C_M independent of j and ε such that

$$(\overline{R}_j^{-1}(I - \mathcal{P}_{j-1})q, (I - \mathcal{P}_{j-1})q) \leq C_M A_H((I - \mathcal{P}_{j-1})q, q) \quad \text{for all } q \in M_j,$$

where

$$\overline{R}_j = R_j + R_j^t - R_j^t A_j R_j. \tag{5.2}$$

The following lemma gives the convergence result of Algorithm V.1.

Lemma V.1 ([23, Lemmas 2.1 and 2.2]) *Assume that R_j satisfies (SA1) or (SA2). Then, there is a constant $\delta < 1$, independent of j and ε, such that the multigrid algorithm defined in Algorithm V.1 satisfies*

$$0 \leq A_H((I - B_jA_j)q, q) \leq \delta A_H(q, q) \quad \textit{for all } q \in M_j.$$

We now define the line Jacobi smoother $\mathcal{J}_j$ and the line Gauss-Seidel smoother $\mathcal{G}_j$ that will be used in our construction of the multigrid preconditioner. Consider the following horizontal strip decomposition of Ω. For $k = 1, \cdots, n-1$, define

$$\Omega_k = \{(x, y) \in \Omega \mid (k-1)h < y < (k+1)h\}$$

and

$$M_{j,k} = \{q \in M_j \mid q = 0 \text{ in } \Omega \setminus \Omega_k\}.$$

Then,

$$\Omega = \bigcup_{k=1}^{n-1} \Omega_k \quad \text{and} \quad M_j = \sum_{k=1}^{n-1} M_{j,k}.$$

Note that the above is a direct sum decomposition of M_j. Define the operator $A_{j,k} : M_{j,k} \to M_{j,k}$ by

$$(A_{j,k}q, \chi) = A_H(q, \chi) \quad \text{for all } \chi \in M_{j,k}.$$

In addition, let $\mathcal{Q}_{j,k} : M_j \to M_{j,k}$ and $\mathcal{P}_{j,k} : M_j \to M_{j,k}$ be the orthogonal projections onto $M_{j,k}$ with respect to the inner products $(\cdot,\cdot)$ and $A_H(\cdot,\cdot)$, respectively. The line Jacobi smoother $\mathcal{J}_j$ and the line Gauss-Seidel smoother $\mathcal{G}_j$ are defined as in the previous chapter by

$$\mathcal{J}_j = \sum_{k=1}^{n-1} A_{j,k}^{-1} \mathcal{Q}_{j,k}$$

and

$$\mathcal{G}_j = \{I - (I - \mathcal{P}_{j,n-1})(I - \mathcal{P}_{j,n-2}) \cdots (I - \mathcal{P}_{j,1})\} A_j^{-1}.$$

The line Jacobi smoother $\mathcal{J}_j$ satisfies the following property, which is similar to condition (M1) in the previous chapter (cf. [21]):

$$\frac{1}{2} A_H(q,q) \le (\mathcal{J}_j^{-1} q, q) \le C \left\{ A_H(q,q) + \frac{\varepsilon}{h_j^2} \|q\|_0^2 \right\} \quad \text{for all } q \in M_j$$

For the line Gauss-Seidel smoother $\mathcal{G}_j$, a similar argument using the above as in Lemma IV.4 gives

$$A_H(q,q) \le (\overline{\mathcal{G}}_j^{-1} q, q) \le C \left\{ A_H(q,q) + \frac{\varepsilon}{h_j^2} \|q\|_0^2 \right\} \quad \text{for all } q \in M_j,$$

where $\overline{\mathcal{G}}_j$ is given from $\mathcal{G}_j$ by the formula (5.2). These smoothing properties for $R_j = \eta \mathcal{J}_j$ with $0 < \eta < 1$ and $R_j = \mathcal{G}_j$, combined with the approximation property

(5.1), give (SA1) and (SA2). The uniform V-cycle preconditioning result in [23] now follows from Lemma V.1.

Theorem V.1 (**[23, Theorem 6.1]**) *Let $R_j = \eta \mathcal{J}_j$ with $0 < \eta < 1$ or $R_j = \mathcal{G}_j$. Then there is a positive number $\delta < 1$, independent of j and ε, such that the multigrid algorithm defined in Algorithm V.1 satisfies*

$$(1-\delta) A_H(q,q) \le A_H(B_j A_j q, q) \le A_H(q,q) \quad \text{for all } q \in M_j.$$

Remark V.1 *In [23], the above uniform result is obtained for the bilinear form $A_H(\cdot,\cdot)$ given by*

$$A_H(p,q) = \int_\Omega \left\{ a(x,y) p_x q_x + b(x,y) p_y q_y \right\} dx\, dy \quad \text{for all } p, q \in H_0^1(\Omega)$$

with variable coefficients $a \equiv a(\cdot,\cdot)$ and $b \equiv b(\cdot,\cdot)$ satisfying the following boundedness conditions:

1. *$a(x,y)$ is uniformly bounded from below and above, i.e., there exist constants $a_{\min}$ and $a_{\max}$ such that*

$$0 < a_{\min} \le a(x,y) \le a_{\max}.$$

2. *$b(x,y)$ is uniformly bounded from above, i.e., there exists a constant $b_{\max}$ such that*

$$0 < b(x,y) \le b_{\max}.$$

3. *The derivatives of $a(x,y)$ are uniformly bounded, i.e., there exists a constant C such that*

$$|\nabla a| \le C,$$

4. *The y derivative of $b(x,y)$ is uniformly bounded in the sense that there exists a constant C_b such that*

$$\frac{|b_y|}{b} \leq C_b.$$

CHAPTER VI

NUMERICAL RESULTS

In this chapter, numerical results for the two-level preconditioner for the uniform rectangular and triangular meshes are reported. We compute the condition numbers of the preconditioner system, that is $B_h^{-1}A_h$. All computations were done with the right hand side function

$$f(x,y) = y(1-y)\sin(x)\sin(1-x)$$

which does not depend on the anisotropy parameter ε.

To compute the condition number, i.e. the ratio of the largest and smallest eigenvalues, in each test case, we use the Lanczos method combined with the Conjugate Gradient method [44]. This algorithm gives a good approximation to the eigenvalues of the symmetric positive definite system being solved by the Conjugate Gradient method while consuming considerably less time than the conventional methods, for example, the power method.

Let $\mathcal{J}_h$ be the line Jacobi smoother defined in Sections IV.B.2 and IV.C.2 associated with the rectangular and triangular meshes, respectively. We consider the following cases where the two-level preconditioner is defined by

$$B_h^{-1} = \mathcal{I}_h B_H^{-1} \mathcal{I}_h^t + \mathcal{J}_h$$

and the coarse level preconditioner B_H^{-1} is given by one of the following:

1. B_H^{-1} is given by the V-cycle preconditioner with line Jacobi smoother given by Algorithm V.1.

2. $B_H^{-1} = 0$, i.e. B_h^{-1} is the line Jacobi preconditioner.

3. $B_H^{-1} = I$.

4. $B_h^{-1} = I$, i.e. no preconditioning is done.

The condition numbers of the preconditioned system for the above cases will be reported respectively in Tables I, II, III, and IV for the rectangular mesh and in Tables V, VI, VII, and VIII for the triangular mesh.

Note that tridiagonal systems need to be inverted in the computation of the line Jacobi smoother $\mathcal{J}_h$ on both the rectangular and triangular meshes, as well as the action of A_h, the Schur complement, on the rectangular mesh. This task is accomplished by using an LAPACK routine [1].

The V-cycle algorithm used in Case 1 involves full coarsening and line smoothing. The transpose $\mathcal{I}_h^t$ of the connection operator can also be viewed as a full coarsening strategy involved with the fine level space Q_h and the coarse level space Q_H. In the V-cycle algorithm in case 1, the scaling factor η of the line Jacobi smoother was taken to be $\eta = .72$ (see Theorem V.1). As was seen in the analysis in Chapter IV, no scaling is required for the line Jacobi smoother $\mathcal{J}_h$ used in the h-level. Tables I and V illustrate that the condition number in this case is bounded uniformly in ε and h, confirming the preconditioning results obtained in Chapter IV.

In Cases 2 and 3, the condition number of the preconditioned system grow in proportion to h^{-2}, for fixed ε, when h is sufficiently small for ε. It is well known that the exactly same condition number growth is observed in isotropic problems. Condition numbers in these cases are reported in Tables II and III for the rectangular mesh and Tables VI and VII for the triangular mesh.

Finally, we remark on Case 4. The condition numbers in the rectangular mesh are independent of the mesh size (Table IV). For the problem on triangular mesh, as in Cases 2 and 3, the condition number grow like h^{-2}, for fixed ε, provided that h is

Table I. Condition numbers for the rectangular mesh with V-cycle preconditioning on the H-level.

ε / h	1.	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
1/4	4.02	2.22	2.70	3.04	3.08	3.09
1/8	5.56	3.23	3.09	3.73	3.91	3.93
1/16	6.50	4.67	3.22	3.87	4.12	4.19
1/32	7.02	5.70	3.63	3.91	4.16	4.24
1/64	7.39	6.36	4.97	3.91	4.18	4.26
1/128	7.54	6.76	5.93	4.51	4.17	4.26
1/256	7.65	6.95	6.52	5.34	4.17	4.26

sufficiently small compared to ε (Table VIII).

Table II. Condition numbers for the rectangular mesh with line Jacobi preconditioner.

h \ ε	1.	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
1/4	1.04e+01	3.96e+00	2.90e+00	2.81e+00	2.80e+00	2.80e+00
1/8	3.95e+01	9.25e+00	4.14e+00	3.70e+00	3.66e+00	3.65e+00
1/16	1.56e+02	2.99e+01	6.03e+00	4.09e+00	3.92e+00	3.91e+00
1/32	6.23e+02	1.14e+02	1.40e+01	4.78e+00	4.05e+00	3.98e+00
1/64	2.49e+03	4.54e+02	5.05e+01	7.41e+00	4.31e+00	4.02e+00
1/128	9.96e+03	1.81e+03	1.98e+02	2.11e+01	5.32e+00	4.13e+00
1/256	3.98e+04	7.24e+03	7.90e+02	8.07e+01	9.49e+00	4.53e+00

Table III. Condition numbers for the rectangular mesh with $B_H = I$.

h \ ε	1.	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
1/4	3.72e+00	2.63e+00	3.06e+00	3.42e+00	3.47e+00	3.47e+00
1/8	1.33e+01	7.00e+00	3.94e+00	4.10e+00	4.30e+00	4.32e+00
1/16	5.21e+01	2.42e+01	6.25e+00	4.51e+00	4.55e+00	4.63e+00
1/32	2.08e+02	9.47e+01	1.36e+01	5.50e+00	4.70e+00	4.69e+00
1/64	8.30e+02	3.77e+02	4.94e+01	8.50e+00	5.08e+00	4.75e+00
1/128	3.32e+03	1.51e+03	1.94e+02	2.10e+01	6.28e+00	4.89e+00
1/256	1.33e+04	6.04e+03	7.74e+02	8.05e+01	1.10e+01	5.37e+00

Table IV. Condition numbers for the rectangular mesh with no preconditioning.

ε	1/4	1/8	1/16	1/32	1/64	1/128
condition no.	1.84e+01	7.68e+01	3.10e+2	1.24e+03	4.98e+03	1.99e+04

Table V. Condition numbers for the triangular mesh with V-cycle preconditioning on the H-level.

h \ ε	1.	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
1/4	6.02	2.86	2.29	2.09	2.03	2.01
1/8	7.21	3.11	2.49	2.17	2.05	2.02
1/16	7.60	3.21	2.69	2.31	2.10	2.03
1/32	7.70	3.25	2.74	2.52	2.20	2.06
1/64	7.73	3.26	2.77	2.66	2.37	2.13
1/128	7.73	3.26	2.78	2.67	2.58	2.25
1/256	7.73	3.26	2.79	2.68	2.65	2.44

Table VI. Condition numbers for the triangular mesh with block Jacobi preconditioner.

h \ ε	1.	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
1/4	1.36e+01	3.86e+00	1.61e+00	1.16e+00	1.05e+00	1.01e+00
1/8	5.25e+01	1.11e+01	2.62e+00	1.37e+00	1.10e+00	1.03e+00
1/16	2.08e+02	3.95e+01	5.91e+00	1.88e+00	1.22e+00	1.06e+00
1/32	8.31e+02	1.53e+02	1.83e+01	3.36e+00	1.50e+00	1.13e+00
1/64	3.32e+03	6.05e+02	6.76e+01	8.51e+00	2.21e+00	1.29e+00
1/128	1.33e+04	2.42e+03	2.65e+02	2.85e+01	4.43e+00	1.66e+00
1/256	5.31e+04	9.68e+03	1.05e+03	1.08e+02	1.25e+01	2.69e+00

Table VII. Condition numbers for the triangular mesh with $B_H = I$.

h \ ε	1.	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
1/4	2.23e+01	7.19e+00	5.14e+00	4.63e+00	4.48e+00	4.43e+00
1/8	2.82e+01	1.69e+01	7.51e+00	5.38e+00	4.99e+00	4.89e+00
1/16	1.05e+02	5.94e+01	1.55e+01	6.91e+00	5.41e+00	5.06e+00
1/32	4.16e+02	2.29e+02	4.47e+01	1.07e+01	6.19e+00	5.31e+00
1/64	1.66e+03	9.08e+02	1.60e+02	2.35e+01	8.00e+00	5.72e+00
1/128	6.64e+03	3.62e+03	6.20e+02	7.32e+01	1.36e+01	6.66e+00
1/256	2.66e+04	1.45e+04	2.46e+03	2.71e+02	3.38e+01	9.22e+00

Table VIII. Condition numbers for the triangular mesh with no preconditioning.

h \ ε	1.	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
1/4	2.87e+01	4.28e+01	2.06e+02	1.94e+03	1.93e+04	1.92e+05
1/8	1.16e+02	1.50+e02	2.77e+02	2.03e+03	1.98e+04	1.98e+05
1/16	4.66e+02	5.86e+02	6.97e+02	2.25e+03	2.02e+04	2.00e+05
1/32	1.87e+03	2.33e+03	2.54e+03	3.55e+03	2.09e+04	2.01e+05
1/64	7.47e+03	9.30e+03	9.95e+03	1.07e+04	2.45e+04	2.03e+05
1/128	2.99e+04	3.72e+04	3.96e+04	4.05e+04	4.90e+04	2.15e+05

CHAPTER VII

CONCLUSION

We have studied numerical methods and iterative solution techniques for a model second-order anisotropic partial differential equation on the unit square. New results have been obtained for the mixed finite element method for this problem. In addition, an efficient and robust two-level preconditioner has been developed for efficient computation of the numerical solution for the mixed finite element problem.

For uniform rectangular and triangular meshes, error estimates for the mixed approximation explicitly giving the behavior of the anisotropy parameter ε have been derived in Chapter III. In Chapter IV, two-level preconditioners have been constructed for the linear system resulting from the anisotropic mixed finite element problem for anisotropically refined rectangular mesh as well as regular rectangular and triangular meshes. These preconditioners have been shown to be uniform both in the anisotropy parameter and the mesh size. Combined with the uniform multigrid preconditioners for the standard finite element problem, e.g. [23], for the uniform rectangular and triangular meshes discussed in Chapter V, our uniform two-level result gives a uniform multi-level preconditioner for the mixed finite element method for the anisotropic problem. The numerical results given in Chapter VI are in agreement with our analysis of the multilevel preconditioner.

As was discussed in Section IV.D, a better approximation is possible by using conforming rectangular meshes that have been anisotropically refined in the regions of boundary layer [51]. This prompts a natural subdomain partition of the domain into regions with and without boundary layer. The current study using uniform rectangular and triangular meshes provides a basis for such a case. Uniform meshes of adequate mesh sizes on each of the subdomains can be used resulting in a globally non-conforming mesh as in the mortar method (see, e.g., [10], [11], or [50]). In the mortar method, the domain is split into subdomains and each subdomain is given a mesh independently of other subdomains. Mixed finite element methods for uniformly positive definite problems on such meshes have been studied in the mortar context in [3] and in a non-mortar context in [4]. Extending these results to anisotropic problems is an area of future research. Multilevel preconditioning for the mortar mixed finite element problem in both the uniform elliptic and the anisotropic cases is also of interest. Multigrid methods for the standard finite element method have been studied in [13] and [45].

The equivalence result for the mesh dependent form established in Proposition IV.1 also holds for piecewise constant ε (Remark IV.3). Moreover, the uniform multigrid result for the standard finite element method by [23] was obtained for certain variable ε. These facts raise a question as to the construction and the properties of the multilevel preconditioners for the mixed system when ε is no longer a constant. As was discussed in Chapter I, applications such as porous media flow modeling which includes simulations of groundwater flow, oil reservoir, and electrical wave propagation in the human heart all involve highly heterogeneous medium. Even though the strategies mentioned in Chapter V based on alternating line relaxation or multiple semicoarsening provides satisfactory computational results in some test cases, no satisfactory analysis of these algorithms are available and poses a challenge yet to be met.

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VITA

Chisup Kim was born in Seoul, Republic of Korea on May 4, 1971 to Sangdae Kim and Nakhee Sung. He studied at Seoul National University from March, 1990 and graduated with a Bachelor of Science in mathematics in February, 1994. After serving his country in the army from January, 1994 to March, 1996, he began his doctoral study in numerical analysis and scientific computation at Texas A&M University in the fall of 1996. He spent the summers of 1999 and 2000 as a summer student at the Center for Applied Scientific Computing at the Lawrence Livermore National Laboratory. The current doctoral thesis on the iteration and approximation of anisotropic problems was defended in June 2001.

Until August, 2003, Chisup Kim will be a Research Associate at Duke University and can be contacted at:

Department of Mathematics

Duke University

Box 90320

Durham, NC 27708-0320, U.S.A.

E-mail address: ckim@math.duke.edu