SHARP L^2-ERROR ESTIMATES AND SUPERCONVERGENCE OF MIXED FINITE ELEMENT METHODS FOR NONFICKIAN FLOWS IN POROUS MEDIA

R. E. EWING, Y. LIN, T. SUN, J. WANG, AND S. ZHANG

ABSTRACT. Compared with [16] and [17], a sharper L^2-error estimate is obtained for the nonFickian flow of fluid in porous media by means of a mixed Ritz-Volterra projection instead of the mixed Ritz projection used in [16] and [17]. Moreover, local L^2 superconvergence for the velocity along the Gauss lines and for the pressure at the Gauss points is derived for the mixed finite element method via the Ritz-Volterra projection, and global L^2 superconvergence for the velocity and the pressure is also investigated by virtue of an interpolation post-processing technique. On the basis of the superconvergence estimates, some useful a-posteriori error estimators are presented for this mixed finite element method.

1. INTRODUCTION

As mentioned in [16] and [17], the nonFickian flow of fluid in porous media is complicated by the history effect which characterizes various mixing length growth of the flow, and can be modeled by an integro-differential equation: Find $u = u(x,t)$ such that

$$\begin{aligned} u_t &= \nabla \cdot \sigma + cu + f && \text{in } \Omega \times J, \\ \sigma &= A(t) \cdot \nabla u - \int_0^t B(t,s) \cdot \nabla u(s) ds && \text{in } \Omega \times J, \\ u &= g && \text{on } \partial\Omega \times J, \\ u &= u_0(x) && x \in \Omega,\ t = 0, \end{aligned} \tag{1.1}$$

where $\Omega \subset R^d$ $(d = 2,3)$ is an open bounded domain with smooth boundary $\partial\Omega$, $J = (0,T)$ with $T > 0$, $A(t) = A(x,t)$ and $B(t,s) = B(x,t,s)$ are two 2×2 or 3×3 matrices, and A is positive definite, c, f, g and u_0 are known smooth functions. Cushman and his colleagues [4, 5, 6, 7, 8, 21] have developed a non-local theory and some applications for the flow of fluid in porous media. Furtado, Glimm, Lindquist, and Pereira [19, 20], Neuman and Zhang [27], and Ewing [12, 13, 14] also studied the history effect of various mixing length growth for flow in heterogeneous porous media. In a recent laboratory experimental investigation of contaminant transport in heterogeneous porous media [30], some nonlocal behavior of dispersion tensors have been observed.

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There is now sizeable literature on the numerical approximations of the problem (1.1). In [29], the method of backward Euler and Crank-Nicolson combined with a certain numerical quadrature rule is employed to deal with the time direction, which aims at reducing the computational cost and storage spaces due to the memory effect. Finite element methods have been also developed for the problem (1.1) during the past ten years [2, 3, 23, 24, 25, 26, 32], in which optimal and superconvergence can be found for the corresponding finite element approximations in various norms, such as L^p with $2 \le p \le \infty$. In particular, the method of using the Ritz-Volterra projection, discovered by Cannon and Lin [2], proved to be a powerful technique behind the analysis.

However, to the best of our knowledge, there are few results except [16, 17, 22] available concerning the mathematical formulation and analysis of the mixed finite element method for (1.1). In [16, 17] the authors dealt with general setting of the problem. But the formulation and anaylsis given in [22] are valid for only a special case, i.e., operator B is proportional to operator A, the reader is refered to [22] for this special case. The mathematical difficulty associated with the analysis of numerical approximations to the solution of (1.1) lies on the integral term added to standard parabolic equations [31, 32].

In the present paper we are concerned with the approximate solutions of (1.1) by mixed finite element methods. Sharper L^2-error estimates than those in [16, 17] are obtained by employing a mixed Ritz-Volterra projection rather than the Ritz projection used in [16, 17]. In addition, local L^2 superconvergence for the velocity along the Gauss lines and for the pressure at the Gauss points is derived, and with the aid of an interpolation post-processing method global L^2 superconvergence is also considered for the velocity and the pressure.

The paper is organized in the following way. In Section 2, we give some necessary preparations, introduce the mixed Ritz-Volterra projection and analyze its approximation properties. In Section 3, we derive a sharper error estimate for the mixed finite element approximations in the L^2-norm. Sections 4 and 5 are devoted to the local and global superconvergence analysis of the mixed finite element method, respectively.

2. The mixed Ritz-Volterra type projection

In this section, we give the mixed finite element approximate formula for the parabolic integro-differential equation (1.1) and the mixed Ritz-Volterra projection. For simplicity, the method will be presented on plane domains.

Let $W := L^2(\Omega)$ be the standard L^2 space on Ω with norm $\|\cdot\|_0$. Denote by

$$\mathbf{V} := H(div, \Omega) = \left\{\sigma \in (L^2(\Omega))^2 : \ \ \nabla\cdot\sigma \in L^2(\Omega)\right\},$$

the Hilbert space equipped with the following norm:

$$\|\sigma\|_{\mathbf{V}} := \left(\|\sigma\|_0^2 + \|\nabla\cdot\sigma\|_0^2\right)^{\frac{1}{2}}.$$

There are several ways to discretize the problem (1.1) based on the variables σ and u; each method corresponds to a particular variational form of (1.1) [16, 17].

Let T_h be a finite element partition of Ω into triangles or quadrilaterals which is quasi-uniform. Let $\mathbf{V}_h \times W_h$ denote a pair of finite element spaces satisfying the

Brezzi-Babuska condition. For example, the elements of Raviart and Thomas [28] would be a good choice for $\mathbf{V}_h$ and W_h. Although our results are based on the use of Raviart-Thomas elements of any order k, their extension to other stable elements can be discussed without any difficulty.

Let us recall from [16] that the weak mixed formulation of (1.1) is given by finding $(u, \sigma) \in W \times \mathbf{V}$ such that

$$\begin{aligned}
&(u_t, w) - (\nabla \cdot \sigma, w) - (cu, w) = (f, w), && \forall w \in W,\\
&(\alpha\sigma, \mathbf{v}) + \int_0^t (M(t,s)\sigma(s), \mathbf{v})ds + (\nabla \cdot \mathbf{v}, u) = \langle g, \mathbf{v} \cdot \mathbf{n} \rangle, && \forall \mathbf{v} \in \mathbf{V},\\
&u(0,x) = u_0(x) \quad \text{in } L^2(\Omega),
\end{aligned} \tag{2.1}$$

where $\alpha = A^{-1}(t)$, $M(t,s) = R(t,s)A^{-1}(s)$ and $R(t,s)$ is the resolvent of the matrix $A^{-1}(t)B(t,s)$ and is given by

$$R(t,s) = A^{-1}(t)B(t,s) + \int_s^t A^{-1}(t)B(t,\tau)\ R(\tau,s)ds, \quad t > s \geq 0.$$

Here $\langle \cdot, \cdot \rangle$ indicates the L^2-inner product on $\partial\Omega$.

The corresponding semi-discrete version is to seek a pair $(u_h, \sigma_h) \in W_h \times \mathbf{V}_h$ such that

$$\begin{aligned}
&(u_{h,t}, w_h) - (\nabla \cdot \sigma_h, w_h) - (cu_h, w_h) = (f, w_h), && \forall w_h \in \mathbf{W}_h,\\
&(\alpha\sigma_h, \mathbf{v}_h) + \int_0^t (M(t,s)\sigma_h(s), \mathbf{v}_h)ds + (u_h, \nabla \cdot \mathbf{v}_h) = \langle g, \mathbf{n} \cdot \mathbf{v}_h \rangle, && \forall \mathbf{v}_h \in \mathbf{V}_h.
\end{aligned} \tag{2.2}$$

The discrete initial condition $u_h(0,x) = u_{0,h}$, where $u_{0,h} \in W_h$ is some appropriately chosen approximation of the initial data $u_0(x)$, should be added to (2.2) for starting. The pair (u_h, σ_h) is a semi-discrete approximation of the true solution of (1.1) in the finite element space $W_h \times \mathbf{V}_h$ [1, 16, 17, 29], where $\sigma_h(0,x)$ is chosen to satisfy the equation (2.2) with $t = 0$; namely, it is related to $u_{0,h}$ as follows:

$$(\alpha\sigma_h(0), \mathbf{v}_h) + (u_{0,h}, \nabla \cdot \mathbf{v}_h) = \langle g_0, \mathbf{n} \cdot \mathbf{v}_h \rangle, \tag{2.3}$$

where $g_0 = g(0,x)$ is the initial value of the boundary data.

In [16], utilizing the mixed Ritz projection we have obtained for the Raviart-Thomas element of the lowest order that

$$||u - u_h||_0^2 + ||\sigma - \sigma_h||_0^2 \leq Ch^2 \left[||u_0||_1^2 + ||\sigma_0||_1^2 + \int_0^t (||u(s)||_2^2 + ||u_t(s)||_2^2)ds \right].$$

Also, we can extend easily the result to the case of any order k (≥ 1) to get

$$||u - u_h||_0^2 + ||\sigma - \sigma_h||_0^2 \leq Ch^{2r} \left[||u_0||_r^2 + ||\sigma_0||_r^2 + \int_0^t (||u(s)||_{r+1}^2 + ||u_t(s)||_{r+1}^2)ds \right], \tag{2.4}$$

for $2 \leq r \leq k+1$. for In fact, we can improve the error estimate by extending the idea from [2, 3] to introduce a new nonlocal projection incorporated with the memory effects, which allows us to obtain a sharper error estimate in regularity than that indicated in (2.4). This new projection is a natural extension of the standard Ritz-Volterra projection in the standard finite element method to the case of the mixed finite element approximations with memory. We refer the readers to [2, 3] and [26] for the analysis and applications of the Ritz-Volttera projection for standard finite element apprroximations to parabolic and hyperbolic integro-differnetial eqautions.

Before the mixed Ritz-Volterra projection is given, we need the following Raviart-Thomas projection [28]:

$$\Pi_h \times P_h : \mathbf{V} \times W \to \mathbf{V}_h \times W_h,$$

which has the properties:

(i) P_h is the local $L^2(\Omega)$ projection;

(ii) Π_h and P_h satisfy

$$(\nabla \cdot (\sigma - \Pi_h \sigma), w_h) = 0,\ w_h \in W_h \quad \text{and} \quad (\nabla \cdot \mathbf{v}_h, u - P_h u) = 0,\ \mathbf{v}_h \in \mathbf{V}_h. \tag{2.5}$$

(iii) the following approximation properties hold:

$$\begin{array}{ll} ||\sigma - \Pi_h \sigma||_0 \le Ch^r ||\sigma||_r, & 1 \le r \le k+1, \\ ||\nabla \cdot (\sigma - \Pi_h \sigma)||_{-s} \le Ch^{r+s} ||\nabla \cdot \sigma||_r, & 0 \le r,\ s \le k+1, \\ ||u - P_h u||_{-s} \le Ch^{r+s} ||u||_r, & 0 \le r,\ s \le k+1. \end{array} \tag{2.6}$$

Definition 2.1. *For $(u, \sigma) \in W \times \mathbf{V}$ we define a pair $(\bar{u}_h, \bar{\sigma}_h) : [0, T] \to W_h \times \mathbf{V}_h$ such that*

$$\begin{array}{ll} \left(\alpha(\sigma - \bar{\sigma}_h) + \displaystyle\int_0^t M(t,s)(\sigma - \bar{\sigma}_h)(s) ds, \mathbf{v}_h \right) + (\nabla \cdot \mathbf{v}_h, u - \bar{u}_h) = 0, & \mathbf{v}_h \in \mathbf{V}_h, \\ (\nabla \cdot (\sigma - \bar{\sigma}_h), w_h) + (c(u - \bar{u}_h), w_h) = 0, & w_h \in W_h, \end{array} \tag{2.7}$$

where $\alpha = A^{-1}$. The pair $(\bar{u}_h, \bar{\sigma}_h)$ is called the mixed Ritz-Volterra projection of (u, σ).

Let

$$\xi := \sigma - \bar{\sigma}_h,\ \eta := u - \bar{u}_h,\ \nu := \Pi_h \sigma - \bar{\sigma}_h,\ \tau := P_h u - \bar{u}_h,\ \rho := u - P_h u.$$

Then (2.7) becomes

$$\begin{array}{ll} \left(\alpha\xi + \displaystyle\int_0^t M(t,s)\xi(s) ds, \mathbf{v}_h \right) + (\nabla \cdot \mathbf{v}_h, \eta) = 0, & \mathbf{v}_h \in \mathbf{V}_h, \\ (\nabla \cdot \xi, w_h) + (c\eta, w_h) = 0, & w_h \in W_h, \end{array} \tag{2.8}$$

or, according to (2.5)

$$\begin{array}{ll} (\alpha\xi, \mathbf{v}_h) + (\nabla \cdot \mathbf{v}_h, \tau) = f(\mathbf{v}_h), & \mathbf{v}_h \in \mathbf{V}_h, \\ (\nabla \cdot \xi, w_h) + (c\tau, w_h) = g(w_h), & w_h \in W_h, \end{array} \tag{2.9}$$

where

$$f(\mathbf{v}_h) := -\left(\int_0^t M(t,s)\xi(s) ds, \mathbf{v}_h \right) \quad \text{and} \quad g(w_h) := -(c\rho, w_h).$$

In order to analyse (ξ, η), let us recall from [10] the following results.

Lemma 2.1. *Let the index k of $\mathbf{V}_h \times W_h$ be at least one and let $0 \le s \le k-1$. Assume that Ω is $(s+2)$-regular [10]. Let $\xi \in \mathbf{V}$, $g \in W' = L^2(\Omega)$ and $f = \{\mathbf{f}_0, f_1\} \in \mathbf{V}'$ with $\mathbf{f}_0 \in (L^2(\Omega))^2$, $f_1 \in L^2(\Omega)$ and*

$$f(\mathbf{v}) = (\mathbf{f}_0, \mathbf{v}) + (f_1, \nabla \cdot \mathbf{v}), \quad \mathbf{v} \in \mathbf{V}.$$

If $z \in W_h$ satisfies the relations

$$\begin{aligned} (\alpha\xi, \mathbf{v}_h) + (\nabla \cdot \mathbf{v}_h, z) &= f(\mathbf{v}_h), \qquad && \mathbf{v}_h \in \mathbf{V}_h, \\ (\nabla \cdot \xi, w_h) + (cz, w_h) &= g(w_h), \qquad && w_h \in W_h, \end{aligned} \tag{2.10}$$

then there exists $h_0 > 0$ sufficiently small such that for all $0 < h \le h_0$,

$$\begin{aligned} ||z||_{-s} \ &\le C\,\{h^{s+1}||\xi||_0 + h^{s+2}||\nabla\cdot\xi||_0 + ||\mathbf{f}_0||_{-s-1} + h^{s+1}||\mathbf{f}_0||_0 \\ &\quad + ||f_1||_{-s} + h^s||f_1||_0 + ||g||_{-s-2} + h^{s+2}||g||_0\} \end{aligned}$$

Lemma 2.2. *Let the index k of $\mathbf{V}_h \times W_h$ be non-negative, and let Ω be $(k+2)$-regular [10]. Let $\xi \in \mathbf{V}$, $g \in W' = L^2(\Omega)$ and $f = \{\mathbf{f}_0, 0\} \in \mathbf{V}'$. If $z \in W_h$ satisfies (2.10), then there exists $h_0 > 0$ sufficiently small such that for all $0 < h \le h_0$,*

$$||z||_{-k} \le C\left\{h^{k+1}\left(||\xi||_0 + ||\nabla\cdot\xi||_0 + ||\mathbf{f}_0||_0 + ||g||_0\right) + ||\mathbf{f}_0||_{-k-1} + ||g||_{-k-2}\right\}.$$

Moreover, we also need the following lemma.

Lemma 2.3. *Assume that the matrix $A(t)$ is positive definite. Then, the norms $||\sigma||_0^2 := (\sigma,\sigma)$ and $||\sigma||_{A^{-1}}^2 := (A^{-1}\sigma,\sigma)$ are equivalent.*

Proof. It is clear that

$$\lambda_L^{-1}||\sigma||_0^2 \le (A^{-1}\sigma,\sigma) \le \lambda_M^{-1}||\sigma||_0^2,$$

where λ_M and λ_L are the smallest and largest eigenvalues of $A(t)$, respectively. □

We are now ready to state and prove our main result in this section.

Theorem 2.1. *For $(u,\sigma) \in W \times \mathbf{V}$ its mixed Ritz-Volterra projection $(\bar{u}_h, \bar{\sigma}_h)$ defined by (2.7) exists and is unique. Moreover, there is a positive constant $C > 0$, independent of $h > 0$ small, such that the error $(u - \bar{u}_h, \sigma - \bar{\sigma}_h)$ can be estimated by*

$$\begin{aligned} &||u - \bar{u}_h||_0 \le C \begin{cases} h|||u(t)|||_2, & \text{if } k = 0, \\ h^r|||u(t)|||_r, & \text{if } k \ge 1 \text{ and } 2 \le r \le k+1, \end{cases} \\ &||\sigma - \bar{\sigma}_h||_0 \le Ch^r|||u(t)|||_{r+1}, \qquad \text{if } 1 \le r \le k+1, \\ &||\nabla\cdot(\sigma - \bar{\sigma}_h)||_0 \le Ch^r|||u(t)|||_{r+2}, \quad \text{if } 0 \le r \le k+1, \end{aligned}$$

where

$$|||u(t)|||_r = ||u(t)||_r + \int_0^t ||u(s)||_r ds, \quad r \in R,\ t \ge 0.$$

Proof. We first prove the existence and uniqueness of the mixed Ritz-Volterra projection. If $M = 0$, then it follows from [1] that $(\bar{u}_h, \bar{\sigma}_h)$ exists uniquely. If M is non-zero, we see that (2.7) in fact can be written as a Volterra system for $(\bar{u}_h, \bar{\sigma}_h)$, i.e.,

$$A_h \begin{pmatrix} \bar{u}_h \\ \bar{\sigma}_h \end{pmatrix} = F_h + \int_0^t B_h(t,s) \begin{pmatrix} \bar{u}_h \\ \bar{\sigma}_h \end{pmatrix} ds,$$

where A_h and B_h are matrices with A_h non-singular and F_h is a vector associated with the solution (u,σ). Hence, the theory of Volterra equations implies that $(\bar{u}_h, \bar{\sigma}_h)$ exists uniquely.

Next we turn our attention to error estimates. It follows from (2.6) and (2.9) that

$$\begin{aligned} &||f||_0 \le C\int_0^t ||\xi||_0 ds, && ||f||_{-1} \le C\int_0^t ||\xi||_{-1} ds,\\ &||g||_0 \le C||\rho||_0, && ||g||_{-1} \le C||\rho||_{-1},\\ &||g||_{-2} \le ||g||_{-1} \le C||\rho||_{-1}, && ||\rho||_{-1} + h||\rho||_0 \le Ch^{r+1}||u||_r. \end{aligned}$$

Now, apply either Lemma 2.1 with $s=0$ or Lemma 2.2 with $k=0$ to (2.9). Then, for h small and for Ω 2-regular we have for $0 \le r \le k+1$ that

$$\begin{aligned} ||\tau||_0 &\le C\left\{h||\xi||_0 + h^{2-\delta_{k0}}||\nabla\cdot\xi||_0 + ||f||_{-1} + h||f||_0 + ||g||_{-2} + h||g||_0\right\}\\ &\le C\left\{h||\xi||_0 + h^{2-\delta_{k0}}||\nabla\cdot\xi||_0 + \int_0^t (||\xi||_{-1} + h||\xi||_0)ds + (||\rho||_{-1} + h||\rho||_0)\right\}\\ &\le C\left\{h|||\xi|||_0 + h^{2-\delta_{k0}}||\nabla\cdot\xi||_0 + \int_0^t ||\xi||_{-1} ds + h^{r+1}||u||_r\right\}, \end{aligned} \tag{2.11}$$

where

$$\delta_{k0} = \begin{cases} 1, & k=0,\\ 0, & k\neq 0. \end{cases}$$

Letting $\varphi \in \left(H^1(\Omega)\right)^2$, then we derive from (2.5) and (2.8) that

$$\begin{aligned} &\left(\alpha\xi + \int_0^t M(t,s)\xi(s)ds, \varphi\right) + (\nabla\cdot\varphi, \eta)\\ &\quad= \left(\alpha\xi + \int_0^t M(t,s)\xi(s)ds, \varphi - \Pi_h\varphi\right) + (\nabla\cdot(\varphi-\Pi_h\varphi), \eta)\\ &\quad+ \left(\alpha\xi + \int_0^t M(t,s)\xi(s)ds, \Pi_h\varphi\right) + (\nabla\cdot\Pi_h\varphi, \eta)\\ &\quad= \left(\alpha\xi + \int_0^t M(t,s)\xi(s)ds, \varphi - \Pi_h\varphi\right) + (\nabla\cdot(\varphi-\Pi_h\varphi), u), \end{aligned}$$

or

$$\begin{aligned} (\alpha\xi, \varphi) &= -\int_0^t (M(t,s)\xi(s), \varphi)ds - (\nabla\cdot\varphi, \eta)\\ &+ \left(\alpha\xi + \int_0^t M(t,s)\xi(s)ds, \varphi - \Pi_h\varphi\right) + (\nabla\cdot(\varphi-\Pi_h\varphi), u) \end{aligned}$$

which, together with (2.6), indicates that

$$\begin{aligned} |(\alpha\xi,\varphi)| &\le C\int_0^t ||\xi(s)||_{-1}ds||\varphi||_1 + ||\eta||_0||\varphi||_1\\ &+Ch|||\xi|||_0||\varphi||_1 + Ch||u||_1||\nabla\cdot(\varphi-\Pi_h\varphi)||_{-1}\\ &\le C\left(\int_0^t ||\varphi||_{-1}ds + ||\eta||_0 + Ch|||\xi|||_0 + Ch||u||_1\right)||\varphi||_1; \end{aligned}$$

that is,

$$||\xi||_{-1} \le C\left\{\int_0^t ||\xi(s)||_{-1}ds + ||\eta||_0 + Ch(|||\xi|||_0 + ||u||_1)\right\}.$$

This, together with Gronwall's lemma, implies that

$$||\xi||_{-1} \le C\left\{||\eta||_0 + Ch(|||\xi|||_0 + ||u||_1)\right\}. \tag{2.12}$$

Substitute (2.12) into (2.11) to obtain

$$||\tau||_0 \le C\left\{\int_0^t ||\eta(s)||_0 ds + h|||\xi|||_0 + h^{2-\delta_{k0}}||\nabla\cdot\xi||_0 + h^{r+1}||u||_r\right\}. \tag{2.13}$$

Therefore, for $0 \le r \le k+1$ we have

$$\begin{aligned} ||\eta||_0 &\le ||\rho||_0 + ||\tau||_0 \\ &\le C\left\{\int_0^t ||\eta(s)||_0 ds + h|||\xi|||_0 + h^{2-\delta_{k0}}|||\nabla\cdot\xi|||_0 + h^r||u||_r\right\}, \end{aligned}$$

and applying Gronwall's lemma leads to

$$||\eta||_0 \le C\left\{h|||\xi|||_0 + h^{2-\delta_{k0}}||\nabla\cdot\xi||_0 + h^r||u||_r\right\}. \tag{2.14}$$

Since, by (2.5), $(\nabla\cdot\nu, w_h) = (\nabla\cdot\xi, w_h)$ for $w_h \in W_h$, it follows from (2.8) and the choice $w_h = \nabla\cdot\nu \in W_h$ that

$$(\nabla\cdot\nu, \nabla\cdot\nu) = (\nabla\cdot\xi, \nabla\cdot\nu) = -(c\eta, \nabla\cdot\nu)$$

or

$$||\nabla\cdot\nu||_0 \le C||\eta||_0, \tag{2.15}$$

so that

$$||\nabla\cdot\xi||_0 \le ||\nabla\cdot\nu||_0 + ||\nabla\cdot(\sigma-\Pi_h\sigma)||_0 \le C(||\eta||_0 + h^q||\nabla\cdot\sigma||_q),\ 0 \le q \le k+1. \tag{2.16}$$

Also, according to (2.8) ν satisfies

$$\begin{aligned} &\left(\alpha\nu + \int_0^t M(t,s)\nu(s)ds, \nu\right) \\ &= \left(\alpha\xi + \int_0^t M(t,s)\xi(s)ds, \nu\right) + \left(\alpha(\Pi_h\sigma - \sigma) + \int_0^t M(t,s)(\Pi_h\sigma-\sigma)(s)ds, \nu\right) \\ &= -(\nabla\cdot\nu, \eta) + \left(\alpha(\Pi_h\sigma-\sigma) + \int_0^t M(t,s)(\Pi_h\sigma-\sigma)(s)ds, \nu\right) \\ &\le ||\nabla\cdot\nu||_0^2 + ||\eta||_0^2 + C|||\Pi_h\sigma - \sigma|||_0||\nu||_0. \end{aligned}$$

Then, we find from Lemma 2.3, (2.15) and the ϵ-type inequality that

$$||\nu||_0^2 - C\int_0^t ||\nu(s)||_0^2 ds \le C(||\eta||_0 + |||\Pi_h\sigma - \sigma|||_0)$$

which, together with Gronwall's lemma and (2.6), implies

$$||\nu||_0 \le C(||\eta||_0 + |||\Pi_h\sigma-\sigma|||_0) \le C(||\eta||_0 + h^m|||\sigma|||_m),\ 1 \le m \le k+1, \tag{2.17}$$

and

$$||\xi||_0 \le ||\nu||_0 + ||\Pi_h\sigma - \sigma||_0 \le C(||\eta||_0 + h^m|||\sigma|||_m),\ 1 \le m \le k+1. \tag{2.18}$$

If (2.16) and (2.18) are substituted into (2.14), then for $0 \le r \le k+1$, $0 \le q \le k+1$, and $1 \le m \le k+1$ it follows that

$$||\eta||_0 \le C\left\{h|||\eta|||_0 + h^r||u||_r + h^{m+1}|||\sigma|||_m + h^{2-\delta_{k0}+q}||\nabla\cdot\sigma||_q\right\}.$$

Thus, for small h we obtain via Gronwall's inequality that

$$\begin{aligned} ||\eta||_0 &\le C\left\{h^r||u||_r + h^{m+1}|||\sigma|||_m + h^{2-\delta_{k0}+q}||\nabla\cdot\sigma||_q\right\}, \\ &\quad 0 \le r,\ q \le k+1,\ 1 \le m \le k+1. \end{aligned}$$

Choose $r = m + 1 = 2 + q - \delta_{k0}$ to gain that

$$||\eta||_0 = \begin{cases} Ch|||u|||_2 & \text{if } k = 0, \\ Ch^r|||u|||_r & \text{if } k \geq 1 \text{ and } 2 \leq r \leq k+1, \end{cases}$$

since $||\sigma||_{r-1} + ||\nabla \cdot \sigma||_{r-2} \leq C||u||_r$.

It then follows immediately that

$$\begin{aligned} ||\xi||_0 &\leq Ch^r|||u|||_{r+1}, && 1 \leq r \leq k+1, \\ ||\nabla \cdot \xi||_0 &\leq Ch^r|||u|||_{r+2}, && 0 \leq r \leq k+1. \end{aligned}$$

Therefore, the proofs of Theorem 2.1 are complete. □

Theorem 2.2. *Let $(\bar{u}_h, \bar{\sigma}_h)$ be the mixed Ritz-Volterra projection of $(u, \sigma) \in W \times \mathbf{V}$ defined by (2.7). Then, there is a positive constant $C > 0$, independent of $h > 0$ small, such that the error $(u - \bar{u}_h, \sigma - \bar{\sigma}_h)$ can be estimated for any positive integer m by*

$$\begin{aligned} ||D_t^m(u - \bar{u}_h)||_0 &\leq C \begin{cases} h|||u(t)|||_{2,m}, & \text{if } k = 0, \\ h^r|||u(t)|||_{r,m}, & \text{if } k \geq 1 \text{ and } 2 \leq r \leq k+1, \end{cases} \\ ||D_t^m(\sigma - \bar{\sigma}_h)||_0 &\leq Ch^r|||u(t)|||_{r+1,m}, \quad \text{if } 1 \leq r \leq k+1, \\ ||D_t^m(\nabla \cdot (\sigma - \bar{\sigma}_h))||_0 &\leq Ch^r|||u(t)|||_{r+2,m}, \quad \text{if } 0 \leq r \leq k+1, \end{aligned}$$

where

$$|||u(t)|||_{r,m} = \sum_{j=0}^{m} ||D_t^j u(t)||_r + \int_0^t \sum_{j=0}^{m} ||D_t^j u(s)||_r ds, \quad r \in R, \ t \geq 0.$$

Proof. Differentiate (2.7), and then the result for $m = 1$ follows from the same arguments as those for Theorem 2.1.

The proof is completed by treating $m \geq 2$ inductively, using the further differentiation of (2.7). □

Corollary 2.1. *Let $(\bar{u}_h, \bar{\sigma}_h)$ be the mixed Ritz-Volterra projection of $(u, \sigma) \in W \times \mathbf{V}$ defined by (2.7). Then,*

$$||u - \bar{u}_h||_\infty \leq Ch^r\left(||u||_{r,\infty} + |||u|||_{r+1}\right), \quad k \geq 1 \text{ and } 1 \leq r \leq k.$$

Proof. we see easily from (2.13) and Theorem 2.1 that

$$||\tau||_0 \leq Ch^{r+1}|||u|||_{r+1} \quad \text{for } k \geq 1 \quad \text{and} \quad 1 \leq r \leq k,$$

and by the inverse inequality that

$$||\tau||_\infty \leq Ch^{-1}||\tau||_0 \leq Ch^r|||u|||_{r+1}.$$

Thus, we have for $k \geq 1$ and $1 \leq r \leq k$ that

$$\begin{aligned} ||u - \bar{u}_h||_\infty &\leq ||u - P_h u||_\infty + ||\tau||_\infty \\ &\leq Ch^r(||u||_{r,\infty} + |||u|||_{r+1}). \end{aligned}$$

□

Remark 2.1 For $k = 0$ we have no order estimate for the quantity $||u - \bar{u}_h||_\infty$. However, using the superconvergence analysis we know from Corollary 5.1 that for the rectangular Raviart-Thomas elements of the lowest order, there holds

$$||u - u_h||_\infty \le Ch,$$

where (u, σ) and (u_h, σ_h) are the solutions of (2.1) and (2.2), respectively.

Theorem 2.3. *Assume that $(\bar{u}_h, \bar{\sigma}_h)$ is the mixed Ritz-Volterra projection of $(u, \sigma) \in W \times \mathbf{V}$ defined by (2.7). Then, there is a positive constant $C_m > 0$, independent of $h > 0$ small, such that for $m \ge 0$*

$$||D_t^m \bar{u}_h||_W + ||D_t^m \bar{\sigma}_h||_{\mathbf{V}} \le C_m \left\{ \sum_{j=0}^{m} (||D_t^j \sigma||_{\mathbf{V}} + ||D_t^j u||_W) + \int_0^t (||\sigma||_{\mathbf{V}} + ||u||_W) ds \right\}. \tag{2.19}$$

Proof. Rewrite (2.7) as

$$\begin{aligned} (\alpha \bar{\sigma}_h, \mathbf{v}_h) + (\nabla \cdot \mathbf{v}_h, \bar{u}_h) &= F(\mathbf{v}_h), \qquad \mathbf{v}_h \in \mathbf{V}_h, \\ (\nabla \cdot \bar{\sigma}_h, w_h) + (c\bar{u}_h, w_h) &= G(w_h), \qquad w_h \in W_h, \end{aligned}$$

where

$$\begin{aligned} F(\mathbf{v}_h) &= \left(\alpha\sigma + \int_0^t M(t,s)(\sigma - \bar{\sigma}_h)(s) ds, \mathbf{v}_h \right) + (\nabla \cdot \mathbf{v}_h, u), \\ G(w_h) &= (\nabla \cdot \sigma, w_h) + (cu, w_h). \end{aligned}$$

$F(\mathbf{v}_h)$ and $G(w_h)$ can be considered as linear functionals of $\mathbf{v}_h$ and w_h defined on $\mathbf{V}_h$ and W_h, respectively. Thus, we have from the stability result of [1] that

$$\begin{aligned} ||\bar{\sigma}_h||_{\mathbf{V}} + ||\bar{u}_h||_W &\le C \left\{ \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{|F(\mathbf{v}_h)|}{||\mathbf{v}_h||_{\mathbf{V}}} + \sup_{w_h \in W_h} \frac{|G(w_h)|}{||w_h||_W} \right\} \\ &\le C \left\{ ||\sigma||_{\mathbf{V}} + \int_0^t ||\sigma||_V ds + ||u||_W + \int_0^t ||\bar{\sigma}_h||_{\mathbf{V}} ds \right\}, \end{aligned}$$

or, by Gronwall's inequality,

$$||\bar{\sigma}_h||_{\mathbf{V}} + ||\bar{u}_h||_W \le C \left\{ ||\sigma||_{\mathbf{V}} + \int_0^t ||\sigma||_{\mathbf{V}} ds + ||u||_W \right\},$$

which demonstrates that (2.19) is true for $m = 0$.

We can also prove (2.19) for $m \ge 1$ by differentiating (2.7) with respect to time t and repeating the same arguments above with mathematical induction. □

Remark 2.2 This stability result (2.19) is needed in the analysis of the backward Euler time-discretization scheme. See [17] for the details.

3. Sharp L^2 error estimates

In this section, we shall show a sharper L^2 error estimate than that one indicated in (2.4) for the time-continuous approximation scheme (2.2), where the regularity requirement is one order lower than that in (2.4), by means of the mixed Ritz-Volterra type projection instead of the mixed Ritz projection used in [16] to obtain (2.4). Here, let us consider the Raviart-Thomas elements of higher order $k \ge 1$ (see [16] for the lowest-order case).

Theorem 3.1. *Assume that (u,σ) and (u_h,σ_h) are the solutions of (2.1) and (2.2), respectively, $||P_h u_0 - u_h(0)|| \le Ch^r||u_0||_r$ and $||\Pi_h\sigma(0) - \sigma_h(0)|| \le Ch^r||u_0||_{r+1}$. Then, we have for $k \ge 1$ that*

$$\begin{aligned} ||u(t)-u_h(t)||_0^2 &\le Ch^{2r}\left\{||u_0||_r^2 + \int_0^t [||u(s)||_r^2 + ||u_t(s)||_r^2]ds\right\}, \qquad 2 \le r \le k+1 \\ ||\sigma(t)-\sigma_h(t)||_0^2 &\le Ch^{2r}\left\{||u_0||_{r+1}^2 + \int_0^t [||u(s)||_{r+1}^2 + ||u_t(s)||_{r+1}^2]ds\right\}, \quad 1 \le r \le k+1. \end{aligned}$$

Proof. Let $(\bar{u}_h,\bar{\sigma}_h)$ be the mixed Ritz-Volttera projection of (u,σ) defined by (2.7), and we rewrite the errors as:

$$\begin{aligned} u - u_h &= (u-\bar{u}_h) + (\bar{u}_h - u_h) := \rho + \rho_h, \\ \sigma - \sigma_h &= (\sigma - \bar{\sigma}_h) + (\bar{\sigma}_h - \sigma_h) := \theta + \theta_h. \end{aligned}$$

Then, we know from Theorems 2.1 and 2.2 that

$$\begin{aligned} &||\rho||_0 \le Ch^r|||u(t)|||_r, && k \ge 1 \quad \text{and} \quad 2 \le r \le k+1, \\ &||\rho_t||_0 \le Ch^r\left(|||u(t)|||_r + |||u_t(t)|||_r\right), && k \ge 1 \quad \text{and} \quad 2 \le r \le k+1, \end{aligned} \tag{3.1}$$

and

$$||\theta(t)||_0 \le Ch^r|||u|||_{r+1}, \quad 1 \le r \le k+1. \tag{3.2}$$

Thus, only $||\rho_h||_0$ and $||\theta_h||_0$ need to be estimated.

It follows from (2.1)-(2.2) and (2.7) that (ρ_h,θ_h) satisfies

$$\begin{aligned} &\left(\alpha\theta_h + \int_0^t M(t,s)\theta_h(s)ds, \mathbf{v_h}\right) + (\nabla\cdot\mathbf{v}_h,\rho_h) = 0, && \mathbf{v}_h \in \mathbf{V}_h, \\ &(\rho_{h,t}, w_h) - (\nabla\cdot\theta_h, w_h) - (c\rho_h, w_h) = -(\rho_t, w_h), && w_h \in W_h. \end{aligned} \tag{3.3}$$

Therefore, setting $w_h = \rho_h$ and $\mathbf{v}_h = \theta_h$ in (3.3) we obtain from their sum that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||\rho_h||_0^2 - (c\rho_h,\rho_h) + ||\theta_h||_{A^{-1}}^2 &= -\left(\int_0^t M(t,s)\theta_h(s)ds, \theta_h\right) - (\rho_t,\rho_h) \\ &\le C\int_0^t ||\theta_h(s)||_0 ds||\theta_h||_0 + ||\rho_t||_0||\rho_h||_0, \end{aligned}$$

and by means of Lemma 2.3 that

$$\frac{1}{2}\frac{d}{dt}||\rho_h||_0^2 + ||\theta_h||_{A^{-1}}^2 \le C\left(||\rho_h||_0^2 + \int_0^t ||\theta_h||_{A^{-1}}^2 ds\right) + \frac{1}{2}\left(||\theta_h||_{A^{-1}}^2 + ||\rho_t||_0^2\right).$$

Integrating from 0 to t leads to

$$||\rho_h||_0^2 + \int_0^t ||\theta_h||_{A^{-1}}^2 ds \le ||\rho_h(0)||_0^2 + \int_0^t \left[||\rho_h||_0^2 + \int_0^s ||\theta_h(s)||_{A^{-1}}^2 ds\right] + \int_0^t ||\rho_t||_0^2 ds$$

which, together with Gronwall's lemma, implies

$$||\rho_h||_0^2 + \int_0^t ||\theta_h(s)||_{A^{-1}}^2 ds \le C\left\{||\rho_h(0)||_0^2 + \int_0^t ||\rho_t||_0^2 ds\right\}. \tag{3.4}$$

It follows from (2.6), Theorem 2.1 and our initial approximation assumption that

$$\begin{aligned} ||\rho_h(0)||_0^2 &= ||\bar{u}_h(0) - u_h(0)||_0^2 \leq ||\bar{u}_h(0) - u_0||_0^2 \\ &+ ||u_0 - P_h u_0||_0^2 + ||P_h u_0 - u_h(0)||_0^2 \\ &\leq Ch^{2r}||u_0||_r^2. \end{aligned} \tag{3.5}$$

Combining (3.1) and (3.5) with (3.4) we gain

$$||\rho_h||_0^2 \leq Ch^{2r} \left\{ ||u_0||_r^2 + \int_0^t [||u(s)||_r^2 + ||u_t(s)||_r^2] ds \right\}. \tag{3.6}$$

In order to get the estimate for $\theta_h(t)$, we first differentiate (3.3) to obtain

$$\left(\alpha_t \theta_h + \alpha \theta_{h,t} + M(t,t)\theta_h + \int_0^t M_t(t,s)\theta_h(s) ds, \mathbf{v}_h \right) + (\nabla \cdot \mathbf{v}_h, \rho_{h,t}) = 0, \ \mathbf{v}_h \in \mathbf{V}_h,$$

and then by setting $\mathbf{v}_h = \theta_h$ in the above equation and $w_h = \rho_{h,t}$ in (3.3) we have that

$$\begin{aligned} ||\rho_{h,t}||_0^2 + (\alpha\theta_{h,t}, \theta_h) + (\alpha_t \theta_h, \theta_h) &= - \left(M(t,t)\theta_h + \int_0^t M_t(t,s)\theta_h(s) ds, \theta_h \right) \\ &+ (c\rho_h, \rho_{h,t}) - (\rho_t, \rho_{h,t}). \end{aligned} \tag{3.7}$$

Since

$$\alpha(\theta_h^2)_t = (\alpha\theta_h^2)_t - \alpha_t \theta_h^2,$$

then

$$\begin{aligned} (\alpha\theta_{h,t}, \theta_h) &= \int_\Omega \alpha \theta_{h,t} \theta_h = \frac{1}{2} \int_\Omega \alpha \frac{d}{dt}(\theta_h^2) \\ &= \frac{1}{2} \int_\Omega \frac{d}{dt}(\alpha\theta_h^2) - \frac{1}{2} \int_\Omega \alpha_t \theta_h^2 \\ &= \frac{1}{2} \frac{d}{dt} ||\theta_h||_{A^{-1}}^2 - \frac{1}{2}(\alpha_t \theta_h, \theta_h). \end{aligned}$$

Hence, (3.7) can be rewritten as

$$\begin{aligned} ||\rho_{h,t}||_0^2 + \frac{1}{2}\frac{d}{dt}||\theta_h||_{A^{-1}}^2 + \frac{1}{2}(\alpha_t \theta_h, \theta_h) &= - \left(M(t,t)\theta_h + \int_0^t M_t(t,s)\theta_h(s) ds, \theta_h \right) \\ &+ (c\rho_h, \rho_{h,t}) - (\rho_t, \rho_{h,t}). \end{aligned}$$

Thus, from the ϵ-inequality we derive that

$$||\rho_{h,t}||_0^2 + \frac{d}{dt}||\theta_h||_{A^{-1}}^2 \leq C \left\{ ||\theta_h||_0^2 + \int_0^t ||\theta_h(s)||_0^2 ds + ||\rho_h||_0^2 + ||\rho_t||_0^2 \right\},$$

and then via integrating from 0 to t, Lemma 2.3 and Gronwall's lemma that

$$||\theta_h||_0^2 \leq C \left\{ ||\theta_h(0)||_0^2 + \int_0^t [||\rho_h(s)||_0^2 + ||\rho_t(s)||_0^2] \right\}. \tag{3.8}$$

It follows from (2.6), Theorem 2.1 and our initial approximation assumption that

$$\begin{aligned} ||\theta_h(0)||_0^2 &= ||\bar{\sigma}_h(0) - \sigma_h(0)||_0^2 \leq ||\bar{\sigma}_h(0) - \sigma(0)||_0^2 \\ &+ ||\sigma(0) - \Pi_h \sigma(0)||_0^2 + ||\Pi_h \sigma(0) - \sigma_h(0)||_0^2 \\ &\leq Ch^{2r}||u_0||_{r+1}^2. \end{aligned} \tag{3.9}$$

If (3.1), (3.6) and (3.9) are substituted into (3.8), then we can obtain

$$||\theta_h||_0^2 \leq Ch^{2r}\left\{||u_0||_{r+1}^2 + \int_0^t [||u(s)||_r^2 + ||u_t(s)||_r^2]ds\right\}.$$

Then, the proofs of Theorem 3.1 are complete via the triangle inequality. □

Remark 3.1 The assumption in the above theorem $||P_h u_0 - u_h(0)||_0 \leq Ch^r||u_0||_r$ and $||\Pi_h\sigma(0) - \sigma_h(0)||_0 \leq Ch^r||u_0||_{r+1}$ is available. In fact, from (2.1) and (2.3) we know that

$$(\alpha(0)(\sigma - \sigma_h)(0), \mathbf{v}_h) + ((u - u_h)(0), \nabla \cdot \mathbf{v}_h) = 0, \quad \mathbf{v}_h \in \mathbf{V}_h. \tag{3.10}$$

When we choose $u_h(0) = P_h u_0$, (3.10) becomes

$$(\alpha(0)(\sigma - \sigma_h)(0), \mathbf{v}_h) = 0, \quad \mathbf{v}_h \in \mathbf{V}_h,$$

since $(u_0 - P_h u_0, \nabla \cdot \mathbf{v}_h) = 0$ according to (2.5). Thus, we have by virtue of (2.6) that

$$(\sigma(0)(\sigma_h(0) - \Pi_h\sigma(0)), \mathbf{v}_h) = (\alpha(0)(\sigma(0) - \Pi_h\sigma(0)), \mathbf{v}_h) \leq Ch^r||u_0||_{r+1}||\mathbf{v}_h||_0$$

which, together with Lemma 2.3, indicates that

$$||\sigma_h(0) - \Pi_h\sigma(0)||_0 \leq Ch^r||u_0||_{r+1}.$$

4. Local L^2 superconvergence on rectangular elements

In the last decade considerable attention has been given to the analysis of superconvergence of mixed finite element approximations to elliptic ([11, 15, 33, 34]) and parabolic ([4, 5]) problems under various norms associated with the Gauss lines for the gradient and the Gauss points for the solution itself. In this section, we will extend these superconvergence results in mixed finite element approximations to our problem of parabolic integro-differential equations.

Following [15] we assume that $\Omega \subset R^2$ is rectangle and define semi-norms on $\mathbf{V}$ and W as follows. Letting $e = [a, b] \times [c, d] \in T_h$, we denote by $(g_1, g_1, \cdots, g_{k+1})$ the Gauss points in $[a, b]$ and $(\hat{g}_1, \hat{g}_2, \cdots, \hat{g}_{k+1})$ the Gauss points in $[c, d]$, and define

$$|||v_1|||_{1,e}^2 := \sum_{j=1}^{k+1} A_j \frac{d-c}{2}\int_a^b |v_1(s, \hat{g}_i)|^2 ds,$$

$$|||v_1|||_{2,e}^2 := \sum_{j=1}^{k+1} A_j \frac{b-a}{2}\int_c^d |v_2(s, g_i)|^2 ds,$$

where $A_j > 0,\ j = 1, 2, \cdots, k+1$, are the coefficients of the Gauss quadrature rule in $[-1, 1]$. Thus, for $\mathbf{v} = (v_1, v_2) \in \mathbf{V}$ and $w \in W$, we define

$$|||\mathbf{v}|||_*^2 := |||v_1|||_1^2 + |||v_2|||_2^2,\ \ |||v_i|||_i^2 := \sum_{e \in T_h} |||v_i|||_{i,e}^2,\ \ i = 1, 2,$$

$$|||w|||_*^2 := \frac{1}{4}\sum_{e \in T_h}\sum_{i,j=1}^{k+1} A_i A_j \ \text{area}(e)|w(g_i, \hat{g}_j)|^2.$$

Clearly, these two semi-norms are equal to the L^2-norm of functions from $\mathbf{V}_h$ and W_h, respectively [11, 15], where $\mathbf{V}_h \times W_h$ is the Raviart-Thomas finite element space of index k (≥ 0). Moreover, let u^I represent the interpolation function of u of degree

k with respect to x and y, respectively, on each element associated with the $(k+1)^2$ Gauss points. First of all, we need the following lemmas.

Lemma 4.1. *Assume that* $\sigma \in \left(H^{k+2}(\Omega)\right)^2 \cap \mathbf{V}$, $u \in H^{k+2}(\Omega)$, *and* u^I *is the interpolation function of* u *defined by* $(k+1)^2$ *Gauss points. Then, we have for some constant* $C > 0$ *that*

$$|||\sigma - \Pi_h \sigma|||_* \leq Ch^{k+2}||\sigma||_{k+2},$$
$$||P_h u - u^I||_0 \leq Ch^{k+2}||u||_{k+2}.$$

Proof. The proof can be found in [11, 15]. □

Lemma 4.2. *Assume that* $\sigma \in \left(H^{k+2}(\Omega)\right)^2 \cap \mathbf{V}$, $u \in H^{k+1}(\Omega)$, c *and* β *are two* $W^{1,\infty}(\Omega)$ *functions. Then we have for some constant* $C > 0$ *that*

$$|(c(P_h u - u), w_h)| \leq Ch^{k+2}||u||_{k+1}||w_h||_0, \qquad w_h \in W_h,$$
$$|(\beta(\Pi_h \sigma - \sigma), \mathbf{v}_h)| \leq Ch^{k+2}||\sigma||_{k+2}||\mathbf{v}_h||_0, \qquad \mathbf{v}_h \in \mathbf{V}_h.$$

Proof. Let $\hat{c} := \int_\Omega c/|\Omega| dx$, where $|\Omega|$ is the measure of Ω. Then,

$$|c(x,t) - \hat{c}(x,t)| \leq Ch||c||_{1,\infty}$$

which, together with the definition of the L^2-projection operator P_h, yields

$$\begin{aligned} |(c(P_h u - u), w_h)| &= |((c - \hat{c})(P_h u - u), w_h)| \\ &\leq Ch||P_h u - u||_0||w_h||_0 \\ &\leq Ch^{k+2}||u||_{k+1}||w_h||_0. \end{aligned}$$

Thus, we obtain the first estimate in Lemma 4.2.

The proof for the second estimate is referred to [11]. □

Theorem 4.1. *Let* $(\bar{u}_h, \bar{\sigma}_h)$ *be the mixed Ritz-Volterra projection of* (u, σ) *defined by (2.7). Then, there exists a positive constant* $C > 0$, *independent of* h, *such that for any* $0 \leq t \leq T$,

$$|||u - \bar{u}_h|||_* + |||\sigma - \bar{\sigma}_h|||_* \leq Ch^{k+2}\left(||u||_{k+2} + ||\sigma||_{k+2} + \int_0^t ||\sigma||_{k+2} ds\right).$$

Proof. We first observe by the equality of the norms $||| \cdot |||_*$ and $|| \cdot ||_0$ for the functions in the finite element spaces W_h and $\mathbf{V}_h$ that

$$\begin{aligned} |||u - \bar{u}_h|||_* &\leq ||||u - P_h u|||_* + ||P_h u - \bar{u}_h||_0, \\ |||\sigma - \bar{\sigma}_h|||_* &\leq |||\sigma - \Pi_h \sigma|||_* + ||\Pi_h \sigma - \bar{\sigma}_h||_0. \end{aligned}$$

Since $u - u^I = 0$ at the $(k+1)^2$ Gauss points in each element e, we have according to Lemma 4.1 that

$$|||P_h u - u|||_* = |||P_h u - u^I|||_* = ||P_h u - u^I||_0 \leq Ch^{k+2}||u||_{k+2}.$$

In addition, from Lemma 4.1 we also know

$$|||\sigma - \Pi_h \sigma|||_* \leq Ch^{k+2}||\sigma||_{k+2}.$$

Hence, it is sufficient to bound $||P_h u - \bar{u}_h||_0$ and $||\Pi_h \sigma - \bar{\sigma}_h||_0$ to complete the proof of Theorem 4.1.

Let $\xi := \Pi_h\sigma - \bar{\sigma}_h$ and $\tau := P_h u - \bar{u}_h$. Then, we see from (2.5) and (2.7) that

$$\begin{aligned} &(\alpha\xi, \mathbf{v}_h) + (\nabla\cdot\mathbf{v}_h, \tau) = F_0(\mathbf{v}_h) + F_1(\mathbf{v}_h), && \mathbf{v}_h \in \mathbf{V}_h, \\ &(\nabla\cdot\xi, w_h) + (c\tau, w_h) = G_0(w_h), && w_h \in W_h, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} F_0(\mathbf{v}_h) &= -\left(\alpha(\sigma - \Pi_h\sigma) + \int_0^t M(t,s)(\sigma - \Pi_h\sigma)(s)ds, \mathbf{v}_h\right), && \mathbf{v}_h \in \mathbf{V}_h, \\ F_1(\mathbf{v}_h) &= -\left(\int_0^t M(t,s)\xi(s)ds, \mathbf{v}_h\right), && \mathbf{v}_h \in \mathbf{V}_h, \\ G_0(w_h) &= -(c(u - P_h u), w_h), && w_h \in W_h. \end{aligned}$$

Since the terms F_0, F_1 and G_0 can be regarded as linear functionals of $\mathbf{v}_h$ and w_h defined on $\mathbf{V}_h$ and W_h, respectively, and we then know from the stability result of [1] that for any fixed time $0 \le t \le T$

$$||\xi||_{\mathbf{V}} + ||\tau||_W \le C\left\{\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{|F_0(\mathbf{v}_h) + F_1(\mathbf{v}_h)|}{||\mathbf{v}_h||_{\mathbf{V}}} + \sup_{w_h\in W_h}\frac{|G_0(w_h)|}{||w_h||_W}\right\}. \tag{4.2}$$

Let

$$F_0(t) = \sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{|F_0(\mathbf{v}_h)|}{||\mathbf{v}_h||_{\mathbf{V}}} \quad \text{and} \quad G_0(t) = \sup_{w_h\in W_h}\frac{|G_0(w_h)|}{||w_h||_W},$$

and notice that

$$\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{|F_1(\mathbf{v}_h)|}{||\mathbf{v}_h||_{\mathbf{V}}} = \sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{\left|\left(\int_0^t M(t,s)\xi(s)ds, \mathbf{v}_h\right)\right|}{||\mathbf{v}_h||_{\mathbf{V}}} \le C\int_0^t ||\xi(s)||_{\mathbf{V}}ds.$$

Therefore, we find from (4.2) that

$$||\xi||_{\mathbf{V}} + ||\tau||_W \le C\left(F_0(t) + G_0(t) + C\int_0^t ||\xi(s)||_{\mathbf{V}}ds\right),$$

and by Gronwall's inequality that

$$||\xi||_{\mathbf{V}} + ||\tau||_W \le C(F_0(t) + G_0(t)). \tag{4.3}$$

Now we apply Lemma 4.2 to $F_0(t)$ and $G_0(t)$ to obtain

$$F_0(t) \le Ch^{k+2}\left(||\sigma||_{k+2} + \int_0^t ||\sigma(s)||_{k+2}ds\right) \quad \text{and} \quad G_0(t) \le Ch^{k+2}||u||_{k+1}$$

which, together with (4.3), indicates

$$||\xi||_{\mathbf{V}} + ||\tau||_W \le Ch^{k+2}(||u||_{k+1} + |||\sigma|||_{k+2}).$$

□

Corollary 4.1. *Let $(\bar{u}_h, \bar{\sigma}_h)$ be the mixed Ritz-Volterra projection of (u, σ). Then,*

$$\begin{aligned} |||D_t(u - \bar{u}_h)|||_* + |||D_t(\sigma - \bar{\sigma}_h)|||_* \; &\le Ch^{k+2}\{||u||_{k+1} + ||u_t||_{k+2} + ||\sigma||_{k+2} \\ &\quad + ||\sigma_t||_{k+2} + \int_0^t [||u(s)||_{k+1} + ||\sigma(s)||_{k+2}]ds\}. \end{aligned}$$

Proof. Differentiating (4.1) with respect to time t, then we see that ξ_t and τ_t satisfy the same equations with the right-hand sides replaced by

$$\begin{aligned}
F_0'(\mathbf{v}_h) &= -(\alpha(\sigma_t - \Pi_h\sigma_t) + (\alpha_t + M(t,t))(\sigma - \Pi_h\sigma), \mathbf{v}_h) \\
&\quad + \left(\int_0^t M_t(t,s)(\sigma - \Pi_h\sigma)(s)ds, \mathbf{v}_h\right), && \mathbf{v}_h \in \mathbf{V}_h \\
F_1'(\mathbf{v}_h) &= -\left(M(t,t)\xi + \int_0^t M_t(t,s)\xi(s)ds, \mathbf{v}_h\right), && \mathbf{v}_h \in \mathbf{V}_h, \\
G_0'(w_h) &= -(c_t(u - P_h u + \tau), w_h) - (c(u - P_h u)_t, w_h), && w_h \in W_h.
\end{aligned}$$

Thus, Corollary 4.1 follows from the same argument above. □

In order to obtain superconvergence results for mixed element approximations for our parabolic integro-differential equations we choose our initial data approximation $(u_h(0), \sigma_h(0)) \approx (u_0(x), A(0)\nabla u_0(x))$ as the mixed elliptic projection:

$$\begin{aligned}
&(\alpha(0)(\sigma_h(0) - \sigma(0)), \mathbf{v}_h) + (\nabla\cdot\mathbf{v}_h, u_h(0) - u_0) = 0, && \mathbf{v}_h \in \mathbf{V}_h, \\
&(\nabla\cdot(\sigma_h(0) - \sigma(0)), w_h) + (c(0)(u_h(0) - u_0), w_h) = 0, && w_h \in W_h.
\end{aligned} \tag{4.4}$$

Theorem 4.2. *Let (u, σ) and (u_h, σ_h) be the solutions of (2.1) and (2.2), respectively, and $(u_h(0), \sigma_h(0))$ is chosen according to (4.4). Then, there exists a positive constant $C > 0$ such that for any $0 \le t \le T$,*

$$\begin{aligned}
&|||u - u_h|||_* + |||\sigma - \sigma_h|||_* \\
\le\ & Ch^{k+2}\left\{||u||_{k+2} + ||\sigma||_{k+2} + \left[\int_0^t (||u||_{k+1}^2 + ||\sigma||_{k+2}^2 + ||u_t||_{k+1}^2 + ||\sigma_t||_{k+2}^2)ds\right]^{1/2}\right\}.
\end{aligned}$$

Proof. First, the errors are decomposed as

$$\begin{aligned}
u - u_h &= (u - \bar{u}_h) + (\bar{u}_h - u_h) := \rho + \rho_h, \\
\sigma - \sigma_h &= (\sigma - \bar{\sigma}_h) + (\bar{\sigma}_h - \sigma_h) := \theta + \theta_h,
\end{aligned}$$

and by Theorem 4.1 that

$$|||\rho|||_* + |||\theta|||_* \le Ch^{k+2}(||u||_{k+2} + |||\sigma|||_{k+2}).$$

Moreover, from (2.7) and (4.4) we derive that

$$\begin{aligned}
&(\alpha(0)\theta_h(0), \mathbf{v}_h) + (\nabla\cdot\mathbf{v}_h, \rho_h(0)) = 0, && \mathbf{v}_h \in \mathbf{V}_h, \\
&(\nabla\cdot\theta_h(0), w_h) + (c(0)\rho_h(0), w_h), && w_h \in W_h,
\end{aligned}$$

which, together with the uniqueness of the solution to (2.7), implies

$$\theta_h(0) = \rho_h(0) = 0. \tag{4.5}$$

Furthermore, from the proof for Corollary 4.1 we know that

$$||\tau_t||_0 \le Ch^{k+2}\left\{|||u|||_{k+1} + |||\sigma|||_{k+2} + ||u_t||_{k+1} + ||\sigma_t||_{k+2}\right\}$$

which, together with the definition of the local L^2-projection operator P_h, demonstrates that

$$\begin{aligned}
|(\rho_t, \rho_h)| &= |(\tau_t, \rho_h)| \\
&\le Ch^{k+2}\left\{|||u|||_{k+1} + |||\sigma|||_{k+2} + ||u_t||_{k+1} + ||\sigma_t||_{k+2}\right\}||\rho_h||_0.
\end{aligned}$$

Noticing that $|||\rho_h|||_* = ||\rho_h||_0$ and $|||\theta_h|||_* = ||\theta_h||_0$ as well as (4.5), we can obtain the desired estimates for ρ_h and θ_h in L^2-norm through the same procedure as that in Theorem 3.1 for ρ_h and θ_h. □

5. Global L^2 superconvergence on quadrilaterals

In [18, 23] superconvergence has been obtained in mixed finite element methods on quadrilaterals for elliptic equations. Here we shall extend these results to our parabolic integro-differential equations. The strategy employed here is that we first examine the superclose accuracy between the interpolation function of the exact solution and the mixed finite element solution of (1.1) by means of integral identities, and then we use a suitable interpolation post-processing method to obtain global superconvergence approximations [23, 24]. As by-products, these superconvergence results can be utilized to form a class of useful a-posteriori error estimators to assess the accuracy of the mixed finite element solutions in applications.

Let $\hat{\mathbf{V}}_h(\hat{e}) \times \hat{W}_h(\hat{e})$ be the standard local Raviart-Thomas rectangular space on the reference element $\hat{e} := [-1,1] \times [-1,1]$ of order k (≥ 0); i.e.,

$$\begin{aligned} \hat{\mathbf{V}}_h(\hat{e}) &:= Q_{k+1,k}(\hat{e}) \times Q_{k,k+1}(\hat{e}), \\ \hat{W}_h(\hat{e}) &:= Q_{k,k}(\hat{e}), \end{aligned}$$

where $Q_{m,n}(\hat{e})$ indicates the space of polynomials of degree no more than m and n in x and y on $\hat{e}$, respectively. On arbitrary convex quadrilateral element $e \in T_h$, the local Raviart-Thomas space is defined by

$$\begin{aligned} \mathbf{V}_h(e) &:= \{\mathbf{q} = G\tilde{\mathbf{q}} \circ \hat{F}_e^{-1} :\ \tilde{\mathbf{q}} \in \hat{\mathbf{V}}_h(\hat{e})\}, \\ W_h(e) &:= \{w = \hat{w} \circ \hat{F}_e^{-1} :\ \hat{w} \in \hat{W}_h(\hat{e})\}, \end{aligned}$$

where $\hat{F}_e$ is the affine map which takes $\hat{e}$ onto e and $G := |\det(M_0)|^{-1} M_0$ with M_0 being the Jacobian matrix (derivative) of $\hat{F}_e$. Of course, $\mathbf{V}_h(e) \subset (C^\infty(e))^2$ and $W_h(e) \subset C^\infty(e)$ are no longer of polynomials on e unless e is a parallelogram.

The global Raviart-Thomas finite element space over the partition T_h is defined in the standard way as follows:

$$\begin{aligned} \mathbf{V}_h &:= \{\mathbf{v} \in H(div;\ \Omega) :\ \mathbf{v}|_e \in \mathbf{V}_h(e),\ \forall e \in T_h\}, \\ W_h &:= \{w \in L^2(\Omega) :\ w|_e \in W_h(e),\ \forall e \in T_h\}. \end{aligned}$$

Let $\tilde{\sigma}$ and $\tilde{u}$ are two vector-valued and scalar-valued functions, respectively, on the reference element $\hat{e}$. Recall that the interpolation functions (or the Raviart-Thomas projection) $\hat{\Pi}_h\tilde{\sigma}$ and $\hat{P}_h\tilde{u}$ over $\hat{e}$ are defined by the following linear systems:

$$\begin{aligned} &\int_{\hat{l}_i} (\tilde{\sigma} - \hat{\Pi}_h\tilde{\sigma}) \cdot \mathbf{n} q ds = 0, && \forall q \in P_k(\hat{l}_i),\ i = 1,2,3,4, \\ &\int_{\hat{e}} (\tilde{\sigma} - \hat{\Pi}_h\tilde{\sigma}) \cdot \phi = 0, && \forall \phi \in Q_{k-1,k}(\hat{e}) \times Q_{k,k-1}(\hat{e}), \quad \text{and} \\ &\int_{\hat{e}} (\tilde{u} - \hat{P}_h\tilde{u}) q = 0, && \forall q \in Q_{k,k}(\hat{e}), \quad \text{respectively,} \end{aligned} \tag{5.1}$$

where $\hat{l}_i$ $(i = 1,2,3,4)$ is one of the four sides of $\hat{e}$, $\mathbf{n}$ is the outward normal vector to $\hat{e}$, and P_r denotes the set of polynomials of total degree no more than r. If $e \in T_h$ is an arbitrary quadrilateral element, σ and u are two vector-valued and scalar-valued functions defined on e, then their interpolation functions $\Pi_h\sigma$ and $P_h u$ on e are defined by

$$\Pi_h\sigma := G(\hat{\Pi}_h(G^{-1}\hat{\sigma})) \quad \text{and} \quad P_h u := \hat{P}_h\hat{u}, \quad \text{respectively,} \tag{5.2}$$

where $\hat{\sigma} := \sigma \circ \hat{F}_e$ and $\hat{u} := u \circ \hat{F}_e$. Then, we have [18]

$$\begin{aligned} &(\nabla\cdot(\sigma - \Pi_h\sigma), w_h) = 0, \qquad &&\forall w_h \in W_h,\\ &(\nabla\cdot\mathbf{v}_h, u - P_h u) = 0, &&\forall \mathbf{v}_h \in \mathbf{V}_h. \end{aligned} \tag{5.3}$$

The semi-discrete mixed finite element method for (1.1) is now defined as: Find $(u_h, \sigma_h) \in W_h \times \mathbf{V}_h$ satisfying

$$\begin{aligned} &(u_{h,t}, w_h) - (\nabla\cdot\sigma_h, w_h) - (cu_h, w_h) = (f, w_h), &&w_h \in W_h,\\ &(\alpha\sigma_h, \mathbf{v}_h) + \int_0^t (M(t,s)\sigma_h(s), \mathbf{v}_h)ds + (u_h, \nabla\cdot\mathbf{v}_h) = \langle g, \mathbf{n}\cdot\mathbf{v}_h\rangle, &&\mathbf{v}_h \in \mathbf{V}_h,\\ &u_h(0) = P_h u_0, \quad \sigma_h(0) = \Pi_h\sigma(0). \end{aligned} \tag{5.4}$$

From (2.1) and (5.4) we derive the following error equation:

$$\begin{aligned} &(u_t - u_{h,t}, w_h) - (\nabla\cdot(\sigma - \sigma_h), w_h) - (c(u - u_h), w_h) = 0, &&w_h \in W_h,\\ &(\alpha(\sigma - \sigma_h), \mathbf{v}_h) + \int_0^t (M(t,s)(\sigma - \sigma_h)(s), \mathbf{v}_h)ds + (u - u_h, \nabla\cdot\mathbf{v}_h) = 0, &&\mathbf{v}_h \in \mathbf{V}_h. \end{aligned} \tag{5.5}$$

From [18, 23] we recall the following lemmas.

Lemma 5.1. *If $P_h u$ is the interpolation function of u defined as in (5.2) and $c \in W^{1,\infty}(\Omega)$, then there exists a constant C such that*

$$|(c(u - P_h u), w_h)| \le Ch^{k+2}||u||_{k+1}||w_h||_0, \quad w_h \in W_h.$$

Lemma 5.2. *If the finite element partition T_h is h^2-uniform ([18])or generalized rectangular ([23]) and $\Pi_h\sigma$ is the interpolation function of σ defined as in (5.2), then there exists a constant C such that for sufficiently smooth β*

$$|(\beta(\sigma - \Pi_h\sigma), \mathbf{v}_h)| \le Ch^{k+2}||\sigma||_{k+2}||\mathbf{v}_h||_0, \quad \mathbf{v}_h \in \mathbf{V}_h.$$

We are now ready to get our main theorem in this section.

Theorem 5.1. *Assume that the finite element partition T_h is h^2-uniform or generalized rectangular and (u_h, σ_h) is the approximate solution of (1.1) defined in (5.4) by using quadrilateral elements of Raviart-Thomas of order k. If the exact solution u and σ satisfies $u \in H^{k+1}(\Omega)$ and σ, $\sigma_t \in (H^{k+2}(\Omega))^2$, then we have*

$$||u_h - P_h u||_0 + ||\sigma_h - \Pi_h\sigma||_0 \le Ch^{k+2}\left[\int_0^t (||u||_{k+1}^2 + ||\sigma||_{k+2}^2 + ||\sigma_t||_{k+2}^2)ds\right]^{1/2}. \tag{5.6}$$

Proof. Let $\rho_h^* := u_h - P_h u$ and $\theta_h^* := \sigma_h - \Pi_h\sigma$. Then, it follows from (5.3) and (5.5) that

$$\begin{aligned} &(\alpha\theta_h^*, \mathbf{v}_h) + \int_0^t (M(t,s)\theta_h^*(s), \mathbf{v}_h)ds + (\rho_h^*, \nabla\cdot\mathbf{v}_h)\\ &\quad = \left(\alpha(\sigma - \Pi_h\sigma) + \int_0^t M(t,s)(\sigma - \Pi_h\sigma)(s)ds, \mathbf{v}_h\right), &&\mathbf{v}_h \in \mathbf{V}_h,\\ &(\rho_{h,t}^*, w_h) - (\nabla\cdot\theta_h^*, w_h) - (c\rho_h^*, w_h) = -(c(u - P_h u), w_h), &&w_h \in W_h. \end{aligned} \tag{5.7}$$

Thus, letting $w_h = \rho_h^*$ and $\mathbf{v}_h = \theta_h^*$ in (5.7) we obtain from Lemmas 2.3, 5.1 and 5.2 as well as the ϵ-type inequality that

$$\frac{1}{2}\frac{d}{dt}||\rho_h^*||_0^2 + ||\theta_h^*||_0^2 \le C\left\{\int_0^t ||\theta_h^*||_0^2 ds + ||\rho_h^*||_0^2 + Ch^{2k+4}(||u||_{k+1}^2 + |||\sigma|||_{k+2}^2)\right\}.$$

Integrating from 0 to t and noticing $\rho_h^*(0) = 0$ yield according to Gronwall's lemma that

$$||\rho_h^*||_0^2 + \int_0^t ||\theta_h^*||_0^2 ds \le Ch^{2k+4}\int_0^t (||u||_{k+1}^2 + |||\sigma|||_{k+2}^2) ds,$$

or

$$||\rho_h^*||_0 \le Ch^{k+2}\left[\int_0^t (||u||_{k+1}^2 + ||\sigma||_{k+2}^2) ds\right]^{1/2}. \tag{5.8}$$

Following the same steps to get the estimate for $\theta_h := \bar{\sigma}_h - \sigma_h$ in Theorem 3.1 we can also obtain

$$||\theta_h^*||_0 \le Ch^{k+2}\left[\int_0^t (||u||_{k+1}^2 + ||\sigma||_{k+2}^2 + ||\sigma_t||_{k+2}^2) ds\right]^{1/2}. \tag{5.9}$$

Combining (5.8) with (5.9) implies (5.6). □

As a by-product of (5.6), we immediately gain the following corollary from the inverse property of the finite element space and the approximation property of the local L^2-projection operator P_h.

Corollary 5.1. *Assume that T_h is h^2-uniform or generalized rectangular and the exact solution u and σ satisfies $u \in W^{k+1,\infty}(\Omega)$ and $\sigma \in (H^{k+2}(\Omega))^2$. Then, we have for the mixed finite element solution u_h defined by (5.4) that*

$$||u - u_h||_\infty \le Ch^{k+1}\left\{||u||_{k+1,\infty} + \left[\int_0^t (||u||_{k+1}^2 + ||\sigma||_{k+2}^2) ds\right]^{1/2}\right\}.$$

In order to improve the accuracy of the finite element approximation to the exact solution on a global scale, a reasonable post-processing method is proposed according to (5.1) and Theorem 5.1 [23, 24]. For this end, we need to define two post-processing interpolation operators Π_{2h} and P_{2h} to satisfy

$$\begin{aligned}
&\Pi_{2h}\Pi_h = \Pi_{2h}, \\
&||\Pi_{2h}\mathbf{v}_h||_0 \le C||\mathbf{v}_h||_0, && \forall \mathbf{v}_h \in \mathbf{V}_h, \\
&||\Pi_{2h}\sigma - \sigma||_0 \le Ch^{k+2}||\sigma||_{k+2}, && \forall \sigma \in (H^{k+2}(\Omega))^2, \\
&P_{2h}P_h = P_{2h}, \\
&||P_{2h}w_h||_0 \le C||w_h||_0, && \forall w_h \in W_h, \\
&||P_{2h}u - u||_0 \le Ch^{k+2}||u||_{k+2}, && \forall u \in H^{k+2}(\Omega).
\end{aligned} \tag{5.10}$$

For easy exposition, we demonstrate our idea mainly for the case of $k = 2$. Thus, we assume that the standard rectangular partition $\hat{T}_h$ has been obtained from $\hat{T}_{2h} = \{\hat{\tau}\}$ with mesh size $2h$ by subdividing each element of $\hat{T}_{2h}$ into four small congruent rectangles. Let $\hat{\tau} := \bigcup_{i=1}^{4} \hat{e}_i$ with $\hat{e}_i \in \hat{T}_h$. Thus, we can define two interpolation operators $\hat{\Pi}_{2h}$ and $\hat{P}_{2h}$ associated with $\hat{T}_{2h}$ of degree at most 3 in x and y on $\hat{\tau}$, respectively, according to the following conditions:

$$\begin{aligned}
&\hat{\Pi}_{2h}\tilde{\sigma}|_{\hat{\tau}} \in Q_{3,3}(\hat{\tau}), && \hat{P}_{2h}\tilde{u}|_{\hat{\tau}} \in Q_{3,3}(\hat{\tau}),\\
&\int_{\hat{l}_i}(\tilde{\sigma}-\hat{\Pi}_{2h}\tilde{\sigma})\cdot\mathbf{n}q\,ds = 0, && \forall q \in P_1(\hat{l}_i),\ i=1,2,\cdots,12,\\
&\int_{\hat{e}_i}(\tilde{\sigma}-\Pi_{2h}\tilde{\sigma}) = 0, && i=1,2,3,4, \quad \text{and}\\
&\int_{\hat{e}_i}(\tilde{u}-\hat{P}_{2h}\tilde{u})q = 0, && \forall q \in Q_{1,1}(\hat{e}_i),\ i=1,2,3,4, \quad \text{respectively,}
\end{aligned} \tag{5.11}$$

where $\hat{l}_i$ $(i=1,2,\cdots,12)$ is one of the twelve sides of the four small elements $\hat{e}_i$ $(i=1,2,3,4)$.

Obviously, the following properties can be easily checked by (5.1) for $k=2$ and (5.11):

$$\begin{aligned}
&\hat{\Pi}_{2h}\hat{\Pi}_h = \hat{\Pi}_{2h}, &&\\
&||\hat{\Pi}_{2h}\hat{\mathbf{v}}_h||_0 \le C||\hat{\mathbf{v}}_h||_0, && \forall \hat{\mathbf{v}}_h \in \hat{\mathbf{V}}_h,\\
&||\hat{\Pi}_{2h}\tilde{\sigma}-\tilde{\sigma}||_0 \le Ch^4||\tilde{\sigma}||_4, && \forall \tilde{\sigma} \in (H^4(\Omega))^2,\\
&\hat{P}_{2h}\hat{P}_h = \hat{P}_{2h}, &&\\
&||\hat{P}_{2h}\hat{w}_h||_0 \le C||\hat{w}_h||_0, && \forall \hat{w}_h \in \hat{W}_h,\\
&||\hat{P}_{2h}\tilde{u}-\tilde{u}||_0 \le Ch^4||\tilde{u}||_4, && \forall \tilde{u} \in H^4(\Omega).
\end{aligned} \tag{5.12}$$

Then, we can define two interpolation operators Π_{2h} and P_{2h} associated with T_{2h} by

$$\Pi_{2h}\sigma := G(\hat{\Pi}_{2h}(G^{-1}\sigma\circ\hat{F}_e)) \quad \text{and} \quad P_{2h}u := \hat{P}_{2h}(u\circ\hat{F}_e), \quad \text{respectively,} \tag{5.13}$$

which satisfy (5.10) by (5.2) and (5.12).

Similarly, we can also define Π_{2h} and P_{2h} for the case of $k \neq 2$.

By virtue of the two interpolation operators Π_{2h} and P_{2h} we immediately gain the following global superconvergence theorem.

Theorem 5.2. *If there is, besides the conditions of Theorem 5.1, $u \in H^{k+2}(\Omega)$, then we have*

$$\begin{aligned}
&||P_{2h}u_h-u||_0 + ||\Pi_{2h}\sigma_h-\sigma||_0\\
&\qquad \le Ch^{k+2}\left\{||u||_{k+2}+||\sigma||_{k+2}+\left[\int_0^t(||u||_{k+1}^2+||\sigma||_{k+2}^2+||\sigma_t||_{k+2}^2)ds\right]^{1/2}\right\}.
\end{aligned}$$

Proof. From one of the properties of the operator P_{2h} in (5.10) we find that

$$P_{2h}u_h - u = P_{2h}(u_h-P_hu)+(P_{2h}u-u).$$

Therefore, it follows from Theorem 5.1 and (5.10) that

$$\begin{aligned}
||P_{2h}u_h-u||_0 &\le C||u_h-P_hu||_0+||P_{2h}u-u||_0\\
&\le Ch^{k+2}\left\{||u||_{k+2}+\left[\int_0^t(||u||_{k+1}^2+||\sigma||_{k+2}^2)ds\right]^{1/2}\right\}.
\end{aligned}$$

Analogously, we can obtain

$$||\Pi_{2h}\sigma_h - \sigma||_0 \le Ch^{k+2}\left\{||\sigma||_{k+2} + \left[\int_0^t (||u||_{k+1}^2 + ||\sigma||_{k+2}^2 + ||\sigma_t||_{k+2}^2)ds\right]^{1/2}\right\}.$$

□

It is of great importance for a mixed finite element method to have a computable a-posteriori error estimator by which we can assess the accuracy of the mixed finite element solution in applications. One way to construct error estimators is to employ certain superconvergence properties of the finite element solutions. In fact, we have

Theorem 5.3. *We have under the conditions of Theorem 5.2 that*

$$||u - u_h||_0 = ||P_{2h}u_h - u_h||_0 + O(h^{k+2}), \tag{5.14}$$

$$||\sigma - \sigma_h||_0 = ||\Pi_{2h}\sigma_h - \sigma_h||_0 + O(h^{k+2}). \tag{5.15}$$

In addition, if there exist positive constants C_1, C_2 *and small* ϵ_1, $\epsilon_2 \in (0,1)$ *such that*

$$||u - u_h||_0 \ge C_1 h^{k+2-\epsilon_1}, \tag{5.16}$$

$$||\sigma - \sigma_h||_0 \ge C_2 h^{k+2-\epsilon_2}, \tag{5.17}$$

then there hold

$$\lim_{h\to 0}\frac{||u - u_h||_0}{||P_{2h}u_h - u_h||_0} = 1, \tag{5.18}$$

$$\lim_{h\to 0}\frac{||\sigma - \sigma_h||_0}{||\Pi_{2h}\sigma_h - \sigma_h||_0} = 1. \tag{5.19}$$

Proof. It follows from Theorem 5.2 and

$$u - u_h = (P_{2h}u_h - u_h) + (u - P_{2h}u_h)$$

that

$$||u - u_h||_0 = ||P_{2h}u_h - u_h||_0 + O(h^{k+2}).$$

Thus, from (5.16) we know

$$\frac{||P_{2h}u_h - u_h||_0}{||u - u_h||_0} + Ch^{\epsilon_1} \ge 1$$

or

$$\underline{\lim}_{h\to 0}\frac{||P_{2h}u_h - u_h||_0}{||u - u_h||_0} \ge 1. \tag{5.20}$$

Similarly, it follows from (5.16) and

$$||P_{2h}u_h - u_h||_0 = ||u - u_h||_0 + O(h^{k+2})$$

that

$$\overline{\lim}_{h\to 0}\frac{||P_{2h}u_h - u_h||_0}{||u - u_h||_0} \le 1$$

which, together with (5.20), leads to (5.18).

Analogously, we can obtain (5.15) and (5.19). □

We know from (5.14) that the computable error quantity $||P_{2h}u_h - u_h||_0$ is the principal part of the mixed finite element error $||u - u_h||_0$, and can be used as a reliable a-posteriori error indicator to assess the accuracy of the mixed finite element solution under the condition (5.16). Also, (5.16) seems to be a reasonable assumption since $O(h^{k+1})$ is the optimal convergence rate of the mixed finite element solution in L^2 norm. The same comments are also valid to (5.15) and (5.17).

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Institute for Scientific Computation, Texas A & M University, College Station, TX 77843-3404

E-mail address: ewing@isc.tamu.edu

Department of Mathematics, University of Alberta, Edmonton, Alberta T6G 2G1 Canada

E-mail address: ylin@math.ualberta.ca

Department of Mathematics, Bowling Green State University, Bowling Green, OH 43403

E-mail address: tsun@bgnet.bgsu.edu

Department of Mathematics, Colorado School of Mines, Golden, CO

E-mail address: jwang@Mines.edu

Department of Mathematics, University of Alberta Edmonton, Alberta T6G 2G1, Canada