On Productivity Index in Pseudo-steady and Boundary-dominated Flow Regimes

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Abstract

We analyze non-stationary, but stabilized flow towards a well in a reservoir with impermeable external boundaries (volumetric reservoir.) Two operating conditions: fixed production rate and given wellbore pressure are considered. Determination of conditions under which the Productivity Index (PI) is time invariant is the focus of our study. For the fixed production rate case the flow regime providing constant PI is called pseudo-steady state. In case of constant wellbore pressure the flow regime characterized by time invariant PI is called boundary-dominated state. It is shown that for any reservoir-well geometry there exists a one-parameter family of initial pressure distributions for which the PI will be the same at any time, if the production rate is kept constant. This pseudo-steady state PI is independent of the actual production rate. Similarly, there exists a one-parameter family of initial pressure distributions assuring boundary-dominated state if the wellbore pressure is kept constant. The boundary-dominated PI does not depend on the actual wellbore pressure. The two families of initial pressure distributions are different and the two values of the PI also differ. For the well located in the center of a closed circular reservoir we derive exact mathematical expressions for both PI-s and compare the numerical values.

Keywords: well productivity, productivity index, pseudo-steady state, boundary-dominated flow, diffusivity equation

1 Introduction

The Productivity Index is probably one of the oldest petroleum engineering concepts (Uren, 1924). It expresses the intuitive feeling that for a given reservoir-well geometry, the ratio of production rate to some

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pressure difference between the reservoir and the well is basically independent from production history or even from actual operating conditions, once the well production is "stabilized".

For a special class of non volumetric reservoirs (with constant pressure at any portion of the external boundary and no-flow on the remaining portions), the stabilized state is the well known steady-state and the PI is closely related to a widely studied mathematical concept: the capacity (Muskat, 1937). In the expression of PI, reservoir engineers may use several options to express the driving force, that is the pressure difference between reservoir and well (Dake, 1978). We prefer to define this "driving force" as the difference between average pressure and wellbore pressure even for steady-state, and use this definition consistently for comparison with other flow regimes. Also, one can consider steady-state as the asymptotic flow regime reached gradually at late times, if the production starts from a constant initial pressure distribution. In accordance with later discussion, however, we prefer to define steady-state as the unique flow regime with the property of time invariance. It is obvious that for a given geometry, external boundary pressure and wellbore pressure, there is one and only one initial pressure distribution starting from which the reservoir is in steady-state already at start, and remains there “forever”.

In this paper we investigate an isolated (“volumetric”) reservoir. Hence time invariance is required only for the PI, since the pressure distribution itself must vary with time. The basic question we ask is: what condition is necessary and sufficient for the initial pressure distribution in order to assure time invariant PI? Recently Helmy and Wattenbarger (1998) have pointed out that time invariance of the Productivity Index does not necessarily imply uniqueness. Therefore we are also interested in the deviation between Productivity Indices for the same reservoir but with different operating conditions at the well.

In the first part we consider pseudo-steady state and the results will be presented in terms of an associated stationary problem. The second part describes the boundary-dominated case. Here the main results are derived from a corresponding eigenvalue-eigenfunction problem. In order to obtain general statements the formulation will be somewhat abstract. From the general treatment, however, practical results are easily obtained, as we show on the example of the simplest model (fully penetrating well located in the center of a circular drainage area.) Accurate calculations show that the two time invariant PI values are numerically close (at least for large drainage area and small well radius) but are always distinct. The underlying conditions for the initial pressure distributions are, however, quite different from each other.

The mathematical model is the well known diffusivity equation describing single-phase flow through porous medium:

$$Div\left(\frac{k(x)}{m}\nabla p\right) = fc\frac{\partial p}{\partial t} \tag{1}$$

where the usual hypotheses are understood: slightly compressible fluid with constant compressibility (c), the porosity of the reservoir (ϕ) and the viscosity of the fluid (μ) are also constant. In general, location x

might be a three dimensional vector and the location dependent permeability (k) might be a diagonal tensor. Equation 1 can be derived combining Darcy's law, a simple equation of state, and the mass conservation equation.

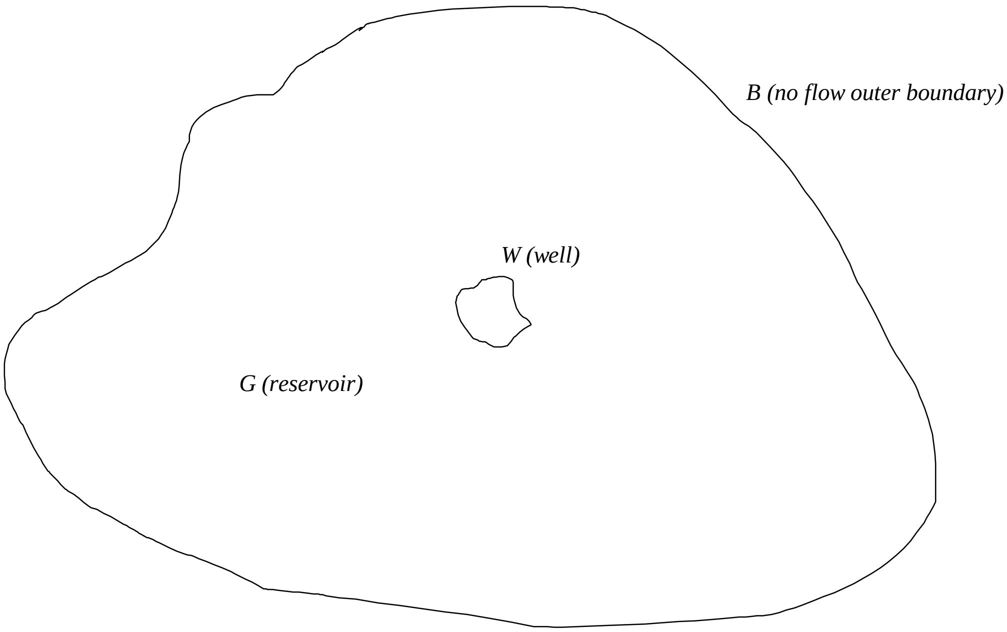

Figure 1. Isolated (volumetric) reservoir and producing well

The connected domain (reservoir) G (see Fig.1) has no-flow outer boundary B, therefore

$$\frac{\partial p}{\partial n} = 0 \quad on\ B \tag{2}$$

where $\frac{\partial}{\partial n}$ denotes the normal (with respect to B) derivative. In the following, V_G denotes the volume of the reservoir.

The well is represented by the internal boundary W. In discussing the constant production rate and constant wellbore pressure cases, the boundary conditions on W will differ from each other. The surface area of the well (inner boundary) is denoted by S_W .

In general, we allow a non-uniform initial pressure distribution $p_i(x)$ and that will be crucial in the following.

Table 1. Dimensionless variables		
Definition	Circular Drainage Area, Constant-rate	Circular Drainage Area, Constant - pressure
$p_D = c_p(p_{ref} - p)$	$c_p = \dfrac{2pk_0 h}{BmQ}$	$c_p = \dfrac{1}{p_{ref} - p_w}$
$q_D = c_q q$	$c_q = \dfrac{1}{Q}$	$c_q = \dfrac{B_o m}{2pk_0 h(p_{ref} - p_w)}$
$Q_D = c_q Q$	see above	see above
$x_D = c_x x$	$c_r = 1/r_w$ (x becomes r)	
$t_D = c_t t$	$c_t = \dfrac{k_0}{fc_t m r_w^2}$	
$k_D = c_k k$	$c_k = 1/k_0$	
$J_D = c_J PI$	$c_J = \dfrac{2pk_0 h}{Bm}$	
	$R_D = r_e / r_w \qquad S_{DW} = 1 \qquad V_{DG} = \dfrac{R_D^2 - 1}{2}$	

Introducing the dimensionless variables (see Table 1) and selecting the constants appropriate for the geometry, physical properties and inner boundary conditions, Eqs. 1 and 2 can be cast into

$$Div(k_D(x_D)\nabla p_D) = \frac{\partial p_D}{\partial t_D} \tag{3}$$

and

$$\frac{\partial p_D}{\partial n_D} = 0 \quad on \quad B \tag{4}$$

2 Pseudo-steady State Flow Regime

We assume that the production rate (Q) is uniformly distributed along the surface W and is kept constant. The dimensionless production rate – as seen from Table 1 – is unity. Then the inner boundary condition becomes:

$$S_{Dw}\frac{\partial p_D}{\partial n_D} = -1 \quad on\,W \tag{5}$$

2.1 Constant-Rate Productivity Index

The dimensionless PI for constant wellbore rate takes the simple form:

$$J_{Dcr} = \frac{1}{[\,\overline{p}_D(t_D)]_W - [\,\overline{p}_D(t_D)]_G} \tag{6}$$

where $[\,\overline{p}_D(t_D)]_W$ is the pressure averaged on the well surface and $[\,\overline{p}_D(t_D)]_G$ is the volumetric average pressure in the reservoir.

2.2 Definition of Pseudo-steady State

We say that the flow regime is pseudo-steady state if the production index, J_{Dcr} remains the same in any time.

2.3 Definition of Auxiliary Problem 1

Let us suppose that $p_{D1}(x_D)$ is a solution of the time-invariant auxiliary problem:

$$Div(k_D(x)\nabla p_{D1}) = \frac{1}{V_{DG}/S_{DW}} \tag{7}$$

$$\frac{\partial p_{D1}}{\partial n} = 0 \quad on\ B \tag{8}$$

$$S_{DW}\frac{\partial p_{D1}}{\partial n} = -1 \quad on\,W \tag{9}$$

It is obvious that the solution to the above Poisson equation with given Neuman conditions (8)-(9) exists and is unique up to an arbitrary constant. To make it unique, we require that its average over the reservoir volume is zero: $[\,\overline{p}_{D1}]_G = 0$.

We note that the physical interpretation of the associated problem is that a hypothetical source is uniformly distributed in the whole reservoir volume. The volume integral of the source equals to the total wellbore production rate.

2.4 Condition for Pseudo-steady State

By definition, in pseudo-steady state the Productivity Index is time independent. Starting from any initial distribution the constant-rate solution can be written in the form of

$$p_D(x_D, t_D) = \frac{1}{V_{DG}} t_D + p_{D1}(x_D) + p_{D2}(x_D, t_D) \tag{10}$$

The first term on the right hand side is uniquely determined by the fixed production rate, and $p_{D1}(x_D)$ is the solution of the Auxiliary Problem 1. The remaining term $p_{D2}(x_D, t_D)$ is the solution of Auxiliary Problem 2 discussed in the Appendix.

Since the first term on the right hand side of Eq. 10 drops out from the driving force and the dimensionless production rate is unity, the PI can be written as:

$$J_{Dcr} = \frac{1}{[\overline{p}_D(t_D)]_w - [\overline{p}_D(t_D)]_G} = \frac{1}{[\overline{p}_{D1}]_W - [\overline{p}_{D1}]_G + [\overline{p}_{D2}(t_D)]_W - [\overline{p}_{D2}(t_D)]_G} \tag{11}$$

From the requirement that J_{Dcr} is constant at any time, we obtain that $[\overline{p}_{D2}(t_D)]_W - [\overline{p}_{D2}(t_D)]_G$ must remain constant with time.

Now our results can be summarized in the following simple way: Assume that the (uniform flux) well production rate is fixed and the Productivity Index does not vary with time (that is the reservoir is in pseudo-steady state.) Then the pressure distribution at any time can differ from $\overline{p}_{D1}(x_D)$ only by an arbitrary constant. In particular, the initial pressure distribution must be of the form:

$$p_{Di}(x_D) = p_{D1}(x_D) + c_1 \tag{12}$$

where the constant c_1 is nothing else, but the average pressure at the initial time t_i. If starting from such a pressure distribution, the reservoir remains in pseudo-steady state.

The corresponding PI is

$$J_{Dcr}(t_D) = J_{Dps} = \frac{1}{[\bar{p}_{D1}]_W - [\bar{p}_{D1}]_G} = \frac{1}{[\bar{p}_{D1}]_W} \tag{13}$$

and it does not depend on time, on c_1 and not even on Q.

It follows from the previous discussion that condition (12) is both necessary and sufficient. In general, of course we do not start production from such a special distribution. Starting from any other distribution, the pressure distribution will asymptotically approach the *form* (12) with vanishing p_{1D}. The PI will tend to the pseudo-steady state value (with exponential rate).

One problem, investigated quite frequently is when the initial pressure distribution is constant. The traditional reservoir engineering term for "transient flow" corresponds to the initial time interval when the PI is still notably different from the pseudo-steady state value. The "time to reach pseudo-steady state" is often defined as the time when the PI is already in a prescribed vicinity (say 1 %) of its limit value.

Another (less frequently investigated but equally important) initial pressure distribution is that corresponding to a pseudo-steady state determined by a previous (usually larger) production rate. We note that the time to reach pseudo-steady state is therefore a misnomer, because the actual deviation from the limit value depends on "how far we were at the start". Intuitively it is obvious, that if the starting pressure distribution is pseudo-steady state with respect to a previous production rate and the new production rate differs from the previous one by a small amount only, then the "time to reach the new pseudo-steady state" will be small. In addition, "transient" does not imply larger PI. In particular, if the production rate has been changed downward, the PI temporarily jumps to a lower value and during the following "transient" regime. it is gradually returning to its previous value.

2.5 Example: Homogeneous Reservoir of Circular Shape in pseudo-steady state

For the circular drainage area with ratio of external radius to wellbore radius, R_D, the dimensionless problem is written in the form:

$$\frac{\partial p_D^2}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial p_D}{\partial r_D} = \frac{\partial p_D}{\partial t_D} \quad 1 \le r_D \le R_D \tag{14}$$

$$\left(\frac{\partial p_D}{\partial r_D}\right)_{r_D = R_D} = 0 \quad t_D \ge 0 \tag{15}$$

$$\left(r_D \frac{\partial p_D}{\partial r_D} \right)_{r_D=1} = -1 \quad t_D \geq 0 \tag{16}$$

The associated time-invariant problem for $p_{D1}(x_D)$ is:

$$\frac{\partial p_{D1}^2}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial p_{D1}}{\partial r_D} = \frac{2}{R_D^2 - 1} \quad 1 \leq r_D \leq R_D \tag{17}$$

$$\left(\frac{\partial p_{D1}}{\partial r_D} \right)_{r_D=R_D} = 0 \tag{18}$$

$$\left(r_D \frac{\partial p_{D1}}{\partial r_D} \right)_{r_D=1} = -1 \tag{19}$$

The solution to the auxiliary problem can be written as:

$$p_{D1}(r_D) = \\ \frac{r_D^2}{2(R_D^2-1)} - \frac{R_D^2}{R_D^2-1}\ln(r_D) - \frac{1+2R_D^2-3R_D^4+4R_D^4\ln(R_D)}{4(R_D^2-1)^2} \qquad 1 \leq r_D \leq R_D \tag{20}$$

because it satisfies Eqs. 17-19 and its average over the reservoir volume is zero.

In order to be in pseudo-steady state, the initial pressure distribution may deviate from p_{D1} only by an arbitrary constant. During the pseudo-steady state operation the time-invariant component of the pressure distribution remains the same, only the “constant” varies (in fact linearly) with time, where the depletion speed is determined by the production rate. In other words, the pressure distribution is continuously and evenly shifted downwards.

According to Eq. 21 the dimensionless Productivity Index in pseudo-steady state, J_{Dps} is

$$J_{Dps} = \frac{1}{p_{D1}(r_D)\big|_{r_D=1}} = \frac{4(R_D^2-1)^2}{-1+4R_D^2-3R_D^4+4R_D^4\ln(R_D)} \tag{21}$$

A well-known approximation, often used in practice is the dimensionless “standard” deliverability[3]:

$$J_{Dst} = \frac{1}{-\frac{3}{4} + \ln(R_D)} \tag{22}$$

that can be obtained from Eq. 15 by assuming $R_D >> 1$.

Similar asymptotic approximations for other reservoir shapes are usually given in the form of shape factors, as introduced by Dietz (1965).

3 Boundary-dominated Flow Regime

We assume that the pressure along the inner boundary (W) is kept constant both with respect to time and to location. (The latter is often considered as the consequence of infinite conductivity in the well.) Then the inner boundary condition becomes:

$$p_{Dw}(x_D, t_D) = p_{Dw} \quad on\, W \tag{23}$$

3.1 Constant-Pressure Productivity Index

The dimensionless PI for constant production rate is calculated from

$$J_{Dcp} = \frac{Q_D(t_D)}{p_{Dw} - [\bar{p}_D(t_D)]_G} \tag{24}$$

where $[\bar{p}_D(t_D)]_G$ is the volumetric average pressure in the reservoir and $Q_D(t_D)$ is the (time varying) production rate:

$$Q_D(t_D) = \int_W \frac{\partial p_D}{\partial n}(x_D, t_D) dS_{DW} \tag{25}$$

3.2 Definition of Boundary-dominated Flow Regime

We say that the flow regime is boundary-dominated if the Productivity Index, J_{Dcp} remains the same in any time.

3.3 Definition of Auxiliary Problem 3

Let us suppose that $p_{D3}(x_D)$ is a solution of the time-invariant auxiliary problem:

$$Div(k_D(x)\nabla p_{D3}) = 0 \tag{26}$$

$$\frac{\partial p_{D3}}{\partial n_D} = 0 \quad on\ B \tag{27}$$

$$p_{D3}(x_D) = 1 \quad on\ W \tag{28}$$

In virtue of the maximum principle, the unique solution is $p_{D3}(x_D) = 1$.

3.4 Condition for Boundary-dominated Flow Regime

Now the solution of the system of Eqs. 3,4 and 23, can be written as the sum of two functions:

$$p_D(x_D, t_D) = p_{D3}(x_D) + p_{D4}(x_D, t_D) = 1 + p_{D4}(x_D, t_D) \tag{29}$$

where $p_{D4}(x_D, t_D)$ is the solution of Auxiliary Problem 4 (see Appendix) and can be written as:

$$[\overline{p}_{D4}(t_D)]_G = \sum_m c_m [\bar{f}_m]_G e^{-(t_D - t_{Di})\, l_m} \tag{30}$$

It follows, that the constant-rate PI can be expressed as:

$$J_{Dcp}(t_D) = \frac{\sum_m c_m [\bar{f}_m]_G e^{-(t_D - t_{Di})\, l_m}\, l_m}{\sum_m c_m [\bar{f}_m]_G e^{-(t_D - t_{Di})\, l_m}} V_{DG} \tag{31}$$

If we assume that the initial pressure distribution is such that

$$p_D(x_D, t_{Di}) = 1 + c_2 f_{m0} \tag{32}$$

(where c_2 is an arbitrary constant) then the constant-rate Productivity Index becomes constant:

$$J_{Dcp}(t_D) = l_{m0} V_{DG} \tag{33}$$

It is crucial to note that in the domain *G,* only the first eigenfunction does not change sign (Mizohata, 1973). Therefore, if the initial distribution is one of the eigenfunctions, it must be the first one.

It follows from the previous discussion that for the constant wellbore pressure case there exists a family of initial pressure distributions such that the Productivity Index does not depend on time. This family is determined by the first eigenfunction of the auxiliary problem:

$$p_{Di}(x_D) = 1 + c_2 f_{m1} \tag{34}$$

where the positive constant c_2 is nothing else, but $[\bar{p}_D(t_{Di})]_G - 1$.

Then the PI is time independent and given by

$$J_{Dcp}(t_D) = J_{Dbd} = l_1 V_{DG} \tag{35}$$

Note that the dimensionless productivity index does not depend on time, on the selected wellbore pressure or on the constant c_2.

Condition (34) is both necessary and sufficient. Of course in general, we do not start production from such a special distribution. Starting from any other distribution, the pressure will asymptotically approach the *form* (34) and the PI will exponentially approach the boundary-dominated value (see A-15).

One problem, investigated occasionally in the literature is when the initial pressure distribution is constant. Then the “transient” period is the initial time interval necessary for the PI to “reach” its limiting value.

Another (equally important) initial pressure distribution is that corresponding to a boundary-dominated state determined by a previous (usually larger) wellbore pressure. We note that the time to reach the boundary-dominated flow regime is not well defined (not only because we never “reach” the limiting PI but also) because the actual deviation from the limiting value depends on "how far we were at the start". Intuitively it is obvious that if the starting pressure distribution is the boundary-dominated one with respect to a previous wellbore pressure, and the new wellbore pressure differs from the previous one only by a small amount, then the length of the transient period, necessary to stabilize again the boundary-dominated state, will be small.

We note that in the strict sense the boundary-dominated regime does not exist, if the well boundary condition is formulated as a time invariant but otherwise non-uniform pressure distribution on the well surface. In such case the PI will vary with time, whatever initial pressure distribution is present in the reservoir (see A-16).

3.5 Example: Homogeneous Reservoir of Circular Shape in Boundary-Dominated State

As previously, we write the dimensionless problem as:

$$\frac{\partial p_D^2}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial p_D}{\partial r_D} = \frac{\partial p_D}{\partial t_D} \quad 1 \le r_D \le R_D \tag{36}$$

$$\left(\frac{\partial p_D}{\partial r_D}\right)_{r_D = R_D} = 0 \quad t_D \ge 0 \tag{37}$$

but with internal boundary condition:

$$p_{Dw}(t_D) = 1 \quad t_D \ge 0 \tag{38}$$

To obtain the first eigenvalue now we have to calculate the first positive root of the following equation:

$$J_0^{'}(z)Y_0\left(\frac{z}{R_D}\right) - J_0\left(\frac{z}{R_D}\right)Y_0^{'}(z) = 0 \tag{39}$$

where $J_0(z)$ is the Bessel function of the first kind of zero order and $Y_0(z)$ is the Bessel function of the second kind of zero order (see e.g. Abramowitz and Stegun, 1972). For finding the root we use Newton's method.

Denoting the first root of Eq. 39 by z_1, the eigenvalue is obtained from

$$l_1 = \frac{z_1^2}{R_D^2} \tag{40}$$

Since for this geometry

$$V_{DG} = \frac{R_D^2 - 1}{2} \tag{41}$$

we obtain

$$J_{Dbd} = z_1^2 \frac{R_D^2 - 1}{2R_D^2} \tag{42}$$

In order to assure boundary-dominated flow, the initial pressure distribution must be of the form:

$$p_{Di}(r_D) = -1 + c_2[J_0^{'}(z_1)Y_0(\frac{z_1}{R_D} r_D) - J_0(\frac{z_1}{R_D} r_D)Y_0^{'}(z_1)] \tag{43}$$

During the boundary-dominated state only the “constant” c_2 varies (in fact exponentially with time, where the depletion speed is determined by the wellbore pressure). In other words, the deviation of the pressure distribution in the reservoir from the constant wellbore pressure is continuously and evenly shrinking.

4 Comparison of the Productivity Indices for Circular Drainage Area

Calculating the standard approximation (Eq. 22), the pseudo-steady state (Eq. 21) and the boundary-dominated (Eq. 42) dimensionless Productivity Indices, we obtain the results shown in Table 2. (The calculations were done in *Mathematica*, 1998.)

Table 2. Comparison of Dimensionless Productivity Indices

Ratio of drainage radius to wellbore radius, R_D	Standard approximation, J_{Dst}	Pseudo-steady state, J_{Dps}	Boundary-dominated, J_{Dbd}
10	0.644087	0.627240	0.601888
100	0.259392	0.259330	0.256797
1,000	0.162397	0.162396	0.161765
10,000	0.118199	0.118199	0.117955
100,000	0.092912	0.092912	0.092794

As seen from the Table, the standard approximation is fairly good for the pseudo-steady state PI, but it is less accurate for the boundary-dominated flow regime. The reason why the pseudo-steady state PI is greater than the corresponding boundary-dominated value is that the origin of the produced fluid is evenly distributed in the reservoir if the flow regime is pseudo-steady state, while in the boundary-dominated case more fluid is coming from the area being further from the well, and hence more energy is dissipated. (By the same token, the steady-state Productivity Index is the least of the three.)

Conclusions

In this work we considered the necessary and sufficient conditions for the time invariance of the Productivity Index of a well producing from an isolated (volumetric) reservoir. The investigation showed that for two distinct operating conditions: given production rate and given wellbore pressure, the nature of the time invariant flow regimes is markedly different.

For the fixed production rate case, in the time invariant (in other words: pseudo-steady state) flow regime the pressure distribution in the reservoir can differ from the solution of Auxiliary Problem 1 only by a constant (representing the average pressure at that particular time point) and the corresponding PI is determined by this solution.

For the given wellbore pressure case, in the time invariant (in other words: boundary-dominated) flow regime the pressure distribution in the reservoir must be the member of another one-parameter family, related to the first solution of an eigenvalue-eigenfunction problem (Auxiliary Problem 4). The corresponding PI is determined by the smallest eigenvalue.

The Productivity Indices, calculated for the two distinct flow regimes, are different, even considering the simplest (circular) drainage area.

The obtained results can be used to describe the (long-time) deliverability of petroleum producing wells under various operating conditions. The insight gained is also significant from the point of view of reservoir simulation, because it opens up the possibility to improve currently used well models.

Nomenclature

B_o =	*formation volume factor,-*
c_T =	*total compressibility, 1/Pa*
c_x =	*1/characteristic length, 1/m*
c_t =	*1/characteristic time, 1/s*
c_p =	*1/characteristic pressure, 1/Pa*
c_q =	*1/characteristic production rate,* s/m^3
c_J =	*1/characteristic productivity index,* $s/(m^3\ Pa)$
c_k =	*1/characteristic permeability,* $1/m^2$
h =	*height, m*
k =	*permeability,* m^2
p =	*pressure, Pa*
q =	*flow rate,* m^3/s
Q =	*well production rate (constant),* m^3/s
r =	*radius, m*

r_w = *wellbore radius, m*

r_e = *drainage radius, m*

t = *time, s*

f = *porosity, -*

m = *viscosity, Pa×s*

V_G = *resrvoir volume,* m^3

S_w = *wellbore surface area,* m^2

PI = *Productivity Index,* $(m^3/s)/Pa$

J_D = *Dimensionless Productivity Index, -*

$[\overline{p}]_W$ = *average over domain W*

c_m = *Fourier coefficient*

l_m = *eigenvalue*

j_m = *eigenfunction*

Subscripts

D *dimensionless*

G *reservoir domain*

B *external boundary*

W *inner boundary (wellbore)*

i *initial*

n *normal*

ref *reference*

st *standard approximation*

ps *pseudo-steady state*

bd *boundary-dominated state*

cr *constant (wellbore) rate*

cp *constant (wellbore) pressure*

Appendix

Auxiliary Problem 2 for the Pseudo-steady State Flow Regime

Let us suppose that $p_{D2init}(x_D)$ is defined as the difference of the initial pressure distribution in the reservoir from the solution of the time-invariant Auxiliary Problem No 1:

$$p_{D2init}(x_D) = p_D(x_D, t_{Di}) - p_{D1}(x_D) \tag{A-1}$$

No we consider the homogeneous transient problem:

$$Div(k_D(x)\nabla p_{D2}) = \frac{\partial p_{D2}}{\partial t} \tag{A-2}$$

$$\frac{\partial p_{D2}}{\partial n_D} = 0 \quad on\ B \tag{A-3}$$

$$\frac{\partial p_{D2}}{\partial n_D} = 0 \quad on\ W \tag{A-4}$$

with initial condition A-1. The unique solution of this problem is denoted by $p_{D2}(x_D, t_D)$. An important property of the function $p_{D2}(x_D, t_D)$ is that with time it tends to a constant, c_1 where the constant is nothing else, but the volumetric average of the initial pressure distribution, $[\bar{p}_D(t_{Di})]_G$. Moreover, the average, $[\bar{p}_{D2}(t_D)]_G$ does not vary with time (it remains c_1 .)

Auxiliary Problem 4 for the Boundary-dominated Flow Regime

Let us define $p_{D4init}(x_D)$ as the difference of the initial pressure distribution in the reservoir from the solution of the time-invariant Auxiliary Problem 3:

$$p_{D4init}(x_D) = p_D(x_D, t_{Di}) - p_{D3}(x_D) = p_D(x_D, t_{Di}) - 1 \tag{A-5}$$

No we consider the homogeneous transient problem:

$$Div(k_D(x)\nabla p_{D4}) = \frac{\partial p_{D4}}{\partial t} \tag{A-6}$$

$$\frac{\partial p_{D4}}{\partial n_D} = 0 \quad on\ B \tag{A-7}$$

$$p_{D4} = 0 \quad on\, W \tag{A-8}$$

with initial condition A1. The unique solution of this problem is denoted by $p_{D4}(x_D, t_D)$ and can be represented in the form of Fourier Series (Egorov and Kondratiev, 1996).

$$p_{D4} = \sum_m c_m j_m(x_D) e^{-(t_D - t_{Di})\, l_m} \tag{A-9}$$

Where $j_m(x_D)$ and l_m denote a corresponding eigenfunction - eigenvalue pair of the problem:

$$Div(k(x)\nabla j_m) = l_m j_m \tag{A-10}$$

$$\frac{\partial j_m}{\partial n} = 0 \tag{A-11}$$

$$j_m = 0 \quad on \quad W \tag{A-12}$$

and c_m is the m-th Fourier coefficient of the function $p_{D4init}(x_D)$. The average value at time t_D is given by

$$[\bar{p}_{D4}(t_D)]_G = \sum_m c_m [\bar{j}_m]_G\, e^{-(t_D - t_{Di})\, l_m} \tag{A-13}$$

and obviously, it tends to zero.

In virtue of the Gauss-Ostrogradsky theorem (see e.g., Egorov and Kondratiev[7]) and because the eigenfunction $j_m(x_D)$ is a solution of problem A-10 to A-12, we obtain

$$V_{DG}\, l_m\, [\bar{j}_m]_G = \int_W \frac{\partial j_m}{\partial n} \partial S_{DW} \tag{A-14}$$

where S_{DW} is the wellbore surface, V_{DG} is the reservoir volume.

Then

$$J_{Dcp}(t_D) = \frac{Q_D(t_D)}{[\bar{p}_D(t_D)]_G - 1} = \frac{Q_D(t_D)}{[\bar{p}_{D4}(t_D)]_G} = \frac{\sum_m c_m [\bar{j}_m]_G \, e^{-(t_D - t_{Di})\, l_m} \; l_m}{\sum_m c_m [\bar{j}_m]_G \, e^{-(t_D - t_{Di})\, l_m}} V_{DG} \qquad \text{(A-15)}$$

If the deviation of the initial pressure distribution from the wellbore pressure is such, that it is orthogonal to all eigenfunctions $j_m(x_D)$ except the first one, $j_1(x_D)$ then all coefficients c_m except c_1 are equal to zero in A-15. Then

$$J_{Dcp}(t_D) = l_1 V_{DG} \qquad \text{(A-16)}$$

(In the main text we discuss why the first eigenvalue-eigenfunction pair has a special significance.)

It is in order to remark that if $p_{D3}(x_D)$ in Eq. 28 is not constant (non-uniform pressure distribution on the well) then

$$J_{Dcp}(t_D) = \frac{\sum_m c_m [\bar{j}_m]_G \, e^{-(t_D - t_{Di})\, l_m} \; l_m}{[\bar{p}_{D3}]_W - [\bar{p}_{D4}]_G + \sum_m c_m [\bar{j}_m]_G \, e^{-(t_D - t_{Di})\, l_m}} V_{DG} \qquad \text{(A-17)}$$

It can be concluded from Ibragimov (1985), that the difference between average on the well and average in the reservoir, $[\bar{p}_{D3}]_W - [\bar{p}_{D4}]_G$ can be zero only if $p_{D3}(x_D)$ is constant in the whole reservoir. Therefore, if the pressure on the well is a given (non-uniform) function of location, $J_{Dcp}(t_D)$ is time dependent for any initial pressure and (in the strict sense) there exists no boundary-dominated flow regime.

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