Limit theorems on ordered random vectors

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Abstract

Let $X_1, X_2, ...$ be independent identically distributed s-dimensional random vectors, whose distribution belongs to the domain of attraction of a stable law. Let's $X_{j,n}, j = 1, 2, \cdots, n$ denote the order statistics built by increase of norms of random vectors $X_1, X_2, \cdots, X_n$, i.e. $|X_{1,n}| \le |X_{2,n}| \le \cdots \le |X_{n,n}|$. We investigate the asymptotic properties of random vectors $T_{nk} = (X_{1n} + \cdots + X_{n-k,n})/|X_{n-k+1,n}|$.

1 Introduction

Let $X_1, X_2, ..., X_n, ...$ be independent identically distributed (i.i.d.) s-dimensional random vectors having common absolute continuous distribution function. Let

$$S(n) = X_1 + X_2 + ... + X_n, \qquad F(x) = P(|X_1| \ge x),$$

and $p(x)$ be density of distribution of vector X_1.

s-dimensional distribution is called to be stable if to every pair of vectors A_1 and A_2 and positive numbers B_1 and B_2 there always correspond a vector A and a positive number B such that for the three independent random vectors X, X_1 and X_2 possessing this distribution, the random vector $B^{-1}(X - A)$ is the sum of vectors $B_1^{-1}(X_1 - A_1)$ and $B_2^{-1}(X_2 - A_2)$.

E. Feldheim and P. Levi [1] have presented that s-dimensional distribution $G_{\alpha,\mu}$ is stable in the above - maintained sense if and only if the logarithm of the characteristic function can be expressed in the form

$$f_{\alpha,\mu}(t) = \begin{cases} -\rho^\alpha(c_1(\phi) + ic_2(\phi)) + i(t,\gamma), & \alpha \ne 1, \quad 0 < \alpha \le 2 \\ -\rho(c_1(\phi) + ic_2'(\phi,\rho)) + i(t,\gamma), & \alpha = 1 \end{cases}$$

where $\rho = |t|$, ϕ is a unit vector of direction in the space R^s, $\gamma = (\gamma_1, \gamma_2, ..., \gamma_s)$ is a constant vector,

$$c_1(\phi) = c \int_{|u|=1} |(t/\rho, u)|^{\alpha} \, \mu(du),$$

$$c_2(\phi) = -c \int_{|u|=1} \mathrm{sgn}(tu) \ \ |(t/\rho, u)|^{\alpha} \, \mathrm{tg}(\alpha\pi/2) \, \mu(du), \quad \alpha \neq 1, \quad 0 < \alpha \leq 2,$$

$$c_2'(\phi, \rho) = c \int_{|u|=1} (t/\rho, u) \ \ln |(t, u)| \, \mu(du), \quad \alpha = 1$$

Here $c > 0$ is a constant, and $\mu(\cdot)$ is a fixed measure on the unit sphere.

If for suitable chosen constants numbers B_n and real vectors A_n the distribution functions of the normalized sums

$$S_n^* = \frac{\sum X_n}{B_n} - A_n$$

converge to $G(x)$, we say that the distribution of vector X_i belongs to the domain of attraction of $G(x)$. It should be noted that only stable laws possess domain of attraction and that the domain of attraction of every stable law is nonempty, since it contains at least the given law.

It is known ([2]) that the distribution of vector X belongs to the domain of attraction of a nondegenerate s-dimensional stable law $G_{\alpha,\mu}$ if and only if

$$\lim_{\rho \to \infty} \frac{P(|X| > k\rho)}{P(|X| > \rho)} = k^{-\alpha} \tag{1.1}$$

for arbitrary $k > 0$, $0 < \alpha < 2$, and

$$\lim_{\rho \to \infty} \frac{P(|X| > \rho, \phi \in M_1)}{P(|X| > \rho, \phi \in M_2)} = \frac{\mu(M_1)}{\mu(M_2)} \tag{1.2}$$

for arbitrary Borel sets M_1, M_2 on the unit sphere, for which $\mu(M_2) \neq 0$.

Many authors have considered the relationship between S_n and the extreme order statistics. We refer to LePage, Woodroofe and Zinn [5], Kesten and Maller [6], Pruitt [7] and Hahn, Mason and Weiner [8] for more discussions and references to relevant literature. It is well known that if distribution of random vector X belongs to the domain of attraction of stable unnormal law, then the maximum term has a non-negligible contribution to the sum S_n. Here we will analyze the total contribution of the first k maximal modulus of summands to S_n. Theorems of the given paper, generalize some results of Darling [3], Arov and Bobrov [4] and Kalinauskaite [9]. We will also find the limit characteristic function of T_{nk}. Obtained results generalize to multivariate case a main result of Teugels [10] and some results of Darling [3].

2 Results

In the multidimensional case it is reasonable to order vectors by increase of norms. Let's $X_{j,n}, j = 1, 2, \cdots, n$ denote the members of variation series built by increase of norms of random vectors $X_1, X_2, \cdots, X_n$, i.e. $|X_{1,n}| \leq |X_{2,n} \leq \cdots \leq |X_{n,n}|$.

Here the main object under consideration is the following random vector

$$T_{n,k} = \frac{X_{1,n} + X_{2,n} + ... + X_{n-k,n} - \bar{a}(n-k)}{|X_{n-k+1,n}|} \tag{2.1}$$

where

$$\bar{a} = \begin{cases} EX_1 & \text{for} \quad 1 < \alpha < 2 \\ 0 & \text{for} \quad 0 \leq \alpha \leq 1 \end{cases}$$

It is established by classic theory that with a definite character of behaviour of "tails", the maximum distributions of S_n exist and form a class of stable laws. The limitations on the behaviour of "tails" of distributions, roughly speaking, are equivalent to the fact that no 'tail' should diminish at infinity more slowly than a power function. The case when "tails" of a distribution diminish more slowly than any power function is different in the peculiarity that for it any linear normalization by constants of sequence of sums S_n leads to a degenerate limiting laws or their absences. This fact was pointed out by Levy. It turns out ([3], [9]) that the limiting approximations of distribution of sums can be found inthis case by using a non-linear setting. The following theorem shows that the same situation for the sequence of sums of the members of our variation series holds.

Theorem 1. If random vectors $X_1, X_2, ..., X_n$ have non-negative components and for all $c > 0$ satisfy the condition

$$\lim_{t\to\infty} \frac{F(ct)}{F(t)} = 1 \tag{2.2}$$

then for $y > 0$

$$\lim_{n\to\infty} P\big(nF(|X_{1,n} + X_{2,n} + ... + X_{n-k+1,n}|) < y\big) = 1 - e^{-y} \sum_{j=0}^{k-1} \frac{y^j}{j!} \tag{2.3}$$

Corollary 1. Under the condition of Theorem 1 we have

$$\lim_{n\to\infty} P(nF(|S_n|) < y) \ = \ 1 - e^{-y}$$

To prove Theorem l we will need the following fact which has also an independent interest.

Theorem 2. If random vectors $X_1, X_2, ...X_n$ satisfy the condition (2.2) then for any fixed k

$$\lim_{n\to\infty} E|T_{nk}|^2 \ = \ 0$$

Corollary 2. Under the conditions of Theorem 2 for any fixed k

$$S_n \;=\; X_{n-k+1,n} + \ldots + X_{n,n} + o_p(1) X_{n-k+1,n}$$

where $o_p(1) \Rightarrow_P 0$, as $n \to \infty$.

Corollary 3. If random vectors satisfy the conditions of Theorem 2 then limiting distribution of random vector T_{nk} as $n \to \infty$ is concentrated in zero.

Until now, recall that we only dealt with fixed k. Let's now consider a sequence k_n such that $k_n \to \infty$ as $n \to \infty$ and $k_n = o(n/\ln n)$ as $n \to \infty$. For such sequences k_n the following two theorems hold.

Theorem 3. Assume $\lim_{t\to\infty} \frac{F(ct)}{F(t)} \;=\; c^{-\alpha}$ for $c > 0, \quad 0 < \alpha < 1$ and X_i has non-negative components. Then

$$\lim_{n\to\infty} E\left(\frac{|T_{nk_n}|}{k_n} - \frac{\alpha\sqrt{s}}{1-\alpha}\right)^2 \;=\; 0$$

Theorem 4. Assume $\lim_{t\to\infty} \frac{F(ct)}{F(t)} \;=\; c^{-\alpha}$ for $c > 0, \quad 1 < \alpha < 2$ and X_i has non-negative components. Then

$$\lim_{n\to\infty} E\left(\frac{|M_{nk_n}|}{k_n} - \frac{\alpha\sqrt{s}}{\alpha-1}\right)^2 \;=\; 0$$

where

$$M_{nk} = \frac{S_n - X_{1,n} - X_{2,n} - \ldots - X_{n-k,n} - kEX_1}{|X_{n-k+1,n}|}$$

Next two theorems concern the limit distribution (characteristic function) of T_{nk}.

Theorem 5. Assume (1) and (2) hold where $\alpha \in (0,1) \cup (1,2)$. Then

$$\lim_{n\to\infty} P(T_{nk} < y) = G_k(y), \qquad y \in R^s$$

where

$$\int_{R^s} e^{i(t,y)} dG_k(y) = \left(1 - \int_{|u|\le 1} \left(e^{i(t,u)} - 1 - \theta i(t,u)\right) d\frac{\mu\left(\frac{u}{|u|}\right)}{|u|^\alpha} \right.$$
$$\left. + \theta \frac{\alpha}{\alpha-1} \int_{|u|=1} i(t,u)\mu(du)\right)^{-k}$$

and $\theta = min(1, [\alpha])$.

Remark. As it follows from Theorem 5 that T_{nk} converges in distribution to a random vector which is a sum of k i.i.d. random vectors with characteristic function

$$\left(1 - \int_{|u|\le 1} \left(e^{i(t,u)} - 1 - \theta i(t,u)\right) d\frac{\mu\left(\frac{u}{|u|}\right)}{|u|^\alpha} + \theta \frac{\alpha}{\alpha-1} \int_{|u|=1} i(t,u)\mu(du)\right)^{-1}$$

An intermediate case corresponding to $\alpha = 1$ can also be studied by these methods:

Theorem 6. Assume (1) and (2) hold where $\alpha = 1$ and $\int_{|u|=1}(t,u)\mu(du) = 0$ for all t. Then as $n \to \infty$

$$E(exp\ i(t, T_{nk})) \to \left(1 - \int_{|u|\leq 1} (e^{i(t,u)} - 1) d\frac{\mu(u/|u|)}{|u|}\right)^{-k}$$

3 Proofs

Lemma. We have

$$E\ exp\ (i(t, T_{nk})) = n\binom{n-1}{k-1} \int_{R^s} \left(\int_{|x|\leq|z|} e^{i(t,\frac{x}{|z|})} p(x)dx \right)^{n-k} \left(\int_{|x|\geq|z|} p(x)dx \right)^{k-1} p(z)dz$$

Proof. Let's consider the set A_{n-k+1} of points $z = (z_1, z_2, ..., z_n)$ such that

$$|z_j| \leq |z_{n-k+1}| \quad \text{for} \quad 1 \leq j \leq n-k+1$$

and

$$|z_j| \geq |z_{n-k+1}| \quad \text{for} \quad n-k+2 \leq j \leq n.$$

Then the joint density $g(z_1, ..., z_n),\ z_j \in R^s$ of the random vectors $X_1, X_2, ..., X_n$ under the condition of the realization of the event A_{n-k+1} is

$$g(z_1, ..., z_n) = \begin{cases} n\ \binom{n-1}{k-1}\ p(z_1)\ \cdots\ p(z_n), & \text{in the domain } A_{n-k+1} \\ 0, & \text{in the remaining cases} \end{cases}$$

Thus

$$Eexp(i(t, T_{nk})) = \int_{R^s} ... \int_{R^s} exp\left(i\left(t, \frac{z_1 + ... + z_{n-k}}{|z_{n-k+1}|}\right)\right) g(z_1, ..., z_n) dz_1 ... dz_n$$

$$= n\binom{n-1}{k-1} \int_{A_{n-k+1}} exp\left(i\left(t, \frac{z_1 + ... + z_{n-k}}{|z_{n-k+1}|}\right)\right) \prod_{j=1}^{n} p(z_j) dz_1 ... dz_n$$

$$= n\binom{n-1}{k-1} \int_{R_s} \left(\int_{|z_1|<|z_{n-k+1}|} exp\left(i\left(t, \frac{z_1}{|z_{n-k+1}|}\right)\right) p(z_1) dz_1 \right)^{n-k}$$

$$\times \left(\int_{|z_1|\geq|z_{n-k+1}|} p(z_1) dz_1 \right)^{k-1} p(z_{n-k+1}) dz_{n-k+1}$$

which yields lemma.

Proof of Theorem 2.

Let's denote $|z| = \beta$. We differentiate the characteristic function of random vector T_{nk} from Lemma l with respect to t_j two times:

$$\frac{\partial^2 Eexp(i(t,T_{nk}))}{\partial t_j^2} = n(n-k)(n-k+1)\binom{n-1}{k-1}\int_{R^s}\left(\int_{|x|\geq\beta} p(x)dx\right)^{k-1}$$

$$\times\left(\int_{|x|\leq\beta} exp\big(i(t,x/\beta)\big)p(x)dx\right)^{n-k-2}\left(\int_{|x|\leq\beta} exp\big(i(t,x/\beta)\big)ix_j\beta^{-1}p(x)dx\right)^2 p(z)dz$$

$$+n(n-k)\binom{n-1}{k-1}\int_{R^s}\left(\int_{|x|\geq\beta} p(x)dx\right)^{k-1}\left(\int_{|x|\leq\beta} exp\big(i(t,x/\beta)\big)p(x)dx\right)^{n-k-1}$$

$$\times\left(\int_{|x|\leq\beta} exp\big(i(t,x/\beta)\big)i^2x_j^2\beta^{-2}p(x)dx\right)p(z)dz$$

Summing the last equations for $1 < j < s$ and then substituting $t = 0$ we obtain

$$E|T_{nk}|^2 = n(n-k)(n-k-1)\binom{n-1}{k-1}\int_{R^s} F^{k-1}(\beta)(1-F(\beta))^{n-k-2}$$

$$\times\sum_{j=1}^{s}\left(\int_{|x|\leq\beta}\beta^{-1}x_j p(x)dx\right)^2 p(z)dz \qquad (3.1)$$

$$+\int_{R^s} F^{k-1}(\beta)(1-F(\beta))^{n-k-1}\left(\int_{|x|\leq\beta}\beta^{-2}\sum_{j=1}^{s}x_j^2 p(x)dx\right)p(z)dz$$

Using the elementary inequality $\sum_{j=1}^{s} z_j \leq \sqrt{|z|}$, from (3.1) we have

$$0 \leq E|T_{nk}|^2 \leq n(n-k)(n-k-1)\binom{n-1}{k-1}s\int_{R^s} F^{k-1}(\beta)(1-F(\beta))^{n-k-2}$$

$$\times\left(\int_{|x|\leq\beta}\beta^{-1}|x|p(x)dx\right)^2 p(z)dz + n(n-k)\binom{n-1}{k-1}s \qquad (3.2)$$

$$\times\int_{R^s} F^{k-1}(\beta)(1-F(\beta))^{n-k-1}\left(\int_{|x|\leq\beta}\beta^{-2}|x|^2 p(x)dx\right)p(z)dz$$

Let's consider

$$\varphi_1(\beta) = \int_{|x|\leq\beta}\beta^{-1}|x|p(x)dx \quad \text{and} \quad \varphi_2(\beta) = \int_{|x|\leq\beta}\beta^{-2}|x|^2 p(x)dx$$

By the substitution $x = \rho_1 u$ we have

$$\varphi_1(\beta) = \int_0^\beta \int_{|u|=1} \beta^{-1}|\rho_1 u| p(\rho_1 u)\rho_1^{s-1} d\rho_1 du$$

Substituting here $\rho_1 = \rho\beta$ we obtain

$$\varphi_1(\beta) = \int_0^1 \int_{|u|=1} \beta^{-1}\rho\beta p(\rho\beta u)\rho^{s-1}\beta^{s-1}\beta d\rho du$$

$$= \int_0^1 \int_{|u|=1} \rho p(\rho\beta u)\rho^{s-1}\beta^{s} d\rho du \ = \ \int_0^1 \rho dP(|X_1| < \rho\beta)$$

Here integration is taken on unit sphere, and the last equality follows from the relation

$$\int_{|u|=1} \rho^{s-1} p(\rho u) du \ = \ \frac{d}{d\rho} P(|X_1| \leq \rho)$$

Farther, the integration by parts gives

$$\varphi_1(\beta) \ = \ -F(\beta) + \int_0^1 F(\rho\beta) d\rho \ = \ F(\beta) \int_0^1 \left(\frac{F(\rho\beta)}{F(\beta)} - 1 \right) d\rho \ = \ F(\beta) A(\beta)$$

where

$$A(\beta) \ = \ \int_0^1 \left(\frac{F(\rho\beta)}{F(\beta)} - 1 \right) d\rho$$

In the same way we have

$$\varphi_2(\beta) = \int_0^\beta \int_{|u|=1} \beta^{-2}|\rho' u|^2 p(\rho' u)(\rho')^{s-1} d\rho' du \ = \ \int_0^1 \int_{|u|=1} \rho^2 p(\rho\beta u)\rho^{s-1}\beta^s d\rho du$$

$$= -F(\beta) + 2\int_0^1 \rho F(\rho\beta) d\rho \ = \ 2F(\beta) \int_0^1 \left(\frac{F(\rho\beta)}{F(\beta)} - 1 \right) d\rho \ = \ 2F(\beta) A(\beta)$$

We first show that $\lim_{\beta\to\infty} A(\beta) = 0$. Note that function $Q(\rho) = max_{0\leq z\leq\rho}\, q(z)$ where $q(z) = z^\varepsilon F(z)$, is a function of regular growth since owing to the condition of Theorem 2 the function q is regularly varying. Now from a Theorem of Karamata (c...) it follows that

$$\lim_{\rho\to\infty} \frac{Q(\rho)}{q(\rho)} = 1$$

This implies that

$$\frac{max_{0\leq\rho\leq 1}\ \rho^\varepsilon F(\rho\beta)}{F(\beta)} \to 1 \qquad\qquad \beta \to \infty$$

Since the expression on the left is bounded for $\beta < \infty$ we obtain

$$\frac{max_{0\le\rho\le1} \;\; \rho^{\varepsilon} F(\rho\beta)}{F(\beta)} \le c, \qquad \frac{F(\rho\beta)}{F(\beta)} \le c\rho^{-\varepsilon}, \qquad 0 \le \rho \le 1,$$

for all β, and the constant c depending only on ε .

Taking into account the fact that

$$\lim_{\beta\to\infty} \int_{\delta}^{1} \left(\frac{F(\beta\rho)}{F(\beta)} - 1\right) d\rho \; = \; 0 \qquad \delta > 0$$

we conclude that $A(\beta) \to 0, \; \beta \to \infty$

Now we get from (3.2) that

$$E|T_{nk}|^2 \le n(n-k)(n-k-1)\binom{n-1}{k-1} s \int_{R^s} F^{k+1}(\beta)(1-F(\beta))^{n-k-2} A^2(\beta) p(z) dz$$

$$+2n(n-k)\binom{n-1}{k-1} s \int_{R^s} F^{k}(\beta)(1-F(\beta))^{n-k-1} A(\beta) p(z) dz \tag{3.3}$$

By Lemma 2 we choose $\varepsilon > 0$ arbitrary, and choose β_0 such that $A(\beta_0) < \varepsilon$ for $\beta \le \beta_0$. Father choose n_0 such that

$$max_{n\ge n_0} \left\{ sn(n-k)(n-k-1)\binom{n-1}{k-1} F^{k}(\beta)(1-F(\beta))^{n-k-2} A^2(\beta), \right.$$

$$\left. 2n(n-k)\binom{n-1}{k-1} sF^{k-1}(\beta)(1-F(\beta))^{n-k-1} A(\beta)) \right\} \le \varepsilon$$

From this and from (3.3) we obtain

$$E|Tnk|^2 \le 2\varepsilon \int_{|z|\le\beta_0} F(\beta) p(z) dz$$

$$+n(n-k)(n-k-1)\binom{n-1}{k-1} s\varepsilon^2 \int_{\beta_0}^{\infty} F^{k+1}(\beta)(1-F(\beta))^{n-k-2} p(z) dz$$

$$+2n(n-k)\binom{n-1}{k-1} s\varepsilon \int_{\beta_0}^{\infty} F^{k}(\beta)(1-F(\beta))^{n-k-1} p(z) dz$$

$$\le 2\varepsilon + n(n-k)(n-k-1)s\binom{n-1}{k-1}\varepsilon^2 \int_0^1 (1-u)^{k+1} u^{n-k-2} du$$

$$+2n(n-k)\binom{n-1}{k-1} s\varepsilon \int_0^1 (1-u)^{k} u^{n-k-1} du$$

$$= 2\varepsilon + k(k+1)s\varepsilon^2 + 2ks\varepsilon$$

Since ε was arbitrary this concludes the proof of Theorem 2.

Proof of Theorem 3. Similar to the proof of Theorem 2 it can be obtained that

$$E|Tnk| \geq (1/\sqrt{s})n(n-k_n)\binom{n-1}{k_n-1}\int_{R^s} F^{k_n-1}(\beta)(1-F(\beta))^{n-k_n-1}$$

$$\times \int_{|x|\geq\beta} \beta^{-1}\sum_{j=1}^{s} x^j p(x)dx \;\; p(z)dz \tag{3.4}$$

From (3.1) and (3.4) we conclude that

$$E\left(\frac{|T_{nk_n}|}{\sqrt{s}k_n} - \frac{\alpha}{1-\alpha}\right)^2 = E\left(\frac{|T_{nk_n}|}{\sqrt{s}k_n}\right)^2 - \frac{2\alpha}{\sqrt{s}(1-\alpha)}E\left(\frac{|T_{nk_n}|}{k_n}\right) + \frac{\alpha^2}{(1-\alpha)^2}$$

$$\leq \frac{n(n-k_n)(n-k_n-1)\binom{n-1}{k_n-1}}{sk_n^2}\int_{R^s} F^{k_n-1}(\beta)(1-F(\beta)^{n-k_n-2}$$

$$\times \sum_{j=1}^{s}\left(\int_{|x|\leq\beta}\beta^{-1}x_j p(x)dx\right)^2 p(z)dz$$

$$+\frac{n(n-k_n)\binom{n-1}{k_n-1}}{sk_n^2}\int_{R^s} F^{k_n-1}(\beta)(1-F(\beta))^{n-k_n-1}\left(\int_{|x|\leq\beta}\beta^{-2}\sum_{j=1}^{s}x_j^2 p(x)dx\right)p(z)dz \tag{3.5}$$

$$-\frac{2\alpha n(n-k_n)\binom{n-1}{k_n-1}}{(1-\alpha)sk_n}\int_{R^s} F^{k_n-1}(\beta)(1-F(\beta))^{n-k_n-1}\left(\int_{|x|\leq\beta}\beta^{-1}\sum_{j=1}^{s}x_j p(x)dx\right)p(z)dz$$

$$+\frac{\alpha^2}{(1-\alpha)^2} = J_{1n} + J_{2n} + J_{3n} + \frac{\alpha^2}{(1-\alpha)^2}$$

It can be easily seen that under the condition of Theorem 3

$$A(\beta) = \int_0^1 \left(\frac{F(t\beta)}{F(\beta)} - 1\right)dt = \frac{\alpha}{1-\alpha} + o(1) \qquad \beta\to\infty$$

Let's estimate J_{1n}. We have

$$J_{1n} \leq \frac{n(n-k_n)(n-k_n-1)(n-1)!}{(k_n-1)!(n-k_n)!k_n^2}\int_{R^s} F^{k_n-1}(\beta)$$

$$\times(1-F(\beta))^{n-k_n-2}\left(\int_{|x|\leq\beta}\beta^{-1}|x|p(x)dx\right)^2 p(z)dz$$

$$= \frac{n!}{(k_n-1)!(n-k_n-2)!k_n^2}\int_{R^s} F^{k_n+1}(\beta)(1-F(\beta))^{n-k_n-2}A^2(\beta)p(z)dz$$

$$= \frac{n!}{(k_n-1)!(n-k_n-2)!k_n^2}\left(\int_{|z|<\beta_0} + \int_{|z|\geq\beta_0}\right)$$

For arbitrarily small $\varepsilon > 0$ and a sufficiently large β_0 we obtain

$$J_{1n} \leq \frac{n!(1-F(\beta_0))^{n-k_n-2}}{(k_n-1)!(n-k_n-1)!k_n^2}$$

$$+\left(\frac{\alpha^2}{(1-\alpha)^2}+\varepsilon\right)\frac{n!}{(k_n-1)!(n-k_n-2)!k_n^2}\int_0^1 u^{n-k_n-2}(1-u)^{k_n+1}du \qquad (3.6)$$

$$= \frac{n!(1-F(\beta_0))^{n-k_n-1}}{(k_n-1)!(n-k_n-1)!k_n^2} + \left(\frac{\alpha^2}{(1-\alpha)^2}+\varepsilon\right)\frac{k_n+1}{k_n}$$

For estimation J_{2n} we use the inequalities

$$J_{2n} \leq \frac{n(n-k_n)(n-1)!}{(k_n-1)!(n-k_n)!k_n^2}\int_{R_s} F^{k_n-1}(\beta)(1-F(\beta))^{n-k_n-1}\varphi_2(\beta)p(z)dz$$

$$\leq \frac{2n!}{(k_n-1)!(n-k_n-1)!k_n^2}\int_{R^s} F^{k_n}(\beta)(1-F(\beta))^{n-k_n-1}A(\beta)p(z)dz$$

$$= \frac{2n!}{(k_n-1)!(n-k_n-1)!k_n^2}\left(\int_{|z|<\beta_0} + \int_{|z|\geq\beta_0}\right) \qquad (3.7)$$

$$\leq \frac{2n!(1-F(\beta_0))^{n-k_n}}{(k_n-1)!(n-k_n)!k_n^2} + \frac{2n!(\alpha/(1-\alpha)+\varepsilon)}{(k_n-1)!(n-k_n-1)!k_n^2}\int_0^1 (1-u)^{k_n}u^{n-k_n-1}du$$

$$= \frac{2n!(1-F(\beta_0))^{n-k_n}}{(k_n-1)!(n-k_n)!k_n^2} + 2\left(\frac{\alpha}{1-\alpha}+\varepsilon\right)\frac{1}{k_n}$$

Similarly

$$J_{3n} \leq \frac{2\alpha n!}{(1-\alpha)(k_n)!(n-k_n-1)!}\left[\int_{|z|\leq\beta_0} F^{k_n-1}(\beta)(1-F(\beta))^{n-k_n-1}\varphi_1(\beta)dF(z)\right.$$

$$\left.+\left(\frac{\alpha}{1-\alpha}-\varepsilon\right)\left(\int_0^\infty F^{k_n}(\beta)(1-F(\beta))^{n-k_n-1}dF(\beta) - \int_0^{\beta_0} F^{k_n}(\beta)(1-F(\beta))^{n-k_n-1}dF(\beta)\right)\right]$$

$$= \frac{2\alpha n!}{(1-\alpha)(k_n)!(n-k_n-1)!}\left(\frac{\alpha}{1-\alpha}-\varepsilon\right)\left(-\int_0^1 u^{k_n}(1-u)^{n-k_n-1}du\right)+J_{4n}$$

where

$$|J_{4n}| = \frac{2\alpha n!}{(1-\alpha)(k_n)!(n-k_n-1)!}\Bigg|\int_{|z|\le\beta_0} F^{k_n-1}(\beta)(1-F(\beta))^{n-k_n-1}$$

$$\times\left[A(\beta)F(\beta)-\left(\frac{\alpha}{1-\alpha}-\varepsilon\right)F(\beta)\right]dF(z)\Bigg| \tag{3.8}$$

$$\le \frac{2\alpha n!(1-F(\beta_0))^{n-k_n}}{(1-\alpha)(n-k_n)(k_n)!(n-k_n-1)!}$$

By the conditions of Theorem 3 it can easily be shown that

$$\frac{n!}{(n-k_n-1)!}(1-F(\beta_0))^{n-k_n} \to 0 \qquad n\to\infty$$

and from (3.6)-(3.8), consequently,

$$J_{1n} \le \left(\frac{\alpha^2}{(1-\alpha)^2}+\varepsilon\right)\frac{k_n+1}{k_n}$$

$$J_{2n} \le 2\left(\frac{\alpha}{1-\alpha}+\varepsilon\right)\frac{1}{k_n}$$

and

$$J_{3n} \le -\frac{2\alpha}{1-\alpha}\left(\frac{\alpha}{1-\alpha}-\varepsilon\right)$$

From this and from (3.5) we finally obtain that

$$0 \le E\left(\frac{|T_{nk_n}|}{\sqrt{s}k_n}-\frac{\alpha}{1-\alpha}\right)^2 \le \frac{\alpha^2}{(1-\alpha)^2}+\varepsilon+\frac{1}{k_n}\left(\frac{\alpha^2}{(1-\alpha)^2}+\varepsilon+2\left(\frac{\alpha}{1-\alpha}+\varepsilon\right)\right)$$

$$-\frac{2\alpha^2}{(1-\alpha)^2}+\frac{2\alpha\varepsilon}{1-\alpha}+\frac{\alpha^2}{(1-\alpha)^2} = \left(1+\frac{2\alpha}{1-\alpha}\right)\varepsilon+\frac{1}{k_n}\left(\frac{\alpha^2}{(1-\alpha)^2}+\frac{2\alpha}{1-\alpha}+3\varepsilon\right)$$

But since $k_n \to \infty$ as $n \to \infty$ and $\varepsilon > 0$ was arbitrary we get from this the proof of Theorem 3.

Here we omit proof of Theorem 4 as it is quite analogous to the proof of Theorem 3.

The proofs of corollaries 1-3 are evident.

Proof of Theorem 1. Choose $\varepsilon > 0$ and $\delta > 0$ arbitrary and then n_0 such that $n \geq n_0$ implies $P(|T_{nk}| > \varepsilon) < \delta$. This is possible owing to the Chebyshev's inequality and Theorem 3. We have

$$P(nF(|X_{n-k+1,n}|) < y) \leq P(nF(|S^*_{n-k+1}|) < y)$$

$$= P(nF(|S^*_{n-k+1}|) < y, |T_{nk}| > \varepsilon) + P(nF(|S^*_{n-k+1}|) < y, |T_{nk}| \leq \varepsilon) \tag{3.9}$$

$$\leq \delta + P(nF(|S^*_{n-k+1}|) < y, |T_{nk}| \leq \varepsilon)$$

where

$$|S^*_{n-k+1}| = |X_{1n} + \cdots + X_{n-k+1,n}| \leq |X_{1n} + \cdots + X_{n-k,n}| + |X_{n-k+1,n}| = |X_{n-k+1,n}| \, (|T_{nk}| + 1)$$

Now we can write

$$P(nF(|S^*_{n-k+1}|) < y, \;\; |T_{nk}| \leq \varepsilon) \leq P(nF(|X_{n-k+1,n}|(\varepsilon + 1)) < y)$$

Consider the difference

$$P(nF(|X_{n-k+1,n}|(1+\varepsilon)) < y) - P(nF(|X_{n-k+1,n}|) < y)$$

$$= P(nF(|X_{n-k+1,n}|(1+\varepsilon)) < y \leq nF(|X_{n-k+1,n}|))$$

$$\leq P(nF(|X_{n-k+1,n}|(1+\varepsilon)) < nF(|X_{n-k+1,n}|))$$

$$\leq P\left(\frac{F(|X_{n-k+1,n}|(1+\varepsilon))}{F(|X_{n-k+1,n}|)} < 1\right)$$

Under the condition of Theorem 1 the last probability tends to zero as $n \to \infty$ and ε in (3.9) can be chosen as a arbitrary small. That's why we can write

$$\lim_{n \to \infty} P(nF(|S^*_{n-k+1}|) < x) = \lim_{n \to \infty} P(nF(|X_{n-k+1,n}|) < x)$$

provided the second limit exists.

Now from the well-known formula for $P(|X_{n-k+1,n}| < y)$ passing to the limit for $n \to \infty$ we have the required fact.

Proof of Theorem 5. In view of Lemma 2.1 proved for the case $0 < \alpha < 1$ one can show that for $1 < \alpha < 2$

$$E(exp\ i(t, T_{nk})) = n\binom{n-1}{k-1}\int_{R^s}\left(\int_{|x|\le\beta} e^{i(t,\frac{x-\bar{a}}{\beta})}p(x)dx\right)^{n-k}F^{k-1}(\beta)p(z)dz \tag{3.10}$$

(i) $0 < \alpha < 1$. Let $\psi(\beta) = \int_{|x|\le\beta} \ e^{i(t,\frac{x}{\beta})}p(x)dx$. Then

$$\psi(\beta) = \int_{|u|\le 1}(exp\ i(t,u) - 1)p(u\beta)d(u\beta) + 1 - F(\beta) \tag{3.11}$$

However using (1) and (2) for large β we can write

$$p(u\beta)d(u\beta) = (1 + o(1))F(\beta)d\frac{\mu\left(\frac{u}{|u|}\right)}{|u|^\alpha}$$

where $o(1)$ depends on β. Let

$$\Phi_1 = \Phi_1(t) = \int_{|u|\le 1}\left(e^{i(t,u)} - 1\right)d\frac{\mu\left(\frac{u}{|u|}\right)}{|u|^\alpha}$$

Apply the Lemma we formally have

$$\lim_{n\to\infty} E(i(t,T_{nk})) = \lim_{n\to\infty} n\binom{n-1}{k-1}\int_0^{nF(\beta)}[1-(1-\Phi_1)F(\beta)+o(F(\beta))]^{n-k}F^{k-1}(\beta)dF(\beta)\mu(u)$$

By the substitution $nF(\beta) = v$ we obtain

$$\lim_{n\to\infty} E(i(t,T_{nk})) =$$

$$\lim_{n\to\infty} n\binom{n-1}{k-1}\int_0^n [1 - \frac{v}{n}(1-\Phi_1) + v\ o(1/n)]^{n-k}v^{k-1}n^{1-k}d(v/n)\int_{|u|=1}\mu(du)$$

$$= \lim_{n\to\infty}\frac{(n-1)!(1-\Phi_1)^{-k}}{(k-1)!(n-k)!n^{k-1}}\int_0^n e^{-v(1-\Phi_1)}\left(v(1-\Phi_1)\right)^{k-1}d(v(1-\Phi_1)) \tag{3.12}$$

Since $\frac{(n-1)!}{(n-k)!}\ n^{k-1}$ as $n\to\infty$ and $\int_0^\infty e^{-t}t^{k-1}dt = (k-1)!$ from (3.12) we obtain the result. To justify the passage to the limit it is sufficient to note that $|\psi(\beta)| < 1$ for any $\beta < \infty$ and $\int_{|z|\le A}\ \psi^{n-k}(\beta)F^{k-1}(\beta)p(z)dz$ is negligible for any bounded A . Integral on the set $|z| \ge nF(\beta)$ tends also to zero.

(ii) $1 < \alpha < 2$. In this case we use the Lemma in form (3.10). Let

$$\eta(\beta)=\int_{|x|\le\beta} exp\ i\big(t,\frac{x-\bar{a}}{\beta}\big)p(x)dx = exp\ i\big(t,-\frac{\bar{a}}{\beta}\big)\psi(\beta)$$

Now we use the estimation for $\psi(\beta)$ from [9] :

$$\psi(\beta)=1+\frac{i(t,\bar{a})}{\beta}-F(\beta)(1-\Phi_2(t))-\frac{\alpha}{\alpha-1}F(\beta)\int_{|u|\le 1} i(t,u)\mu(du)+o(F(\beta))$$

where

$$\Phi_2(t)=\int_{|u|\le 1}\Big(e^{i(t,u)}-1-i(t,u)\Big)d|u|^{-\alpha}\mu\big(\frac{u}{|u|}\big)$$

From this and since $\beta^{-2}=o(F(\beta))$ we find

$$\eta(\beta)=(1-\beta^{-1}i(t,\bar{a})+o(F(\beta)))\psi(\beta)$$

$$=1-F(\beta)\Big(1-\Phi^2(t)+\frac{\alpha}{\alpha-1}\int_{|u|=1} i(t,u)\mu(du)\Big)+o(F(\beta))$$

Now, in the same way as (i) we can write

$$\lim_{n\to\infty} E(expi(t,T_{nk})=\lim_{n\to\infty} n\binom{n-1}{k-1}\int_0^{nF(\beta)}\int_{|u|=1}(1-F(\beta))\Big(1-\Phi_2(t)+$$

$$\frac{\alpha}{\alpha-1}\Big(\int_{|u|=1} i(t,u)\mu(du)\Big)+o(F(\beta))\Big)^{n-k}F^{k-1}(\beta)dF(\beta)\mu(u)$$

$$=\lim_{n\to\infty} n\binom{n-1}{k-1}\int_0^n\Big(1-\frac{v}{n}\big(1-\Phi_2(t)+\frac{\alpha}{\alpha-1}\int_{|u|=1} i(t,u)\mu(du)\big)$$

$$+vo(1/n)\Big)^{n-k}v^{k-1}n^{-k}dv\int_{|u|=1}\mu(du)$$

$$+\lim_{n\to\infty}\frac{n\binom{n-1}{k-1}}{n^k}\int_0^n exp\Big(-v\big(1-\Phi_2(t)+\frac{\alpha}{\alpha-1}\int_{|u|=1} i(t,u)\mu(du)\big)\Big)v^{k-1}dv$$

Finally, we have

$$\lim_{n\to\infty} E(exp\ i(t,T_{nk}))=\Big(1-\Phi_2(t)+\frac{\alpha}{\alpha-1}\int_{|u|=1} i(t,u)\mu(du)\Big)^{-k},\qquad 1<\alpha<2$$

Theorem 5 is proved.

The proof of Theorem 6 is similar and is omitted.

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