ERROR CONTROL, LOCAL GRID REFINEMENT, AND EFFICIENT SOLUTION ALGORITHMS FOR SINGULARLY PERTURBED PROBLEMS

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Error control, local grid refinement, and efficient solution algorithms for singularly perturbed problems *

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Abstract

We consider boundary value problems for both singularly perturbed elliptic equations of second order with boundary layers and problems with singular solutions. In the former class two important problems, namely convection-dominated and reaction-dominated problems, have been considered. In the latter class we consider problems with singular solutions due to corners of the boundary or discontinuity of the coefficients. Our goal is to develop, implement, and test an efficient computational methodology for numerically solving such problems with guaranteed accuracy. This has been achieved by utilizing the recent advances in finite element approximations, *a posteriori* error estimators and indicators, local refinement techniques, and efficient solution methods for solving the resulting algebraic system of equations.

A series of numerical experiments showing the results of the computations for various model problems are presented. From these numerical experiments one concludes that the proposed techniques provide a reliable and efficient solution framework for a large class of steady-state problems.

1 Introduction

In this paper we shall discuss adaptive techniques for error control in the finite element approximations of the following model diffusion-convection-reaction problem: Find u satisfying

$$-\nabla \cdot A\nabla u + \mathbf{b} \cdot \nabla u + cu = f \ \text{ in } \ \Omega, \ \ u = 0 \text{ on } \partial\Omega. \tag{1}$$

Here Ω is a bounded polygonal domain in $\mathcal{R}^2$ with boundary $\partial\Omega$, A is a symmetric, bounded and uniformly positive definite matrix in Ω , $\mathbf{b} = \mathbf{b}(x) \in \mathcal{R}^2$ is a given vector and $c = c(x)$ and $f(x)$ are given functions. We assume that the problem has unique solution in $H_0^1(\Omega)$. A sufficient condition for the existence and uniqueness of the solution is, for example, $-0.5\nabla \cdot \mathbf{b}(x) + c(x) \geq c_0 > 0$ in Ω, where c_0 is a constant. For more precise conditions we refer to [15, 16, 23]. Our goal is to design, implement, and experimentally verify a computational methodology for the case when A is small relative to c and $\mathbf{b}$. Thus, we shall consider the simplest case when $A = \varepsilon I$, with $\varepsilon > 0$ sufficiently small, i.e., the so-called singularly perturbed problems. We shall discuss numerical methods for two main cases: (1) reaction dominated diffusion-reaction problem, i.e., $\mathbf{b} \equiv 0$; (2) convection-dominated convection-diffusion problem, i.e., $c \equiv 0$. In the first case, in general, we have boundary layer along the whole boundary $\partial\Omega$, while in the second case we may have internal layers and a boundary layer only along the part of the boundary characterized by $\mathbf{b} \cdot \mathbf{n} \geq 0$. Here $\mathbf{n}$ is the outer unit normal vector to the boundary $\partial\Omega$.

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Boundary value problems for singularly perturbed equations of this type often appear in modeling steady-state processes of diffusion in reactive flows, heat and mass transfer, flows in porous media, etc. For example, singularly perturbed convection-diffusion equations model stationary processes in flows when the heat transfer coefficient is small compared with the typical velocity of the medium. If the diffusion coefficient ε equals zero, such elliptic equations degenerate into first-order transport equations.

There are many techniques for solving singularly perturbed elliptic equations. One question is how to find a "good" mesh for a given problem. This in turn is related to the question of how accurately we can obtain information about the unknown solution from both the given data (domain, coefficients of the differential equation, boundary conditions and boundary data) and the approximate solution using some numerical or analytical technique.

Some of the most important developments in this direction are the asymptotic analysis of the solution and various techniques for obtaining expansions with respect to the small parameter. This information then can be used to construct *a priori* the type of the mesh. For large classes of problems this will lead to the so-called Shishkin meshes (see, for example, the monographs of Miller, O'Riordan, and Shishkin [18], H.-O. Ross, M. Stynes, and L. Tobiska [20], or the survey paper of H.-O. Ross [19]). This type of technique is very successful for steady-state singularly perturbed problems and leads to excellent results when combined with relevant computational methods. Mostly the grids are piece-wise uniform, which can also reduce the storage requirement and the solution cost. However, *a priori* fitted meshes have certain limitations when it comes to embedding the numerical method in a general computational procedure for multidimensional multicomponent problems. In such cases a *posteriori* error analysis and adaptive grid refinement are essentially the only practical alternative.

In this paper, we use a purely computational approach. This means, that neither the numerical method nor the initial computational grid use any information about the structure of the solution or its behavior. The grid is chosen during the solution process in a fully adaptive way, namely, refined in the areas where the *a posteriori* error indicator suggests. This approach, proposed in the works of Babuska and Rheinboldt (see, e.g., [3, 4]), has been subject to intensive research and development in the past decade (see, [1, 2, 6, 7, 8, 9, 10, 11, 13, 23, 25, 26]) and is related to local grid refinement, *a posteriori* error estimators and indicators. We point out that our aim is to control the error in the whole domain. This is much simpler than the case when one is interested in the solution only on some subdomains (or in some particular linear functionals of the solution, say the total flux across certain manifold or its value at a prescribed point). Such adaptive strategies can be found in [5, 11].

Our computational strategy is based on the finite element method for solving the problem (1). Namely, we rewrite (1) into the weak form: find $u \in H_0^1(\Omega)$ which satisfies

$$a(u,v) \equiv (\varepsilon \nabla u, \nabla v) + (\mathbf{b} \cdot \nabla u, v) + (cu, v) = (f, v)$$

for all $v \in H_0^1(\Omega)$. Here $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$.

The domain Ω is split into triangles, called finite elements and denoted by K. The triangulation is denoted by $\mathcal{T}_h$ and the diameter of $K \in \mathcal{T}_h$ is denoted by h_K. Next, we introduce the space V_h a finite dimensional subspace of $H_0^1(\Omega)$ of piece-wise linear functions over the partition $\mathcal{T}_h$. The finite element solution u_h is computed by Galerkin method. Namely, we introduce an approximation $a_h(u,v)$ of the form $a(u,v)$ in V_h and find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, v_h) \text{ for all } v_h \in V_h.$$

For the case when there is no convection or small convection $a_h(u,v)$ is simply $a(u,v)$ evaluated over V_h. For convection-dominated problems this approximation gives strongly oscillating numerical results. Regarding finite element schemes for convection-dominated problems our interests are in methods that yield solutions satisfying the discrete maximum principle. Such schemes are also known as monotone schemes. A well-known sufficient condition for a scheme to be monotone is that the corresponding stiffness matrix be an M–matrix. This motivated our choice of the Tabata scheme

[21] for the case of convection-dominated problems. This scheme is based on upstream weighting approximation of the convection term and produces good computational results on quite general grids. In this scheme an upwind element K_i is assigned to each vertex P_i (a vertex of K_i): K_i is an upwind element for P_i if the side opposite to P_i in K_i meets the oriented half line with end at P_i and direction of $\mathbf{b}(P_i)$. Thus, the approximation of the convection term is given by the following procedure. Given nodal basis function ϕ_i,

$$(\mathbf{b} \cdot \nabla u_h, \phi_i) = m(P_i)\mathbf{b}(P_i) \cdot \nabla u_h\big|_{K_i},$$

where $m(P_i)$ is the area of the barycentric domain around P_i (one third the area sum of the triangles having P_i as vertex). Multiplying by $v_h(P_i)$ and summing over all internal mesh nodes gives the desired approximation to the convection term and the approximation $a_h(u_h, v_h)$ of the bilinear form $a(u, v)$. The derivation and analysis of this scheme can be found in [15]. Other good choices of monotone schemes are the stream-line upwind Galerkin method (SUPG scheme) of Franca, Frey, and Hughes [14] and the scheme of Xu and Zikatanov [24], which constructs a finite element discretization by an appropriate averaging of the differential equation coefficients on the element edges.

Our goal is to implement a grid refinement strategy based on local error estimators which will produce an approximation u_h to the solution u within a given tolerance δ in a certain norm. We use three different norms to measure the error $u - u_h$. Namely, the discrete maximum norm of the first derivatives (section 2.1), the discrete energy norm (sections 2.2 and 2.4) which is equivalent to the norm in the Sobolev space H^1, and the averaged L^2-norm of the gradient (in fact this is the gradient multiplied by the conductivity matrix, so this measures the error of the heat flux, section 2.3).

All of the techniques presented are embedded into multilevel solution techniques for the corresponding system of linear equations. We add the following multigrid settings to the general solution procedure and notations described above . Starting from a quasi-uniform mesh of moderate size the grid refinement strategy yields a sequence of nested triangulations $\mathcal{T}_k, k = 0, \cdots, J$, where $\mathcal{T}_J \equiv \mathcal{T}_h$. In terms of these nested triangulations, we define corresponding nested approximation spaces

$$V_0 \subset V_1 \subset \ldots \subset V_J \equiv V_h,$$

which are used to define multilevel preconditioners. All computational results in this article have been obtained by using *Richardson* and *Jacobi* smoothers. The theoretical justification of multigrid algorithm for the case of reaction dominated problems on locally refined grids can be found in [12].

The proposed techniques have been tested on various analytical examples. The computational results show that the considered technique gives an efficient methodology for solving convection-dominated and reaction-dominated problems and problems with corner singularities.

Our current implementation is for finite element and finite volume element approximations of 2-D second order convection-diffusion problems on arbitrary triangulations of the domain and linear finite elements. A work on 3-D problems is in progress.

2 A posteriori error estimators and indicators

The behavior of the physical process is greatly affected by local properties (coefficients, sources, and boundary data) as well as the singularities due to corners, boundary layers or nonlinear behavior. For such cases it is essential that the numerical method has capabilities to resolve the local behavior of the solution. In the context of the finite element method there are two main techniques for the error reduction. One is based on increasing the order of the algebraic polynomials used in the approximation process, the so called "$p-$version" of the finite element method (p–refinement), while the other uses polynomials of the same degree, but adaptively refines the grid (by decreasing the mesh size h), the so called "h–version" adaptive refinement (h–refinement).

Here we present four well known and widely used error indicators based on the h–version of the finite element method. Namely, we present indicators based on: the second derivatives, the residual, Zienkiewicz-Zhu technique, and hierarchical refinement.

2.1 Error estimator based on second derivative (SD)

This technique (see [13]) aims to control the size of certain second derivatives in the maximum norm (denoted by *SD Refinement*). To be more precise the method tries to control the gradient of the error in the maximum norm. The error control is based on an optimal *a priori* estimate of the form

$$\|\nabla(u-u^h)\|_{\infty,\Omega} \leq C_0 \max_{K\in T_h} h_K \sup_{|\alpha|=2,\ x\in K} |D^\alpha u(x)| \ , \tag{2}$$

where D^α is the multi–index notation for derivatives of order $|\alpha|$, C_0 is a positive constant, assumed to be known. The goal to control the right hand side of (2) with a tolerance δ leads to a choice of h_K such that $C_0 h_K \sup_{|\alpha|=2,\ x\in K} |D^\alpha u(x)| \sim \delta$. The quantity $\sup_{|\alpha|=2,\ x\in K} |D^\alpha u(x)|$ is approximated locally by using certain local difference quotients of u^h

$$D_H^2(u^h;K) := \max_{x\in K} D_H^2 u^h(x) \approx \sup_{|\alpha|=2,\ x\in K} |D^\alpha u(x)| \ .$$

We use the following approximation of D_H^2

$$D_H^2(u^h;K) = \max_{|\alpha|=1,\ K'\in N(K)} \frac{|D^\alpha u^h(P_K) - D^\alpha u^h(P_{K'})|}{|P_K - P_{K'}|} \ , \tag{3}$$

where $N(K)$ is the set of neighboring to K finite elements, P_K is the center of gravity of K. If $h_K D_H^2(u^h;K) \leq \delta$ then the triangle is not refined, otherwise we split it into 4 equal triangles. This algorithm is justified theoretically in [13] and the problem of estimating the constant C_0 is considered in order to make the error control fully quantitative.

2.2 Residual based (RB) error estimator

Estimates of this type are first introduced by Babuska and Rheinboldt [3]. Here we describe the residual method (denoted by *RB Refinement*) as given by Becker *et. al.* in [10], [11], and Verfürth [23]. This method is based on equilibrating certain residuals. The main idea we illustrate on the model problem (1) with $A = \epsilon I$ and $\mathbf{b} = 0$. Using the Galerkin orthogonality $a_h(e, v_h) = 0$ for all $v_h \in V_h$, where $e = u - u_h$, and integration by parts yields the identity

$$a_h(e,z) = \sum_{K\in T_h} \left\{ (f - cu_h + \epsilon \triangle u_h, z - z_h)_K - \frac{1}{2}(\mathbf{n}\cdot[\epsilon\nabla u_h], z - z_h)_{\partial K} \right\}$$

for any $z \in H_0^1$, where $[\nabla u_h]$ denotes the jump of ∇u_h across the element boundary and $z_h \in V_h$ is a suitable approximation of z. Applying Hölder's inequality on each element leads to

$$|a_h(e,z)| \leq \sum_{K\in T_h} \rho_K \omega_K$$

with the local residuals ρ_K and weights ω_K defined by

$$\begin{aligned} \rho_K &:= h_K \|f - cu_h + \epsilon \triangle u_h\|_K + \tfrac{1}{2} h_K^{1/2} \|\mathbf{n}\cdot[\epsilon\nabla u_h]\|_{\partial K}, \\ \omega_K &:= max\left\{ h_K^{-1} \|z - z_h\|_K, h_K^{-1/2} \|z - z_h\|_{\partial K} \right\}. \end{aligned} \tag{4}$$

The local approximation properties of the finite elements ensure that there is a $z_h \in V_h$ such that $\omega_K \leq C_{i,K}||\nabla z||_K$, for all $z \in H_0^1(\Omega)$, where $C_{i,K}$ is an interpolation constant of size $C_{i,K} \approx 1$. Taking $z = e$ leads to the following error bound in the energy norm

$$||e||_1^2 \equiv (\nabla e, \nabla e) + (e, e) \leq C_S^2 C_I^2 \sum_{K \in T_h} \rho_K^2,$$

where C_S measures the stability properties of the dual problem $(\nabla\phi, \nabla z) = (e, \phi)$ in terms of the global H^2-*a priori* estimate $||z||_{H^2(\Omega)} \leq C_S||e||$. The mesh generation aims to equilibrate the local residuals ρ_K, i.e. for a given tolerance δ, the elements $K \in T_h$ are refined according to the criteria $\rho_K \approx \delta/(C_S C_I\sqrt{N})$, where N is the number of the finite elements in the mesh T_h.

2.3 Zienkiewicz-Zhu (ZZ) estimator

This error estimator is due to Zienkiewicz and Zhu [25, 26] known also as *ZZ estimator*. It is not only reasonably accurate but also very easy to compute. These characteristics make it very useful in practical engineering analysis. The estimator allows the global energy norm error and the local errors to be well estimated. A brief description of the method (as given in [25, 26]) follows.

The velocity vector (heat flux) can defined by $\sigma = -A\nabla u$. Let u_h be the finite element approximation of u. Then the finite element approximation of the velocity is given by $\sigma_h = -A\nabla u_h$ and the errors in the solution u and the velocity σ are denoted by $e = u - u_h$, $e_\sigma = \sigma - \sigma_h$. The error in the *energy norm* will be

$$||e||_E^2 = (A\nabla e, \nabla e) = (A^{-1}e_\sigma, e_\sigma). \tag{5}$$

A good approximation for the velocity σ in V_h can be obtained, for example, by nodal averaging of σ_h, i.e., the velocity at a given node is computed by averaging the velocities at the elements that share the considered node. Having the velocities at the nodes we denote our velocity approximation in V_h by σ^*. It is intuitively "obvious" that σ^* will be a better approximation of σ than σ_h, i.e. taking $e_\sigma \approx \sigma^* - \sigma_h$ we can evaluate $||e||_E$.

Although the norm $||e||_E$ is defined on the whole domain, we see that we can also express it as a sum of element contributions $||e||_E^2 = \sum_{i=1}^N ||e||_i^2$, where N is the number of the finite elements in $\mathcal{T}_h$. If we introduce the relative error $\eta = ||e||_E/||u||$ then the refinement strategy will be (as always) to equilibrate the errors $||e||_i$ over the different triangles, i.e. given the error η locally, refine the grid $\mathcal{T}_h$ in order to get $||e||_i \approx \eta||u||/\sqrt{N}$.

2.4 Hierarchical (HB) refinement

This last estimator is based on the use of hierarchical bases functions (denoted *HB refinement*). A brief description, as given in [8] follows.

For this technique, a finite element space is enriched by adding to it certain hierarchical basis functions. The new space is denoted by $\bar{V}_h = V_h \oplus W_h$, where W_h is the space corresponding to the span of the additional basis functions. The goal is to get estimate of the form

$$C_1||u - u_h||_1 \leq ||e_h||_1 \leq C_2||u - u_h||_1 \ , \tag{6}$$

where $u_h \in V_h$ is the approximate solution, $e_h \in W_h$ is the computed a *posteriori* error estimator and $||\cdot||_1$ is the energy norm. The a posteriori error estimator e_h is defined by: Find $e_h \in W_h$ such that

$$a(e_h, v) = f(v) - a(u_h, v) \quad \text{for all } v \in W_h. \tag{7}$$

In order to get (6), the following assumptions are made. First some notion of convergence is assumed $||u - u_h||_1 \to 0$ as $h \to 0$. Although the solution $\bar{u}_h \in \bar{V}_h$ is not explicitly computed in this method, it enters into the a *posteriori* analysis by assuming the following *saturation assumption*

$$||u - \bar{u}_h||_1 \leq \beta||u - u_h||_1 \ , \tag{8}$$

where $\beta < 1$ is independent of h. Because of the higher order of approximation in the space $\bar{V}_h$, usually one can anticipate $\beta = O(h^r)$, for some $r > 0$. The second assumption is that $\bar{V}_h$ has a hierarchical decomposition $\bar{V}_h = V_h \oplus W_h$. This means that for every $u \in \bar{V}_h$ there is a unique decomposition $u = v + w$, where $v \in V_h$ and $w \in W_h$. For the subspaces V_h and W_h the *strengthen Cauchy inequality* is assumed, i.e. for all $v \in V_h$ and $w \in W_h$

$$|a_h(v, w)| \leq \gamma ||v||_1 \; ||w||_1 \; , \tag{9}$$

where $\gamma < 1$ is independent of h.

An outline of the proof of (6) is given below. Using standard finite element orthogonality relationships and the saturation assumption one can show

$$(1 - \beta^2)||u - u_h||_1^2 \leq ||\bar{u}_h - u_h||_1^2 \leq ||u - u_h||_1^2 \; ,$$

i.e. $e = \bar{u}_h - u_h$ is a good representation to the error. The goal is to show that the more easily computed function e_h also yields a good approximation. In [8], Bank and Smith prove:

$$(1 - \beta^2)(1 - \gamma^2)||u - u_h||_1^2 \leq ||\bar{e}_h||_1^2 \leq ||u - u_h||_1^2 \; .$$

The right inequality is a simple consequence of the orthogonality relation $a_h(u - u_h - e_h, v) = 0$ for all $v \in W_h$ for the choice $v = e_h$. In the proof of the left inequality in addition to orthogonality relations and (8) also (9) is used (given in [8]).

3 Computational results

3.1 Reaction-diffusion problems

We will demonstrate the local refinement strategies performance on the reaction dominated diffusion problem (1) where $A = \varepsilon I$, $\mathbf{b} \equiv 0$, $c = 2$, Ω is the unit square and f is such that we have the following exact solution: $u(x_1, x_2) = x_1 x_2 (1 - exp(\frac{x_1 - 1}{\varepsilon}))(1 - exp(\frac{x_2 - 1}{\varepsilon}))$. For ε small, the solution has a boundary layer near the boundary $x_1 = 1$ and $x_2 = 1$. The local refinement strategies intuitively are supposed to produce much finer grids in the subdomain where the boundary layer is located. For $\varepsilon = 0.01$ the results are summarized in Table 1. Quadrature that preserves cubic polynomials was used in the computation of the discrete L^2 and H^1 norms and 13-point Gaussian quadrature (degree of precision 7) was used in the computation of the RHS.

level	# nodes	# slaves	$\|e\|_{max}$	$\|e\|_{L^2}$	$\|e\|_{H^1}$
0	25	0	0.26563	0.16796	13.12243
1	81	0	0.16181	0.12131	9.41188
2	289	0	0.05351	0.07652	6.43875
3	479	29	0.01053	0.03558	4.16403
4	1010	86	0.00174	0.01190	2.42569
5	2729	203	0.00043	0.00328	1.28376
6	8506	440	0.00005	0.00088	0.65283
Globally uniform grid					
5	16641	0	0.00041	0.00327	1.28750

Table 1: Results for Problem (1) for $\varepsilon = 0.01$ using SD-refinement

Figures 1 and 2 give the computational results for reaction-diffusion problem solved using *SD*-error estimator with 5 levels of refinement. As expected, the grid is refined in the boundary layer.

The error indicator determines correctly where the error is. It is equilibrated on level 5 as seen in Figure 2. The relative error $||e||/||u||$ on level 6 in discrete maximum, L^2 and H^1 norms is correspondingly 0.0005, 0.0028 and 0.11. For comparison we have also computed an approximation by using a uniform grid. On level 5 this grid has 16641 nodes and the error in L^2-norm is 0.003276, while the error in H^1-norm is 1.28. The accuracy is comparable to the accuracy of the locally refined grid on level 5 with 2729 nodes. Note, that the locally refined grid has an order of magnitude less grid points than the uniform grid.

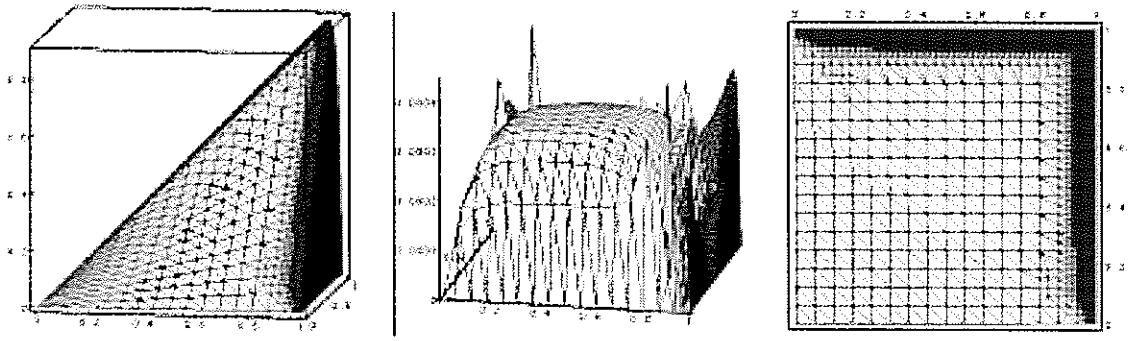

Figure 1: Reaction-diffusion problem with $\varepsilon = 0.01$; the approximate solution (left), the error (middle) and the mesh (right) obtained for level 5 with 2729 grid points and 203 slave nodes

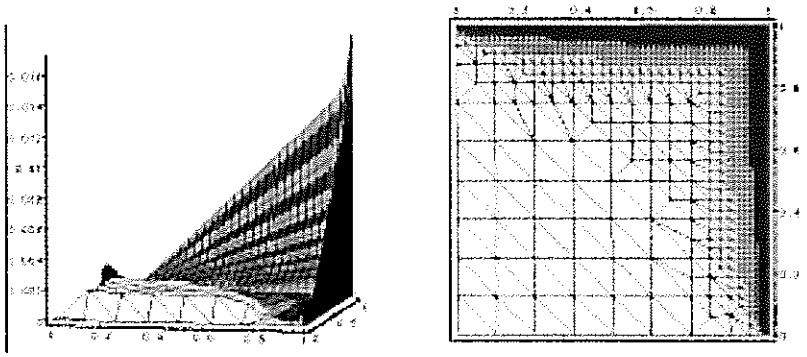

Figure 2: Reaction-diffusion problem with $\varepsilon = 0.001$; the error (left) and the mesh (right) obtained for level 5 with 2385 grid points by SD-refinement (0.3 tolerance)

The other error indicators give similar results. There are small differences in the obtained meshes (for more examples and meshes we refer to our Technical Report [17]). However, qualitatively the meshes are refined in the area where the solution has some type of singularity (or very steep gradient) and quantitatively they have almost the same number of nodes. However, the more singular the problem is the larger the error of the finite element approximation is. Comparing the Figures 1 and 2 we see that while the solutions of reaction-diffusion problems with $\epsilon = 0.01$ on a grid with about 2800 nodes has maximal error of 0.05% the solution for the same problem with $\epsilon = 0.001$ on a grid with about 2400 nodes has a maximal error of about 2%. Similar results were obtained by using other types of refinement. For example, the meshes on level 5 using *HB* and *ZZ refinement*, are given on Figure 3.

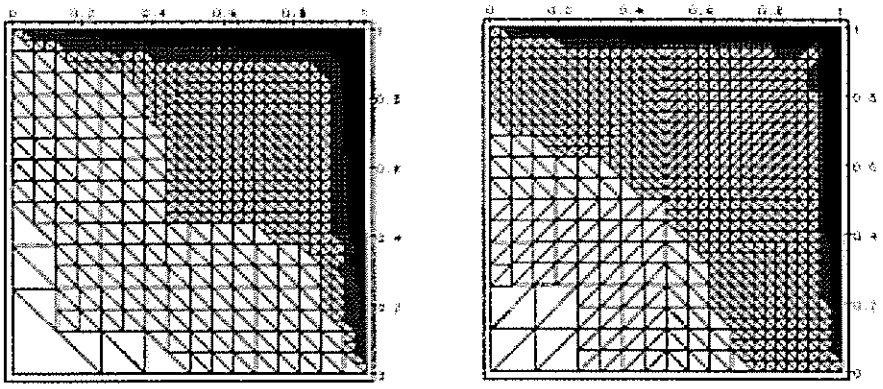

Figure 3: Reaction-diffusion problem with $\varepsilon = 0.001$; meshes obtained by HB-refinement (left) 2013 grid points and 194 slave nodes and ZZ-refinement (right) 2031 grid points and 194 slave nodes at level 5

3.2 Convection-diffusion problems

Figures 4 and 5 show the computational results obtained on convection dominated problems by Tabata's scheme. Namely, we solve the problem (1) with Dirichlet data on the whole boundary $\partial\Omega$, $A = \varepsilon I$, $\mathbf{b} \equiv (2,1)$, $c = 0$, $f = 0$ in a trapezoid. On the left and on the upper edges of Ω we have zero boundary conditions while on the rest of the boundary the solution is equal to 1. The solution develops a boundary layer at the upper half of the right edge of Ω and an interior layer along the line $x_1 = 2x_2$. Also there are two corner singularities at the origin and at the upper right corner due to the discontinuity of the Dirichlet data. Figures 4 and 5 represent the computational meshes and the solutions for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-5}$, correspondingly. The computations are obtained using RB-refinement for the case $\varepsilon = 10^{-3}$ and ZZ refinement for $\varepsilon = 10^{-5}$. As seen from the figures the error estimator produces finer grids in the regions where the solution has a boundary layer or singularity. The computed solutions are monotone so no oscillations due to the numerical approximation are produced.

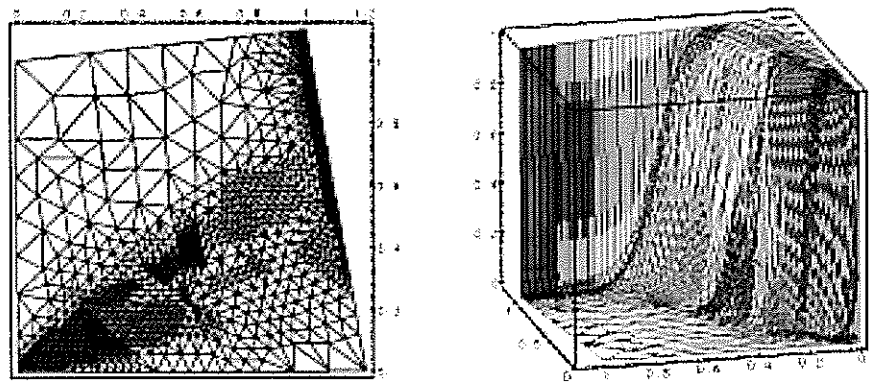

Figure 4: Convection-diffusion problem with $\varepsilon = 10^{-3}$; the mesh (left) with 1209 grid points and 234 slave nodes and solution (right) obtained at level 4

We have also experimented and produced similar results with another monotone scheme proposed by Xu and Zikatanov in [24], which also appears to be computationally effective in solving convection-

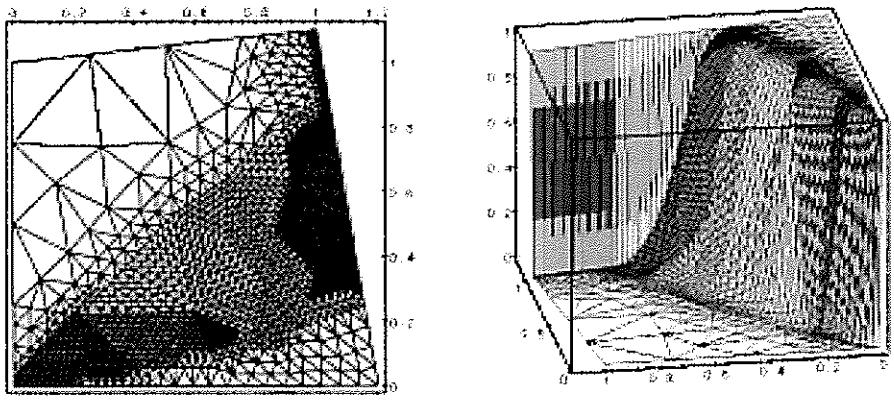

Figure 5: Convection-diffusion problem with $\varepsilon = 10^{-5}$; the mesh (left) with 1761 grid points plus 164 slave nodes and the solution (right) obtained at level 4

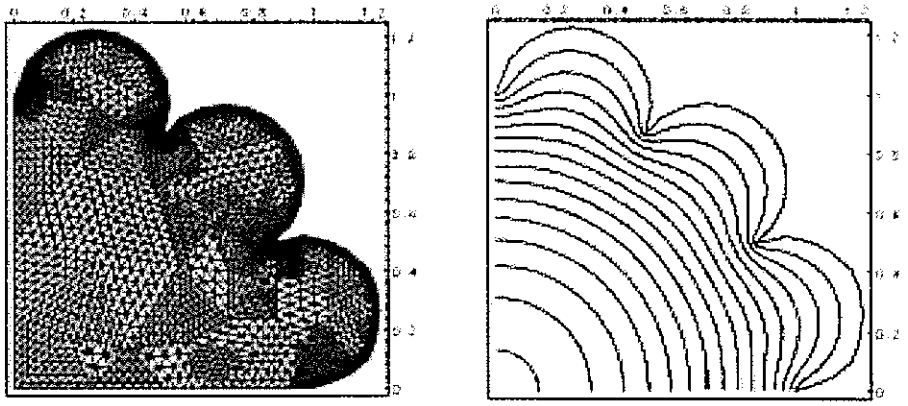

Figure 6: The mesh (left) with 2562 grid points and 410 slave nodes and the level curves (right)

dominated problems. In this scheme the coefficients of the differential equation are "properly" averaged on element edges so that the resulting finite element stiffness matrix is an M-matrix. For a detailed description and analysis of this scheme we refer to [24].

3.3 Problems with corner singularities

Finally, we consider elliptic problems with corner singularities. Our first problem is the model problem with $A = I$, $\mathbf{b} \equiv 0$, $c = 0$, $f = 1$ in a "rose"-shaped domain with homogeneous Dirichlet boundary conditions. Due to the symmetry we can consider only a quarter of the domain (in fact we could have considered only $1/12$ of the domain). The isolines of the computed unknown solution and the corresponding grid are given on Figure 6. Here we have used RB refinement. As expected, the grid is refined around the corners since the solution has corner singularities.

Next, we solve the problem (1) with $A = I$, $\mathbf{b} \equiv 0$, $c = 0$, $f \equiv 0$. Ω is the L–shaped domain as shown on Figure 7. Here the singularity is located at the origin. For this problem we take Dirichlet boundary data so that the exact solution is $u(x_1, x_2) = r^{2/3} \sin \frac{2\theta}{3}$, where $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ are the polar coordinates in the plane (x_1, x_2). Computational meshes were produced using ZZ-estimator. Table 2 gives the relative errors for the refined grid on the consecutive refinement levels. The relative errors (in different norms) for uniform refinement are given as well. Plots of mesh and

the error on level 4 are given on Figure 7.

Finally, we consider the Laplace equation in a slit domain. Namely, in the square $\Omega = \{(-1,1) \times (-1,1)\}$, we make a cut along the positive x_1-axis. Thus, the boundary now consists of all four sides of Ω plus the points in the interval $\{(x_1,0) : 0 < x_1 < 1\}$. Dirichlet boundary conditions are prescribed on the whole boundary of the slit domain so that the exact solution is $u(x_1,x_2) = r^{1/2}\sin(\frac{\theta}{2})$. The singularity at the origin is $O(r^{1/2})$. Figure 8 gives the computational mesh on level 9 obtained using SD-refinement and the corresponding error. For this level the relative errors in maximum, L^2- and H^1-norms are 0.59%, 0.12% and 1.07%, correspondingly.

Both problems have symmetry, which allows to use only half of the domain. In fact, we have intentionally used the whole domain in order to see how the refinement procedure works. We see that the obtained meshes are very symmetric.

Locally refined grid					
level	# nodes	# slaves	L^2-error	H_1-error	max-error
1	153	0	0.0087	0.0375	0.0224
2	447	20	0.0038	0.0243	0.0147
3	608	48	0.0016	0.0156	0.0094
4	647	62	0.0008	0.0104	0.0060
5	659	70	0.0006	0.0077	0.0038
6	664	74	0.0005	0.0066	0.0025
Globally uniform grid					
3	2145	0	0.0016	0.0154	0.0094

Table 2: Numerical results for the homogeneous Poisson equation with Dirichlet boundary conditions and solution $u = r^{2/3}\sin\frac{2\theta}{3}$ in L-shaped domain; the results are obtained using ZZ refinement.

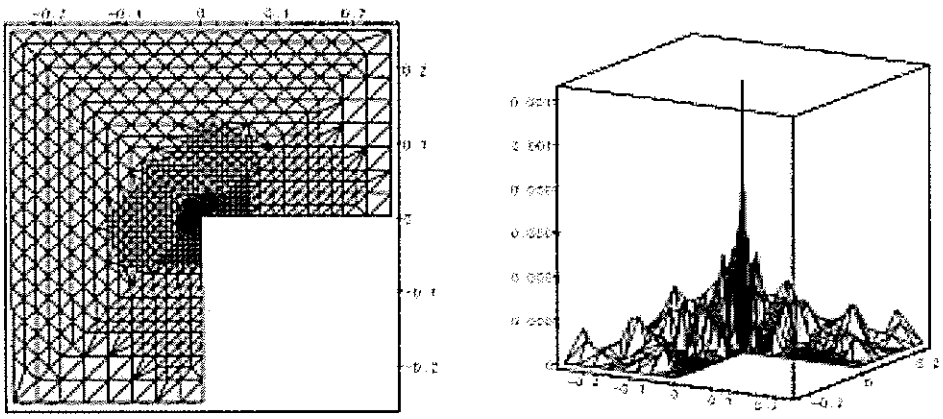

Figure 7: Poisson equation in L-shaped domain with solution $u = r^{2/3}\sin(\frac{2\theta}{3})$ computed by ZZ refinement; the mesh (left) with 659 grid points and 70 slave nodes and the error (right) for level 5.

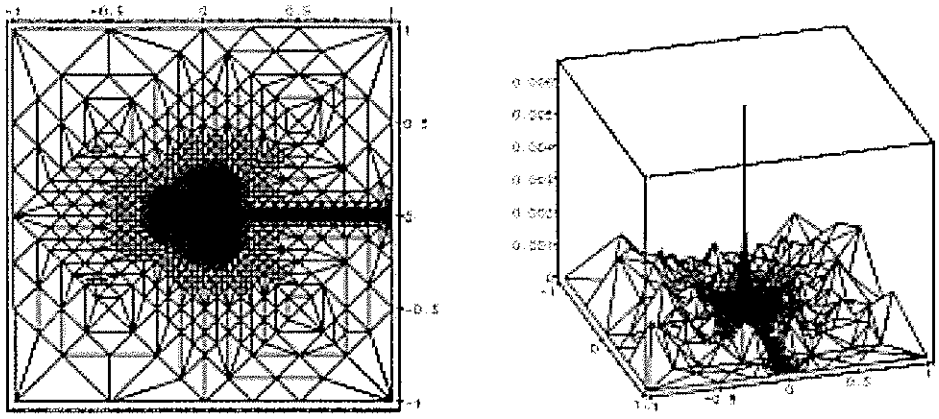

Figure 8: Poisson equation in a slit domain with solution $u = r^{1/2}\sin(\frac{\theta}{2})$; the mesh (left) with 2064 grid points and 556 slave nodes and the error (right) for level 9.

4 Conclusions

We have presented a computational technology for adaptively solving singularly perturbed second order elliptic problems and problems with singular solutions. We start with an initial triangulation which describes adequately the given problem (domain, coefficients, boundary conditions, and right hand side). During the solution process the grid is refined (based on a criteria formulated from one of the four error estimators) until maximum refinement level is reached or the local error is found below a given threshold δ. Obviously, the grids obtained from the four error estimators differ slightly, but in all cases they are refined in the areas of singular behavior of the solution and produce comparable results.

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