Least-squares streamline diffusion finite element approximations to singularly perturbed convection-diffusion problems

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Abstract

In this paper we introduce and study a least-squares finite element approximation for singularly perturbed convection-diffusion equations of second order. By introducing the flux (diffusive plus convective) as a new unknow the problem is written in a mixed form as a first order system. Further, the flux is augmented by adding the lower order terms with a small parameter. The new first order system is approximated by the least-squares finite element method using the minus one norm approach of Bramble, Lazarov, and Pasciak [2]. Further, we estimate the error of the method and discuss its implementation and the numerical solution of some test problems.

1 Introduction

Mathematical models in physics and engineering often lead to differential equations with coefficients that may differ by several orders of magnitude. Such problems can be found when modeling processes in chemical kinetics, transport of heat and mass, plate bending etc. Important characteristic of such problems is that the solution are highly localized by exhibiting boundary and internal layers, point and line singularities etc. The aim of a numerical technique for solving such problems is to find a mesh which resolves these localized phenomena. This in turn is related to the question how accurately (and inexpensively) one can obtain information about the solution. A reasonable approach should include both a priori analysis of the problem and its solution and a posteriori analysis of the computational results in order to verify their accuracy and subsequently improve the results by refining the mesh.

The a priori analysis can be used to a priori construct the mesh. For the state-of-the-art research in this direction we refer to the monographs of Miller, O'Riordan, and Shishkin [13], H.-O. Ross, M. Stynes, and L. Tobiska [16]. The a posteriori analysis is used to construct the "best" mesh for the solution of a particular problem within given tolerance for the error. Practically, this means that starting with a very coarse initial mesh, further in the solution process, the mesh is refined in a fully adaptive way, namely new grid points are added in the areas where the a posteriori error estimators and indicators suggest. For studies in this direction we refer to the monographs of I. Babuška, O. C. Zienkiewicz [1] and R. Verfürth [19].

The aim of the present paper is to derive and study unconditionally stable approximations of singularly perturbed problems of second order based on least-squares finite element method. Using

*This research was done at the Center of Applied Scientific Computing (CASC), Lawrence Livermore National Laboratory while R. Lazarov was on a sabbatical leave from TAMU; the financial and technical support of CASC and the partial financial support of National Science Foundation Grant No. DMS-9626567 are greatly appreciated

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a minus one inner product we derive stable approximations of the problem in mixed form. Further, we derive a priori error estimates under minimal smoothness of the solution.

We consider the following singularly perturbed problem: find $w \in H_0^1(\Omega)$, such that

$$Lw \equiv -\nabla \cdot (\epsilon \nabla w + \mathbf{b}w) + c_0 w = f \quad \text{in} \quad \Omega, \quad \text{and} \quad w = 0 \quad \text{on} \quad \partial\Omega. \tag{1}$$

We assume that Ω is bounded domain in $\mathbf{R}^d$, $d = 2, 3$ with Lipschitz boundary $\partial\Omega$ and $c_0(x)$ and $\mathbf{b}(x)$ (a vector column) satisfy the following condition:

$$c_0(x) - \frac{1}{2}\nabla \cdot \mathbf{b}(x) \geq \beta_0 = \text{const} > 0 \text{ for all } x \in \Omega. \tag{2}$$

Without loss of generality, we can take $\beta_0 = 1$. This condition guarantees that the bilinear form defined on $H_0^1(\Omega)$ by

$$A(w, v) = (\epsilon \nabla w, \nabla v) + (\mathbf{b}w, \nabla v) + (c_0 w, v). \tag{3}$$

defined on $H_0^1(\Omega)$ is coercive.

The weak form of (1) will be: find $w \in H_0^1(\Omega)$, such that

$$A(w, v) = (f, v) \quad \text{for all } v \in H_0^1(\Omega). \tag{4}$$

By Lax-Milgram theorem, this problem has a unique solutions in $H_0^1(\Omega)$ for any $f \in L^2(\Omega)$, and the solution satisfies the a priori estimate $\epsilon\|\nabla w\|^2 + \|w\|^2 \leq \|f\|^2$.

We assume that ϵ is a positive, but small parameter, i.e. $0 < \epsilon \ll |\mathbf{b}|$. Our goal is to develop a numerical method, based on the least-squares approximation, which is unconditionally stable and convergent under minimal regularity of the solution u.

There are several known discretization methods for convection-dominated diffusion problems. Our approach is based on the one of the most powerful techniques for such problems: the streamline diffusion finite element method originating in the paper of Hughes and Brooks [9]. This methods has the advantage over the more classical Galerkin method and the artificial diffusion methods in the fact that it allows to obtain convergence in a norm that contains and additional term $\delta^{1/2}\|\mathbf{b} \cdot \nabla u\|$. Then taking $\delta \simeq h$ we get a suboptimal convergence rate of $O(h^{3/2})$ for sufficiently smooth solution u. For the analysis of the streamline diffusion finite element method we refer to [11] and for various extensions to incompressible Euler equations and Navier-Stokes equations to [12, 17].

Another approach is based on the mixed formulation. In many applications, the flux (diffusive plus convective) plays an essential role. Introducing the physical flux as a new unknown the problem (1) is written in the mixed form

$$\underset{\sim}{\lambda}_0 = -\epsilon \nabla w - \mathbf{b}w, \quad \nabla \cdot \underset{\sim}{\lambda}_0 + c_0 w = f. \tag{5}$$

Here $\underset{\sim}{\lambda}_0 \cdot \mathbf{n}$ has a meaning of the amount of heat transferred through a unit surface area with a unit normal (vector to the surface) $\mathbf{n}$.

One can numerically approach the problem (1) by the Galerkin method, either applied to the original equation (1) or to its mixed form (5). In both cases the analysis of the method when ϵ is very small (or $\epsilon \to 0$) is not well understood. It is well known that the Galerkin method for the equation (1) is not stable (unless the mesh step-size h is sufficiently small) and stabilization or special treatment of the convective term is necessary.

There are two competing techniques for approximation of the mixed system (5): (a) mixed finite element method based on its weak formulation (see, for example, [4]); (b) least-squares finite element method based on the least-squares formulation of (5) (see, for example, [2, 5, 6, 7, 14, 15]). Both approaches produce symmetric algebraic systems, but while the former produces indefinite system the latter leads to a positive definite one.

We here derive a stable least-squares finite element approximation of the problem (1) based on a modification of the system (5). Namely, following [14] we introduce a corrected flux by adding a weighted steam-line derivative with a small parameter δ and thus modify the mixed system. In general, this approach leads to stable Galerkin schemes with relatively good approximation properties (see, e.g. [9, 10, 11, 12]). Following [2], we introduce a least-squares method which uses the sum of the weighted L^2-norm of the equation for the new flux and the discrete H^{-1}-norm of the differential equation. For a computable H^{-1}-norm for the singularly perturbed problem we use the algebraically stabilized version of the hierarchical basis method (see, e.g. [18]). The new moment of the paper is that we modify this method to suit to a singularly perturbed problem and track the dependence of the coercivity constant on the small parameter ϵ. The use of the discrete H^{-1}-norm allows us to obtain quasi-optimal error estimates for H^2- and H^3-regular solutions. The method of [14] has the same convergence rate for H^3-regular solutions, but does not provide an estimate when the solution is only H^2-regular. This deficiency is common for some of the least-squares methods (see, e.g. [6, 15]).

Finally, in the last section we provide computational results that illustrate the asymptotic convergence rates of the method on a model problem with a smooth solution.

2 Streamline-diffusion approximation of the mixed system

First attempt is to correct the flux $\underline{\lambda}_0$ in such a way that adds more from the streamline derivative. Namely, we add to the flux the term $\mathbf{b}Lu$ weighted by small parameters $\delta > 0$ to get a new flux denoted by $\underline{\lambda}$

$$\underline{\lambda} = -\epsilon\nabla w - \mathbf{b}w + \delta\mathbf{b}Lw. \tag{6}$$

If we take divergence of the equation (6), add c_0u to the left hand side, and take into account that $\nabla\cdot(\mathbf{b}Lw) = \nabla\cdot(\mathbf{b}f)$, we get the following equation for $\underline{\lambda}$ and w:

$$\nabla\cdot\underline{\lambda} + c_0w = f + \delta\nabla\cdot(\mathbf{b}f) \equiv f_\delta. \tag{7}$$

The equations (6) and (7) were obtained from the original equation (1) and therefore the solution $(\underline{\lambda}, w)$ of (6) and (7) will satisfy (1).

Unfortunately, (7) is not an equation of second order (since Lw contains term $-\epsilon\Delta w$) and it does not fit into our approximation framework. In order to avoid this inconvenience, we make a modification of the equation. Namely, instead of the flux $\underline{\lambda}$ defined by (6), we introduce the truncated flux $\underline{\sigma}$ by replacing the operator Lw in (6) by its truncated version

$$\Lambda w \equiv -\nabla\cdot(\mathbf{b}w) + c_0w,$$

i.e. instead of the flux defined by (6) we consider the following “truncated flux”

$$\underline{\sigma} = -\epsilon\nabla u - \mathbf{b}u + \delta\mathbf{b}\Lambda u \quad \text{i.e.} \quad \underline{\sigma} = -\epsilon\nabla u - \mathbf{b}u + \delta\mathbf{b}\big(-\nabla\cdot(\mathbf{b}u) + c_0u\big)$$

but keep the right hand side of equation (7) the same: $f_\delta(x)$. Therefore, instead of the problem (1), we consider the following mixed system: find $\underline{\sigma} \in H(\mathrm{div};\Omega)$ and $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} \underline{\sigma} + \epsilon\nabla u + \mathbf{b}u - \delta\mathbf{b}\Lambda u &= 0 \quad \text{in } \Omega, \\ \nabla\cdot\underline{\sigma} + c_0u = f + \delta\nabla\cdot(\mathbf{b}f) &\equiv f_\delta \quad \text{in } \Omega. \end{aligned} \tag{8}$$

Obviously, the solution $(\underline{\lambda}, w)$ of the problem (6), (7) is not equal to the solution $(\underline{\sigma}, u)$ of the problem (8). One can easily estimate the difference $U = u - w$. Indeed, (8) can be reduced to

$$\nabla\cdot(-\epsilon\nabla u - \mathbf{b}u + \delta\mathbf{b}\Lambda u) + c_0u = f_\delta \quad \text{in } \Omega \tag{9}$$

while (6) and (7) will lead to

$$\nabla \cdot (-\epsilon \nabla w - \mathbf{b} w + \delta \mathbf{b} L w) + c_0 w = f_\delta \quad \text{in } \Omega. \tag{10}$$

By subtracting (10) from (9), we get the following problem for $U = u - w$:

$$\nabla \cdot (-\epsilon \nabla U - \mathbf{b} U + \delta \mathbf{b} \Lambda U) + c_0 U = \delta \epsilon \nabla \cdot (\mathbf{b} \Delta w) \text{ in } \Omega, \quad U = 0 \text{ on } \partial\Omega.$$

We multiply this equation by U and integrate over Ω. Taking into account the boundary condition we get the following inequality for U

$$\epsilon \|\nabla U\|^2 + \|U\|^2 + \delta \|\mathbf{b} \cdot \nabla U\|^2 + \delta (c_0 - \nabla \cdot \mathbf{b}), U^2) \le \delta \epsilon (\Delta w, \mathbf{b} \cdot \nabla U).$$

Let $\max_\Omega \|c_0 - \nabla \cdot \mathbf{b}\| = \beta_1$ and assume that $\delta > 0$ is sufficiently small so that $\beta_1 \delta \le 1/2$. Therefore

$$\epsilon \|\nabla U\|^2 + \frac{1}{2}\|U\|^2 + \delta \|\mathbf{b} \cdot \nabla U\|^2 \le \delta \epsilon (\Delta v, \mathbf{b} \cdot \nabla U) \frac{\delta}{2} \|\mathbf{b} \cdot \nabla U\|^2 + \frac{\delta \epsilon^2}{2} \|\Delta u\|^2$$

or

$$2\epsilon \|\nabla U\|^2 + \|U\|^2 + \delta \|\mathbf{b} \cdot \nabla U\|^2 \le \delta \epsilon^2 \|\Delta w\|^2. \tag{11}$$

We now introduce the following norm in $H_0^1(\Omega)$:

$$\|u\|_{1,*}^2 = \epsilon \|\nabla u\|^2 + \|u\|^2 + \delta \|\mathbf{b} \cdot \nabla u\|^2. \tag{12}$$

The estimate (11) essentially says that $\|U\|_{1,*} \le \epsilon \sqrt{\delta} \|\Delta w\|$, where w is the solution of the original problem (1). We shall approximate the problem (8) by finite element method on a grid with grid-size h. In this case one chooses $\delta \simeq h$ and assumes that the diffusion parameter $\epsilon < h$, which results in the following estimate for the difference between the solutions of the problem (1) and (8): $\|U\|_{1,*} \le C h^{3/2} \|\Delta w\|$. This difference is in general smaller than the error of the finite element approximation of the problem (8). Further in this paper we shall consider the problem (8).

We show that the problem (9) (which is equivalent to (8)) has unique solution in $H_0^1(\Omega)$. We first introduce the bilinear form

$$A(u,v) = \epsilon(\nabla u, \nabla v) + (\mathbf{b} u, \nabla v) + \delta(\mathbf{b} \cdot \nabla u, \mathbf{b} \cdot \nabla v) + \delta\big((\nabla \cdot \mathbf{b} - c_0)u, \mathbf{b} \cdot \nabla v\big) + (c_0 u, v),$$

which is obviously bounded in the norm $||.||_{1,*}$. Next, we prove that for sufficiently small $\delta > 0$ the form $A(u,v)$ is coercive in the norm $||.||_{1,*}$. Indeed, for $v = u$ we have

$$\begin{aligned} A(u,u) &= \epsilon \|\nabla u\|^2 + \left(\left(c_0 - \tfrac{1}{2}\nabla \cdot \mathbf{b}\right) u, u\right) + \delta \|\mathbf{b} \cdot \nabla u\|^2 + \delta((\nabla \cdot \mathbf{b} - c_0)u, b \cdot \nabla u) \\ &\ge \epsilon \|\nabla u\|^2 + \|u\|^2 + \delta \|\mathbf{b} \cdot u\|^2 - \delta \beta_1 \|u\| \, \|\mathbf{b} \cdot \nabla u\| \\ &\ge \epsilon \|\nabla u\|^2 + \tfrac{1}{2}\|u\|^2 + \left(1 - \tfrac{\delta \beta_1^2}{2}\right) \delta \|\mathbf{b} \cdot \nabla u\|^2. \end{aligned}$$

Thus, if $\delta \beta_1^2 \le 1$, then

$$2A(u,u) \ge \epsilon \|\nabla u\|^2 + \|u\|^2 + \delta \|\mathbf{b} \cdot \nabla u\|^2 \equiv \|u\|_{1,*}^2,$$

which is the required coercivity. The boundness of the bilinear and linear forms in the same norm follows easily. Then by Lax-Milgram lemma the problem (8) has unique solution $u \in H_0^1(\Omega)$.

3 Least-squares form of the streamline-diffusion system

We apply H^{-1}-norm least-squares method for the system (8): namely, we seek the minimum of the quadratic functional

$$J(\underline{\sigma}, u) = \epsilon^{-1}\|\underline{\sigma} + \epsilon\nabla u + \mathbf{b}u - \delta\mathbf{b}\Lambda u\|^2 + \|\nabla\cdot\underline{\sigma} + c_0 u - f_\delta\|^2_{-1,*} \tag{13}$$

over the space $H(\text{div};\Omega)\times H^1_0(\Omega)$. Here $\|\cdot\|_{-1,*}$ is the dual of the norm (12) and is defined by

$$\|f\|_{-1,*} = \sup_{v\in H^1_0} \frac{(f,v)}{\|v\|_{1,*}} \text{ for all } f\in H^{-1}(\Omega).$$

This norm is equivalent to the norm defined by $||f||^2_{-1,*} = (Tf, f)$. Here the operator $T:\ H^{-1}\to H^1_0$ is the solution operator of the problem: find $u\in H^1_0$ such that

$$D_*(u,v) \equiv \int_\Omega (\epsilon\nabla u\nabla v + uv + \delta\mathbf{b}\cdot\nabla u\ \mathbf{b}\cdot\nabla v)\,dx = \int_\Omega fv \text{ for all } v\in H^1_0. \tag{14}$$

The minimum $(\underline{\sigma}, u)\in H(\text{div};\Omega)\times H^1_0(\Omega)$ of the quadratic functional (13) will satisfy the following integral identity:

$$\mathcal{K}(\underline{\sigma}, u; \underline{\chi}, v) = (Tf_\delta, \nabla\cdot\underline{\chi} + c_0 v) \text{ for all } (\underline{\chi}, v)\in H(\text{div};\Omega)\times H^1_0(\Omega), \tag{15}$$

where

$$\mathcal{K}(\underline{\sigma}, u; \underline{\chi}, v) \equiv \epsilon^{-1}(\underline{\sigma} + \epsilon\nabla u + \mathbf{b}u - \delta\mathbf{b}\Lambda u, \underline{\chi} + \epsilon\nabla v + \mathbf{b}v - \delta\mathbf{b}\Lambda v) + (T(\nabla\cdot\underline{\sigma} + c_0 u), \nabla\cdot\underline{\chi} + c_0 v).$$

We show that $\mathcal{K}(\cdot;\cdot)$ is bounded and coercive in the norm $\|\cdot\|_{1,*}$. It is important to find the constants of equivalence.

Lemma 3.1 *For sufficiently small δ the bilinear form $\mathcal{K}$ is bounded from below by $\|v\|^2_{1,*} + \|\nabla\cdot\underline{\chi}\|^2_{-1,*}$, namely*

$$\mathcal{K}(\underline{\chi}, v; \underline{\chi}, v) \geq c_0\left(\|v\|^2_{1,*} + \|\nabla\cdot\underline{\chi}\|^2_{-1,*}\right) \tag{16}$$

for all $\underline{\chi}\in H(\text{div};\Omega)$ and $v\in H^1_0(\Omega)$ with a constant $c_0>0$ independent of ϵ.

Proof 1 *We start with the term $\epsilon\|\nabla u\|^2$:*

$$\begin{aligned}
\epsilon(\nabla v, \nabla v) &= (\underline{\chi} + \epsilon\nabla v + \mathbf{b}v - \delta\mathbf{b}\Lambda v, \nabla v) - (\underline{\chi} + \mathbf{b}v - \delta\mathbf{b}\Lambda v, \nabla v)\\
&\leq \epsilon^{-1}\|\underline{\chi} + \epsilon\nabla v + \mathbf{b}v - \delta\mathbf{b}\Lambda v\|^2 + \frac{\epsilon}{4}\|\nabla v\|^2 - (\underline{\chi}, \nabla v)\\
&\quad + \frac{1}{2}(\nabla\cdot\mathbf{b}, v^2) + \delta(\mathbf{b}\Lambda v, \nabla v)\\
&\leq \epsilon^{-1}\|\underline{\chi} + \epsilon\nabla v + \mathbf{b}v - \delta\mathbf{b}\Lambda v\|^2 + \frac{\epsilon}{4}\|\nabla v\|^2 + (\nabla\cdot\underline{\chi} + c_0 v, v)\\
&\quad - \left(\left(c_0 - \frac{1}{2}\nabla\cdot\mathbf{b}\right), v^2\right) + \delta(\Lambda v, \mathbf{b}\cdot\nabla v)\\
&\leq \epsilon^{-1}\|\underline{\chi} + \epsilon\nabla v + \mathbf{b}v - \delta\mathbf{b}\Lambda v\|^2 + \frac{\epsilon}{4}\|\nabla v\|^2 - \|v\|^2\\
&\quad + \|\nabla\cdot\underline{\chi} + c_0 v\|_{-1,*}\|v\|_{1,*} - \delta\|\mathbf{b}\cdot\nabla v\|^2 + \delta((c_0 - \nabla\cdot\mathbf{b}), v^2).
\end{aligned} \tag{17}$$

We now choose δ such that $\delta\max_{x\in\Omega}|c_0(x) - \nabla\cdot\mathbf{b}(x)| \leq \frac{1}{4}$ and transfer the terms $\epsilon\|\nabla v\|^2, \|v\|^2$ and $\|\mathbf{b}\cdot\nabla v\|^2$ to the left hand side to get

$$\|v\|^2_{1,*} \leq 2\epsilon^{-1}\|\underline{\chi} + \epsilon\nabla v + \mathbf{b}v - \delta\mathbf{b}\Lambda v\|^2 + 2\|\nabla\cdot\underline{\chi} + c_0 v\|^2_{-1,*}. \tag{18}$$

Further,

$$\|\nabla\cdot\underline{\chi}\|_{-1,*} = \sup_{\phi\in H_0^1(\Omega)} \frac{(\nabla\cdot\underline{\chi},\phi)}{\|\phi|_{1,*}} \le \sup\frac{(\nabla\cdot\underline{\chi}+c_0v,\phi)}{\|\phi\|_{1,*}} + \sup\frac{(c_0v,\phi)}{\|\phi\|_{1,*}} \\ \le \|\nabla\cdot\underline{\chi}+c_0v\|_{-1,*} + \beta_2\|v\|, \tag{19}$$

where $\beta_2 = \max_{\Omega}|c_0(x)| =$ *const* ≥ 0. *The* $||v||$*-term is bounded by (18) so that (19) yields*

$$\|\nabla\cdot\underline{\chi}\|^2_{-1,*} \le 2(1+\beta_2^2)\|\nabla\cdot\underline{\chi}+c_0v\|^2_{-1,*} + 4\beta_2^2\epsilon^{-1}\|\underline{\chi}+\epsilon\nabla v+\mathbf{b}v-\delta\mathbf{b}\Lambda v\|^2.$$

This estimate combined with (18) gives the coercivity estimate (16), which completes the proof of the Lemma 3.1.

Remark 3.1 *The constant in coercivity estimate of the bilinear form* $\mathcal{K}$ *depends on the maximum value of the coefficient* $c_0(x)$*. If this coefficient is relatively large (i.e.* $c_0(x)\ge\beta_2>>1$*) then one should scale the zero order term in the norm with the constant* β_2*.*

Once the coercivity of $\mathcal{K}$ has been established, we need to show the boundness of $\mathcal{K}$ for $v\in H_0^1(\Omega)$ and $\underline{\chi}\in H(\mathrm{div};\Omega)$.

Lemma 3.2 *There is a constant* c_1*, independent of* ϵ *and such that the bilinear form* $\mathcal{K}$ *satisfies the following estimate*

$$\mathcal{K}(\underline{\chi},v;\underline{\chi},v) \le c_1\left(\|v\|^2_{1,*}+\|\nabla\cdot\underline{\chi}\|^2_{-1,*}+\epsilon^{-1}\|\underline{\chi}\|^2+\epsilon^{-1}\|v\|^2+\epsilon^{-1}\delta^2\|\mathbf{b}\cdot\nabla v\|^2\right) \tag{20}$$

for all $\underline{\chi}\in H(\mathrm{div};\Omega)$ *and* $v\in H_0^1(\Omega)$.

Proof 2 *The proof follows immediately from the definition of the bilinear form* $\mathcal{K}$*.*

Remark 3.2 *The boundness of the linear functional of the right hand side in the norm* $\|\nabla\cdot\underline{\chi}\|_{-1,*}+\|v\|_{1,*}$ *is immediate since* $\|\nabla\cdot\underline{\chi}+c_0v\|_{-1,*}\le\|\nabla\cdot\underline{\chi}\|_{-1,*}+||c_0v\|_{-1,*}\le\|\nabla\cdot\underline{\chi}\|_{-1,*}+\beta_2\|v\|$.

4 Finite element method for the least-squares formulation

Let $W_h\subset H_0^1(\Omega)$ and $\mathbf{V}_h\subset H(\mathrm{div};\Omega)$ be finite element spaces of piece-wise linear functions over the quasi-uniform triangulation $\mathcal{T}_h$ of the domain Ω. We assume that for some integer $s\ge 1$ the following approximation properties of the spaces W_h and $\mathbf{V}_h$ are available:
(H.1) for any $\underline{\eta}\in H^r(\Omega)^d\cap H(\mathrm{div};\Omega)$ and $1\le r\le s$

$$\inf_{\underline{\chi}\in\mathbf{V}_h}\left(\|\underline{\eta}-\underline{\chi}\|+h\|\nabla\cdot(\underline{\eta}-\underline{\chi})\|\right)\le Ch^r\|\underline{\eta}\|_{r,\Omega};$$

(H.2) for any $w\in H_0^1(\Omega)\cap H^2(\Omega)$

$$\inf_{v\in W_h}(\|w-v\|+h\|\nabla(w-v)\|)\le Ch^{r+1}\|w\|_{r+1,\Omega};$$

(H.3) there is an orthogonal projection operator $Q_h: L^2(\Omega)\to W_h$, which is bounded with respect to the norm in $H^1(\Omega)$, i.e.

$$\|Q_hv\|_{1,\Omega}\le C\|v\|_{1,\Omega} \text{ for all } v\in H^1(\Omega).$$

Most of the known finite element spaces satisfy the above assumptions. In general, the solution u has low regularity and it makes sense to consider only the case $s=1$ and both W_h and $\mathbf{V}_h$ consist of piece-wise linear functions over the triangulation $\mathcal{T}_h$.

In order to define the finite element approximation to the problem (15), we need a definition of a discrete analog of the norm $\|\cdot\|_{-1,*}$. Here we borrow the concept of computable discrete H^{-1}-norm from [2, 3]. Similarly to the continuous case, we first define the operator T_h as a solution operator of the problem: find $v_h \in W_h$ such that

$$D_*(v_h, \phi) = (f, \phi) \text{ for all } \phi \in W_h, \tag{21}$$

where the bilinear form $D_*(v_h, \phi)$ is defined by (14). Then the operator T_h is defined by $v_h = T_h f$.

Let B_h be a preconditioner for T_h, which is symmetric, positive definite and spectrally equivalent to T_h in the L^2-inner product, *i.e.* there are constants $0 < c_1 \le c_2$ independent of h, such that

$$c_1(B_h v_1, v) \le (T_h v, v) \le c_2(B_h v, v) \text{ for all } v \in W_h.$$

Remark 4.1 *One possibility to construct preconditioner B_h for the operator T_h corresponding to the bilinear form $D_*(u,\, v) = \big((\epsilon I + \mathbf{b}\mathbf{b}^T)\nabla u,\, \nabla v\big) + (u,\, v)$, for $u,\, v \in W_h$ is an algebraically stabilized version of the hierarchical basis method, which is known to be more robust with respect to various problem parameters. For more details, cf., [18].*

Finally, following [2, 3] we define $\tilde{T}_h = h^2 I + B_h$ and introduce the finite element approximation of the least-squares mixed method: find $u_h \in W$ and $\underline{\sigma}_h \in \mathbf{V}_h$ such that

$$\mathcal{K}_h(\underline{\sigma}_h, u_h; \underline{\chi}, v) = (\tilde{T}_h f_\delta, \nabla \cdot \underline{\chi} + c_0 v) \tag{22}$$

for all $\underline{\chi} \in \mathbf{V}_h$ and $v \in W_h$. Here

$$\begin{aligned} \mathcal{K}_h(\underline{\sigma}_h, u_h; \underline{\chi}, v) \;\; &\equiv (\tilde{T}_h(\nabla \cdot \underline{\sigma}_h + c_0 u_h), \nabla \cdot \underline{\chi} + c_0 v) \\ &+\epsilon^{-1}(\underline{\sigma}_h + \epsilon \nabla u_h + \mathbf{b} u_h - \delta \mathbf{b} \Lambda u_h, \underline{\chi} + \epsilon \nabla v + \mathbf{b} v - \delta \mathbf{b} \Lambda v) \end{aligned} \tag{23}$$

is the discrete analog of the bilinear form $\mathcal{K}(\cdot,\cdot)$.

The least-squares method (22) leads to a symmetric and positive definite algebraic problem. The symmetry is by construction and the positive definiteness follows from the coercivity of the bilinear form $\mathcal{K}(\cdot,\cdot)$ in the semi-norm $\|v\|_{1,*}^2 + \|\nabla \cdot \underline{\chi}\|_{-1,*}^2$.

5 Error estimate of the least-squares method

Let $\underline{\epsilon}_h = \underline{\sigma} - \underline{\sigma}_h$ and $e_h = u - u_h$ then $\mathcal{K}_h(\underline{\epsilon}_h, e_h; \underline{\chi}, v) = 0$ for all $\underline{\chi} \in \mathbf{V}_h$ and $v \in W_h$. The coercivity of the form $\mathcal{K}_h$ then implies:

$$\mathcal{K}_h(\underline{\epsilon}_h, e_h; \underline{\epsilon}_h, e_h) \le \inf_{\chi \in \mathbf{V}_h, v \in W_h} \mathcal{K}_h(\underline{\sigma} - \underline{\chi}, u - v; \underline{\sigma} - \underline{\chi}, u - v).$$

Thus, the task to bound the error $\|\nabla \cdot \underline{\epsilon}_h\|_{-1,*} + \|e_h\|_{1,*}$ reduces to best approximation of the solution from the finite element space $\mathbf{V}_h \times W_h$ in the energy norm defined from the form $\mathcal{K}$. Hence it is sufficient the estimate of the error for a suitable chosen function $\underline{\chi}$ and v from $\mathbf{V}_h$ and W_h close to $\underline{\sigma}$ and u.

Denote by $\underline{\Sigma}_h = \underline{\sigma} - \underline{\chi}$ and $U_h = u - v$, where $\underline{\chi} \in \mathbf{V}_h$ and $v \in W_h$. Then by the definition of the form $\mathcal{K}_h$ and the operator $\tilde{T}_h$ we have:

$$\begin{aligned} \mathcal{K}_h(\underline{\Sigma}_h, U_h; \underline{\Sigma}_h, U_h) \;\; &= (\tilde{T}_h(\nabla \cdot \underline{\Sigma}_h + c_0 U_h), \nabla \cdot \underline{\Sigma}_h + c_0 U_h) \\ &\quad +\epsilon^{-1}(\underline{\Sigma}_h + \epsilon \nabla U_h + \mathbf{b} U_h - \delta \mathbf{b} \Lambda U_h, \underline{\Sigma}_h + \epsilon \nabla U_h + \mathbf{b} U_h - \delta \mathbf{b} \Lambda U_h) \\ &\le h^2 \|\nabla \cdot \underline{\Sigma}_h + c_0 U_h\|^2 + \Big(B_h(\nabla \cdot \underline{\Sigma}_h + c_0 U_h), \nabla \cdot \underline{\Sigma}_h + c_0 U_h\Big) \\ &\quad +\epsilon^{-1}\|\underline{\Sigma}_h\|^2 + \epsilon \|\nabla U_h\|^2 + \epsilon^{-1}\|\mathbf{b} U_h\|^2 + \delta^2 \epsilon^{-1} \|\mathbf{b} \Lambda U_h\|^2. \end{aligned}$$

One easily deducts that

$$\Big(B_h(\nabla\cdot\underline{\Sigma}_h + c_0U_h), \nabla\cdot\underline{\Sigma}_h + c_0U_h\Big) \le C\Big(\epsilon^{-1}\|\underline{\Sigma}_h\|^2 + \|U_h\|^2\Big),$$

so we get the estimate

$$\mathcal{K}_h(\underline{\Sigma}_h, U_h; \underline{\Sigma}_h, U_h) \le C\Big(\epsilon^{-1}\|\underline{\Sigma}_h\|^2 + \epsilon^{-1}\|U_h\|^2 + (\epsilon^{-1}\delta^2+\epsilon)\|\nabla U_h\|^2 + h^2\|\nabla\cdot\underline{\Sigma}_h\|^2\Big). \tag{24}$$

Using the boundness of the bilinear form $\mathcal{K}_h$ from below we get:

$$\|\nabla\cdot\underline{\epsilon}_h\|^2_{-1,*} + \|e_h\|^2_{1,*} \le C\Big(\epsilon^{-1}\|\underline{\Sigma}_h\|^2 + \epsilon^{-1}\|U_h\|^2 + (\epsilon^{-1}\delta^2+\epsilon)\|\nabla U_h\|^2 + h^2\|\nabla\cdot\underline{\Sigma}_h\|^2\Big). \tag{25}$$

And finally, using the inequality

$$\epsilon^{-1}\|\underline{\epsilon}_h\|^2 \le C\Big(\mathcal{K}_h(\underline{\epsilon}_h, e_h; \underline{\epsilon}_h, e_h) + \|e_h\|^2 + \delta^2\|\mathbf{b}\cdot\nabla e_h\|^2\Big)$$

we get an L^2-estimate for $\underline{\epsilon}_h$:

$$\|\underline{\epsilon}_h\|^2 \le C\Big(\|\underline{\Sigma}_h\|^2 + \|U_h\|^2 + (\delta^2+\epsilon^2)\|\nabla U_h\|^2 + h^2\epsilon\|\nabla\cdot\underline{\Sigma}_h\|^2\Big). \tag{26}$$

These estimates are quite similar to the estimates of [14]. The main difference is that the term $\|\nabla\cdot\underline{\Sigma}_h\|$ from [14] is replaced by $h\|\nabla\cdot\underline{\Sigma}_h\|$ in (24), which will result in reducing the regularity requirement for the convergence of the method.

Now choose $\underline{\chi} \in \mathbf{V}_h$ and $v \in W_h$ such that the estimates of the hypothesis (H.1) and (H.2) are satisfied with $r = 1$. By (25) and (26) we conclude:

Theorem 5.1 *The finite element solution u_h of the streamline diffusion least-squares method (22) converges in the norm $\|\cdot\|_{1,*}$ at a rate $\mathcal{O}(h\epsilon^{-\frac{1}{2}} + \delta h\epsilon^{-\frac{1}{2}} + h)$ if u is H^2-regular. If u is H^3-regular the convergence rate is $\mathcal{O}(h^2\epsilon^{-\frac{1}{2}} + \delta h\epsilon^{-\frac{1}{2}} + h^2)$. In particular, if one chooses $\delta \simeq h$, for $h \simeq \epsilon$ the familiar $\mathcal{O}(h^{\frac{3}{2}})$ L^2–error estimate from the classical streamline diffusion method is recovered. Similarly, the streamline derivative $\mathbf{b}\cdot\nabla u$ is then computed with an $\mathcal{O}(h^{\frac{1}{2}})$ error in L^2.*

For H^3–regular solution, one can choose $h \simeq \sqrt{\epsilon}$; then the error will be of order $\mathcal{O}(\delta + h) = \mathcal{O}(h)$ for both $\underline{\sigma}$ and u.

We remark, that in the H^2–regular case, one may simply let $\delta = 0$, and still get the same error estimates of order $\mathcal{O}(h^{\frac{1}{2}})$, which is the estimate proved in [8] for upwind mixed co-volume schemes. The least squares method in the present case does not exploit any upwinding, instead it requires a sufficiently small mesh $h \simeq \epsilon$.

The error estimate for u may seem to be quite unsatisfactory since a reasonable convergence rate can be achieved only for small step-size $h \simeq \epsilon$, which is quite a restrictive assumption. In fact, one would like to be able to compute with much larger step-sizes (as we did in our computations), say $h \simeq \sqrt{\epsilon}$. None of the methods provides good error estimates for the solution u for this case for H^2-regular solutions. However, the present method has one strong point, namely direct continuous approximation of the flux. Therefore, streamline methods should be used for solutions that are are expected to be sufficiently regular.

6 Implementation and numerical experiments

In this section we present some numerical results that illustrate the error behavior of the studied streamline diffusion least–squares finite element method. We consider the same problem as in [14],

$$\nabla\cdot(-\epsilon a\nabla w - \mathbf{b}w) + c_0 w = f(x,y) \quad (x,y) \in \Omega = (0,1)^2. \tag{27}$$

The exact solution was chosen $w = x(1-x)y(1-y)$ and Dirichlet boundary conditions were imposed. The coefficients of the operator were: $a = \text{diag}(a_1, a_2)$, $a_1 = 1 + 10x^2 + y^2$, $a_2 = 1 + x^2 + 10y^2$, $c_0 = 1$ and $\mathbf{b} = (b_1, b_2)$ where $b_1 = \cos\alpha(1 - x\cos\alpha), b_2 = \sin\alpha(1 - y\sin\alpha)$, for angles $\alpha = -\frac{\pi}{4}, 0, \frac{\pi}{4}$. Obviously, $\nabla \cdot \mathbf{b} = -1$ and the condition (2) is satisfied. Note that the problem (27) differs slightly from (1), namely, ϵI is replaced by $\epsilon \text{diag}(a_1, a_2)$.

We used for both variables the space of piecewise bilinear functions W_h (zero boundary conditions) and $\mathbf{V}_h$ (no boundary conditions) on a square mesh (composing the triangulation $\mathcal{T}_h$) of mesh-size $h = h_x = h_y = 2^{-L}$ for $L = 4, 5, 6, 7$.

The least–squares method leads to the following system of linear algebraic equations

$$\left(\begin{bmatrix} \mathbf{A} & B^T \\ B & M \end{bmatrix} + [B_0,\ C_0]^T \,\Theta\, [B_0,\ C_0] \right) \begin{bmatrix} \underline{\sigma}_h \\ u_h \end{bmatrix} = \begin{bmatrix} [B_0,\ C_0]^T \,\Theta\, \mathbf{r}.\mathbf{h}.\mathbf{s}. \end{bmatrix}. \tag{28}$$

We have used the same notations $\underline{\sigma}_h$ and u_h for the vectors of the coefficients in the presentation of $\underline{\sigma}_h$ and u_h in the basis of $\mathbf{V}_h$ and W_h. Here, the matrices $\mathbf{A}$, B, C, B_0 and C_0 are sparse and assembled explicitly. More specifically, they are computed from the bilinear forms:

- the bilinear form $\epsilon^{-1}\left(a^{-1}\left(\underline{\chi} + \epsilon a \nabla\varphi + \mathbf{b}\varphi - \delta\Lambda\varphi\right),\ \underline{\theta} + \epsilon a\nabla\psi + \mathbf{b}\psi - \delta\Lambda\psi\right)$ defines the matrix $\mathcal{A} = \begin{bmatrix} \mathbf{A} & B^T \\ B & M \end{bmatrix}$, where $\underline{\theta} = (\theta_1,\ \theta_2)$ and $\underline{\chi} = (\chi_1,\ \chi_2)$, and θ_i and χ_i run over a basis in the respective components of $\mathbf{V}_h = (V_{1,h},\ V_{2,h})$, and similarly φ and ψ run over a basis of W_h. We have chosen as already mentioned bilinear elements for all components ($V_{1,h},\ V_{2,h}$ and W_h);

- the matrix B_0 is computed from the bilinear form $(\text{div}\,\underline{\theta},\ \psi)$, where $\underline{\theta} = (\theta_1,\ \theta_2)$ runs over the basis of $\mathbf{V}_h$ and ψ runs over the basis of W_h;

- C_0 is the mass matrix $(c_0\varphi,\ \psi)$ where φ and ψ run over the basis of W_h.

The operator Θ is an algebraically stabilized (AMLI) multilevel preconditioner for the form $D_*(\cdot, \cdot)$. Details, on the AMLI methods are found in [18]. The vector $\mathbf{r}.\mathbf{h}.\mathbf{s}$ has components $(f_\delta,\ \varphi)$, where φ runs over the basis of W_h. Recall, that $f_\delta = f + \delta\nabla \cdot (\mathbf{b}f)$.

The least–squares system (28) is solved by the preconditioned conjugate gradient method, with an AMLI preconditioner coming from the explicitly available sparse matrix $\mathcal{A} + h^{-2}[B_0,\ C_0]^T[B_0,\ C_0]$.

In the tables below we report the following error related quantities: $\delta_0 \equiv \|I_h w - u_h\|$, $\delta_1 \equiv \|I_h w - u_h\|_1$, $\delta_{\text{SD}} \equiv \|I_h w - u_h\|_{\text{SD},h} \equiv \|\mathbf{b} \cdot \nabla(I_h w - u_h)\|$, $\Delta_0 \equiv \|\mathbf{I}_h \underline{\lambda}_0 - \underline{\sigma}_h\|$, $\Delta_{\text{div}} \equiv \|\nabla \cdot (\mathbf{I}_h \underline{\lambda}_0 - \underline{\sigma}_h)\|$. Here, w and $\underline{\lambda}_0 = -\epsilon a \nabla w - \mathbf{b}w$ are the exact solution and the exact continuous flux; I_h and $\mathbf{I}_h$ stand for nodal interpolation in the finite element space W_h (for the scalar function u) and in $\mathbf{V}_h$ (for the vector function $\underline{\lambda}_0$); u_h is the finite element solution we have computed together with $\underline{\sigma}_h$. Note that $\underline{\sigma}_h$ is the approximate solution corresponding to the modified continuous flux $\underline{\sigma} = \underline{\lambda}_0 - \delta\mathbf{b}\,(\nabla \cdot (\mathbf{b}u) - c_0 u)$, where $\delta = h$ is the streamline–diffusion parameter that we have used in the test. Hence, we cannot get better than first order approximation to the flux $\underline{\lambda}_0$.

It is clearly seen, that the error behavior for all quantities (except δ_0) is of approximate first order. For the L^2–error, δ_0, between the interpolant and the finite element solution one may see a superconvergent behavior of order higher than second in some tables (see, Tables 2, 5 and 8). For all tests, δ_0, is of higher than first order.

Table 1: Error behavior for $\alpha = 0$, $\epsilon = 10^{-2}$

	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$	$\approx$ order
δ_0	4.73e-4	2.18e-4	1.09e-4	4.24e-5	> 1
δ_1	1.02e-2	5.00e-3	2.53e-3	1.24e-3	1
δ_{SD}	4.63e-3	2.17e-3	1.16e-3	5.81e-4	1
Δ_0	3.26e-3	1.73e-3	8.82e-4	3.45e-4	1
Δ_{div}	2.42e-2	1.33e-2	7.19e-3	3.97e-3	1
# unknowns	675	2 883	11 907	48 387	

Table 2: Error behavior for $\alpha = 0$, $\epsilon = 10^{-4}$

	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$	$\approx$ order
δ_0	3.78e-3	1.22e-3	2.78e-4	4.11e-5	> 2
δ_1	3.09e-2	1.31e-2	4.78e-3	1.58e-3	1
δ_{SD}	2.01e-2	1.00e-2	3.88e-3	1.13e-3	1
Δ_0	2.35e-3	1.02e-3	6.35e-4	3.87e-4	1
Δ_{div}	2.70e-2	1.42e-2	6.76e-3	3.21e-3	1
# unknowns	675	2 883	11 907	48 387	

Table 3: Error behavior for $\alpha = 0$, $\epsilon = 10^{-6}$

	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$	$\approx$ order
δ_0	1.89e-2	8.16e-3	2.75e-3	1.31e-3	1
δ_1	9.31e-2	5.60e-2	2.31e-2	1.20e-2	1
δ_{SD}	4.96e-2	3.16e-2	1.67e-2	1.00e-2	1
Δ_0	1.15e-2	5.06e-3	1.74e-3	9.14e-4	1
Δ_{div}	5.25e-2	3.35e-2	1.76e-2	1.06e-2	1
# unknowns	675	2 883	11 907	48 387	

Table 4: Error behavior for $\alpha = \frac{\pi}{4}$, $\epsilon = 10^{-2}$

	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$	$\approx$ order
δ_0	5.50e-4	3.08e-4	2.63e-4	6.40e-5	1
δ_1	1.03e-2	5.30e-3	2.78e-3	1.34e-3	1
δ_{SD}	5.19e-3	2.73e-3	1.49e-3	7.13e-4	1
Δ_0	3.81e-3	1.96e-3	9.93e-4	3.88e-4	1
Δ_{div}	2.50e-2	1.34e-2	7.19e-3	4.50e-3	1
# unknowns	675	2 883	11 907	48 387	

Table 5: Error behavior for $\alpha = \frac{\pi}{4}$, $\epsilon = 10^{-4}$

	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$	$\approx$ order
δ_0	4.46e-3	1.15e-3	2.15e-4	2.98e-5	> 2
δ_1	3.26e-2	1.28e-2	4.44e-3	1.52e-3	1
δ_{SD}	2.07e-2	8.24e-3	2.79e-3	8.83e-4	1
Δ_0	3.48e-3	1.40e-3	8.06e-4	4.43e-4	1
Δ_{div}	2.42e-2	1.19e-2	5.84e-3	2.93e-3	1
# unknowns	675	2 883	11 907	48 387	

Table 6: Error behavior for $\alpha = \frac{\pi}{4}$, $\epsilon = 10^{-6}$

	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$	$\approx$ order
δ_0	3.17e-2	1.32e-2	2.95e-3	7.50e-4	2
δ_1	0.142	6.54e-2	2.41e-2	9.13e-3	> 1
δ_{SD}	7.20e-2	3.96e-2	1.54e-2	5.87e-3	1
Δ_0	2.10e-2	9.53e-3	2.06e-3	5.15e-4	2
Δ_{div}	7.23e-2	4.03e-2	1.53e-2	6.08e-3	1
# unknowns	675	2 883	11 907	48 387	

Table 7: Error behavior for $\alpha = -\frac{\pi}{4}$, $\epsilon = 10^{-2}$

	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$	$\approx$ order
δ_0	1.48e-3	6.34e-4	3.07e-4	1.21d-4	> 1
δ_1	1.24e-2	5.94e-3	2.97e-3	1.38d-3	1
δ_{SD}	9.95e-3	4.85e-3	2.45e-3	1.13d-3	1
Δ_0	7.93e-3	3.99e-3	2.00e-3	7.85d-4	1
Δ_{div}	3.96e-2	2.04e-2	1.04e-2	4.95d-3	1
# unknowns	675	2 883	11 907	48 387	

Table 8: Error behavior for $\alpha = -\frac{\pi}{4}$, $\epsilon = 10^{-4}$

	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$	$\approx$ order
δ_0	1.61e-2	3.70e-3	8.27e-4	1.55d-4	> 2
δ_1	7.45e-2	2.37e-2	8.24e-3	2.60d-3	> 1
δ_{SD}	5.71e-2	1.73e-2	5.97e-3	1.87d-3	> 1
Δ_0	1.87e-2	5.93e-3	2.35e-3	1.08d-3	1
Δ_{div}	6.95e-2	2.82e-2	1.26e-2	5.85d-3	1
# unknowns	675	2 883	11 907	48 387	

Table 9: Error behavior for $\alpha = -\frac{\pi}{4}$, $\epsilon = 10^{-6}$

	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$	$\approx$ order
δ_0	3.32e-2	3.11e-2	1.08e-2	2.42e-3	> 1
δ_1	0.148	0.139	5.28e-2	1.71e-2	> 1
δ_{SD}	0.115	0.108	3.98e-2	1.29e-2	> 1
Δ_0	3.55e-2	3.33e-2	1.19e-2	2.83e-3	1
Δ_{div}	0.115	0.109	4.29e-2	1.45e-2	1
# unknowns	675	2 883	11 907	48 387	

References

[1] I. Babuška, O. C. Zienkiewicz: Feedback, adaptivity, and a posteriori estimates in finite elements: Aims, theory, and experience, Accuracy estimates and adaptive refinements in finite element computations. Wiley, New York, 1986

[2] J. H. Bramble, R. D. Lazarov and J. E. Pasciak: A least-squares approach based on a discrete minus one inner product for first order systems. Math. Comp., **66** (1997) 935-955

[3] J. H. Bramble and J. E. Pasciak: Least-squares methods for Stokes equations based on a discrete minus one inner product. J. Comput. Appl. Math., **74** (1996) 155-173.

[4] F. Brezzi and M. Fortin: *Mixed and Hybrid Finite Element Methods*. Springer-Verlag, New York, 1991

[5] Z. Cai, R. D. Lazarov, T. A. Manteuffel, and S. F. McCormick: First order systems least-squares for second-order partial differential equations: Part I. SIAM J. Numer. Anal., **31** (1994) 1785-1799

[6] Z. Cai, T. A. Manteuffel, and S. F. McCormick: First order systems least-squares for second-order partial differential equations: Part II. SIAM J. Numer. Anal., **34** (1997) 425-454

[7] G. F. Carey, A. I. Pehlivanov, and P. S. Vassilevski: Least–squares mixed finite elements for non–selfadjoint elliptic problems: II, Performance of block–ILU factorization methods. SIAM J. Sci. Comput., **16** (1995), 1126-1136

[8] So–Hsiang Chou and P. S. Vassilevski: A general mixed co–volume framework for constructing conservative schemes for elliptic problems. Math. Comp., **68** (1999) (to appear)

[9] T.J. Hughes and A. Brooks: A Multidimensional Upwind Scheme with no Cross-wind Diffusion. AMD **34**, *Finite element methods for convection dominated flows*, T.J.Hughes (ed.), AMSE, New York, 1979

[10] T. J. R. Hughes and L. P. Franca: A new finite element formulation for computational fluid dynamics. V. Circumventing the Babuška-Brezzi condition: a stable Petrov-Galerkin formulation of the Stokes problem accommodating equal-order interpolations. Comput. Meth. Appl. Mech. Engrg., **59** (1986) 85-99

[11] C. Johnson, U. Nävert, and J. Pitkäranta: Finite element methods for linear hyperbolic problems. Comput. Meth. Appl. Mech. Engrg., **45** (1984) 285-312

[12] C. Johnson and J. Saranen: Streamline diffusion methods for the incompressible Euler and Navier-Stokes equations. Math. Comp., **47** (1986) 1-18

[13] J. J. H. Miller, E. O'Riordan, G. I. Shishkin: Fitted Numerical Methods for Singular Perturbation Problem. World scientific, Singapore 1996

[14] R. D. Lazarov, L. Tobiska, and P. S. Vassilevski: Stream-line diffusion least-squares mixed finite element methods for convection-diffusion problems. East-West J. Numer. Math., **5** (1997) 321-335

[15] A. I. Pehlivanov and G. F. Carey: Error estimates for least-square mixed finite elements. RAIRO MMMNA, **28** (1994) 43-65

[16] H.-O. Ross, M. Stynes, and L. Tobiska: Numerical Methods for Singularly Perturbed Differential Equations. Springer, 1996

[17] L. Tobiska and R. Verürth: Analysis of stream!line diffusion finite element methods for Stokes and Navier-Stokes equations. SIAM J. Numer. Anal., **33** (1996) 107–127

[18] P. S. Vassilevski: On two ways of stabilizing the hierarchical basis methods. SIAM Review, **39** (1997) 18–53

[19] R. Verfürth: A Review of a Posteriori Error Estimators and Adaptive Mesh Refinement Techniques. Teubner-Wiley, Stuttgart, 1996