A new a–posteriori error estimator in adaptive direct boundary element methods. Part I: The Dirichlet problem*

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Abstract

In this paper we propose a new a–posteriori error estimator for a weakly singular integral equation concerned with a direct boundary element approach for a Dirichlet problem with a second order elliptic partial differential operator. The method is based on an approximate solution of a second kind Fredholm integral equation by a Neumann series to estimate the error of a previous computed solution of an arbitrary boundary element method, for example a Galerkin method, collocation or qualocation. Due to the solution of this error equation the proposed estimator provides a high accuracy. Since our method is based on standard techniques which are available in every boundary element code, it is easy to implement.

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1 Introduction

In this paper we describe and analyse a new a–posteriori error estimator for an approximate solution t_h of the weakly singular boundary integral equation

$$Vt = (\sigma I + K)g\,, \tag{1.1}$$

which results from a direct boundary integral approach for a second order partial differential equation with given Dirichlet boundary conditions.
After applying a boundary element method to compute an approximate solution t_h of (1.1), e.g. by a Galerkin or a collocation scheme, one is interested to estimate the error $e_h := t - t_h$ in a suitable norm $||e_h||$. This information, combined with certain localization techniques, can be used afterwards to decrease the error by using an adaptive mesh refinement.

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As in finite element methods most existing error estimators for boundary element methods are based on variants of the error equation for (1.1),

$$V e_h = r_h := (\sigma I + K) g - V t_h \,. \tag{1.2}$$

Provided the continuity of $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and of the inverse V^{-1} there immediately follows the global error inclusion

$$c_1 \cdot ||r_h||_{H^{1/2}(\Gamma)} \leq ||e_h||_{H^{-1/2}(\Gamma)} \leq c_2 \cdot ||r_h||_{H^{1/2}(\Gamma)} \,, \tag{1.3}$$

that means, $\eta := ||r_h||_{H^{1/2}(\Gamma)}$ defines an error estimator, see [5, 6, 8, 16, 17, 26, 27]. After proving an estimate of the kind (1.3) two problems occur. First, in the practical application one is interested in the value of $||e_h||_{H^{-1/2}(\Gamma)}$, so one needs information about the constants c_1 and c_2. The second problem is the localization: To drive an adaption process it is necessary to have some local information about the error distribution. There are two ways to construct local error estimators. The first one is to localize the residual itself, this was done in [1, 5, 6, 8, 26]. The second possibility is to prove localized versions of (1.3), see [16, 17, 23, 27].

A very simple way to obtain an error estimator is to solve the error equation (1.2) approximately with higher accuracy than the original method itself, e.g. by using trial functions of higher polynomial degree or an a refined mesh. Then, to prove an inclusion as (1.3), one needs the so–called saturation assumption, see [2].

Another type of estimators are based on averaging or recovery methods. For that kind of error estimation some superconvergence results of the computed solution in certain points or a post–processed superconvergence solution are necessary, see [28, 30] for the finite element method, and [18, 19, 22] for the boundary element method. If a superconvergent solution $\hat{t}_h$ is available, one can define $\eta := ||\hat{t}_h - t_h||_{H^{-1/2}(\Gamma)}$. If it is possible to prove superconvergence, in the most cases the constants c_1 and c_2 are near by 1. Moreover, if one can prove pointwise superconvergence the localization of the error estimator is inherent.

Here we present a new approach to approximate the error $t - t_h$ with high accuracy. Our method is based on the solution of an appropriate error equation, which is simpler to solve than the original boundary integral equation (1.1) or the corresponding error equation (1.2). Since our alternative error equation is a second kind Fredholm integral equation involving the adjoint double layer potential we can apply a Neumann series for the solution process. To get an accurate approximation of the error, only a few Neumann iterates are necessary to compute, i.e., it will be shown that even without applying the Neumann iteration we can compute an approximate error yielding an almost optimal error estimator. Since the proposed error estimator is based on standard components of boundary element methods, in particular the discretization of standard boundary integral operators as the adjoint double layer potential, it is easy to implement.

The paper is organized as follows. In Section 2 we will recall a Galerkin boundary element method for the solution of (1.1) and give some basic definitions of error estimators. Section 3 is devoted to the formulation of an equivalent error equation and global error estimators based on a Neumann series are given. In Section 4 we describe a numerical implementation scheme and prove corresponding error estimates. A numerical example in Section 5 confirms the theoretical results.

Throughout the paper, by c we will denote a general constant which may have different values at difference occurrences.

2 Preliminaries

2.1 Boundary element methods

For a bounded domain $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ with a Lipschitz continuous boundary Γ and an elliptic second order partial differential operator L we consider the homogeneous Dirichlet boundary value problem

$$\left.\begin{array}{rcll} Lu(x) & = & 0 & \text{for } x \in \Omega\,, \\ u(x) & = & g(x) & \text{for } x \in \Gamma\,. \end{array}\right\} \tag{2.1}$$

Note that we may consider inhomogeneous partial differential equations in (2.1) as well, but for simplicity in the presentation we consider the homogeneous case only. Since in this paper we are interested in the construction of an error estimator for the weakly singular integral equation (1.1), the consideration of Dirichlet boundary conditions in (2.1) is sufficient.
If a fundamental solution $U^*(x, y)$ of L is known, the solution of the boundary value problem (2.1) is given by the representation formula

$$u(x) \;=\; \int\limits_\Gamma U^*(x,y)t(y)ds_y - \int\limits_\Gamma g(y)T^*(x,y)ds_y \quad \text{for } x \in \Omega\,, \tag{2.2}$$

where $T^*(x, y) = T_y U^*(x, y)$ using the conormal derivative operator T_y for $y \in \Gamma$. In the representation formula (2.2) the density $t(y) = T_y u(y)$ is unknown, hence we have to solve the boundary integral equation

$$(Vt)(x) \;=\; (\sigma(x)I + K)g(x) \;=:\; f(x) \quad \text{for } x \in \Gamma \tag{2.3}$$

with

$$\sigma(x) \;=\; \left\{\begin{array}{cl} 1 & \text{for } x \in \Omega, \\ \alpha(x)/2\pi & \text{for } x \in \Gamma, \\ 0 & \text{for } x \in \mathbb{R}^n\backslash\overline{\Omega} \end{array}\right.$$

and $\alpha(x)$ denotes the interior angle in $x \in \Gamma$. In (2.3) we used the standard notations for the single layer potential V and the double layer potential K,

$$(Vt)(x) \;=\; \int\limits_\Gamma U^*(x,y)t(y)ds_y\,, \quad (Ku)(x) \;=\; \int\limits_\Gamma u(y)T^*(x,y)ds_y\,. \tag{2.4}$$

It is well known (see [7]) that the boundary integral operators

$$V \,:\, H^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma), \qquad K \,:\, H^{1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma)$$

are continuous for $s \in [-\frac{1}{2}, \frac{1}{2}]$. Moreover, V satisfies a Gårdings inequality, i.e., there exist a compact operator $C : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ such that

$$\mathrm{Re}\left(\langle (V+C)t, t\rangle_{L^2(\Gamma)}\right) \;\geq\; c \cdot ||t||^2_{H^{-1/2}(\Gamma)} \tag{2.5}$$

holds for all $t \in H^{-1/2}(\Gamma)$ with some positive constant c. Hence, V is invertible and bijective, see [21].
Throughout the paper we assume that the Dirichlet boundary value problem (2.1) has a unique solution $u \in H^{\rho}(\Omega)$ with some $\rho > \frac{3}{2}$. Hence, the boundary integral equation (2.3) will have a unique solution $t \in H^{\sigma}(\Gamma)$ for some $\sigma > 0$.
To solve the boundary integral equation (2.3) numerically, we consider a family of regular triangulations Γ_h of Γ into boundary elements Γ_k with local mesh size h_k and a global mesh size $h := \max_k h_k$. With respect to Γ_h we then introduce a family of trial spaces

$$Z_h \;:=\; \mathrm{span}\{\varphi_k^{\nu}\}_{k=1}^{N} \subset H^{-1/2}(\Gamma) \tag{2.6}$$

of discontinuous splines of polynomial order ν, for example of piecewise constant trial functions ($\nu = 0$). Note that there hold the approximation property in Z_h [15], i.e.,

$$\inf_{\tau_h \in Z_h} ||\tau - \tau_h||_{H^{-1/2}(\Gamma)} \;\leq\; c \cdot h^{s+\frac{1}{2}} \cdot ||\tau||_{H^s(\Gamma)} \tag{2.7}$$

for all $\tau \in H^s(\Gamma)$ with $-\frac{1}{2} \leq s \leq \nu + 1$.
The Galerkin variational formulation of (2.3) is to find $t_h \in Z_h$ such that

$$\langle V t_h, \tau_h\rangle_{L^2(\Gamma)} \;=\; \langle f, \tau_h\rangle_{L^2(\Gamma)} \quad \text{for all } \tau_h \in Z_h\,. \tag{2.8}$$

For the stability of the Galerkin scheme (2.8) and the convergence see for example [10, 21, 25], in particular we get the quasi–optimal error estimate

$$||t - t_h||_{H^{-1/2}(\Gamma)} \;\leq\; c \cdot \inf_{\tau_h \in Z_h} ||t - \tau_h||_{H^{-1/2}(\Gamma)} \tag{2.9}$$

and, combining this with the approximation property (2.7) there follows convergence,

$$||t - t_h||_{H^{-1/2}(\Gamma)} \;\leq\; c \cdot h^{s+\frac{1}{2}} \cdot ||t||_{H^s(\Gamma)} \quad \text{with } s = \min\{\sigma, \nu+1\}\,. \tag{2.10}$$

2.2 Error estimators

The error of the Galerkin solution t_h of (2.8) is defined as

$$e_h(x) \;:=\; (t - t_h)(x) \quad \text{for } x \in \Gamma\,. \tag{2.11}$$

The aim is now to define an *estimator* η of a suitable error norm $||e_h||$ and to localize η. An estimator η is called *error estimator* if there exist positive constants c_1 and c_2 independent of approximation parameters (as the mesh size h) such that an inclusion

$$c_1 \cdot \eta \leq ||e_h|| \leq c_2 \cdot \eta \tag{2.12}$$

is fulfilled. The error estimator is called *asymptotically exact* if there holds

$$\lim_{h \to 0} \frac{||e_h||}{\eta} = 1 \,. \tag{2.13}$$

Local error estimators with respect to a boundary element Γ_k will be denoted by η_k. After computing η_ℓ for all boundary elements Γ_ℓ we will refine all elements Γ_k where

$$\eta_k \;\geq\; \theta \cdot \max_\ell \;\eta_\ell \tag{2.14}$$

holds with some appropriate chosen refinement parameter θ.

3 A new a–posteriori error estimator

As mentioned in the introduction, one way to obtain an error estimator is to solve the error equation (1.2) with higher accuracy than the original method itself, but this is very expansive in general. One possible way out is to derive another equation for the error with known right hand side, but with an operator on the left hand side which is easier to invert as the original operator V itself. Since we will not use the properties of the Galerkin formulation (2.8) itself, our technique can be applied directly to any other discretization technique, such as collocation or qualocation. Hence we assume in the following that t_h is some boundary element solution of the boundary integral equation (2.3), for example the Galerkin solution of (2.8).
According to (2.2) we define the approximate solution of the boundary value problem (2.1) as

$$u_h(x) \;=\; \int_\Gamma U^*(x,y) t_h(y) ds_y - \int_\Gamma g(y) T^*(x,y) ds_y \quad \text{for } x \in \Omega \tag{3.1}$$

possessing the Cauchy data

$$\tilde{g}(x) \;:=\; u_h(x), \quad \tilde{t}(x) \;:=\; (T_x u_h)(x) \qquad \text{for } x \in \Gamma \,. \tag{3.2}$$

Due to its definition, u_h is a solution of the Dirichlet boundary value problem

$$L u_h(x) = 0 \quad \text{for } x \in \Omega \,, \qquad u_h(x) = \tilde{g}(x) \quad \text{for } x \in \Gamma \,.$$

From this we conclude that $\tilde{t}$ is a solution of the boundary integral equation

$$(V\tilde{t})(x) \;=\; (\sigma(x) I + K)\tilde{g}(x) \quad \text{for } x \in \Gamma \,. \tag{3.3}$$

Now we can prove a relation between the error $t - t_h$ and the computable function $\tilde{t} - t_h$. For this we define the adjoint double layer potential operator to be

$$(K't)(x) \;=\; \int_\Gamma T_x^*(x,y) t(y) ds_y \quad \text{for } x \in \Gamma \,. \tag{3.4}$$

From [7] it is known that

$$K' \;:\; H^{-1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma) \tag{3.5}$$

is continuous for all $s \in [-\frac{1}{2}, \frac{1}{2}]$.

Lemma 3.1 *The error $t - t_h$ of the boundary element solution t_h is a solution of the boundary integral equation*

$$((1-\sigma(x))I - K')(t-t_h)(x) \;=\; (\tilde{t} - t_h)(x) \quad \textit{for } x \in \Gamma\,. \tag{3.6}$$

Proof. Using the jump relation of the double layer potential for $x \to \Gamma$ we get from the continuous representation formula (2.2)

$$g(x) \;=\; (Vt)(x) + ((1-\sigma(x))I - K)g(x) \quad \text{for } x \in \Gamma$$

and from the approximate representation formula (3.1)

$$\tilde{g}(x) \;=\; (Vt_h)(x) + ((1-\sigma(x))I - K)g(x) \quad \text{for } x \in \Gamma\,.$$

Hence we have

$$(V(t-t_h))(x) \;=\; (g - \tilde{g})(x) \quad \text{for } x \in \Gamma\,. \tag{3.7}$$

Thus, using (2.3), (3.3), (3.7) and $KV = VK'$ [14] we get

$$\begin{aligned} V(\tilde{t} - t_h)(x) &= V(\tilde{t} - t)(x) + V(t - t_h)(x) \\ &= (\sigma(x)I + K)(\tilde{g} - g)(x) + (g - \tilde{g})(x) \\ &= ((1-\sigma(x))I - K)(g - \tilde{g})(x) \\ &= ((1-\sigma(x))I - K)V(t - t_h)(x) \\ &= V((1-\sigma(x))I - K')(t - t_h)(x) \end{aligned}$$

for $x \in \Gamma$. From the bijectivity of the single layer potential V the assertion follows. ■

To compute the error $t - t_h$ from (3.6) we need to have the invertibility of $(1-\sigma)I - K'$. To prove this, we define the hypersingular integral operator

$$(Du)(x) \;=\; -T_x \int_\Gamma T_y U^*(x,y)u(y)ds_y \quad \text{for } x \in \Gamma \tag{3.8}$$

with $D : H^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma)$ for $s \in [-\frac{1}{2}, \frac{1}{2}]$, see [7].

Lemma 3.2 *The operator $(1-\sigma)I - K' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is bijective. The inverse is given by the Neumann series*

$$((1-\sigma)I - K')^{-1} \;=\; \sum_{\ell=0}^{\infty} (\sigma I + K')^{\ell}\,, \tag{3.9}$$

where the spectral radius of the operator $\sigma I + K'$ is bounded by

$$\rho(\sigma I + K') \;\leq\; c_K \;<\; 1 \tag{3.10}$$

with some positive constant c_K which depends on Γ only.

Proof. Due to $VK = K'V$ [14] the eigenvalues of K coincide with those of K'. Hence it is sufficient to consider K only. Let us denote the eigenvalues of $\sigma I + K$ by λ_k and the eigenfunctions by v_k:

$$(\sigma I + K)v_k = \lambda_k v_k \quad \Rightarrow \quad ((1-\sigma)I - K)v_k = (1-\lambda_k)v_k \,. \tag{3.11}$$

It is well known that all eigenvalues α_k of the operator VD are real and non–negative. Using the relation

$$VD \;=\; ((1-\sigma)I - K)(\sigma I + K) \;:\; H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

and (3.11) we obtain that the eigenvalues of VD can be written in the form $\alpha_k = \lambda_k(1-\lambda_k)$. From $\alpha_k \in \mathbb{R}^+$ there follows that all eigenvalues λ_k of the operator $\sigma I + K$ are real and they are in the interval $[0,1]$. All possible eigenvalues $\alpha_{k_0} = 0$ belong to all eigenfunctions v_{k_0} of the homogeneous Neumann boundary value problem

$$(Lv_{k_0})(x) = 0 \;\text{ in } \Omega\,, \quad (T_x v_{k_0})(x) = 0 \;\text{ on } \Gamma\,, \tag{3.12}$$

Hence we obtain either $\lambda_{k_0} = 0$ or $\lambda_{k_0} = 1$. Since the solutions v_{k_0} of (3.12) satisfy

$$(\sigma(x)I + K)v_{k_0}(x) \;=\; 0 \quad \text{for } x \in \Gamma,$$

we conclude that $\lambda_{k_0} = 0$ when $\alpha_{k_0} = 0$ and therefore $|\lambda_k| < 1$ for all k and

$$\rho(\sigma I + K') \;=\; \rho(\sigma I + K) \;\leq\; c_K \;<\; 1$$

with a constant c_K which depends on Γ only, see also [9, 12, 24].
The eigenvalues of $(1-\sigma)I - K'$ are given by $\mu_k = 1 - \lambda_k$ with $1 \geq \mu_k \geq 1 - c_K > 0$. Therefore, the operator $(1-\sigma)I - K' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is invertible and its inverse is given by the Neumann series (3.9). ■

From (3.10) it follows that there holds

$$||(\sigma I + K')v||_{H^{-1/2}(\Gamma)} \;\leq\; c_K \cdot ||v||_{H^{-1/2}(\Gamma)} \qquad \text{for all } v \in H^{-1/2}(\Gamma)\,. \tag{3.13}$$

Applying Lemma 3.1 and Lemma 3.2 one can represent error of the boundary element solution t_h by

$$e_h := t - t_h \;=\; \sum_{\ell=0}^{\infty} (\sigma I + K')^{\ell}(\tilde{t} - t_h)\,. \tag{3.14}$$

Using the properties of $(1-\sigma)I - K'$ we get already equivalence inequalities between the error (3.14) and the computable function $\tilde{t} - t_h$:

Theorem 3.1 *There hold the equivalence inequalities*

$$||\tilde{t} - t_h||_{H^{-1/2}(\Gamma)} \;\leq\; ||e_h||_{H^{-1/2}(\Gamma)} \;\leq\; \frac{1}{1-c_K} \cdot ||\tilde{t} - t_h||_{H^{-1/2}(\Gamma)} \tag{3.15}$$

with a positive constant $c_K < 1$ as given in Lemma 3.2.

Proof. From the boundedness of $(1-\sigma)I - K'$ and using $\mu_{\max} = 1$ we get

$$\begin{aligned} ||\tilde{t} - t_h||_{H^{-1/2}(\Gamma)} &= ||((1-\sigma)I - K')(t - t_h)||_{H^{-1/2}(\Gamma)} \\ &\leq ||t - t_h||_{H^{-1/2}(\Gamma)} = ||e_h||_{H^{-1/2}(\Gamma)} . \end{aligned}$$

Using (3.14) and (3.13) we get

$$||t - t_h||_{H^{-1/2}(\Gamma)} \leq \sum_{\ell=0}^{\infty} c_K^{\ell} \cdot ||\tilde{t} - t_h||_{H^{-1/2}(\Gamma)} \leq \frac{1}{1 - c_K} \cdot ||\tilde{t} - t_h||_{H^{-1/2}(\Gamma)},$$

which completes the proof. ∎

Hence we can define a global error estimator for the boundary element solution t_h as

$$\eta^{(0)} := ||\tilde{t} - t_h||_{H^{-1/2}(\Gamma)} . \tag{3.16}$$

Note that (3.16) already defines an almost optimal error estimator which can be controlled by determining the constant c_K. To improve the equivalence inequalities (3.15), we can apply some iterations of the Neumann series (3.9) to compute

$$e_h^{(q)}(x) := \sum_{\ell=0}^{q} (\sigma I + K')^{\ell} (\tilde{t} - t_h)(x) \qquad \text{for } q > 0 . \tag{3.17}$$

Theorem 3.2 *There hold the equivalence inequalities*

$$\frac{1}{1 + c_K^{q+1}} \cdot ||e_h^{(q)}||_{H^{-1/2}(\Gamma)} \leq ||e_h||_{H^{-1/2}(\Gamma)} \leq \frac{1}{1 - c_K^{q+1}} \cdot ||e_h^{(q)}||_{H^{-1/2}(\Gamma)} . \tag{3.18}$$

Proof. Using (3.14), (3.17) and (3.13) we get

$$\begin{aligned} ||e_h - e_h^{(q)}||_{H^{-1/2}(\Gamma)} &= ||\sum_{\ell=q+1}^{\infty} (\sigma I + K')^{\ell} (\tilde{t} - t_h)||_{H^{-1/2}(\Gamma)} \\ &= ||(\sigma I + K')^{q+1} e_h||_{H^{-1/2}(\Gamma)} \leq c_K^{q+1} \cdot ||e_h||_{H^{-1/2}(\Gamma)} . \end{aligned}$$

Hence the assertion follows by applying the triangle inequality twice. ∎

Applying Theorem 3.2 we are able to define the global error estimator

$$\eta^{(q)} := ||e_h^{(q)}||_{H^{-1/2}(\Gamma)} = ||\sum_{\ell=0}^{q} (\sigma I + K')^{\ell} (\tilde{t} - t_h)||_{H^{-1/2}(\Gamma)} \tag{3.19}$$

satisfying the equivalence inequalities (2.12) with constants as given in Theorem 3.2. Moreover, since $c_K < 1$ we get

$$\lim_{q \to \infty} \frac{||e_h||_{L^2(\Gamma)}}{\eta^{(q)}} \leq \lim_{q \to \infty} \frac{1}{1 - c_K^{q+1}} = 1 ,$$

i.e., the error estimator (3.19) is asymptotically exact for $q \to \infty$.

To compute the Sobolev norm appearing in (3.19) we can use the equivalent energy norm given by

$$||v||_V \;:=\; \sqrt{\langle Vv,v\rangle_{L^2(\Gamma)}} \;\sim\; ||v||_{H^{-1/2}(\Gamma)} \quad \text{for all } v\in H^{-1/2}(\Gamma)\,. \tag{3.20}$$

An alternative computation of (3.19) can be done by applying multilevel techniques as described in [4].
Let us denote the boundary elements by Γ_k, $k=1,\ldots,N$. To obtain local error indicators, we define

$$\eta_k^{(q)} := \sqrt{\langle Ve_h^{(q)}, e_h^{(q)}\rangle_{L^2(\Gamma_k)}}\,, \quad k=1,\ldots,N\,. \tag{3.21}$$

From the continuity and bijectivity of V and from the relation

$$\sum_{k=1}^{N}\left(\eta_k^{(q)}\right)^2 = \sum_{k=1}^{N}\langle Ve_h^{(q)}, e_h^{(q)}\rangle_{L^2(\Gamma_k)} = \langle Ve_h^{(q)}, e_h^{(q)}\rangle\,,$$

there immediately follows the estimate

$$c_1\cdot\sum_{k=1}^{N}\left(\eta_k^{(q)}\right)^2 \le ||e_h^{(q)}||^2_{H^{-1/2}(\Gamma)} \le c_2\cdot\sum_{k=1}^{N}\left(\eta_k^{(q)}\right)^2 \tag{3.22}$$

with constants c_1 and c_2 which depend on V only. Finally estimate (3.22) implies

$$\frac{c_1}{(1+c_K^{q+1})^2}\cdot\sum_{k=1}^{N}\left(\eta_k^{(q)}\right)^2 \;\le\; ||e_h||^2_{H^{-1/2}(\Gamma)} \;\le\; \frac{c_2}{(1-c_K^{q+1})^2}\cdot\sum_{k=1}^{N}\left(\eta_k^{(q)}\right)^2 \tag{3.23}$$

by Theorem 3.2.

4 Practical implementation

For the computation of the estimated error $e_h^{(0)} = \tilde{t} - t_h$ we first have to compute the Cauchy datum $\tilde{t}$ of u_h as given in (3.1). Applying the conormal derivative operator T_x to (3.1) and taking the limit $x\to\Gamma$ we get

$$\begin{aligned}\tilde{t}(x) \;&=\; \sigma(x)t_h(x) + T_x\int_\Gamma U^*(x,y)t_h(y)ds_y - T_x\int_\Gamma g(y)T^*(x,y)ds_y\\ &=\; (\sigma(x)I+K')t_h(x) + (Dg)(x) \quad \text{for } x\in\Gamma \end{aligned} \tag{4.1}$$

using the jump relation of the adjoint double layer potential and definition (3.8) of the hypersingular integral operator. Note that $\tilde{t}\in L^2(\Gamma)$ due to the assumption $t\in H^\sigma(\Gamma), \sigma>0$, the regularity of t_h and the mapping properties of K'. Using (4.1) we now can compute the initial error function

$$e_h^{(0)}(x) \;=\; (\tilde{t}-t_h)(x) \;=\; (\sigma(x)I+K')t_h(x) + (Dg)(x) - t_h(x)\,. \tag{4.2}$$

Due to Theorem 3.1 we can use $e_h^{(0)}$ to compute the global error estimator (3.16). To get the improved equivalence inequalities as in Theorem 3.2 we have to apply the Neumann series (3.17). From a practical point of view we have to introduce some finite dimensional approximation to evaluate (3.17). For this reason we define as in (2.6) a trial space $Z_{\tilde{h}}$ of discontinuous splines of polynomial order ν with respect to a refined triangulation $\Gamma_{\tilde{h}}$ with $\tilde{h} < h$ sufficiently small. Note that one may also define $Z_{\tilde{h}}$ over the triangulation Γ_h by using polynomial trial functions of higher degree. Now we define the L^2–Galerkin projection $G_{\tilde{h}}u = u_{\tilde{h}} \in Z_{\tilde{h}}$ for a given $u \in L^2(\Gamma)$ such that

$$\langle u_{\tilde{h}}, v_{\tilde{h}} \rangle_{L^2(\Gamma)} \;=\; \langle u, v_{\tilde{h}} \rangle_{L^2(\Gamma)} \qquad \text{for all } v_{\tilde{h}} \in Z_{\tilde{h}}\,. \tag{4.3}$$

From (4.3) it is obvious that $G_{\tilde{h}}$ is bounded,

$$||G_{\tilde{h}}u||_{L^2(\Gamma)} \;\leq\; ||u||_{L^2(\Gamma)} \tag{4.4}$$

and that there holds

$$||(I - G_{\tilde{h}})u||_{L^2(\Gamma)} \;\leq\; ||u||_{L^2(\Gamma)} \qquad \text{for all } u \in L^2(\Gamma)\,. \tag{4.5}$$

Moreover, applying the Aubin–Nitsche trick [11, 13], we get the error estimate

$$||(I - G_{\tilde{h}})u||_{H^{-1/2}(\Gamma)} \;\leq\; c \cdot \tilde{h}^{1/2} \cdot ||u||_{L^2(\Gamma)}\,. \tag{4.6}$$

If we write (3.17) for $\ell = 1, \ldots, q$ as

$$e_h^{(\ell)} \;=\; e_h^{(\ell-1)} + z_h^{(\ell)}, \qquad z_h^{(\ell)} \;=\; (\sigma I + K')z_h^{(\ell-1)} \tag{4.7}$$

with $z_h^{(0)} = e_h^{(0)}$, we can define, using (4.3) and $\tilde{e}_h^{(0)} = G_{\tilde{h}}e_h^{(0)}$, the computable sequence

$$\tilde{e}_h^{(\ell)} \;=\; \tilde{e}_h^{(\ell-1)} + \tilde{z}_h^{(\ell)}, \qquad \tilde{z}_h^{(\ell)} \;=\; G_{\tilde{h}}(\sigma I + K')\tilde{z}_h^{(\ell-1)} \tag{4.8}$$

for $\ell = 1, \ldots, q$ and with $\tilde{z}_h^{(0)} = \tilde{e}_h^{(0)}$.
Hence we can define a computable error estimator as

$$\tilde{\eta}^{(q)} \;=\; ||\tilde{e}_h^{(q)}||_{H^{-1/2}(\Gamma)} \qquad \text{for some } q \geq 0\,. \tag{4.9}$$

To show the equivalence inequalities (2.12) for the error estimator (4.9) we will use Theorem 3.2. Therefore we have to estimate the difference $\tilde{e}_h^{(q)} - e_h^{(q)}$, which is obviously based on the difference $\tilde{z}_h^{(q)} - z_h^{(q)}$.

Lemma 4.1 *For each $\ell = 1, \ldots, q$ there holds the error estimate*

$$||\tilde{z}_h^{(\ell)} - z_h^{(\ell)}||_{H^{-1/2}(\Gamma)} \;\leq\; c_K^\ell \cdot ||\tilde{e}_h^{(0)} - e_h^{(0)}||_{H^{-1/2}(\Gamma)} + c \cdot \tilde{h}^{1/2} \cdot ||e_h^{(0)}||_{L^2(\Gamma)}\,. \tag{4.10}$$

Proof. Let us first define

$$\hat{z}_h^{(\ell)} \;=\; (\sigma I + K')\tilde{z}_h^{(\ell-1)} \qquad \text{for } \ell = 1, \ldots, q\,.$$

Then there follows

$$\begin{aligned}||\tilde{z}_h^{(\ell)} - z_h^{(\ell)}||_{H^{-1/2}(\Gamma)} &\leq ||\tilde{z}_h^{(\ell)} - \hat{z}_h^{(\ell)}||_{H^{-1/2}(\Gamma)} + ||\hat{z}_h^{(\ell)} - z_h^{(\ell)}||_{H^{-1/2}(\Gamma)} \\ &= ||\tilde{z}_h^{(\ell)} - \hat{z}_h^{(\ell)}||_{H^{-1/2}(\Gamma)} + ||(\sigma I + K')(\tilde{z}_h^{(\ell-1)} - z_h^{(\ell-1)})||_{H^{-1/2}(\Gamma)} \\ &\leq ||\tilde{z}_h^{(\ell)} - \hat{z}_h^{(\ell)}||_{H^{-1/2}(\Gamma)} + c_K \cdot ||\tilde{z}_h^{(\ell-1)} - z_h^{(\ell-1)}||_{H^{-1/2}(\Gamma)}.\end{aligned}$$

The first term can be estimated by (4.6)

$$\begin{aligned}||\tilde{z}_h^{(\ell)} - \hat{z}_h^{(\ell)}||_{H^{-1/2}(\Gamma)} &= ||(G_{\tilde{h}} - I)(\sigma I + K')\tilde{z}_h^{(\ell-1)}||_{H^{-1/2}(\Gamma)} \\ &\leq c \cdot \tilde{h}^{1/2} \cdot ||(\sigma I + K')\tilde{z}_h^{(\ell-1)}||_{L^2(\Gamma)} \\ &\leq c \cdot c_K \cdot \tilde{h}^{1/2} \cdot ||\tilde{z}_h^{(\ell-1)}||_{L^2(\Gamma)}\end{aligned}$$

and by the continuity of $\sigma I + K'$. Now, using (4.4), we get

$$\begin{aligned}||\tilde{z}_h^{(j)}||_{L^2(\Gamma)} &= ||G_{\tilde{h}}(\sigma I + K')\tilde{z}_h^{(j-1)}||_{L^2(\Gamma)} \leq ||(\sigma I + K')\tilde{z}_h^{(j-1)}||_{L^2(\Gamma)} \\ &\leq c_K \cdot ||\tilde{z}_h^{(j-1)}||_{L^2(\Gamma)} \leq c_K^j \cdot ||\tilde{z}_h^{(0)}||_{L^2(\Gamma)} = c_K^j \cdot ||\tilde{e}_h^{(0)}||_{L^2(\Gamma)} \\ &\leq c_K^j \cdot ||e_h^{(0)}||_{L^2(\Gamma)}\end{aligned}$$

for all $j = 1, \ldots, q$. Hence we have

$$||\tilde{z}_h^{(\ell)} - z_h^{(\ell)}||_{H^{-1/2}(\Gamma)} \leq c_K \cdot ||\tilde{z}_h^{(\ell-1)} - z_h^{(\ell-1)}||_{H^{-1/2}(\Gamma)} + c \cdot \tilde{h}^{1/2} \cdot c_K^\ell \cdot ||e_h^{(0)}||_{L^2(\Gamma)}$$

and, using this estimate recursively,

$$\begin{aligned}||\tilde{z}_h^{(\ell)} - z_h^{(\ell)}||_{H^{-1/2}(\Gamma)} &\leq c_K^\ell \cdot ||\tilde{e}_h^{(0)} - e_h^{(0)}||_{H^{-1/2}(\Gamma)} + c \cdot \tilde{h}^{1/2} \cdot \sum_{j=1}^{\ell} c_K^j \cdot ||e_h^{(0)}||_{L^2(\Gamma)} \\ &\leq c_K^\ell \cdot ||\tilde{e}_h^{(0)} - e_h^{(0)}||_{H^{-1/2}(\Gamma)} + c \cdot \tilde{h}^{1/2} \cdot c_K \cdot \frac{1 - c_K^\ell}{1 - c_K} \cdot ||e_h^{(0)}||_{L^2(\Gamma)} .\end{aligned}$$

■

Lemma 4.2 *For the estimated error function $\tilde{e}_h^{(q)}$ defined recursively by (4.8) there holds the error estimate*

$$||\tilde{e}_h^{(q)} - e_h^{(q)}||_{H^{-1/2}(\Gamma)} \leq c \cdot q \cdot \tilde{h}^{1/2} \cdot ||e_h^{(0)}||_{L^2(\Gamma)}. \tag{4.11}$$

Proof. For any $\ell = 1, \ldots, q$ we have

$$||\tilde{e}_h^{(\ell)} - e_h^{(\ell)}||_{H^{-1/2}(\Gamma)} \leq ||\tilde{e}_h^{(\ell-1)} - e_h^{(\ell-1)}||_{H^{-1/2}(\Gamma)} + ||\tilde{z}_h^{(\ell)} - z_h^{(\ell)}||_{H^{-1/2}(\Gamma)} ,$$

from which follows, note that $z_h^{(0)} = e_h^{(0)}, \tilde{z}_h^{(0)} = \tilde{e}_h^{(0)}$,

$$||\tilde{e}_h^{(q)} - e_h^{(q)}||_{H^{-1/2}(\Gamma)} \leq \sum_{\ell=0}^{q} ||\tilde{z}_h^{(\ell)} - z_h^{(\ell)}||_{H^{-1/2}(\Gamma)} .$$

Applying Lemma 4.1 gives

$$\begin{aligned} &||\tilde{e}_h^{(q)} - e_h^{(q)}||_{H^{-1/2}(\Gamma)} \\ &\qquad \leq \left(\sum_{\ell=0}^{q} c_K^{\ell} \right) ||\tilde{e}_h^{(0)} - e_h^{(0)}||_{H^{-1/2}(\Gamma)} \, + \, c \cdot \tilde{h}^{1/2} \cdot q \cdot ||e_h^{(0)}||_{L^2(\Gamma)} \\ &\qquad \leq \frac{1}{1 - c_K} \cdot ||\tilde{e}_h^{(0)} - e_h^{(0)}||_{H^{-1/2}(\Gamma)} \, + \, c \cdot \tilde{h}^{1/2} \cdot q \cdot ||e_h^{(0)}||_{L^2(\Gamma)} \, . \end{aligned}$$

Now (4.11) follows from (4.6). ■

Using the previous results now we can prove final equivalence inequalities (2.12) for the error estimator (4.9).

Theorem 4.1 *Let $\tilde{e}_h^{(q)}$ be defined recursively via (4.8). Then there hold the equivalence inequalities*

$$\begin{aligned} &\frac{1}{1 + c_K^q} \cdot \left\{ ||\tilde{e}_h^{(q)}||_{H^{-1/2}(\Gamma)} - c \cdot q \cdot \tilde{h}^{1/2} \cdot ||e_h^{(0)}||_{H^{-1/2}(\Gamma)} \right\} \\ &\qquad \leq ||e_h||_{H^{-1/2}(\Gamma)} \leq \\ &\qquad\qquad \frac{1}{1 - c_K^q} \cdot \left\{ ||\tilde{e}_h^{(q)}||_{H^{-1/2}(\Gamma)} + c \cdot q \cdot \tilde{h}^{1/2} \cdot ||e_h^{(0)}||_{H^{-1/2}(\Gamma)} \right\} . \end{aligned}$$

Therefore, if $\tilde{h} < h$ is sufficiently small, $\tilde{\eta}^{(q)}$ as defined in (4.9) is a global error estimator satisfying (2.12).

Proof. Using Theorem 3.2, the triangle inequality and Lemma 4.2, we get

$$\begin{aligned} ||e_h||_{H^{-1/2}(\Gamma)} &\leq \frac{1}{1 - c_K^q} \cdot \left\{ ||\tilde{e}_h^{(q)}||_{H^{-1/2}(\Gamma)} + ||\tilde{e}_h^{(q)} - e_h^{(q)}||_{H^{-1/2}(\Gamma)} \right\} \\ &\leq \frac{1}{1 - c_K^q} \cdot \left\{ ||\tilde{e}_h^{(q)}||_{H^{-1/2}(\Gamma)} + c \cdot q \cdot \tilde{h}^{1/2} \cdot ||e_h^{(0)}||_{L^2(\Gamma)} \right\} . \end{aligned}$$

The lower inequality follows by applying the same arguments. ■

The local error estimators are now defined by

$$\tilde{\eta}_k^{(q)} \;=\; \sqrt{\langle V \tilde{e}_h^{(q)}, \tilde{e}_h^{(q)} \rangle_{L^2(\Gamma_k)}} \, , \tag{4.12}$$

and, similar to (3.22), it can be shown that there holds the estimate

$$c_1 \sum_{k=1}^{N} \left(\widetilde{\eta}_k^{(q)} \right)^2 \leq \widetilde{\eta}^{(q)} \leq c_2 \sum_{k=1}^{N} \left(\widetilde{\eta}_k^{(q)} \right)^2 \, , \tag{4.13}$$

where c_1 and c_2 are the same constants as in (3.22).

At the end of this section we will give some remarks concerning the computational costs of the proposed error estimator as well as we want to comment some variants of our technique applicable in some special cases.

Remark 4.1 *The application of the error estimator (4.9) requires basically the computation of $\tilde{t}$ with respect to a refined triangulation $\Gamma_{\tilde{h}}$ by using the representation formula (4.1). An alternative approach to compute $\tilde{t}$ with the same numerical effort, namely $O(N)$ per degree of freedom in $Z_{\tilde{h}}$, would be to use finite differences to approximate the conormal derivative operator T by evaluating the representation formula (3.1).*

Remark 4.2 *As stated in Theorem 4.1 we can use $\tilde{\eta}^{(0)}$ to be the global error estimator, i.e., without applying the Neumann series (4.8). If one is able to estimate the constant c_K in Lemma 3.2, e.g. by a power method to find the maximal eigenvalue of $\sigma I + K$, one can define a modified error estimator as*

$$\hat{\eta}^{(0)} := \frac{1}{1 - c_K} \cdot \tilde{\eta}^{(0)} \tag{4.14}$$

satisfying (2.12) with a lower constant $c_1 = 1$.

Remark 4.3 *If the adjoint double layer potential operator K' is more regular as stated in (3.5), e.g. in the case of a (piecewise) C^∞ boundary Γ or a polygonal boundary Γ, the error estimate (4.6) changes to be*

$$||(I - G_{\tilde{h}})u||_{H^{-1/2}(\Gamma)} \leq c \cdot \tilde{h}^{s+\frac{1}{2}} \cdot ||u||_{H^s(\Gamma)} \tag{4.15}$$

with $u \in H^s(\Gamma)$ for some $s > 0$. Note that for $n = 3$ one has to use at least piecewise linear trial functions to ensure $e_h^{(0)} \in H^s(\Gamma), s > 0$. Moreover, one can use the simpler computable L^2 norm to define all global and local error estimates.

Remark 4.4 *The numerical amount of work of the proposed error estimator is comparable to residual based methods, where the residual has to be computed in certain quadrature points, see [6]. As it will be seen from the numerical example we are able to approximate the Galerkin error very accurate by applying some few Neumann iterations only. Moreover, the implementation of the proposed error estimator in direct boundary element methods can be done easily, since it is based on standard components which are already available in boundary element codes.*

5 Numerical results

In our numerical example we consider a Dirichlet boundary value problem for the two–dimensional Laplacian where Ω is the L shaped domain as sketched in Figure 5.1:

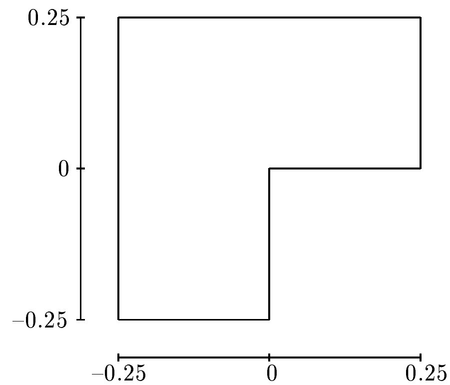

Figure 5.1: L shaped domain Ω

Note that Ω satisfies the condition diam $\Omega < 1$ which is sufficient to get positive definiteness of the single layer potential for the two–dimensional Laplacian [10]. The Dirichlet data g are given in such a way that the solution of (2.1) is

$$u(x) \;=\; u(r,\varphi) \;=\; r^{\frac{2}{3}} \cdot \sin(\frac{2}{3}\varphi) \tag{5.1}$$

when using polar coordinates. This is a standard example for adaptive boundary element methods [6, 8, 17], even in the case, when the corner singularity is known a priori for using a suitable mesh grading [3, 29]. For the solution (5.1) of the Dirichlet boundary value problem (2.1) we have $u \in H^\rho(\Omega)$ with $\rho < \frac{5}{3}$ and hence $t \in H^\sigma(\Gamma)$ with $\sigma < \frac{1}{6}$. The Galerkin variational formulation (2.8) is discretized by using piecewise constant trial functions ($\nu = 0$) and analytical integration formulae. For the solution of the resulting symmetric and positive definite system a preconditioned [20] conjugate gradient scheme is used. In the case of an uniform refinement we get from (2.10) the order of convergence to be $\frac{2}{3}$. This is confirmed by the numerical results given in Table 5.1, where the order of convergence at the mesh level ℓ is computed via

$$\text{order} \;=\; \frac{\log ||e_{h_\ell}||_{H^{-1/2}(\Gamma)} - \log ||e_{h_{\ell+1}}||_{H^{-1/2}(\Gamma)}}{\log N_{\ell+1} - \log N_\ell}. \tag{5.2}$$

N	$\|\|e_h\|\|_V$	order
8	6.79 –2	
16	4.08 –2	0.72
32	2.59 –2	0.66
64	1.65 –2	0.65
128	1.06 –2	0.64
256	6.74 –3	0.65
512	4.30 –3	0.65
1024	2.75 –3	0.65
2048	1.75 –3	0.65

Table 5.1: Uniform refinement

In the case of a regular solution $t \in H^1(\Gamma)$ one would expect an order of convergence $\frac{3}{2}$. This motivates to use an adaptive scheme to improve the order of convergence. For the definition of the error estimator $\tilde{\eta}^{(q)}$ we defined the trial space $Z_{\tilde{h}}$ of piecewise constant trial functions with $\tilde{h} = h/8$. The representation formula (4.1) as well as all integrals appearing in the Neumann iteration (4.8) are computed again by using analytic formulae. The local error estimators are defined by (4.12). For the adaptive mesh refinement we used the criteria (2.14) with $\theta = 0.05$. In Table 5.2 we give the results in the case of an adaptive refinement based on the proposed error estimator $\tilde{\eta}^{(q)}$ for $q = 0$ and $q = 5$.

N	$\tilde{\eta}^{(0)}$	$\tilde{\eta}^{(5)}$	$\|\|e_h\|\|_V$	order
8	3.26 –2	6.42 –2	6.79 –2	
16	1.91 –2	3.77 –2	4.08 –2	0.73
28	1.24 –2	2.44 –2	2.62 –2	0.79
42	7.95 –3	1.57 –2	1.66 –2	1.13
50	5.19 –3	1.02 –2	1.07 –2	2.52
68	3.40 –3	6.69 –3	6.92 –3	1.42
102	2.17 –3	4.27 –3	4.37 –3	1.14
122	1.43 –3	2.82 –3	2.83 –3	2.43
176	9.14 –4	1.80 –3	1.80 –3	1.23
220	6.14 –4	1.21 –3	1.16 –3	1.97
294	4.15 –4	8.17 –4	7.46 –4	1.52
434	2.63 –4	5.18 –4	4.82 –4	1.12

Table 5.2: Adaptive refinement

Note that in both cases the error estimator $\tilde{\eta}^{(q)}$ defines the same adaptive mesh refinement while, due to Theorem 4.1, the estimated error $\tilde{\eta}^{(5)}$ is more accurate than the estimated error $\tilde{\eta}^{(0)}$. On the other hand, since $\tilde{\eta}^{(0)}$ does not require any Neumann iteration in general, its application requires much less work than $\tilde{\eta}^{(5)}$.

Note that the proposed error estimator produces not only an adaptively refined mesh providing an almost optimal convergence of the Galerkin solution t_h, moreover, it gives a very accurate estimate of the real Galerkin error.

Figure 5.2 shows the error in the energy norm for the uniform and the adaptive mesh with respect to the degree of freedoms and with respect to the computing time. Note that in the uniform refinement case we used an extrapolation to get all values for $N > 2048$. In the uniform case the computing time includes the discretization and solution process at one level only while in the adaptive case the total computing time includes discretization, solution and error estimation over all previous mesh levels.

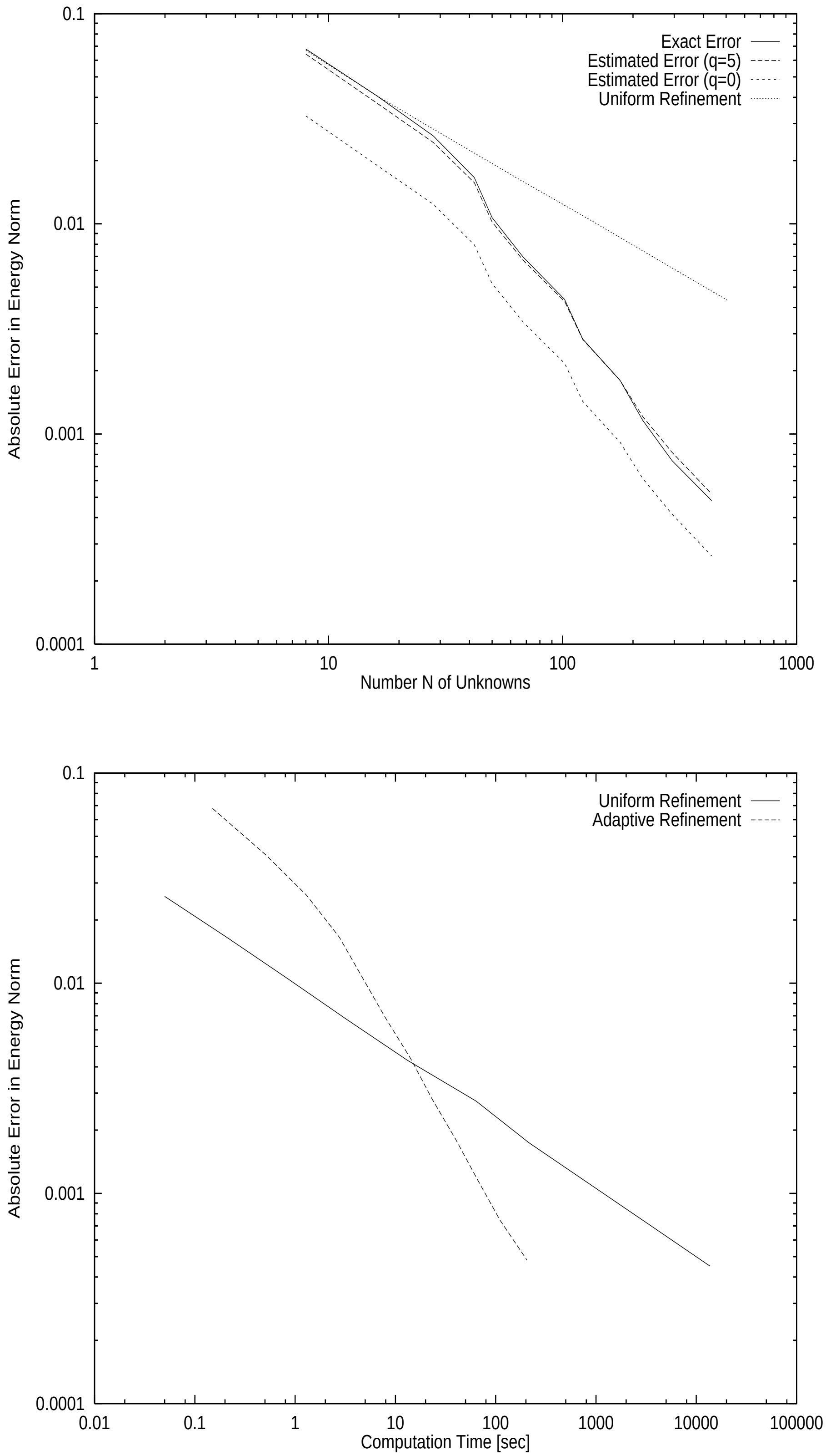

Figure 5.2: Adaptive refinement vs. uniform refinement

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