Computing manuscript No.
(will be inserted by the editor)

Finite Volume Element Approximations of Nonlocal in Time One-dimensional Flows in Porous Media

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Revised version: July 17, 1998

Abstract Various finite volume element approximations for one-dimensional parabolic integro-differential equations in 1-D are derived and studied. These types of equations arise in modeling reactive flows or material with memory effects. Our main goal is to develop a general framework for obtaining finite volume element approximations and studying their error analysis. We consider the lowest-order (linear and L-splines) finite volume elements although higher-order volume elements can be considered as well under this framework. It is proved that finite volume element approximations are convergent with optimal order in H^1-norms, suboptimal in the L^2-norm and super-convergent in a discrete H^1-norm.

Key words finite volume method, parabolic equation, integro-differential equation

1 Introduction

Various processes in the natural sciences and engineering lead to the following problem: find $u = u(x,t)$ such that

$$u_t + \mathcal{A}u + \int_0^t \mathcal{B}(t,s)u(s)ds = f, \qquad x \in \Omega, \qquad 0 < t \leq T, \tag{1}$$
$$u = u_0(x), \qquad x \in \Omega, \qquad t = 0.$$

Here Ω is a bounded domain in R^d $(d = 2,3)$ with a boundary $\partial\Omega$, $T > 0$, $\mathcal{A}$ is a second-order strongly elliptic and positive definite operator, $\mathcal{B}$ is a second-order differential operator with smooth coefficients, and f and u_0

are known functions. Dirichlet or Neumann boundry conditions are incorporated in the definitions of the operators $\mathcal{A}$ and $\mathcal{B}$. The problem (1) is an abstract form of an initial boundary value problem for parabolic integro-differential equation.

Mathematical formulations of this kind appear in various engineering models, such as nonlocal reactive flows in porous media [11] and [12], radioactive nuclear decay in fluid flows [32], non-Newtonian fluid flows, or viscoelastic deformations of materials with memory [30]. One very important characteristic of all these models is that they all express a conservation of a certain quantity (mass, momentum, heat, etc.) in any moment for any subdomain. This in many applications is the most desirable feature of the approximation method when it comes to numerical solution of the corresponding initial boundary value problem.

In this paper, we consider a one-dimensional formulation of the problem (1): namely, find $u = u(x,t)$ such that

$$\begin{aligned} u_t - (au_x)_x + \int_0^t (b(t,s)u_x(s))_x ds = f, \qquad 0 < x < 1, \qquad 0 < t \leq T, \\ u(0,t) = u(1,t) = 0, \qquad 0 < t \leq T, \\ u(x,0) = u_0(x), \qquad 0 < x < 1, \end{aligned} \tag{2}$$

where $a = a(x) \geq a_0 > 0$, $b = b(x,t,s)$, $f = f(x,t)$ and $u_0(x)$ are known functions which are assumed to be smooth so that problem (2) has a unique solution in a certain Sobolev space. For more references concerning the existence and uniqueness of the solution we refer the readers to [30]. In most of this paper, we assume that the unique solution (2) exists and is as smooth as needed.

However, in many applications one needs to deal with piece-wise smooth coefficients a and b. In such problems, we assume that a has finite jumps at a fixed number of points, i.e. the left and right limits exist. In this case, we also assume that the function b has jumps at the same points. Therefore, b could be presented in the form $b(x,t,s) = a(x)B(x,t,s)$, where $B(x,t,s)$ is a smooth function.

In the last decade, various discretization methods based on finite element approximations in space and special quadratures in time have been developed and studied for this type of problems (see, e.g. [6,25–27,29,33,36, 37]). The main tools in the analysis are the Ritz and Ritz-Volterra projections, which were instrumental in deriving optimal oder error estimates in L^p, $2 \leq p < \infty$, and H^1-norms, in maximum-norm and super-convergence estimates (see for details [6,7,25–27]).

The finite element approximations do not conserve exactly the flux over each element (or volume). In the asymptotic limit (i.e. for small step-sizes) this is not a serious problem since the method is convergent. However, this could be a disadvantage of the method when relatively coarse grids are used since it then does not reflect the local conservative properties of the mathematical model. For many applications this property might be crucial and most desirable. Numerical methods which have this property for every space cell are called locally conservative. Our main goal in this paper is to derive discretization schemes for the above problem which are locally conservative. This is done in the framework of the Petrov-Galerkin method, namely, the solution space consists of continuous piece-wise linear functions over a certain triangulation of the domain, while the test space consists of piecewise constants over a different (called dual or finite volume) partition of the domain. This approach has been applied consistently to various elliptic and parabolic problems in the monograph of R. Li and Z. Chen [24].

We illustrate the main points of our approach on the following abstract Petrov-Galerkin formulation. Let S and S^* be a pair of Sobolev spaces, $a(u,v)$ be a bilinear form defined on $S \times S^*$, and $f(v)$ be a linear form defined on S^*. We consider the problem of finding $u \in S$ such that $a(u,v) = f(v)$ for all $v \in S^*$, assuming that the bilinear form $a(u,v)$ is weakly coercive and continuous on $S \times S^*$ and the linear form $f(v)$ is continuous on S^*, i.e.

$$C|u||_S \geq \sup_{v \in S^*} \frac{a(u,v)}{||v||_{S^*}} \geq c||u||_S \text{ and } |l(v)| \leq C||v||_{S^*}.$$

Here c, C are some constants and $||.||_{S^*}$ and $||.||_S$ are some norms in S and S^*, respectively.

Both the finite element and the finite volume methods can be viewed as particular approximations of this abstract framework. Namely, let S_h and S_h^* be finite-dimensional subspaces of S and S^*, respectively, for which the bilinear form $a(.,.)$ is weakly coercive and bounded for some norms in S_h and S_h^*.

In the finite element method, we set $S = S^* = H_0^1$, then introduce a partition of the domain into finite elements and construct the finite element spaces $S_h = S_h^* \subset H_0^1$ of piece-wise polynomials over the partition.

In the finite volume method, we introduce two different partitions of the domain into finite elements and finite volumes. Then $S^* = L^2$ and the finite-dimensional spaces S_h and S_h^* can be chosen as piece-wise linear and piece-wise constants over the partitions of the domain, respectively, so the bilinear form is well defined on $S_h \times S_h^*$. In this case, the equality $a(u,v) = f(v)$ expresses the balance of some substance (mass, heat, etc) over each subdomain of the partition. We shall call S_h solution space while S_h^* is called a test space.

In this paper, the outlined general framework has been applied to the class of integro-differential equations detailed above. The solution space S_h is constructed from the finite element approximation of S, i.e. the functions are piece-wise polynomials over a certain partition T_h of the interval $(0,1)$ into finite elements. The test space S_h^* consists of piece-wise constants over a different partition T_h^* of the interval $(0,1)$ into subintervals called finite volumes. The main efforts have been directed to characterize the finite dimensional spaces S_h and S_h^* and to show the weak coercivity and the boundness of the bilinear form $a(.,.)$ on $S_h \times S_h^*$. Once these fundamentals are established, next we derive the discretization schemes and study their approximation properties in various norms under certain smoothness assumptions on the solution.

To the best of the author's knowledge, the finite volume element approximations of the problem (2) have not been studied before. We apply the general framework outlined above to the problem (2) and use the previous results in the area of finite volumes (see, e.g. [3,8,17,24,28]), finite differences (see, e.g. [31,34,35]) and finite elements (see e.g. [6,24–29,33, 37]) for elliptic and parabolic equations. What needs to be done in the context of the general transient integro-differential equation is to add a time discretization and to derive absolutely stable schemes. The stability of an implicit scheme is a rather simple consequence of the construction and the weak coercivity of the elliptic part. In order to obtain error estimates of optimal order in both H^1 and L^2-norms, we had to introduce a new variant of the Ritz-Volterra projection (in our context it should be called rather Petrov-Volterra projection), which was used by Cannon and Lin in [6] for finite element approximations to similar types of problems. Thus, the essential part of the analysis is reduced to the error estimates for the Ritz-Volterra projection in various norms.

The error estimates of these schemes are local in the sense that the constant grows exponentially with the time t. Long time stability is an important characteristic of the solution for many applications. Schemes which reflect this property have been studied in [1] and [2,37] for smooth and integrable kernels, respectively. In [37] the Ritz projection has been used, while in [1,2] semi-group theory, the Ritz-Volterra projection technique, and resolvent estimates has been applied.

This paper is devoted to one-dimensional problems and uses finite elements of the lowest order. Extensions to higher-order elements and other types of schemes such as discontinuous Petrov-Galerkin methods are discussed in [15]. In §1, we consider discretizations for which the space S_h consists of linear finite elements and L-splines. In §2, we present an important part of our analysis: extension of the concept of the Ritz-Volterra projection V_h introduced in [6] for finite element discretizations to the framework of finite volume discretizations. Finally in §3, we estimate the error of the finite volume element approximations derived in the previous sections.

2 The Lowest-Order Finite Volumes

2.1 Notations and Some Preliminary Results

We shall use the standard notations for Sobolev spaces $W^{k,p}$ for $1 \leq p \leq \infty$ and $H^k = W^{k,2}$ of functions defined on $(0,1)$ for k an integer. The norm in $W^{k,p}$ is defined as

$$\|u\|_{k,p} = \left(\int_0^1 \sum_{i=0}^{k} |D_x^i u|^p dx \right)^{1/p} \quad \text{for } 1 \leq p < \infty$$

and with the standard definition for $p = \infty$. The space H_0^1 consists of those functions in H^1 which vanish at the endpoints $x = 0$ and $x = 1$.

Next, we introduce a notation for the partition of the unit interval $[0, 1]$ into N-subintervals by the points

$$0 = x_0 < x_1 < x_2 < \cdots < x_j < \cdots < x_N = 1.$$

We define $h_j = x_j - x_{j-1}$, $I_j = [x_{j-1}, x_j]$, $j = 1, 2, \cdots, N$, $h = \max_j h_j$. We assume that the partition is quasi-uniform, i.e. there is a positive constant $c_0 > 0$ such that $h_j \geq c_0 h$ for all $j = 1, 2, \cdots, N$. This partition is denoted by $T_h = \cup_{j=1}^N I_j$ and the subintervals I_j are called finite elements.

The dual partition T_h^* is now constructed as follows. Set $x_{j-1/2} = (x_{j-1} + x_j)/2$, $j = 1, 2, \cdots, N$,

$$0 = x_0 < x_{1/2} < x_{3/2} < \cdots < x_{j-1/2} < \cdots < x_{N-1/2} = 1.$$

Then $T_h^* = \cup_{j=0}^N I_j^*$, where $I_j^* = [x_{j-1/2}, x_{j+1/2}]$, $j = 1, 2, \cdots, N-1$, $I_0^* = [0, x_{1/2}]$ and $I_N^* = [x_{N-1/2}, x_N]$. The subintervals I_j^* are often called finite volumes.

The space S_h^* of piece-wise constant functions over T_h^* is defined by

$$S_h^* = \{v \in L^2(0,1) : \; v|_{I_j^*} \text{ is constant}, j = 1, \cdots, N-1, \quad \text{and} \quad v|_{I_0^* \cup I_N^*} = 0\}.$$

Over this partition of the domain, we shall introduce two different spaces for S_h: one is based on the linear finite element interpolant over the partition T_h and the second one is based on the so-called L-splines, i.e. local solutions of the differential equation $Lu \equiv (au_x)_x = 0$ on the partition T_h. While the solution space of piece-wise linear functions can be used for smooth coefficients $a(x)$, the second one, based on L-splines, produces schemes with harmonic averaging of the coefficient $a(x)$ and can be used for problems with rough coefficient $a(x)$. A detailed description of these spaces is given below. We assume that the space S_h consists of continuous functions which, over each interval of the partition T_h, are either linear functions or L-splines. The functions in S_h are entirely determined by their values at the points x_j. Then a local basis of "hat"-functions $\phi_j(x)$ exists, i.e. $\phi_j(x_j) = 1$ and $\phi_j(x_i) = 0$ for $i \neq j$.

The characteristic function χ_j of $I_j^* = [x_{j-1/2}, x_{j+1/2}]$, defined by

$$\chi_j(x) = \begin{cases} 1, \, x_{j-1/2} \leq x \leq x_{j+1/2}, \\ 0, \, \text{otherwise}, \end{cases}$$

form a basis for the space S_h^*. Thus, for any $u_h \in S_h$ and $v_h \in S_h^*$ we have

$$u_h = \sum_{j=1}^{N-1} u_j \phi_j(x) \quad \text{and} \quad v_h = \sum_{j=1}^{N-1} v_j \chi_j(x)$$

with $u_j = u_h(x_j)$ and $v_j = v_h(x_j)$.

The existence of a local basis allows us to introduce easily the interpolation operators: $I_h : C_0(0,1) \to S_h$ as $I_h u = \sum_{j=1}^{N-1} u(x_j)\phi_j(x)$ and $I_h^* : C_0(0,1) \to S_h^*$ as $I_h^* u = \sum_{j=1}^{N-1} u(x_j)\chi_j(x)$. We shall need various semi-norms in the spaces H_0^1 and $H_0^1 \cap H^2$, which are related to the partition T_h. Namely, we define

$$||w||_{0,h} = \left(\sum_{j=1}^{N-1} h_j w_j^2 \right)^{1/2}, \qquad |w|_{1,h} = \left(\sum_{j=1}^{N} (w_j - w_{j-1})^2 h_j^{-1} \right)^{1/2},$$

$$||w||_{1,h} = \left(|w|_{1,h}^2 + ||w||_{0,h}^2 \right)^{1/2}, \qquad |||w|||_0 = (I_h w, I_h^* w)^{1/2},$$

$$|w|_{1,h}^* = \left(\sum_{j=1}^{N} w_x^2(x_{j-1/2}) h_j \right)^{1/2}.$$

Here $(\cdot,\cdot)$ denotes the standard L^2-inner product of functions defined on $(0,1)$. Obviously, the semi-norms $||.||_{0,h}$ and $|||.|||_0$ are equivalent norms on S_h with constants of equivalence independent of h. Similarly, $|.|_{1,h}$ and $|.|_{1,h}^*$ are equivalent norms on S_h. The norms $\|.\|_{0,h}$ and $\|.\|_{1,h}$ use only the values of the function at the grid points and therefore $\|u\|_{0,h} = \|I_h u\|_{0,h}$ and $\|u\|_{1,h} = \|I_h u\|_{1,h}$.

Since the functions w_h from S_h have generalized derivatives, their norm $\|u_h\|_1$ is well defined and there are independent of h constants $c_0, c_1 > 0$ such that

$$c_0 ||w_h||_{1,h} \le ||w_h||_1 \le c_1 ||w_h||_0, \qquad w_h \in S_h.$$

A basis for the finite volume element approximation will be the integral equality obtained by integrating (2) over the volume $I_j^* = [x_{j-1/2}, x_{j+1/2}]$, which expresses conservation of the physical quantity (mass, heat, etc) over each finite volume in T_h^*. Restricting this equality to u in the space S_h we get the following

$$\int_{x_{j-1/2}}^{x_{j+1/2}} u_{h,t} dx - \left(a u_{h,x} + \int_0^t b(t,s) u_{h,x}(s) \right) \Bigg|_{x_{j-1/2}}^{x_{j+1/2}} = \int_{x_{j-1/2}}^{x_{j+1/2}} f dx, \tag{3}$$
$$u_h(0) = u_{0,h} \in S_h,$$

where $u_{0,h}$ is an appropriate approximation of the initial data u_0 in S_h.

Now we introduce two different constructions of the space S_h.

2.2 Finite Volume Method with Linear Elements

The finite element space S_h consists of piece-wise linear functions, i.e.

$$S_h = \{ v \in C_0(0,1) : \ v|_{I_j} \text{ is linear function for } j = 1, \cdots, N \}.$$

Obviously, the functions

$$\phi_j(x) = \begin{cases} 1 - h_j^{-1}|x - x_j|, \ x_{j-1} \le x \le x_j, \\ 1 + h_{j+1}^{-1}|x - x_j|, x_j \le x \le x_{j+1}, \\ 0, \qquad\qquad \text{otherwise}, \end{cases}$$

for $j = 1, \ldots, N-1$ form a basis for S_h.

We define the bilinear forms $a(\cdot,\cdot)$ and $b(t,s;\cdot,\cdot)$ on various pairs of spaces. First, for $u, v \in H_0^1(0,1)$ we use the standard definition

$$a(u,v) = \int_0^1 a u_x v_x ds \quad \text{and} \quad b(t,s;u,v) = \int_0^1 b(t,s) u_x v_x dx. \tag{4}$$

Next, for $u \in H_0^1(0,1) \cap H^2(0,1)$ and $v_h \in S_h^*$, we extend the definition of $a(\cdot,\cdot)$ formally to

$$a(u,v_h) = \sum_{j=1}^{N-1} \left(a_{j-1/2} u_x(x_{j-1/2}) - a_{j+1/2} u_x(x_{j+1/2}) \right) v_j.$$

Using summation by parts and taking into account that $v_0 = v_N = 0$, we come to the following definition of $a(u,v_h)$ and $b(t,s;u,v_h)$ for $u \in H_0^1(0,1) \cap H^2(0,1)$ and $v_h \in S_h^*$:

$$\begin{aligned} a(u,v_h) &= \sum_{j=1}^{N} a_{j-1/2} u_x(x_{j-1/2})(v_j - v_{j-1}), \\ b(t,s;u,v_h) &= \sum_{j=1}^{N} b_{j-1/2}(t,s) u_x(x_{j-1/2})(v_j - v_{j-1}). \end{aligned} \tag{5}$$

For $u_h \in S_h$ the value $u_{h,x}(x_{j-1/2})$ is well defined, and this allows us to use the definition (5) for $(u_h, v_h) \in S_h \times S_h^*$ as well. Thus, the semi-discrete finite volume element method (3) can be rewritten as to find $u_h(t) \in S_h$ for $t > 0$ such that

$$(u_{h,t}, v_h) + a(u_h, v_h) + \int_0^t b(t,s;u_h(s),v_h) ds = (f, v_h), \qquad v_h \in S_h^*, \tag{6}$$

with $u_h(0) = u_{0,h} \in S_h$.

The so-called "lumped" mass semi-discrete approximation of (2) is: find $u_h(t) \in S_h$ such that

$$(I_h^* u_{h,t}, v_h) + a(u_h, v_h) + \int_0^t b(t,s;u_h(s),v_h) ds = (f, v_h), \qquad v_h \in S_h^*. \tag{7}$$

Taking into account the definition of I_h^*, we see that this produces the scheme

$$\begin{aligned}\frac{h_j+h_{j+1}}{2}u_{j,t} - &\left(a_{j+1/2}\frac{u_{j+1}-u_j}{h_{j+1}} - a_{j-1/2}\frac{u_j-u_{j-1}}{h_j}\right)\\ &+\int_0^t \left(b_{j+1/2}(t,s)\frac{u_{j+1}(s)-u_j(s)}{h_{j+1}}\right.\\ &-\left. b_{j-1/2}(t,s)\frac{u_j(s)-u_{j-1}(s)}{h_j}\right) ds\\ &= \tilde{f}_j, \quad \text{for} \quad t>0 \end{aligned} \tag{8}$$

for $j=1,2,\cdots,N-1$, where $\tilde{f}_j = \int_{x_{j-1/2}}^{x_{j+1/2}} f(x,t)dx$, $a_{j-1/2} = a(x_{j-1/2})$ and $b_{j-1/2}(t,s) = b(x_{j-1/2},t,s)$. Consequently, the above equation is the standard three-point finite difference discretization of the problem (2).

We can rewrite these schemes as systems of ordinary differential equations. Let $u_h = \sum_{j=1}^{N-1} u_j(t)\phi_j(x)$ and $U = (u_1,u_2,\cdots,u_{N-1})^T$, then the vector-function $U=U(t)$ satisfies:

$$M_hU_t + A_hU(t) + \int_0^t B_h(t,s)U(s)ds = F_h(t), \quad t>0. \tag{9}$$

Here the mass matrix M_h is diagonal for the scheme (7) and tridiagonal for the scheme (6) and A_h and B_h are symmetric tridiagonal matrices. We also have to add an initial condition given in the form $U(0)$, in which is related to the initial approximation $u_h(0) = u_{0,h} \in S_h$ in (6).

In order to define the fully-discrete approximation of (2), we discretize the time by taking $t_n = n\Delta t$, $\Delta t>0$, $n=1,2,\cdots$ and using a numerical quadrature for the integral

$$\int_0^t g(s)ds \approx \sum_{k=1}^n \omega_{n,k}g^k, \qquad g^k = g(t_k),$$

where $\{\omega_{n,k}\}$ are the integration weights and the following error estimate is valid:

$$|\int_0^{t_n} gds - \sum_{k=1}^n \omega_{n,k}g(t_k)| \le C\Delta t \int_0^{t_n} (|g|+|g'|)ds.$$

Then the fully discrete backward Euler finite volume element approximation of (2) is: find $u_h^n \in S_h$ $(n=1,2,\cdots)$ such that for all $v_h \in S_h^*$

$$\left(\frac{u_h^n - u_h^{n-1}}{\Delta t}, v_h\right) + a(u_h^n,v_h) + \sum_{k=0}^{n-1}\omega_{n,k}b_{n,k}(u_h^k,v_h) = (f^n,v_h), \tag{10}$$

for $n=1,\ldots$, $u_h^0 = u_{0,h}$, where $u_{0,h}$ is some approximation of the initial data and $b_{n,k}(u_h^k,v_h) = b(t_n,t_k;u_h^k,v_h)$.

In a similar fashion, one can define the fully discrete "lumped mass" scheme and Crank-Nicolson scheme.

2.3 Finite Volume Method with L-spines

Now, we define the space S_h in the following way:

$$S_h = \{v \in C(0,1) : (av_x)_x|_{I_j} = 0, \quad j = 1, \cdots, N, \quad v(0) = v(1) = 0\}.$$

Here and in the text that follows further the solution of the equation $(au_x)_x = 0$ is understood in a weak sense. Note that the velocity (flux) $Q(x) = -(au_x)(x)$ has first derivative zero almost everywhere and therefore is a constant function. Thus, if the coefficient $a(x)$ has a jump at some point then the first derivative u_x should have also a jump at that point so that their product is a continuous function. In order to introduce a basis in S_h, we first define the following harmonic means of the coefficient $a(x)$ over the partition T_h:

$$a^H_{j-1/2} = h_j \left(\int_{x_{j-1}}^{x_j} \frac{ds}{a(s)} \right)^{-1}$$

for $j = 1, \ldots .N-1$; apparently, the following functions form a basis in S_h:

$$\phi_j(x) = \begin{cases} a^H_{j-1/2} \dfrac{1}{h_j} \displaystyle\int_{x_{j-1}}^{x} \frac{ds}{a(s)}, & \text{for} \quad x_{j-1} \le x \le x_j, \\ a^H_{j+1/2} \dfrac{1}{h_{j+1}} \displaystyle\int_{x}^{x_{j+1}} \frac{ds}{a(s)}, & \text{for} \quad x_j \le x \le x_{j+1}, \\ 0, & \text{otherwise}, \end{cases} \tag{11}$$

for $j = 1, \ldots, N-1$. On each subinterval of T_h, the derivative of $\phi_j(x)$ exists in generalized sense and has the following property:

$$(a\phi_{j,x})(x_{j-1/2}) = a^H_{j-1/2}/h_j, \quad \text{and} \quad (a\phi_{j,x})(x_{j+1/2}) = -a^H_{j+1/2}/h_{j+1}.$$

Therefore, we come to the following simple representation for the fluxes at the end-points of the finite volumes of the partition T_h^*:

$$(au_{h,x})(x_{j-1/2}) = a^H_{j-1/2} \frac{u_j - u_{j-1}}{h_j}, \quad j = 1, ..., N.$$

In order to introduce the finite volume element method, we first define the bilinear forms $a(\cdot,\cdot)$ and $b(t,s;\cdot,\cdot)$ on $S_h \times S_h^*$:

$$\begin{aligned} a(u_h, v_h) &= \sum_{j=1}^{N} \frac{1}{h_j} a^H_{j-1/2} (u_j - u_{j-1})(v_j - v_{j-1}), \\ b(t, s; u_h, v_h) &= \sum_{j=1}^{N} \frac{1}{h_j} b_{j-1/2}(t,s) \frac{a^H_{j-1/2}}{a(x_{j-1/2})} (u_j - u_{j-1})(v_j - v_{j-1}) \\ &\equiv \sum_{j=1}^{N} \frac{1}{h_j} a^H_{j-1/2} B_{j-1/2}(t,s)(u_j - u_{j-1})(v_j - v_{j-1}). \end{aligned} \tag{12}$$

Here we have used the fact that in the case of discontinuous coefficient, $a(x)$ and the kernel $b(x,t,s)$ in the integral term is of the form $b(x,t,s) = a(x)B(x,t,s)$, where $B(x,t,s)$ is a smooth function of its arguments.

Then the semi-discrete finite volume method with L-splines is given by (6) where the bilinear forms $a(\cdot,\cdot)$ and $b(t,s;\cdot,\cdot)$ are defined in (12). Similarly, the "lumped" mass semi-discrete method and the fully discrete backward Euler method are given by (7) and (10), respectively. This time for the lumped mass approximation, we use the trapezoidal rule over each finite element. It is obvious that the complexity of these schemes is the same as those obtained by using linear elements.

This type of approximation of self-adjoint second-order ordinary differential equations was first used by Tikhonov and Samarskii (see, e.g. [34,35]) to construct finite difference schemes of arbitrary order of accuracy for equations with piece-wise coefficients. The interesting feature of these schemes is that they have high order of accuracy on non-uniform meshes and for equations with jumps in the coefficients. Further, this approach has been extended in [16] to equations with coefficients in certain Sobolev classes.

Remark 1 It is obvious from the matrix presentation of the discretizations (9) that M_h and A_h are symmetric and positive definite matrices and matrix $B_h(t,s)$ is symmetric and has entries that are uniformly bounded with respect to the variables t, s. Therefore, both the semi-discrete and discrete schemes have unique solutions, which are stable in L^2-norm in x and L^∞-norm in t (see, e.g. [5]).

2.4 Some Auxiliary Results and Inequalities

Before discussing the error estimates of the finite volume approximations derived above we shall need some useful inequalities related to the bilinear forms a and b and the finite element interpolant of the solution $u(x,t)$.

Lemma 1 *There exists positive constants C_0, $C_1 > 0$, independent of h, such that*

$$a(w_h, I_h^* w_h) \geq C_0 |w_h|_{1,h}^2, \qquad w_h \in S_h, \tag{13}$$

$$|a(w_h, I_h^* v_h)| \leq C_1 |w_h|_{1,h}\, |v_h|_{1,h}, \qquad w_h,\ v_h \in S_h. \tag{14}$$

Proof. Let $w_h = \sum_{j=1}^{N-1} w_j \phi_j(x)$ and $v_h = \sum_{j=1}^{N-1} v_j \phi_j(x)$; then for linear elements we have

$$a(w_h, I_h^* v_h) = \sum_{j=1}^{N} a_{j-1/2} \frac{w_j - w_{j-1}}{h_j} \frac{v_j - v_{j-1}}{h_j}.$$

Hence, (13) and (14) follow by taking $C_0 = a_0 = \min_x a(x)$ and $C_1 = a_1 = \max_x a(x)$. Note that this inequality is also valid for the scheme obtained by using L-splines for the space S_h. In this case, $a_{j+1/2}$ is replaced by the harmonic averages $a_{j+1/2}^H$ and the inequality follows in the same manner with slightly differrent constants.

Remark 2 The estimate (13) essentially implies that the bilinear form $a(\cdot,\cdot)$ is weakly coercive in $S_h \times S_h^*$ where S_h is equipped with the norm $|\cdot|_1$. Indeed,

$$\sup_{v_h \in S_h^*} a(w_h, v_h) \geq a(w_h, I_h^* w_h) \geq C_0 |w_h|_{1,h}^2.$$

This guarantees the solvability of the discrete problem and its optimal stability.

Next, we study the properties of the interpolant $I_h u$. Here we distinguish two cases: (1) $a(x)$ is a sufficiently smooth function (here it is enough to assume that $a \in W^{k,\infty}$ where $k \geq 1$); then the solution is at least in $W^{k,2}$; (2) $a(x)$ has a finite number of jump discontinuities and between these points of discontinuity it is sufficiently smooth; the flux $Q = -au_x$ is a smooth function. Now we consider these two cases in the next two lemmas.

Lemma 2 *Let the space S_h consist of piece-wise linear functions and $u \in H_0^1 \cap W^{k,p}$ for some $1 \leq p \leq \infty$ and $2 \leq k \leq 3$. Then there exists a positive constant $C > 0$, independent of h, such that*

$$|a(u - I_h u, I_h^* v_h)| \leq Ch^{k-1} |u|_{k,p} |v_h|_{1,q}, v_h \in S_h,\ q = p/(p-1).$$

Proof. Since

$$a(u - I_h u, I_h^* v_h) = \sum_{j=1}^{N} a_{j-1/2} (u - I_h u)_x (x_{j-1/2})(v_j - v_{j-1}) \qquad (15)$$

it is enough to estimate $l(u) = (u - I_h u)_x (x_{j-1/2})$. Thus (15) follows from the Bramble-Hilbert lemma by showing that the linear functional $l(u)$ is bounded in $W^{k,1}[x_{j-1}, x_j]$ for $k \geq 2$ and vanishes for polynomials of degree 2. To estimate the constant in the inequality (15) one can use a Taylor expansion. In the case $k = 3$ this is

$$\begin{aligned} u_x(x_{j-1/2}) - \frac{u_j - u_{j-1}}{h_j} = \frac{-1}{2h_j} \Bigg\{ & \int_{x_{j-1/2}}^{x_{j-1}} (\xi - x_{j-1})^2 u_{xxx}(\xi) d\xi \\ & + \int_{x_j}^{x_{j-1/2}} (\xi - x_j)^2 u_{xxx}(\xi) d\xi \Bigg\}. \end{aligned} \qquad (16)$$

Then, Cauchy-Schwarz inequality yields

$$|a(u - I_h u, I_h^* v_h)| \leq Ch^2 \sum_{j=1}^{N} |u|_{k,p} \ |v_{h,x}|_q \leq Ch^2 |u|_{3,p} |v_h|_{1,q},$$

which completes the proof.

Note that if the coefficient $a(x)$ has one continuous derivative, then using the Sobolev norm of $Q(x) = -a(x)u_x(x)$ one gets

$$|u|_{2,p} \leq C||Q||_{1,p}$$

and the right hand side in the error estimate can be expressed through the the Sobolev norm of the $Q(x)$. This equivalent representation of the error is not important for smooth coefficient $a(x)$. However, this approach is essential when the coefficient $a(x)$ has jumps, the case treated by L-splines. Below we show that the error is of first order in this case, too.

Lemma 3 *Let the space S_h consist of L-splines and its basis be defined by (11). Assume that $Q \in W^{1,p}$ for $1 \leq p \leq \infty$. Then there exists a positive constant $C > 0$, independent of h, such that*

$$|a(u - I_h u, I_h^* v_h)| \leq Ch|Q|_{1,p}|v_h|_{1,q}, \qquad v_h \in S_h,$$

for $q = p/(p-1)$.

Proof. Here we have used the following convention:

$$a(u - I_h u, I_h^* v_h) = a(u, I_h^* v_h) - a(I_h u, I_h^* v_h),$$

where $a(u, I_h^* v_h)$ is defined by (5) and $a(I_h u, I_h^* v_h)$ is defined by (12). Then taking into account that

$$\frac{u_j - u_{j-1}}{h_j} = \frac{1}{h_j} \int_{x_{j-1}}^{x_j} \frac{Q(x)}{a(x)} dx,$$

we get the following representation

$$a(u - I_h u, I_h^* v_h) = -\sum_{j=1}^{N} (v_j - v_{j-1}) l(Q). \tag{17}$$

where

$$l(Q) = Q_{j-1/2} - \frac{a_{j-1/2}^H}{h_j} \int_{x_{j-1}}^{x_j} \frac{Q(x)}{a(x)} dx.$$

Obviously the expression $l(Q)$ is a linear functional in Q, which is bounded for Q in the Sobolev space $W^{1,p}(x_{j-1}, x_j)$ and vanishes for Q being a constant. Thus, by the Bramble-Hilbert lemma

$$|l(Q)| \leq Ch^{1-1/p} \left(\int_{x_{j-1}}^{x_j} |Q'(x)|^p dx \right)^{1/p}.$$

Then, the Cauchy-Schwarz inequality yields the required estimate (17).

Remark 3 It is obvious from (17) that if the coefficient $a(x)$ is smooth, then $a_{j-1/2}^H = a_{j-1/2} + Ch^2$ and one easily gets the estimate (15).

3 Optimal-Order Error Estimates for the Finite Volume Method

A key point in the error analysis of the finite element method for parabolic problems plays the decomposition of the error into two parts: $u - u_h = (u - R_h u) + (R_h u - u_h)$, where $R_h u \in S_h$ is the Ritz projection defined by $a(u - R_h u, v_h) = 0$ for all $v_h \in S_h$. A direct use of Ritz projection in the finite element analysis of parabolic and hyperbolic integro-differential equations, with two or more elliptic operators of the same order, leads to suboptimal error estimates. In order to offset this deficiency Cannon and Lin [6] introduced the so-called Ritz-Volterra projection $V_h u$, which we shall use as a main tool in the error analysis of the finite volume schemes.

3.1 Ritz-Volterra Projection V_h

Here we introduce the Ritz-Volterra projection operator V_h in the context of the finite volume element method for the equation (2): namely, for $u(t)$ in $H_0^1 \cap H^2$ for any $t > 0$ we define its Ritz-Volterra projection $V_h u \in S_h$ for $t \geq 0$ by

$$a(u - V_h u, v_h) + \int_0^t b(t, s; u(s) - V_h u(s), v_h) ds = 0, \qquad v_h \in S_h^*. \quad (18)$$

Note, that $V_h u$ defined over the partition T_h. We begin our analysis with the existence of the Ritz-Volterra projection and its error in the H^1-norm.

Theorem 1 *Let $u(t)$ be in H_0^1, let $Q(t) = -a(x)u_x(x,t)$ be in H^1 for all $t \geq 0$, and let $u(t)$ be differentiable in t. Then the Ritz-Volterra projection $V_h u$ of u defined by (18) exists, is unique, and there is a positive constant $C > 0$, independent of h, such that for $t > 0$*

$$|u(t) - V_h u(t)|_{1,h} \equiv |I_h u - V_h u|_1 \leq Ch \left(||Q(t)||_1 + \int_0^t ||Q(s)||_1 ds \right), \quad (19)$$

and

$$|D_t(u(t) - V_h u(t))|_{1,h} \leq Ch \left(||Q(t)||_1 + ||Q_t(t)||_1 + \int_0^t ||Q(s)||_1 ds \right). \quad (20)$$

Proof. This theorem gives an optimal-order convergence under minimal assumptions on the regularity of the solution for both methods: the one based on linear finite elements and the one based on L-splines. However, the scheme based on linear finite elements requires smoothness of the coefficient $a(x)$, while the scheme obtained by using L-splines has first-order convergence for a discontinuous coefficient $a(x)$.

The existence and uniqueness of the Ritz-Volterra projection follow from Lemma 1. Indeed, let $V_h u = \sum_{j=1}^{N-1} V_j(t)\phi_j(x)$ and $V = (V_1, V_2, \cdots, V_{N-1})^T$, then V satisfies

$$A_h V(t) + \int_0^t B_h(t,s) V(s) ds = F_h(t),$$

where $F_h = (F_1, F_2, \cdots, F_{N-1})^T$, $F_j = a(u, \chi_j) + \int_0^t b(t, s; u(s), \chi_j) ds$. Since A_h is a non-singular matrix by Lemma 1 the theory of Volterra equations implies that $V(t)$ exists and is unique (see, e.g. [5]).

Denote by $v_h = I_h u - V_h u \in S_h$. Then the estimate (19) follows from (15) of Lemma 2 and (17) of Lemma 3. Indeed,

$$\begin{aligned}
C_0 \|v_h\|_1^2 &= C_0 |I_h u - V_h u|_1^2 \le a(I_h u - V_h u, I_h^*(I_h u - V_h u)) \\
&= a(I_h u - V_h u, I_h^* v_h) + \int_0^t b(t, s; I_h u(s) - V_h u(s), I_h^* v_h) ds \\
&\quad - \int_0^t b(t, s; I_h u(s) - V_h u(s), I_h^* v_h) ds \\
&= a(I_h u - u, I_h^* v_h) + \int_0^t b(t, s; I_h u(s) - u(s), I_h^* v_h) ds \\
&\quad - \int_0^t b(t, s; I_h u(s) - V_h u(s), I_h^* v_h) ds \\
&= C \left\{ h \left(||Q||_1 + \int_0^t ||Q(s)||_1 ds \right) + \int_0^t |I_h u(s) - V_h u(s)|_1 ds \right\} |v_h|_1,
\end{aligned}$$

and therefore,

$$|I_h u(t) - V_h u(t)|_1 \le Ch \left(||Q||_1 + \int_0^t ||Q(s)||_1 ds) \right) + C \int_0^t |I_h u(s) - V_h u(s)|_1 ds.$$

Thus Gronwall's inequality implies (19).

To prove (20) we differentiate (18) with respect to time and obtain the following identity for any $v_h \in S_h^*$:

$$\begin{aligned}
a((u - V_h u)_t, v_h) &+ b(t, t; u - V_h u, v_h) \\
&+ \int_0^t b_t(t, s; u(s) - V_h u(s), v_h) ds = 0.
\end{aligned}$$

Then by Lemma 2 and the above identity for $v_h = D_t(I_h u - V_h u)$ we get

$$\begin{aligned}
c_0 |D_t(I_h u - V_h u)|_1^2 &\le a(D_t(I_h u - V_h u), I_h^* v_h) \\
&= a(D_t(I_h u - u), I_h^* v_h) + b(t, t; I_h u - u, I_h^* v_h) \\
&\quad + \int_0^t b_t(t, s; I_h u(s) - u(s), I_h^* v_h) ds \\
&\quad - b(t, t; I_h u - V_h u, I_h^* v_h) + \int_0^t b_t(t, s; I_h u(s) - V_h u(s), I_h^* v_h) ds \\
&\le Ch \left(|Q_t|_1 + |Q|_1 + \int_0^t |Q(s)|_1 ds \right) |I_h v_h|_1 \\
&\quad + \left(|I_h u - V_h u|_1 + \int_0^t |I_h u(s) - V_h u(s)|_1 ds \right) |I_h v_h|_1.
\end{aligned}$$

Then by (19) we conclude that

$$|D_t(I_hu - V_hu)|_1 \le Ch\left(|Q|_1 + |Q_t|_1 + \int_0^t |Q(s)|_1 ds\right).$$

which leads to (20).

3.2 Error Estimate of the Finite Volume Element Method

In this section we derive an error estimate for the finite volume solution u_h in the discrete H^1 norm. Namely, we prove the following theorem.

Theorem 2 *Let $u(t)$ and $u_h(t)$ be the solution of problem (2) and its finite volume element approximation defined by (6), respectively. Then there exists a positive constant $C > 0$, independent of h, such that for $t \ge 0$*

$$C||u-u_h||_1 \le ||u_0 - u_{0,h}||_{1,h} + h\left(|Q(0)|_1 + \left(\int_0^t |Q_t(s)|_1^2 ds\right)^{1/2}\right) \quad (21)$$

Proof. As usual we decompose the error into two parts:

$$u - u_h = (u - V_hu) + (V_hu - u_h) = \rho(t) + \theta(t).$$

Theorem 1 gives us an estimate for $\rho(t)$:

$$||\rho(t)||_{1,h} + \int_0^t ||\rho_t(s)||_{1,h} ds \le Ch\int_0^t (||Q(s)|| + ||Q_t(s)||_1)ds. \quad (22)$$

Now we estimate $\theta(t)$. From (2), (6) and (18) it follows that $\theta(t)$ satisfies

$$(\theta_t, v_h) + a(\theta, v_h) + \int_0^t b(t,s;\theta(s), v_h) = -(\rho_t, v_h), \qquad v_h \in S_h^*. \quad (23)$$

To prove (21), we set $v_h = I_h^*\theta_t(t)$ and apply Lemma 1

$$\begin{aligned} |||I_h^*\theta_t|||_0^2 + \frac{1}{2}\frac{d}{dt}a(\theta, I_h^*\theta) &= (\rho_t, I_h^*\theta_t) - \frac{d}{dt}\int_0^t b(t,s;\theta(s), I_h^*\theta(t))ds \\ &\quad + b(t,t;\theta(t), I_h^*\theta(t)) + \int_0^t b_t(t,s;\theta(s), I_h^*\theta(t))ds \\ &\le \frac{1}{2}|||I_h^*\theta_t|||_0^2 + C\left(||\rho_t||^2 + |\theta|_1^2 + \int_0^t |\theta(s)|_1^2 ds\right) \\ &\quad + \frac{d}{dt}\int_0^t b_t(t,s;\theta(s), I_h^*\theta(t))ds. \end{aligned}$$

After integration in t we get

$$\int_0^t |||I_h^*\theta_t|||_0^2 ds + |\theta|_1^2 \le C\left(|\theta(0)|_1^2 + \int_0^t (||\rho_t||^2 + |\theta(s)|_1^2) ds\right) + \int_0^t b_t(t,s;\theta(s), I_h^*\theta(t)) ds \le \frac{1}{2}|\theta|_1^2 + C\left(|\theta(0)|_1^2 + \int_0^t (||\rho_t||^2 + |\theta(s)|_1^2) ds\right),$$

and then by Gronwall's inequality

$$|\theta(t)|_1^2 \le C\left(|\theta(0)|_1^2 + \int_0^t ||\rho_t||^2 ds\right). \tag{24}$$

Since

$$|\theta(0)|_1 \le |V_h u(0) - u(0)|_{1,h} + ||u(0) - u_h(0)||_1 \tag{25}$$

then (21) follows by inequality (19) and our assumptions.

4 Higher-Order Estimates for Linear Elements

4.1 Superconvergence in H^1-norm for Ritz-Volterra Projection

Our next goal is to derive a higher-order error estimate for the Ritz-Volterra projection based on linear finite elements. Namely, we prove

Theorem 3 *Let $V_h u$ be the Ritz-Volterra projection of u and assume that $u \in H^3 \cap H_0^1$. Then there exists a positive constant $C > 0$, independent of h, such that*

$$|u(t) - V_h u(t)|_{1,h} = |I_h u(t) - V_h u(t)|_{1,h} \le Ch^2\left(|u|_3 + \int_0^t |u(s)|_3 ds\right), \quad t \ge 0, \tag{26}$$

$$|u(t) - V_h u(t)|_{1,h}^* \le Ch^2\left(|u|_3 + \int_0^t |u(s)|_3 ds\right), \quad t \ge 0. \tag{27}$$

Proof. The proof follows from the same argument as that used in Theorem 1 except that (15) of Lemma 2 will be used with $p = q = 2$ and $k = 3$:

$$|u(t) - V_h u(t)|_{1,h} = |I_h u - V_h u|_{1,h} \le Ch^2\left(|u|_3 + \int_0^t |u(s)|_3 ds\right), \quad t \ge 0.$$

Since $|v|_{1,h} = |v|^*_{1,h}$ for $v \in S_h$ then by the triangle inequality and the previous estimate we get

$$\begin{aligned}|u(t) - V_h u(t)|^*_{1,h} &\le |u(t) - I_h u(t)|^*_{1,h} + |I_h u(t) - V_h u(t)|^*_{1,h} \\ &\le Ch^2 |u|_3 + |I_h u(t) - V_h u(t)|^*_{1,h},\end{aligned}$$

which combined with (27) gives the required result.

We remark that $|\cdot|^*_{1,h}$ is a discrete semi-norm for the gradient evaluated at the points $x_{j-1/2}$. In the engineering literature, these points are often called optimal stress points. In the finite element terminology these are the so-called superconvergence points for the gradient.

For smooth $a(x)$ we can get the same error estimate for the solution of the scheme based on L-splines. Indeed, in this case the estimate follows immediately from the conclusions of Remark 3. In the case of rough coefficient $a(x)$ we believe that we cannot have superconvergence. This can be seen from the presentation of the error in $u - I_h u$ by (17).

4.2 Error Estimates in L^2-norm for Ritz-Volterra Projection

The error estimates (26) and (27) will produce estimates for the Ritz-Volterra projection in L^2-norm as well. However, in the case of linear elements we can prove a second-order convergence in a weaker norm of the solution $u(x,t)$. Namely, we prove the following result:

Theorem 4 *Let $V_h u$ be defined by (18) and assume that $u \in W^{3,1}(0,1) \cap H^1_0$. Then there exists a positive constant $C > 0$, independent of h, such that*

$$||u(t) - V_h u(t)||_0 \le Ch^2 \left(|u|_{3,1} + \int_0^t |u(s)|_{3,1} ds \right), \qquad t \ge 0. \tag{28}$$

Proof. The proof is based on a duality argument. Let $w \in H^2 \cap H^1_0$ such that

$$a(w,v) = (u - V_h u, v), \qquad v \in H^1_0.$$

Then by the elliptic regularity $||w||_2 \le C||u - V_h u||_0$. Then taking $v = u - V_h u$ in the above equation and using the Ritz-Volterra projection we see that

$$\begin{aligned}||u - V_h u||_0^2 &= a(u - V_h u, w) = a(u - V_h u, w) + \int_0^t b(t,s;u(s) - V_h u(s), w) ds \\ &\quad - \int_0^t b(t,s;u(s) - V_h u(s), w) ds \\ &= a(u - V_h u, w - I_h w) + \int_0^t b(t,s;u(s) - V_h u(s), w - I_h w) ds \\ &+ a(u - V_h u, I_h w - I_h^* w) + \int_0^t b(t,s;u(s) - V_h u(s), I_h w - I_h^* w) ds \\ &\quad - \int_0^t b(t,s;u(s) - V_h u(s), w) ds.\end{aligned}$$

For the first two terms on the right hand side, we use Theorem 1 interpolation estimates and elliptic regularity to get

$$\begin{aligned} a(u - V_h u, w - I_h w) &+ \int_0^t b(t, s; u(s) - V_h u(s), w - I_h w) ds \\ &\le Ch \left(|u|_2 + \int_0^t |u(s)|_2 ds \right) \left(\sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} (w_x - \frac{w_j - w_{j-1}}{h_j})^2 dx \right)^{1/2} \\ &\le Ch^2 \left(|u|_2 + \int_0^t |u(s)|_2 ds \right) ||u - V_h u||_0. \end{aligned}$$

Here we have used again the Bramble-Hilbert lemma to estimate the integral term by $||w||_2$ and the elliptic regularity ensuring that $\|w\|_2 \le C||u - V_h u||_0$. Using integration by parts for the last term we get

$$\begin{aligned} \int_0^t b(t, s; u(s) - V_h u(s), w) ds & \\ &\le | \int_0^t (u(s) - V_h u(s), (b w_x)_x) ds| \\ &\le C \int_0^t ||u(s) - V_h u(s)||_0 ds ||w||_2 \\ &\le C \left(\int_0^t ||u(s) - V_h u(s)||_0 ds \right) ||u - V_h u||_0. \end{aligned}$$

Now we estimate the remaining two terms. First we present them in the form

$$\begin{aligned} a(u - V_h u, I_h w - I_h^* w) &+ \int_0^t b(t, s; u(s) - V_h u(s), I_h w - I_h^* w) ds \\ = \sum_{j=1}^{N} \{ & \int_{x_{j-1}}^{x_j} (a - a_{j-1/2})(u - V_h u)_x dx \\ &+ \int_0^t (b(x, t, s) - b_{j-1/2}(t, s))(u - V_h u)_x dx dt \} \frac{w_j - w_{j-1}}{h_j} \\ + \sum_{j=1}^{N} \{ & \int_{x_{j-1}}^{x_j} a_{j-1/2}(u_j - u_{j-1} - h_j u_x(x_{j-1/2})) \\ &+ \int_0^t (b_{j-1/2}(t, s)((u_j - u_{j-1} - h_j u_x(x_{j-1/2}) ds \}) \frac{w_j - w_{j-1}}{h_j} \equiv N_1 + N_2. \end{aligned}$$

The first term can be bounded using Lemma 2, the Sobolev embedding inequality and elliptic regularity:

$$|N_1| \le Ch^2 |a|_{1,\infty} \left(|u|_2 + \int_0^t |u(s)|_2 ds \right) ||u - V_h u||_0.$$

To estimate the term N_2 we use a Taylor expansion and Lemma 2

$$|N_2| \le Ch^2 \left(|u|_{3,1} + \int_0^t |u(s)|_{3,1} ds \right) ||u - V_h u||_0.$$

Combining all these estimates we get

$$||u - V_h u||_0 \le Ch^2 \left(|u|_2 + \int_0^t |u(s)|_2 ds \right)$$
$$+Ch^2 \left(|u|_{3,1} + \int_0^t |u(s)|_{3,1} ds \right) + C \int_0^t ||u(s) - V_h u(s)||_0 ds$$

and the proof is complete by Gronwall's inequality and $|u|_2 \le C|u|_{3,1}$.

Theorem 5 *Assume that $u(t)$ is sufficiently smooth. Then there is a constant $C_k > 0$, independent of h, such that the following estimates hold:*

$$|D_t^k(u - V_h u)|_1 \le C_k h \left\{ \sum_{l=0}^{k} |D_t^l u|_2 + \int_0^t \sum_{l=0}^{k-1} |D_t^l u(s)|_2 ds \right\}, \tag{29}$$

$$|D_t^k(u - V_h u)|_0 \le C_k h^2 \left\{ \sum_{l=0}^{k} |D_t^l u|_{3,1} + \int_0^t \sum_{l=0}^{k-1} |D_t^l u(s)|_{3,1} ds \right\}, \tag{30}$$

$$|D_t^k(u - V_h u)|_{1,h}^* \le C_k h^2 \left\{ \sum_{l=0}^{k} |D_t^l u|_3 + \int_0^t \sum_{l=0}^{k-1} |D_t^l u(s)|_3 ds \right\}. \tag{31}$$

Proof. Using again the identity (21) for $w_h = D_t(I_h u - V_h u)$, we get

$$\begin{aligned}
&c_0|D_t(I_h u - V_h u)|_1^2 \le a(D_t(I_h u - V_h u), I_h^* v_h) \\
&= a(D_t(I_h u - u), I_h^* v_h) + b(t, t; I_h u - u, I_h^* v_h) \\
&+ \int_0^t b_t(t, s; I_h u(s) - u(s), I_h^* v_h) ds - b(t, t; I_h u - V_h u, I_h^* v_h) \\
&+ \int_0^t b_t(t, s; I_h u(s) - V_h u(s), I_h^* v_h) ds \\
&\le Ch \left(|u_t|_2 + |u|_2 + \int_0^t |u(s)|_2 ds \right) |I_h v_h|_1 \\
&\quad + \left(|I_h u - V_h u|_1 + \int_0^t |I_h u(s) - V_h u(s)|_1 ds \right) |I_h v_h|_1.
\end{aligned}$$

Then by Theorem 1, we conclude that

$$|D_t(I_h u - V_h u)|_1 \le Ch \left(|u|_2 + |u_t|_2 + \int_0^t |u(s)|_2 ds \right).$$

Thus, the required estimate (29) for $k = 1$ follows from the above inequality and

$$|D_t(u - V_h u)|_1 \le |D_t(u - I_h u)|_1 + |D_t(I_h u - V_h u)|_1.$$

The estimates (30),(31), as well as those for higher time-derivatives, can be proved using the same arguments.

4.3 Error Estimates for the Finite Volume Element Method

In this section we obtain higher order error estimates for the finite volume element approximation. The analysis is a modification of the finite element analysis of integro-differential equations of parabolic and hyperbolic type (see, e.g. [6,7,25–27]) and uses the Ritz-Volterra projection and the results established in Lemmas 3 and 3, and Theorems 1-5.

Theorem 6 *Let $u(t)$ and $u_h(t)$ be the solution of problem (2) and its finite volume element solution defined by (6), respectively. Then there exists a positive constant $C > 0$, independent of h, such that for $t \geq 0$*

$$||u - u_h||_0 \leq C \left\{ ||u_0 - u_{0,h}|| + h^2 \left(|u_0|_{3,1} + \int_0^t |u_t(s)|_{3,1} ds \right) \right\}, \quad (32)$$

$$||u - u_h||^*_{1,h} \leq C \left\{ |u_0 - u_{0,h}|_{1,h} + h^2 \left(|u_0|_3 + \int_0^t |u_t(s)|_3 ds \right) \right\}. \quad (33)$$

Proof. Ee decompose the error by $u - u_h = (u - V_h u) + (V_h u - u_h) = \rho(t) + \theta(t)$. From Theorem 5 we have

$$||\rho(t)||_0 + \int_0^t ||\rho_t(s)||_0 ds \leq Ch^2 \left(|u_0|_{3,1} + \int_0^t |u_t(s)|_{3,1} ds \right), \quad (34)$$

$$||\rho(t)||_1 + \int_0^t ||\rho_t(s)||_1 ds \leq Ch \left(|u_0|_2 + \int_0^t |u_t(s)|_2 ds \right), \quad (35)$$

$$||\rho(t)||^*_{1,h} + \int_0^t ||\rho_t(s)||^*_{1,h} ds \leq Ch^2 \left(|u_0|_3 + \int_0^t |u_t(s)|_3 ds \right). \quad (36)$$

First, we see from (2), (6) and (18) that $\theta(t)$ satisfies (23). Since $\theta(t) \in S_h$, we let $v_h = I_h^* \theta$ in (23) to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |||\theta(t)|||_0^2 + a_0 |\theta|_1^2 &\leq ||\rho_t|| \, ||I_h^* \theta|| + C \left(\int_0^t |\theta(s)|_1 ds \right) |\theta(t)|_1 \\ &\leq \frac{a_0}{2} |\theta|_1^2 + C \int_0^t |\theta(s)|_1^2 ds + ||\rho_t|| \, |||\theta|||_0. \end{aligned}$$

Now integrating from 0 to t, we find that

$$\begin{aligned} |||\theta(t)|||_0^2 + \int_0^t |\theta(s)|_1^2 ds &\leq C \left(|||\theta(0)|||_0^2 + \int_0^t ||\rho_t|| \, |||\theta|||_0 \right) \\ &\quad + C \int_0^t \int_0^\tau |\theta(s)|_1^2 ds d\tau, \end{aligned}$$

and then from Gronwall's inequality that

$$\begin{aligned} |||\theta(t)|||_0^2 + \int_0^t |\theta(s)|_1^2 ds &\leq C \left(|||\theta(0)|||_0^2 + \int_0^t ||\rho_t|| \, |||\theta|||_0 \right) \\ &\leq \frac{1}{2} \sup_{0<s<t} |||\theta(s)|||_0^2 + C |||\theta(0)|||_0^2 + C \left(\int_0^t ||\rho_t(s)|| ds \right)^2. \end{aligned}$$

Thus we have easily that

$$|||\theta(t)|||_0 + \left(\int_0^t |\theta(s)|_1^2 ds\right)^{1/2} \leq C\left(|||\theta(0)|||_0 + \int_0^t ||\rho_t(s)||ds\right). \quad (37)$$

Since

$$|||\theta(0)|||_0 \leq |||V_h u_0 - u_0|||_0 + |||u_0 - u_{0,h}|||_0 \leq Ch^2|u_0|_{3,1} + ||u_0 - u_{0,h}||_0,$$

the required estimate (32) follows from the above analysis, (34), and the triangle inequality.

Finally, from (24) and (34) we also have that

$$|\theta(t)|_1^2 \leq C\left(|\theta(0)|_1^2 + \int_0^t ||\rho_t||_0^2 ds\right) \leq C\left(|\theta(0)|_1^2 + h^2(|u_0|_{3,1} + \int_0^t |u_t|_{3,1} ds)\right)$$

and

$$|\theta(0)|_{1,h}^* \leq Ch^2|u_0|_3 + |u_0 - u_{0,h}|_{1,h}^*$$

so that (33) follows from

$$|u - u_h|_{1,h}^* \leq |u - V_h u|_{1,h}^* + |V_h u - u_h|_{1,h}$$

and the above analysis. The proof is complete.

In order to estimate the error of the lumped mass finite volume element approximation (7), we need an estimate for error of the quadrature which produced the lumped approximation. This error is estimated in the following lemma:

Lemma 4 *There exists a positive constant $C > 0$ such that*

$$|(w_h - I_h^* w_h, I_h^* v_h)| \leq Ch^2|w_h|_1 \, |v_h|_1 \quad \textit{for all} \quad w_h,\ v_h \in S_h. \quad (38)$$

Let

$$w_h = \sum_{j=1}^{N-1} w_j\phi_j, \qquad v_h = \sum_{j=1}^{N-1} v_j\phi_j, \qquad I_h v_h = \sum_{j=1}^{N-1} v_j\chi_j$$

we find from a simple calculation that

$$\begin{aligned}
(w_h - I_h^* w_h, I_h^* v_h) &= \sum_{j=1}^{N-1} v_j(w_h - I_h^* w_h, \chi_j) = \sum_{j=1}^{N-1} v_j\left(\sum_{k=1}^{N-1}\int_{x_{k-1/2}}^{x_{k+1/2}} (w_h - I_h^* w_h)dx\right) \\
&= \sum_{j=1}^{N-1} v_j\left(\frac{3(h_j + h_{j+1})}{8}w_j + \frac{h_j}{8}w_{j-1} + \frac{h_{j+1}}{8}w_{j+1} - \frac{h_j + h_{j+1}}{2}w_j\right) \\
&= \frac{1}{8}\sum_{j=1}^{N-1} v_j\left\{\frac{w_{j+1} - w_j}{h_{j+1}}h_{j+1}^2 - \frac{w_j - w_{j-1}}{h_j}h_j^2\right\} \\
&= \frac{1}{8}\sum_{j=1}^{N-1}\frac{w_j - w_{j-1}}{h_j}h_j^2(v_j - v_{j-1}) \leq Ch^2|w_h|_1|v_h|_1.
\end{aligned}$$

We are now ready to state and prove the error estimates for the lumped mass method.

Theorem 7 *Assume that $u(t)$ and $u_h(t)$ are the solution of problem (2) and its lumped mass finite volume element solution defined by (7). Then there exists a positive constant $C > 0$, independent of h, such that for $t \geq 0$*

$$||u - u_h||_0 \leq C \left\{ ||u_0 - u_{0,h}|| + h^2 \left(|u_0|_{3,1}^2 + \int_0^t |u_t(s)|_{3,1}^2 ds \right)^{1/2} \right\} \quad (39)$$

$$||u - u_h||_1 \leq C \left\{ ||u_0 - u_{0,h}||_1 + h \left(|u_0|_2^2 + \int_0^t |u_t(s)|_2^2 ds \right)^{1/2} \right\}, \quad (40)$$

$$||u - u_h||_{1,h}^* \leq C \left\{ |u_0 - u_{0,h}|_{1,h}^* + h^2 \left(|u_0|_3^2 + \int_0^t |u_t(s)|_3^2 ds \right)^{1/2} \right\}. \quad (41)$$

Proof. As before, we write $u - u_h = (u - V_h u) + (V_h u - u_h) = \rho(t) + \theta(t)$. Then the estimates for ρ is the same as in Theorem 5, and θ satisfies now

$$(\theta_t, v_h) + a(\theta, v_h) + \int_0^t b(t, s; \theta(s), v_h) = -(\rho^*, v_h), \qquad v_h \in S_h^*. \quad (42)$$

with $\rho^* = u_t - I_h^* D_t V_h u = D_t(u - V_h u) + (D_t V_h u - I_h^* D_t V_h u) = \rho_1 + \rho_2$.

Set $v_h = I_h^* \theta$ in (42) to obtain

$$\frac{1}{2} \frac{d}{dt} |||\theta(t)|||_0^2 + a(\theta, I_h^* \theta) \leq (\rho_1 + \rho_2, I_h^* \theta) + C \left(\int_0^t |\theta(s)|_1 ds \right) |\theta(t)|_1.$$

Since $|(\rho_1, I_h^* \theta)| \leq ||D_t(u - V_h u)||\ |||\theta|||_0$, Lemma 4 and Theorem 5 provide

$$|(\rho_2, I_h^* \theta)| \leq Ch^2 |V_h u|_1 |\theta|_1 \leq Ch^2 \left(|u|_2 + \int_0^t |u(s)|_2 ds \right) |\theta|_1,$$

where we have used $|V_h u|_1 \leq C(|u|_1 + \int_0^t |u(s)| ds)$. Thus, we obtain easily that

$$\frac{1}{2} \frac{d}{dt} |||\theta(t)|||_0^2 + a_0 |\theta|_1^2 \leq \frac{a_0}{2} |\theta|^2 + |||D_t(u - V_h u)||\ |||\theta|||_0 + C \int_0^t |\theta(s)|_2^2 ds$$
$$+ Ch^4 \left(|u|_2^2 + \int_0^t |u(s)|_2^2 ds \right).$$

The rest of the proof is the same as that of Theorem 6 for the estimate (32). The remaining estimates (40) and (41) are established in a similar way.

Theorem 8 *Assume that $u(t)$ and $u_h^n(t)$ are the solutions of problem (2) and its backward Euler finite volume element solution (10), respectively.*

Then there exists a positive constant $C > 0$, independent of h, such that for $0 \le t_n \le T$

$$||u(t_n) - u_h^n||_0 \le C \left\{ ||u_0 - u_{0,h}|| + h^2 \left(|u_0|_{3,1} + \int_0^{t_n} |u_t(s)|_{3,1} ds \right) \right\} + C\Delta t \int_0^{t_n} (|u_t(s)|_1 + |u_{tt}(s)|) ds, \tag{43}$$

$$||u(t_n) - u_h^n||_1 \le C \left\{ ||u_0 - u_{0,h}||_1 + h \left(|u_0|_2 + \left(\int_0^{t_n} |u_t(s)|_2^2 ds \right)^{1/2} \right) \right\} + C\Delta t \int_0^{t_n} (|u_t(s)|_1 + |u_{tt}(s)|) ds, \tag{44}$$

$$||u(t_n) - u_h^n||_{1,h}^* \le C \left\{ |u_0 - u_{0,h}|_{1,h}^* + h^2 \left(|u_0|_3 + \int_0^{t_n} |u_t(s)|_3 ds \right) \right\} + C\Delta t \int_0^{t_n} (|u_t(s)|_1 + |u_{tt}(s)|) ds. \tag{45}$$

Proof. Let $u(t_n) - u_h^n = \rho^n + \theta^n$, where $\rho^n = u(t_n) - V_h u(t_n)$ and $\theta^n = V_h u(t_n) - u_h^n$, then from (18) and (24) we have

$$(\partial\theta^n, v_h) + a(\theta^n, v_h) + \sum_{k=1}^{n-1} \Delta t b_{n,k}(\theta^n, v_h), = -(\tau^n, v_h) + q^n(v_h),\ v_h \in S_h^*, \tag{46}$$

where

$$\tau^n = \partial\rho^n + u_t(t_n) - \partial u(t_n),$$
$$q^n(v_h) = \sum_{k=1}^{n-1} \Delta t b_{n,k}((V_h u)(t_k), v_h) - \int_0^t b(t_n, s; V_h u(s), v_h) ds.$$

Set $v_h = I_h^* \theta^n$ in (46) and use Cauchy inequality and numerical quadrature error estimate to get

$$\frac{|||\theta^n|||_0^2 - |||\theta^{n-1}|||_0^2}{2\Delta t} + c_0 |\theta^n|_1^2 \le (\tau^n, I_h^*\theta^n) + q^n(I_h^*\theta^n) - \sum_{k=1}^{n-1} \Delta t b_{n,k}(\theta^n, I_h^*\theta^n)$$
$$\le \frac{c_0}{2} |\theta^n|_1^2 + C \sum_{k=1}^{n-1} \Delta t |\theta^n|_1^2 + ||\tau^n||\ |||\theta^n|||_0 + C \left(\Delta t \int_0^{t_n} |D_t V_h u(s)|_1 ds \right)^2.$$

Thus, using Theorem 5 for the last term on the right hand side, summing on n and then employing Gronwall's inequality, we obtain

$$|||\theta^n|||_0^2 \le C \left\{ |||\theta^0|||_0^2 + \left(\Delta t \int_0^{t_n} |D_t u(s)|_1 ds \right)^2 \right\} + \sum_{k=1}^{n-1} \Delta t ||\tau^n||\ |||\theta^n|||_0$$

$$\leq \frac{1}{2}\max_{1\leq k\leq n}|||\theta^k|||_0^2 + C\left(\sum_{k=1}^{n-1}\Delta t||\tau^n||\right)^2$$
$$+C\left\{|||\theta^0|||_0^2 + \left(\Delta t\int_0^{t_n}|D_t u(s)|_1 ds\right)^2\right\},$$

from which it follows that

$$|||\theta^n|||_0 \leq C\left(\sum_{k=1}^{n-1}\Delta t||\tau^n|| + |||\theta^0|||_0^2 + \Delta t\int_0^{t_n}|D_t u(s)|_1 ds\right).$$

A simple calculation together with Theorem 5 shows that

$$\Delta t\sum_{k=1}^{n-1}||\tau_n|| \leq C\left(h^2\int_0^{t_n}|u|_{3,1}ds + \Delta t\int_0^{t_n}|u_{tt}(s)|ds\right)$$

and $|||\theta^0|||_0 \leq Ch^2|u_0|_{3,1} + ||u_0 - u_{0,h}||$, hence (43) follows from the above analysis and Theorem 5. The proof of (44) and (45) are done in a similar fashion.

Remark 4 (a) The lumped mass finite volume element approximations can be analysed in a similar way. (b) If second-order numerical quadrature formulae are used to discretize the time integral terms, then a Crank-Nicolson type scheme will have optimal-order convergence in both space and time. Finite element schemes of this type have been abalyzed in [6,7,10,27,33]. (c) For storage saving and computational speed up, some combined numerical quadrature rules proposed in [10] and [33] can be ed as well. The error analysis can be done in the framework developed in this section by using the error estimates from [10] and [33].

Acknowledgments
The research of the first and the second authors has been partially supported by the EPA Grant # R 825207-01-1. The third author has been supported in parts also by NSERC (Canada) The third author thanks the Institute for Scientific Computations and the Department of Mathematics at Texas A & M University for the financial support and hospitality during his sabbatical leave from the University of Alberta.

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