AN MODIFIED NONLINEAR GALERKIN METHOD FOR THE VISCOELASTIC FLUID MOTION EQUATIONS

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Abstract

In this article we first provide *a priori* estimates of the solution for the nonstationary two-dimensional viscoelastic fluid motion equations with periodic boundary condition. We then present an modified nonlinear Galerkin method for solving such equations. By comparing the convergence rates of the proposed method with the standard Galerkin method, we conclude that the modified nonlinear Galerkin method is better than the standard Galerkin method because the former can save a large amount of computational work and maintain the convergence rate of the latter.

1. Introduction

In this paper we consider Oldroyd's mathematical model of two-dimensional viscoelastic fluid motion. Such model (see [1]) can be defined by the rheological relation

$$k_0\sigma + k_1\frac{\partial\sigma}{\partial t} = \eta_0\xi + \eta_1\frac{\partial\xi}{\partial t},\ \ k_1\sigma(0,x) = \eta_1\xi(0,x), \tag{1.1}$$

where σ is the stress tensor and ξ is the strain tensor, and k_0 , k_1 , η_0 , η_1 are positive constants. In fact $\xi = (\xi_{i,j})$ is the 2×2 matrix with components

$$\xi_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

with $u = (u_1, u_2)$ being the rate of the fluid motion. If $\eta_0 k_1 = k_0\eta_1$ in (1.1), we shall obtain Newton's model of uncompressible viscous fluid [1, 2, 11].

Relation (1.1) and the motion equation in Cauchy form leads us to the initial boundary value problem

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u - \nu\triangle u - \int_0^t \rho\exp\{-\delta(t-s)\}\triangle u ds - \nabla p = F, \tag{1.2}$$

$$\operatorname{div} u = 0\ \ (t\geq 0, x\in\Omega)\,, \tag{1.3}$$

$$u = 0\ \ (t\geq 0, x\in\partial\Omega); \tag{1.4}$$

$$u(x,0) = u_0(x)\quad (x\in\overline{\Omega}),\ (p,1) = \int_\Omega p(x,t)dx = 0\,, \tag{1.5}$$

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where Ω is an open bounded domain in R^2 with boundary $\partial\Omega$ and $\overline{\Omega} = \Omega \bigcup \partial\Omega$, p is the pressure of the fluid, F is the prescribed external force and

$$\nu = \frac{\eta_1}{k_1},\ \rho = \frac{1}{k_1^2}(\eta_0 k_1 - k_0\eta_1),\ \delta = \frac{k_0}{k_1}.$$

The last condition in (1.5) is introduced for uniqueness of the pressure p. Problem (1.2)-(1.5) is a generalization of the first initial-boundary value problem for the Navier-Stokes equations. See [1, 2, 7, 8, 11, 13] for details of the physical background and its mathematical modeling.

The problem (1.2)-(1.5) was first investigated in the articles of Oskolkov and his pupils (see [8]), where Ladyzhenskaja's methods (see [9]) were applied. These investigations were continued in the article of Agranovich and Sobolevskii [4]. In these papers the authors obtained some sufficient conditions for the existence of problem (1.2). Similar results also appeared in [2] for a nonlinear (nonlinear rheological relation) viscoelastic model. Recently Sobolevskii investigated the behavior of the solution of problem (1.2)-(1.5) when $t \to \infty$ [13], where it is assumed that $u_0(x) \in H^2(\Omega)^2 \bigcap H_0^1(\Omega)^2$ with $\operatorname{div} u_0 = 0$ and $F(x,t)$ satisfies the stabilization conditions that for some $0 \le \delta_0 \le \delta$, $0 \le \alpha \le 1$, $M > 0$, $\phi(x) \in L^2(\Omega)^2$ and arbitrary $0 \le s \le t$:

$$\|e^{\delta_0 t}(F(x,t) - \phi(x))\|_{L^2(\Omega)^2} \le M\,,$$
$$\|e^{\delta_0 t}F(x,t) - e^{\delta_0 s}F(x,s)\|_{L^2(\Omega)^2} \le M|t-s|^{\alpha}\,. \tag{1.6}$$

In this case the highest derivatives belong to $C(R^+; L^2(\Omega)^2)$. The convergence of (u,p) to (v,q) together with their highest derivatives in the $L^2(\Omega)^2$ norm is established. Here (v,q) is a solution of the boundary value problem

$$-(\nu + \frac{\rho}{\delta})\triangle v + (v\cdot\nabla)v - \nabla q = \phi\,, \tag{1.7}$$
$$\operatorname{div} v = 0\,(x \in \Omega)\,; u = 0\,(x \in \partial\Omega), \tag{1.8}$$
$$(q,1) = 0\,;\ \phi(x) = \lim_{t\to\infty} F(x,t)\,. \tag{1.9}$$

The result is obtained under an assumption that $\rho \ge 0$ [13].

In this article we consider numerical methods solving the problem (1.2)-(1.3) under the initial condition (1.5) and the periodic-boundary condition:

$$u\,,\ p \text{ and the first derivatives of } u \text{ are } \Omega\text{-periodic}, \tag{1.10}$$

where $\Omega = (0,L_1)\times(0,L_2)$. We present an modified nonlinear Galerkin method which is a generalization of the nonlinear Galerkin methods (see [3, 5, 6, 10, 12]). Throughout this paper we assume that

$$\begin{aligned} &u_0(x) \in H^1_{per}(\Omega)^2 \text{ with } \operatorname{div} u_0 = 0,\\ &\int_\Omega u_0(x)dx = 0 \quad , F(x,t) \in L^\infty(R^+; L^2(\Omega)^2) \end{aligned} \tag{1.11}$$

without the stabilization conditions (1.6).

This paper is organized as follows. In section 2, we give the functional setting of the viscoelastic fluid motion equations and some preliminary concepts. In section 3, we demonstrate the existence and uniqueness of the solution of our problem. In section 4, we give some *a priori* estimates of the solution. In section 5, we present an modified nonlinear Galerkin method for solving the viscoelastic fluid motion equations (1.2)-(1.3), (1.5) and (1.10). Finally in section 6, we provide the convergence rates of the modified nonlinear Galerkin method and the Galerkin method. The results show that the modified nonlinear Galerkin method can save a large amount of computational time with the same convergence rate of standard Galerkin methods.

Only the sepcial case of the peroidic boundary condition and exponential kernel are considered in this paper, which simplify our analysis a great deal. The case of non-periodic boundary condition and general fading memory kernel will be dealt with elsewhere due to substantial difficulties of obtaining a priori regularity of the solutions for large time attractors in $H^1(\Omega)$ as well as other estimates. Nevertheless the analysis in this pape provides us with a framework that is useful for the analysis of other problems of a similar nature, in particular for the viscoelastic fluid modeling using the integral constitutive laws [7].

2. Functional Setting of the Viscoelastic Fluid Motion Equations

For the mathematical setting of problem (1.2)-(1.3) with the initial-boundary value conditions (1.5), (1.10), we consider a Hilbert space H which is a closed subspace of $L^2_{per}(\Omega)^2$ given by

$$H = \{v \in L^2_{per}(\Omega)^2; \operatorname{div} v = 0 \text{ and } \int_\Omega v(x)dx = 0\}.$$

The space H is endowed with the scalar product and norm of $L^2(\Omega)^2$ denoted by $(.,.)$ and $|.|$. Another useful space is V, a closed subspace of $H^1_{per}(\Omega)^2$:

$$V = \{v \in H^1_{per}(\Omega)^2; \operatorname{div} v = 0 \text{ and } \int_\Omega v(x)dx = 0\}$$

endowed with the scalar product $((u,v)) = (\nabla u, \nabla v)$ and the norm $\|u\| = ((u,u))^{1/2}$. We refer the reader to Temam [14-15] for more details on these spaces.

Let P be the orthonormal projection in $L^2(\Omega)^2$ onto H. We define the Stokes operator

$$Au = -P\Delta u \quad \forall u \in D(A) = V \cap H^2_{per}(\Omega)^2,$$

and the bilinear operator

$$B(u,v) = P[(u \cdot \nabla)v] \quad \forall u, v \in V.$$

The Stokes operator A is an unbounded positive self-adjoint closed operator in H with domain $D(A)$ dense in H and its inverse A^{-1} is compact in H (see [14, 15]). Consequently, there exists an orthonormal basis of H consisting of the eigenvectors w_j of A :

$$Aw_j = \lambda_j w_j, 0 < \lambda_1 \le \lambda_2 \le \cdots, \lambda_j \to \infty \text{ (as } j \to \infty). \tag{2.1}$$

Furthermore, we can also define the s-order power A^s of $A, \forall s \in R$. The space $D(A^s)$ is a Hilbert space when equipped with the scalar product $(A^s \cdot, A^s \cdot)$ and norm $|A^s \cdot|$ (see [14, 15]). In particular, we have

$$D(A^{1/2}) = V, \ (A^{1/2}u, A^{1/2}v) = ((u,v)), \ |A^{1/2}u| = \|u\|.$$

We define a trilinear form on $V \times V \times V$ by

$$b(u,v,w) = < B(u,v), w >_{V',V} \quad \forall u, v, w \in V.$$

It is easy to verify that b and B satisfy the following important properties (see [14, 15]) :

$$b(u,v,w) = -b(u,w,v) \quad \forall u,v,w \in V, \ b(u,u,Au) = 0 \quad \forall u \in D(A). \tag{2.2}$$

$$|b(u,v,w)| \le c_0|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2} \quad \forall u,v,w\in V, \tag{2.3}$$

$$|b(u,v,w)| \le c_0|u|^{1/2}|Au|^{1/2}\|v\||w| \quad \forall u\in D(A), v\in V, w\in H, \tag{2.4}$$

$$|b(u,v,w)| \le c_0|u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2}|w| \quad \forall u\in V, v\in D(A), w\in H, \tag{2.5}$$

$$|b(u,v,w)| \le c_0|u|\|v\||w|^{1/2}|Aw|^{1/2} \quad \forall u\in H, v\in V, w\in D(A), \tag{2.6}$$

where we denote by c_0 some positive constant.

Under the above notations, problem (1.2)-(1.3) with the initial-boundary value condition (1.5) and (1.10) is equivalent to the following abstract problem:

$$\frac{du}{dt} + \nu Au + \rho\int_0^t \exp\{-\delta(t-s)\}Au ds + B(u,u) = f, \tag{2.7}$$

$$u(0) = u_0\,, \tag{2.8}$$

where $f = PF$.

Abstract equation (2.7) contains the integral operator representing the memory effects. In order to prove the existence of the solution of the abstract problem (2.7)-(2.8), we need some properties of such an integral operator and the general Gronwall Lemma [4].

Lemma 2.1 Assume that $\alpha \ge 0$ and $\gamma(t)\in L^1_{loc}(R^+)$. Then for any $t>0$, there holds

$$\int_0^t\int_0^s \exp\{-\alpha(s-\tau)\}\gamma(\tau)\gamma(s)d\tau ds \ge 0\,. \tag{2.9}$$

Proof. Using integration by parts, we find

$$\int_0^t\int_0^s \exp\{-\alpha(s-\tau)\}\gamma(\tau)\gamma(s)d\tau ds = \int_0^t e^{-2\alpha s}\frac{d}{ds}|\int_0^s e^{\alpha\tau}\gamma(\tau)d\tau|^2 ds$$

$$= e^{-2\alpha t}|\int_0^t e^{\alpha t}\gamma(\tau)d\tau|^2 + 2\alpha\int_0^t e^{-2\alpha s}|\int_0^s e^{\alpha\tau}\gamma(\tau)d\tau|^2 ds \ge 0\,,$$

which implies (2.9). #

General Gronwall Lemma Let g, f, h, y be four locally integrable positive functions on $[t_0,\infty)$ that satisfy

$$0 \le y(t) + \int_{t_0}^t f(s)ds \le C + \int_{t_0}^t h(s)ds + \int_{t_0}^t g(s)y(s)ds \quad \forall t\in[t_0,\infty)\,, \tag{2.10}$$

where $C\ge 0$ is a constant. Then,

$$0 \le y(t) + \int_{t_0}^t f(s)ds \le (C + \int_{t_0}^t h(s)ds)\exp\{\int_{t_0}^t g(s)ds\} \quad \forall t\ge t_0\,. \tag{2.11}$$

Proof. We have from (2.10) that

$$0 \le y(t) \le C + \int_{t_0}^t h(s)ds + \int_{t_0}^t g(s)y(s)ds \quad \forall t\in[t_0,\infty),$$

which implies that

$$\frac{d}{dt}\left\{\left(\int_{t_0}^t g(s)y(s)ds\right)\exp\{-\int_{t_0}^t g(s)ds\}\right\} \le \left(C + \int_{t_0}^t h(s)ds\right)g(t)\exp\{-\int_{t_0}^t g(s)ds\}.$$

Thus it follows from integration from t_0 to t that

$$\begin{aligned}\left(\int_{t_0}^t g(s)y(s)ds\right)\exp\{-\int_{t_0}^t g(s)ds\} &= \int_{t_0}^t \left(C+\int_{t_0}^s h(\tau)d\tau\right) g(s)\exp\{-\int_{t_0}^s g(\tau)\tau\}ds \\ &\le \left(C+\int_{t_0}^t h(s)ds\right)\int_{t_0}^t g(s)\exp\{-\int_{t_0}^s g(\tau)\tau\}ds \\ &\le \left(C+\int_{t_0}^t h(s)ds\right)\left(1-\exp\{-\int_{t_0}^t g(\tau)d\tau\}\right).\end{aligned}$$

Thus we obtain

$$\int_{t_0}^t g(s)y(s)ds \le \left(C+\int_{t_0}^t h(s)ds\right)\left(\exp\{\int_{t_0}^t g(\tau)d\tau\}-1\right).$$

(2.11) follows by combining the above inequality and (2.10). #

3. Existence and Uniqueness

In this section, we prove the existence and uniqueness of the solution for problem (2.7)-(2.8). Hereafter, we always assume that

$$\rho \ge 0\,,\; 0<\delta_0<\frac{1}{2}\min\{\delta,\nu\lambda_1\}\,. \tag{3.1}$$

Theorem 3.1 Assume that $f\in L^\infty(R^+;H)$. Then for $u_0\in H$, the system (2.7)-(2.8) admits a unique solution $u\in L^\infty(R^+;H)\cap L^2(0,T;V), \forall T>0$.

Proof. First, the existence of a solution of problem (2.7)-(2.8) in $L^\infty(0,T;H)\cap L^2(0,T,V)$, $\forall T>0$ is demonstrated by the Faedo-Galerkin method.

For each m we look for an approximate solution u_m of the form

$$u_m(t)=\sum_{i=1}^m g_{im}(t)w_i$$

satisfying

$$\begin{aligned}(\frac{du_m}{dt},v)+a(u_m,v)+\rho\int_0^t \exp\{-\delta(t-s)\}((u_m(s),v))ds& \\ +b(u_m,u_m,v)=(f,v)\quad \forall v\in H_m\,,&\end{aligned} \tag{3.2}$$

$$u_m(0)=P_m u_0\,, \tag{3.3}$$

where P_m is the orthogonal projector in H onto H_m defined by

$$H_m=\mathrm{Span}\,\{w_1,...,w_m\}\,.$$

Since A and P_m commute, the relation (3.2) is also equivalent to

$$\frac{du_m}{dt}+\nu Au_m+\rho\int_0^t \exp\{-\delta(t-s)\}Au_m(s)ds+P_mB(u_m,u_m)=P_mf\,. \tag{3.4}$$

The existence and uniqueness of a solution u_m defined on a maximum interval $[0,T_m)$ follows from standard theorems on ODEs. The a priori estimates that we derive below guarantee that $T_m = \infty$. Also, they will allow us to study the limit $m \to \infty$ and to obtain the following convergence result :

$$\begin{aligned} & u_m \to u \text{ in } L^2(R^+; H) \text{ weak-star} \\ & \text{and in } L^\infty(0,T;V) \text{ strongly, for all } T > 0 \text{ as } m \to \infty. \end{aligned} \tag{3.5}$$

Let $v = e^{2\delta_0 t} u_m(t)$ in (3.2) and use (2.1), we obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}|e^{\delta_0 t}u_m|^2 + \nu\|e^{\delta_0 t}u_m\|^2 + \rho\int_0^t \exp\{-\alpha_0(t-s)\}((e^{\delta_0 s}u_m(s), e^{\delta_0 t}u_m(t)))ds \\ & \qquad = \delta_0|e^{\delta_0 t}u_m|^2 + (e^{\delta_0 t}f, e^{\delta_0 t}u_m)\,, \end{aligned} \tag{3.6}$$

where $\alpha_0 = \delta - \delta_0$. Since

$$\lambda_1|u|^2 \le \|u\|^2 \quad \forall u \in V \tag{3.7}$$

we have

$$(e^{\delta_0 t}f, e^{\delta_0 t}u_m) \le \lambda_1^{-1/2}|e^{\delta_0 t}f|\|e^{\delta_0 t}u_m\| \le \frac{\nu}{4}\|e^{\delta_0 t}u_m\|^2 + \frac{1}{\nu\lambda_1}|e^{\delta_0 t}f|^2\,. \tag{3.8}$$

Hence, we infer from (3.6)-(3.8) that

$$\begin{aligned} & \frac{d}{dt}|e^{\delta_0 t}u_m|^2 + \frac{\nu}{2}\|e^{\delta_0 t}u_m\|^2 + (\nu\lambda_1 - 2\delta_0)|e^{\delta_0 t}u_m|^2 \\ & \qquad + 2\rho\int_0^t \exp\{-\alpha_0(t-s)\}((e^{\delta_0 s}u_m(s), e^{\delta_0 t}u_m(t)))ds \le \frac{2}{\nu\lambda_1}|e^{\delta_0 t}f|^2\,, \end{aligned}$$

which with (3.1) yields

$$\begin{aligned} & \frac{d}{dt}(e^{\delta_0 t}|u_m|^2) + \frac{\nu}{2}e^{\delta_0 t}\|u_m\|^2 \le \frac{2}{\nu\lambda_1}|e^{\delta_0 t}f|^2 \\ & \qquad - 2\rho\int_0^t \exp\{-\alpha_0(t-s)\}((e^{\delta_0 s}u_m(s), e^{\delta_0 t}u_m(t)))ds\,. \end{aligned} \tag{3.9}$$

Integrating (3.9) and using Lemma 2.1, we obtain

$$|e^{\delta_0 t}u_m(t)|^2 + \frac{\nu}{2}\int_0^t \|e^{\delta_0 s}u_m(s)\|^2 ds \le |u_m(0)|^2 + \frac{1}{\nu\lambda_1\delta_0}f_\infty^2(e^{2\delta_0 t} - 1)\,, \tag{3.10}$$

$$|u_m(t)|^2 + \frac{\nu}{2}\int_0^t \|e^{\delta_0(s-t)}u_m(s)\|^2 ds \le |e^{-\delta_0 t}u_0|^2 + \frac{1}{\nu\lambda_1\delta_0}f_\infty^2(1 - e^{-2\delta_0 t}). \tag{3.11}$$

Here we have used the fact $|u_m(0)| = |P_m u_0| \le |u_0|$, where $f_\infty = \sup\{|f(t)|; t \ge 0\}$. Therefore, we have demonstrated the following results.

Lemma 3.1 The sequence u_m remains in a bounded set of $L^\infty(R^+; H)\cap$ $L^2(0,T;V)$ for all $T > 0$ as $m \to \infty$.

Due to (2.1)-(2.2), we have

$$\|B(\phi,\phi)\|_{V'} \le c_0|\phi|\|\phi\| \quad \forall \phi \in V\,. \tag{3.12}$$

Therefore $B(u_m, u_m)$ and $P_m B(u_m, u_m)$ remain bounded in $L^2(0,T;V')$. Hence, (3.4) and Lemma 3.1 yield the result.

Lemma 3.2 The derivatives $\dfrac{du_m}{dt}$ remains bounded in $L^2(0,T;V')$ for all $T > 0$.

By weak compactness it follows from Lemma 3.1 that there exists $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ $\forall T>0$ and a subsequence still denoted by u_m, such that for all $T>0$

$$u_m \to u \text{ in } L^\infty(0,T;H) \text{ weak-star and in } L^2(0,T;V)\,,$$

$$\frac{du_m}{dt} \to \frac{du}{dt} \text{ in } L^2(0,T;V') \text{ weakly, as } m\to\infty. \tag{3.13}$$

Due to Lemma 3.2 and a classical compactness theorem (see Temam [14-15]), we also have

$$u_m \to u \text{ in } L^2(0,T;H) \text{ strongly, as } m \to \infty\,. \tag{3.14}$$

This is sufficient to pass to the limit in (3.2)-(3.4), which implies (2.7)-(2.8). For (2.7) we simply observe that (3.5) implies that

$$u_m(t) \to u(t) \text{ as } m\to\infty$$

weakly in V' or even in H ,$\forall t\in[0,T]$. By (2.7), $\frac{du}{dt}\in L^2(0,T;V')$ and by Lemma II.3.2 in Temam [14], u is in $C([0,T];H)$.

To complete the proof of Theorem 3.1, it remains to check the strong convergence result in (3.5), let us introduce the expression

$$X_m = \frac{1}{2}|u_m(T)-u(T)|^2 + \nu\int_0^T \|u_m(t)-u(t)\|^2dt$$

$$+\rho\int_0^T\int_0^t \exp\{-\delta(t-s)\}((u_m(s)-u(s),u_m(t)-u(t)))dsdt\,.$$

Then, we note that it suffices to show

$$\lim_{m\to\infty} X_m = 0 \text{ for all } T>0\,. \tag{3.15}$$

Indeed, (3.13) gives

$$\lim_{m\to\infty}|u_m(t)-u(t)| = 0, \text{ for all } t\geq 0\,, \tag{3.16}$$

$$\lim_{m\to\infty}\int_0^T \|u_m(t)-u(t)\|^2dt = 0, \text{ for all } T>0\,. \tag{3.17}$$

We now prove (3.15). Integrating (3.6) between zero and T, we obtain

$$\frac{1}{2}|u_m(T)|^2 + \nu\int_0^T\|u_m\|^2dt + \rho\int_0^T\int_0^t \exp\{-\delta(t-s)\}((u_m(s),u_m(t)))dsdt$$

$$= \frac{1}{2}|u_m(0)|^2 + \int_0^T (f(t),u_m(t))dt\,,$$

so that X_m can be rewritten :

$$X_m = -(u_m(T),u(T)) + \frac{1}{2}(|u(T)|^2+|u_m(0)|^2) - \nu\int_0^T\{2((u_m(t),u(t))) - \|u(t)\|^2\}dt$$

$$-\rho\int_0^T\int_0^t \exp\{-\delta(t-s)\}\{2((u_m(s),u(t))) - ((u(s),u(t)))\}dsdt$$

$$+\int_0^T (f(t),u_m(t))dt\,. \tag{3.18}$$

By use of (3.13), we can pass to the limit in (3.18) and we obtain

$$\lim_{m\to\infty} X_m = -\frac{1}{2}|u(T)|^2 + \frac{1}{2}|u_0|^2 - \nu\int_0^T \|u\|^2 dt$$
$$- \rho\int_0^T\int_0^t \exp\{-\delta(t-s)\}((u(s),u(t)))dsdt + \int_0^T (f,u)dt\,,$$

which via (2.7) implies that this limit is zero. Hence, (3.15) is proved.

Finally, we need prove the uniqueness of solution $u(t)$ of (2.7)-(2.8). Let u_1, u_2 be two solutions of (2.7)-(2.8). Then, $w = u_1 - u_2$ satisfies

$$\frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 + b(w,u_1,w)$$
$$+ \rho\int_0^t \exp\{-\delta(t-s)\}((w(s),w(t)))ds = 0\,, \tag{3.19}$$

where $w(0) = 0$. Thanks to (2.2)-(2.3), we have

$$|b(w,u_1,w)| \le c_0\|u_1\|\|w\||w| \le \frac{\nu}{2}\|w\|^2 + \frac{1}{2\nu}c_0^2\|u_1\|^2|w|^2\,. \tag{3.20}$$

Thus, (3.19)-(3.20) gives

$$\frac{d}{dt}|w|^2 + \rho\int_0^t \exp\{-\delta(t-s)\}((w(s),w(t)))ds \le \nu^{-1}c_0^2\|u_1\|^2|w|^2\,. \tag{3.21}$$

Integrating (3.21) and using (2.9), we obtain

$$|w(t)|^2 \le \int_0^t \nu^{-1}c_0^2\|u_1(s)\|^2|w(s)|^2 ds\,. \tag{3.22}$$

Since $u_1 \in L^2(0,T;V)$ for each $T > 0$, it follows from the usual Gronwall lemma (see [14]) that $w(t) = 0$ for each t, namely, $u_1(t) = u_2(t)$. #

4. *A Priori* Estimates

In this section, we will give some *a priori* estimates on the solution of the problem (2.7)-(2.8). The *a priori* estimate in H is obtained by using the exact same arguments as those in the derivation of (3.11). It is

$$|u(t)|^2 + \frac{\nu}{2}\int_0^t e^{-2\delta_0(t-s)}\|u(s)\|^2 ds \le e^{-2\delta_0 t}|u_0|^2 + \frac{f_\infty^2}{\nu\lambda_1\delta_0}\,. \tag{4.1}$$

Moreover, we will obtain the following energy-type equality

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu|Au|^2 + b(u,u,Au)$$
$$+ \rho\int_0^t \exp\{-\delta(t-s)\}(Au(s),Au(t))ds = (f,Au) \tag{4.2}$$

by taking the scalar product of (2.7) with Au. ¿From (2.2) and

$$|(f,Au)| \le |f||Au| \le \frac{\nu}{4}|Au|^2 + \frac{1}{\nu}|f|^2\,, \tag{4.3}$$

we have the following energy-type inequality

$$\frac{d}{dt}\|u\|^2 + \frac{3}{2}\nu|Au|^2 + 2\rho\int_0^t e^{-\delta(t-s)}(Au(s), Au(t))ds \le \frac{2}{\nu}|f|^2 \quad \forall t \ge 0\,. \tag{4.4}$$

Thus, we derive from (4.4) and (3.7) that

$$\frac{d}{dt}\|e^{\delta_0 t}u\|^2 + \frac{\nu}{2}|e^{\delta_0 t}Au|^2 + (\nu\lambda_1 - 2\delta_0)\|e^{\delta_0 t}u\|^2$$
$$+ 2\rho\int_0^t e^{-\alpha_0(t-s)}(e^{\delta_0 s}Au(s), e^{\delta_0 t}Au(t))ds \le \frac{2}{\nu}|e^{\delta_0 t}f|^2\,. \tag{4.5}$$

Integrating (4.5) from 0 to t and using (2.9) and (3.1), we obtain

$$\|e^{\delta_0 t}u(t)\|^2 + \frac{\nu}{2}\int_0^t |e^{\delta_0 s}Au(s)|^2 ds \le \|u_0\|^2 + \frac{f_\infty^2}{\nu\delta_0}e^{2\delta_0 t}\,,$$

which yields the *a priori* estimate of the solution $u(t)$ in V

$$\|u(t)\|^2 + \frac{\nu}{2}\int_0^t e^{-2\delta_0(t-s)}|Au(s)|^2 ds \le e^{-2\delta_0 t}\|u_0\|^2 + \frac{f_\infty^2}{\nu\delta_0}\,. \tag{4.6}$$

5. Modified Nonlinear Galerkin Method

Let us explain briefly the idea of constructing a finite dimensional approximate dynamical system. We select a cut-off value n and define

P_n: the projection operator of H onto $H_n = \text{Span}\{w_1, \cdots, w_n\}$;

$Q_n : Q_n = I - P_n$.

Therefore, we can write

$$u = p + q, p = P_n u \in H_n, q = Q_n u \in H \setminus H_n,$$

where p, corresponding to the small eigenvalues, represents the large eddies of the flow, while q, corresponding to the large eigenvalues, represents the small eddies of the flow. Now we apply respectively P_n and Q_n to (2.7):

$$\frac{dp}{dt} + \nu Ap + \rho\int_0^t e^{-\delta(t-s)}Ap(s)ds + P_n B(p+q, q+p) = P_n f, \tag{5.1}$$

$$\frac{dq}{dt} + \nu Aq + \rho\int_0^t e^{-\delta(t-s)}Aq(s)ds + Q_n B(p+q, q+p) = Q_n f. \tag{5.2}$$

By virtue of (2.1) and (4.6), we obtain

$$|q(t)| \le \lambda_{n+1}^{-1/2}\|q(t)\| \le \lambda_{n+1}^{-1/2}\|u(t)\| = \lambda_{n+1}^{-1/2}(\|u_0\|^2 + \frac{f_\infty^2}{\nu\delta_0})^{1/2} \quad \forall t \ge 0,$$

namely, $q(t)$ only carries a small part of the kinematic energy. It is then reasonable to apply the Taylor approximate expansion to the term $Q_n B(p+q, p+q)$ in (5.2),namely,

$$Q_n B(p+q, p+q) \sim Q_n(B(p,p) + B(p,q) + B(q,p)).$$

This leads us to approximate (5.1)-(5.2) by the modified nonlinear Galerkin system:

$$\frac{dy}{dt} + \nu Ay + \rho\int_0^t e^{-\delta(t-s)}Ay(s)ds + P_n B(y+z, y+z) = P_n f,$$

$$\frac{dz}{dt} + \nu Az + \rho \int_0^t e^{-\delta(t-s)} Az(s)ds + Q_n[B(y+z,y) + B(y,z)] = Q_n f.$$

Since the latter is an infinite dimensional system, we introduce a finite dimensional version of the system above called the modified nonlinear Galerkin (MNG) method: Find $u_{app}(t) = u_O(t) = y(t) + z(t), y(t) \in H_n, z(t) \in H_N \setminus H_n$ such that

$$\frac{dy}{dt} + \nu Ay + \rho \int_0^t e^{-\delta(t-s)} Ay(s)ds + P_n B(y+z, y+z) = P_n f, \tag{5.3}$$

$$\frac{dz}{dt} + \nu Az + \rho \int_0^t e^{-\delta(t-s)} Az(s)ds + Q_n^N[B(y+z,y) + B(y,z)] = Q_N^n f, \tag{5.4}$$

$$y(0) = P_n u_0, z(0) = Q_N^n u_0, \tag{5.5}$$

where $N > n, Q_n^N = P_N - P_n = Q_n - Q_N, y(t)$ is called the approximate large eddies, $z(t)$ is called the approximate small eddies [14, 15].

It is well-known that the Galerkin method consists in finding $u_{app}(t) = u_G(t)$ such that

$$\frac{du_G}{dt} + \nu Au_G + \rho \int_0^t e^{-\delta(t-s)} Au_G(s)ds + P_N B(u_G, u_G) = P_N f, \tag{5.6}$$

$$u_G(0) = P_N u_0. \tag{5.7}$$

Moreover, if we neglect the small term $P_n B(z,z)$ in (5.3) and neglect the small terms $\frac{dz}{dt}$ and $Q_N^n[B(y,z) + B(z,y)]$ in (5.4), then we obtain the nonlinear Galerkin method.

It is easy to see that similar arguments given in section 4 can be used to show

$$\|\phi(t)\|^2 + \nu \int_0^t e^{-2\delta_0(t-s)} |A\phi(s)|^2 ds \le e^{-2\delta_0 t}\|u_0\|^2 + \frac{f_\infty^2}{\nu\delta_0}, \tag{5.8}$$

where $\phi = u_G$ or u_O.

Finally, we conclude this section by giving some properties of P_n and H_n. According to (2.1) and the definition of P_n, H_n and $D(A^s)$, the following estimates hold [14, 15]:

$$\begin{aligned} P_n Au = AP_n u, Q_n Au = AQ_n u \quad & \forall u \in D(A), \\ \lambda_1^{s_2 - s_1}|A^{s_1}u| \le |A^{s_2}u| \qquad & \forall u \in D(A^{s_2}), s_1 \le s_2, \\ \lambda_{n+1}^{s_2 - s_1}|A^{s_1}q| \le |A^{s_2}q| \qquad & \forall q \in D(A^{s_2}) \setminus H_n, s_1 \le s_2, \\ |A^{s_2}p| \le \lambda_n^{s_2 - s_1}|A^{s_1}p| \qquad & \forall p \in H_n, s_1 \le s_2, \\ |A^s p|^2 + |A^s q|^2 = |A^s(p+q)|^2 \qquad & \forall p \in H_n, q \in D(A^s) \setminus H_n. \end{aligned} \tag{5.9}$$

6. Convergence Analysis

In this section, we shall provide some important convergence results for the numerical solutions $u_G(t), u_O(t)$.

Theorem 6.1 The approximate solutions $u_G(t), u_O(t)$ satisfy the following rates of convergence:

$$\begin{aligned} |u(t) - u_G(t)|^2 + \frac{\nu}{4}\int_0^t e^{-2\delta_0(t-s)}\|u(s) - u_G(s)\|^2 ds \\ \le M_1^2(G(t)+1)\lambda_{N+1}^{-1} \quad \forall t \ge 0, \end{aligned} \tag{6.1}$$

$$|u(t)-u_O(t)|^2+\frac{\nu}{4}\int_0^t e^{-2\delta_0(t-s)}\|u(s)-u_O(s)\|^2ds$$
$$\le M_1^2(2G(t)+1)\lambda_{N+1}^{-1}+M_1^2G(t)\lambda_1\lambda_{n+1}^{-2}\quad \forall t\ge 0, \tag{6.2}$$

where

$$M_1^2=\|u_0\|^2+\frac{f_\infty^2}{\nu\delta_0},\ G(t)=\frac{16}{\nu^2\lambda_1}c_0^2M_1^2\exp(\frac{4}{\nu}c_0^2\int_0^t\|u(s)\|^2ds)\,.$$

Proof We set

$$E_G(t)=P_Nu(t)-u_G(t)\quad \text{for the Galerkin method,}$$
$$E_O(t)=P_Nu(t)-u_O(t)=P_nE_O(t)+Q_N^nE_O(t)\quad \text{for the MNG method,}$$

where $E_G(0)=E_O(0)=0$. Then (2.7) and (5.6) yield

$$(\frac{d}{dt}E_G,v)+\nu((E_G,v))+\rho\int_0^t e^{-\delta(t-s)}((E_G(s),v))ds+b(E_G,u,v)+b(u_G,E_G,v)$$
$$+b(Q_Nu,u,v)+b(u_G,Q_Nu,v)=0\quad \forall v\in H_N. \tag{6.3}$$

Taking $v=E_G$ in (6.3) and using (2.5), one finds

$$\frac{1}{2}\frac{d}{dt}|E_G|^2+\nu\|E_G\|^2+\rho\int_0^t e^{-\delta(t-s)}((E_G(s),E_G(t)))ds$$
$$+b(E_G,u,E_G)+b(u_G,Q_Nu,E_G)+b(Q_Nu,u,E_G)=0. \tag{6.4}$$

Thanks to (2.1)-(2.6) and (5.9), we have

$$|b(E_G,u,E_G)|\le c_0|E_G|\|E_G\|\|u\|\le\frac{\nu}{8}\|E_G\|^2+2\nu^{-1}c_0^2\|u\|^2|E_G|^2, \tag{6.5}$$

$$|b(Q_Nu,u,E_G)|\le c_0|Q_Nu|\|E_G\||u|^{1/2}|Au|^{1/2}$$
$$\le\frac{\nu}{8}\|E_G\|^2+2\nu^{-1}\lambda_1^{-1}c_0^2|Au|^2|Q_Nu|^2, \tag{6.6}$$
$$|b(u_G,Q_Nu,E_G)|\le\frac{\nu}{8}\|E_G\|^2+2\nu^{-1}\lambda_1^{-1}c_0^2|Au_G|^2|Q_Nu|^2. \tag{6.7}$$

Combining (6.4) with (6.5)-(6.7) yields

$$\frac{d}{dt}|E_G|^2+\frac{5}{4}\nu\|E_G\|^2+2\rho\int_0^t e^{-\delta(t-s)}((E_G(s),E_G(t)))ds$$
$$\le\frac{4}{\nu}c_0^2\|u\|^2|E_G|^2+\frac{4c_0^2}{\nu\lambda_1}(|Au|^2+|Au_G|^2)|Q_Nu|^2. \tag{6.8}$$

Due to (3.1) and (3.7), there holds

$$\nu\|v\|^2-2\delta_0|v|^2\ge(\nu\lambda_1-2\delta_0)|v|^2\ge 0\quad \forall v\in V\,. \tag{6.9}$$

Hence, (6.8) yields

$$\frac{d}{dt}|e^{\delta_0t}E_G|^2+\frac{\nu}{4}\|e^{\delta_0t}E_G\|^2+2\rho\int_0^t e^{-\alpha_0(t-s)}((e^{\delta_0s}E_G(s),e^{\delta_0t}E_G(t)))ds$$
$$\le\frac{4}{\nu}c_0^2\|u\|^2|e^{\delta_0t}E_G|^2+\frac{4c_0^2}{\nu\lambda_1}(|e^{\delta_0t}Au|^2+|e^{\delta_0t}Au_G|^2)|Q_Nu|^2\,. \tag{6.10}$$

Integrating (6.10) and using (2.9), we obtain

$$|e^{\delta_0 t}E_G(t)|^2+\frac{\nu}{4}\int_0^t\|e^{\delta_0 s}E_G(s)\|^2ds$$
$$\le\frac{4c_0^2}{\nu\lambda_1}\int_0^t(|e^{\delta_0 s}Au(s)|^2+|e^{\delta_0 s}Au_G(s)|^2)|Q_Nu(s)|^2ds$$
$$+\frac{4c_0^2}{\nu}\int_0^t\|u(s)\|^2|e^{\delta_0 s}E_G(s)|^2ds\,. \tag{6.11}$$

Similarly, (2.7) and (5.3)-(5.4) yield

$$\frac{1}{2}\frac{d}{dt}|E_O|^2+\nu\|E_O\|^2+\rho\int_0^t e^{-\delta(t-s)}((E_O(s),E_O(t)))ds+b(E_O,u,E_O)$$
$$+b(u_O,Q_Nu,E_O)+b(Q_Nu,u,E_O)+b(z,z,Q_N^nE_O)=0. \tag{6.12}$$

Referring to the estimates (6.5)-(6.7) and using (5.9), we also have

$$|b(E_O,u,E_O)|\le\frac{\nu}{8}\|E_O\|^2+\frac{2}{\nu}c_0^2\|u\|^2|E_O|^2\,, \tag{6.13}$$

$$|b(u_O,Q_Nu,E_O)+b(Q_Nu,u,E_O)|$$
$$\le\frac{\nu}{8}\|E_O\|^2+4\nu^{-1}\lambda_1^{-1}c_0^2(|Au_O|^2+|Au|^2)|Q_Nu|^2, \tag{6.14}$$

$$|b(z,z,Q_n^NE_O)|\le\frac{\nu}{8}\|E_O\|^2+2\nu^{-1}c_0^2|z|^2\|z\|^2$$
$$\le\frac{\nu}{8}\|E_O\|^2+\frac{2}{\nu}c_0^2\lambda_{n+1}^{-2}\|u_O\|^2|Au_O|^2\,. \tag{6.15}$$

Combining (6.12) with (6.13)-(6.15) yields

$$\frac{d}{dt}|E_O|^2+\frac{5}{4}\nu\|E_O\|^2+2\rho\int_0^t e^{-\delta(t-s)}((E_O(s),E_O(t)))ds$$
$$\le\frac{4}{\nu}c_0^2\|u\|^2|E_O|^2+\frac{8c_0^2}{\nu\lambda_1}(|Au|^2+|Au_O|^2)|Q_Nu|^2$$
$$+4\nu^{-1}c_0^2\lambda_{n+1}^{-2}\|u_O\|^2|Au_O|^2. \tag{6.16}$$

Thanks to (6.9), we obtain from (6.16) that

$$\frac{d}{dt}|e^{\delta_0 t}E_O|^2+\frac{5}{4}\nu\|e^{\delta_0 t}E_O\|^2+2\rho\int_0^t e^{-\alpha_0(t-s)}((e^{\delta_0 s}E_O(s),e^{\delta_0 t}E_O(t)))ds$$
$$\le\frac{8c_0^2}{\nu\lambda_1}(|e^{\delta_0 t}Au|^2+|e^{\delta_0 t}Au_O|^2)|Q_Nu|^2$$
$$+\frac{4c_0^2}{\nu}\lambda_{n+1}^{-2}|e^{\delta_0 t}Au_O|^2\|u_O\|^2+\frac{4}{\nu}c_0^2\|u\|^2|e^{\delta_0 t}E_O|^2\,. \tag{6.17}$$

Integrating (6.17) and using (2.9), we have

$$
\begin{aligned}
&|e^{\delta_0 t}E_O(t)|^2 + \frac{\nu}{4}\int_0^t \|e^{\delta_0 s}E_O(s)\|^2 ds \\
&\quad \le \frac{8c_0^2}{\nu\lambda_1}\int_0^t (|e^{\delta_0 s}Au(s)|^2 + |e^{\delta_0 s}Au_O(s)|^2)|Q_N u(s)|^2 ds \\
&\quad + 4\nu^{-1}c_0^2\lambda_{n+1}^{-2}\int_0^t |e^{\delta_0 s}Au_O(s)|^2\|u_O(s)\|^2 ds \\
&\quad + 4\nu^{-1}c_0^2\int_0^t \|u(s)\|^2|e^{\delta_0 s}E_O(s)|^2 ds. \qquad (6.18)
\end{aligned}
$$

Thanks to (4.6) and (5.8)-(5.9), one has that for all $t \ge 0$

$$|Q_N u|^2 \le \lambda_{N+1}^{-1}\|Q_N u\|^2 \le \lambda_{N+1}^{-1}M_1^2\,,\ \|u_O\|^2 \le M_1^2\,. \qquad (6.19)$$

$$\|u(t)\|^2 + \frac{\nu}{2}\int_0^t e^{-2\delta_0(t-s)}|Au(s)|^2 ds \le M_1^2\,, \qquad (6.20)$$

$$\frac{\nu}{2}\max\{\int_0^t e^{-2\delta_0(t-s)}|Au_G(s)|^2 ds\,,\ \int_0^t e^{-2\delta_0(t-s)}|Au_O(s)|^2 ds\} \le M_1^2\,. \qquad (6.21)$$

Hence, (6.11) and (6.18) can be written as

$$
\begin{aligned}
&|E_G(t)|^2 + \frac{\nu}{4}\int_0^t e^{-2\delta_0(t-s)}\|E_G(s)\|^2 ds \\
&\quad \le \frac{16c_0^2M_1^4}{\nu^2\lambda_1\lambda_{N+1}} + \frac{4}{\nu}c_0^2\int_0^t e^{-2\delta_0(t-s)}\|u(s)\|^2|E_G(s)|^2 ds\,, \qquad (6.22)
\end{aligned}
$$

$$
\begin{aligned}
&|E_O(t)|^2 + \frac{\nu}{4}\int_0^t e^{-2\delta_0(t-s)}\|E_O(s)\|^2 ds \\
&\quad \le \frac{8}{\nu^2\lambda_1}c_0^2M_1^4\left(\frac{4}{\lambda_{N+1}} + \frac{\lambda_1}{\lambda_{n+1}^2}\right) + \frac{4}{\nu}c_0^2\int_0^t e^{-2\delta_0(t-s)}\|u(s)\|^2|E_O(s)|^2 ds\,. \qquad (6.23)
\end{aligned}
$$

Applying the general Gronwall Lemma to (6.22), (6.23) respectively, we obtain

$$|E_G(t)|^2 + \frac{\nu}{4}\int_0^t e^{-2\delta_0(t-s)}\|E_G(s)\|^2 ds \le M_1^2G(t)\lambda_{N+1}^{-1}\,, \qquad (6.24)$$

$$|E_O(t)|^2 + \frac{\nu}{4}\int_0^t e^{-2\delta_0(t-s)}\|E_O(s)\|^2 ds \le M_1^2G(t)(2\lambda_{N+1}^{-1} + \lambda_1\lambda_{n+1}^{-2})\,. \qquad (6.25)$$

Using again (4.6) and (5.9), we derive

$$|Q_N u(t)|^2 + \frac{\nu}{4}\int_0^t e^{-2\delta_0(t-s)}\|Q_N u(s)\|^2 ds \le M_1^2\lambda_{N+1}^{-1}\,. \qquad (6.26)$$

Hence, (6.24)-(6.25) and (6.26) imply (6.1)-(6.2). #

Theorem 6.2 If $\bar{u}_0$ and f are sufficiently small such that

$$M_1^2 = \|\bar{u}_0\|^2 + \frac{f_\infty^2}{\nu\delta_0} \le \frac{\nu^2\lambda_1}{32c_0^2}\,, \qquad (6.27)$$

then

$$|u(t)-u_G(t)|^2+\frac{\nu}{8}\int_0^t e^{-2\delta_0(t-s)}\|u(s)-u_G(s)\|^2ds \le M_1^2(G(0)+1)\lambda_{N+1}^{-1} \quad \forall t\ge 0\,, \tag{6.28}$$

$$|u(t)-u_O(t)|^2+\frac{\nu}{8}\int_0^t e^{-2\delta_0(t-s)}\|u(s)-u_O(s)\|^2ds \le M_1^2(2G(0)+1)\lambda_{N+1}^{-1}+M_1^2G(0)\lambda_1\lambda_{n+1}^{-2} \quad \forall t\ge 0\,. \tag{6.29}$$

Proof. If (6.27) holds, then we derive from (3.7), (6.20) that for all $\phi\in L^2_{loc}(R^+;V)$

$$\frac{\nu}{8}\int_0^t \|e^{\delta_0 s}\phi(s)\|^2ds-\frac{4}{\nu}c_0^2\int_0^t\|u(s)\|^2|e^{\delta_0 s}\phi(s)|^2ds \ge (\frac{\nu\lambda_1}{8}-\frac{4c_0^2M_1^2}{\nu})\int_0^t|e^{\delta_0 s}\phi(s)|^2ds\ge 0\,. \tag{6.30}$$

Hence, (6.22) and (6.23) yield

$$|E_G(t)|^2+\frac{\nu}{8}\int_0^t e^{-2\delta_0(t-s)}\|E_G(s)\|^2ds\le\frac{16c_0^2M_1^4}{\nu^2\lambda_1\lambda_{N+1}}\,, \tag{6.31}$$

$$|E_O(t)|^2+\frac{\nu}{8}\int_0^t e^{-2\delta_0(t-s)}\|E_O(s)\|^2ds\le\frac{8}{\nu^2\lambda_1}c_0^2M_1^4(\frac{4}{\lambda_{N+1}}+\frac{\lambda_1}{\lambda_{n+1}^2})\,, \tag{6.32}$$

which and (6.26) imply (6.28)-(6.29). #

Remark 6.1 Recall from section 5 that the Galerkin method consists in solving a nonlinear problem in a large space H_N, while the MNG method consists in solving similar nonlinear subproblem in a small space H_n and solving a linear subproblem in a large space $H_N\setminus H_n$. Hence, the MNG method is simpler and cheaper than the Galerkin method.

Remark 6.2 We recall that for the eigenvalue of the 2-D Stokes operator A, $\lambda_n\sim n$ (see [14, 15]). For an arbitrary given finite time interval $[0,T]$ and all $t\in[0,T]$, Theorem 6.1 and Theorem 6.2 show that if we choose n such that

$$n=O(N^{1/2}) \tag{6.33}$$

then the MNG method is of the convergence rate of same order as the Galerkin method, while the rate of convergence of standard nonlinear Galerkin method [3, 10, 12] is of $n=O(N^{2/3})$. Hence, for n such chosen in (6.33), the MNG method can save a large amount of computation time while maintaining the same convergence rate of the Galerkin method.

Remark 6.3 The formulation of our modified nonlinear Galerkin method is somewhat more expensive compared to the standard nonlinear Galerkin method for problem (1.2) without memory [3, 10, 12] mainly due to the absence of the term $\frac{dz}{dt}$. However the nature of problem with memory makes the additional term $\frac{dz}{dt}$ not computational significant due to the large amount of computation generated by the memory term. Therefore this seems make our modified nonlinear Galerkin method attractive when compared with other methods for viscoelastic fluid flows with memory.

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