Long-Time Stability of Finite Element Approximations for Parabolic Equations with Memory[1]

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Abstract

In this paper we derive the sharp long-time stability and error estimates of finite element approximations for parabolic integro-differential equations. First, the exponential decay of the solution as $t \to \infty$ is studied, and then the semi-discrete and fully discrete approximations are considered using the Ritz-Volterra projection. Other related problems are studied as well. The main feature of our analysis is that the results are valid for both smooth and non-smooth (weakly singular) kernels.

1. Introduction.

In this paper we continue the work by Thomée and Wahlbin [23] and study the long-time stability and error estimates of finite element approximations for parabolic integro-differential equations. For simplicity we consider the following parabolic integro-differential equation: find $u = u(x,t)$ such that

$$\begin{aligned} & u_t + Au + \int_0^t K(t-s)Bu(s)ds = f(t), \text{ in } Q_\infty, \\ & u = 0, \text{ on } \partial\Omega \times (0,\infty), \\ & u(x,0) = u_0(x), \quad x \in \Omega. \end{aligned} \tag{1.1}$$

where $Q_\infty = \Omega \times (0,\infty)$, $\Omega \subset R^d$ $(d \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $K(t)$ is a non-negative memory kernel (other kernels can be handled in the same way by taking the absolute value $|K(t)|$ in the analysis), and f is a known function. A is a symmetric positive definite second order elliptic operator,

$$A = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}) + a(x)I, \qquad a(x) \geq 0,$$

$$a_{ij}(x) = a_{ji}(x),\ i,j = 1,\cdots,d, \qquad a_1 \sum_{i=1}^{d} \xi_i^2 \geq \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \geq a_0 \sum_{i=1}^{d} \xi_i^2, \quad a_0, a_1 > 0,$$

[1] This work is supported in part by NSERC (Canada)
AMS(MOS) *subject classification* : 65N30, 45K05.
Keywords: Parabolic integro-differential, Long-time, Error estimates, Finite element, Stability.

and B is any second order operator,

$$B = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(b_{ij}(x)\frac{\partial}{\partial x_j}) + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i} + b(x)I,$$

with smooth coefficients in x and t.

Let S_h be finite dimensional subspaces of $H_0^1(\Omega)$ such that for some integer $l \geq 2$,

$$\inf_{\chi \in S_h} (||w-\chi|| + h||w-\chi||_1) \leq Ch^r||w||_r, \quad w \in H^r \cap H_0^1, \quad 1 \leq r \leq l. \tag{1.2}$$

Let $u_h \in S_h$ be the semi-discrete finite element solution of (1.1): i.e., $u_h(t) \in S_h$ such that

$$\begin{aligned} &(u_{h,t}, v) + A(u_h, v) + \int_0^t K(t-s)\, B(u_h(s), v)ds = (f, v), \quad v \in S_h, \\ &u_h(0) = u_{0,h} \in S_h, \end{aligned} \tag{1.3}$$

where $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$ are the bilinear forms associated with the operator A and B, respectively, on $H_0^1(\Omega)\times H_0^1(\Omega)$, $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. $u_{0,h}$ is an appropriate approximation of u_0 in S_h and $u_{h,t}$ denotes the time derivative of u_h.

It is well-known that the error estimates [3, 4, 5, 7, 8, 9, 13, 14, 16, 22] are, in general, of the following form:

$$||u(t) - u_h(t)|| \leq C(T)h^r||u||^*, \quad 0 \leq t \leq T, \tag{1.4}$$

where $||u||^*$ is an appropriate norm of u and the asymptotic constant $C(T)$ grows exponentially with T, for example $C(T) = C_0 \exp(C_0 T)$ for some $C_0 > 0$. This type of error estimates, although it is optimal with respect to the order of convergence, will not give us any information on the long time calculation of whether or not u_h is still a good approximation of u for large time T.

Thomée and Wahlbin [23] considered the weak point of the above type error estimates and stability. Roughly they show under some assumptions on A, B, and

$$|K(t)| \leq Me^{-\alpha t}, \quad M > 0, \quad \alpha > 0, \tag{1.5}$$

and with $||u||_a = \sqrt{A(u,u)}$,

$$|B(u,v)| \leq c_0||u||_a||v||_a, \quad u, v \in H_0^1(\Omega), \quad \text{and} \quad c_0 < \alpha/M, \tag{1.6}$$

that $C(T)$ in (1.4) is independent of $T > 0$.

We see clearly from (1.5) that the results are not valid for a singular kernel. Also we mention that the exponential decay rates in [23] for the backward Euler and second order Euler schemes are not sharp. For complete references concerning finite element approximations to the solution of problem (1.1) we refer the readers to [3, 4, 5, 7, 8, 9, 13, 14, 16, 22], and to [10, 18] for mathematical modeling and well-posedness of the problems (1.1) and other related problems.

In this paper we improve the above mentioned two weak aspects of [23]. That is, similar stability and error estimates will be derived for finite element approximations of problems with singular kernels and the exponential decay rates given below are sharp.

In order to describe our results briefly, let us list the following assumptions on the data:

A1 : Let $||\psi||_a^2 = A(\psi,\psi)$ for $\psi \in H_0^1(\Omega)$. $||\cdot||_a$ is a norm equivalent to $||\cdot||_1$ on $H_0^1(\Omega)$. There exists $c_0 > 0$ such that

$$|B(u,v)| \leq c_0 \lambda_0^{\beta/2-1} ||u||_a \, ||v||_a \quad \text{for} \quad u, v \in H_0^1(\Omega),$$

where β is the order of the operator B (i.e, $\beta = 0$ if $b_{ij} = b_i = b = 0$ and etc), and $\lambda_0 = \lambda_0(\Omega, a_0, a_1) > 0$ is the first eigenvalue of the elliptic problem

$$A\phi - \lambda\phi = 0, \quad x \in \Omega \qquad \text{and} \qquad \phi = 0, \quad x \in \partial\Omega.$$

A2: There exists a $\nu_0 > 0$ such that for $0 \leq \nu < \nu_0$,

$$K_\nu = \int_0^\infty e^{\nu t} K(t) dt < \infty \quad \text{and} \qquad c_0 K_0 \lambda_0^{\beta/2-1} < 1;$$

A3: Let $\mu = \max\{\mu_0\}$, where μ_0 are the solutions of

$$1 - c_0 K_{\mu_0} \lambda_0^{\beta/2-1} \geq \frac{\mu_0}{\lambda_0}, \quad 0 < \mu_0 < \min\{\nu_0, \lambda_0\};$$

We will see below that the assumptions **(A1)-(A3)** are not needed if the kernel $K(t)$ is positive definite and the operator B is non-negative symmetric (Section 2).

The main results of this paper are as follows: If u and u_h are solutions of (1.1) and its semi-discrete finite element approximation (1.3), then there exists a constant $C_0 > 0$, independent of h and time $t \geq 0$, that

$$(1.8) \quad ||u(t) - u_h(t)|| \leq C_0 h^r \left\{ ||u_0||_r \hat{R}(t) + |||u|||_{r,R} + \int_0^t e^{-\mu(t-s)} |||u_t|||_{r,R} ds \right\}, \quad t \geq 0,$$

where $\hat{R}(t)$ and $|||u|||_{r,R}$ for $u(\cdot,t) \in H^r(\Omega)$ are defined by

$$\begin{aligned} \hat{R}(t) &= e^{-\mu t} + \int_0^t e^{-\mu(t-s)} \left(R(s) + \int_0^s R(s-\tau) R(\tau) d\tau \right) ds, \quad t \geq 0, \\ |||u|||_{r,R} &= ||u(t)||_r + \int_0^t R(t-s) ||u(s)||_r ds + \int_0^t R(t-s) \left(\int_0^s R(s-\tau) ||u(\tau)||_r d\tau \right) ds, \end{aligned}$$

and where μ is defined in **(A3)** and $R(t)$ is the resolvent of the kernel $K(t)$.

For backward Euler finite element approximations, let us denote $\Delta t > 0$ and $t_n = n\Delta t$, $n = 0, 1, 2, \cdots,$. Let g be defined for $t \geq 0$. Then the numerical quadrature $q_n(g)$ is defined by

$$(1.9) \qquad q_n(g) = \sum_{j=0}^{n-1} \omega_{nj} g^j \approx \int_0^{t_n} K(t_n - s) g(s) ds,$$

where $g^j = g(t_j)$, and $\{\omega_{nj}\}$ are appropriate weightings associated with the kernel $K(t)$ to be specified later in Section 4. Also we define

$$(1.10) \qquad q_{n,B}(g,\chi) = \sum_{j=0}^{n-1} \omega_{nj} B(g^j, \chi), \quad \chi \in S_h.$$

The backward Euler discrete finite element approximation, $u_h^n \in S_h$, are thus defined by

$$(1.11)\qquad \begin{aligned} &(\frac{u_h^n - u_h^{n-1}}{\Delta t}, \chi) + A(u_h^n, \chi) + q_{n,B}(u_h^j, \chi) = (f^n, \chi), \qquad \chi \in S_h, \\ &u_n^0 = u_{0,h} \in S_h, \end{aligned}$$

where $f^n \approx f(t_n)$. Denote the quadrature error by

$$(1.12)\qquad E_{n,B}(g) = \sum_{j=0}^{n-1} \omega_{nj} B g^j - \int_0^{t_n} K(t_n - s) B g(s) ds.$$

Then we will prove in Section 4 that in general it holds for $n \geq 0$ that:

$$(1.13)\qquad \begin{aligned} ||u(t_n) - u_h^n|| \ \leq\ & C_0 h^r \left\{ ||u_0||_r \hat{R}(t_n) + |||u(t_n)|||_{r,R} + \int_0^{t_n} e^{-\mu(t_n - s)} |||u_t|||_{r,R} ds \right\} \\ & + C_0 \Delta t \sum_{j=0}^{n} e^{-\mu(t_n - s)} Q_j(u), \end{aligned}$$

where $Q_j(u)$ is a function related to $||E_{n,B}(u)||$ to be defined in Section 4.

This paper is organized as the follows. In section 2 the exponential decays of the solutions are considered which are the basic elements for the next sections. In section 3 semi-discrete finite element approximations are considered, i.e., the long-time stability and error estimates are given. In section 4 the backward Euler scheme is considered. Finally in section 5 the asymptotic behavior of the numerical solution to the steady-state u^∞ is discussed.

In the next section we need the following version of Gronwall's inequality.

Lemma 1.1. Assume that the non-negative functions $f(t)$, $g(t)$ and $K(t) \in L^1(0, \infty)$ are such that

$$(1.14)\qquad f(t) \leq g(t) + \int_0^t K(t-s) f(s) ds, \qquad t \geq 0,$$

then

$$(1.15)\qquad f(t) \leq g(t) + \int_0^t R(t-s) g(s) ds, \qquad t \geq 0,$$

where $R(t)$ is the resolvent of $K(t)$ and is defined by

$$(1.16)\qquad R(t) = K(t) + \int_0^t K(t-s) R(s) ds, \qquad t \geq 0.$$

Proof: See [1]. ■

Let $R_h : H_0^1(\Omega) \to S_h$ be the Ritz projection: For $w \in H_0^1(\Omega)$,

$$(1.17)\qquad A(w - R_h w, \chi) = 0, \qquad \chi \in S_h,$$

$$(1.18)\qquad ||R_h w||_{W^{1,p}(\Omega)} \leq C ||w||_{W^{1,p}(\Omega)}, \qquad 2 \leq p \leq \infty.$$

2. Exponential Stability Estimates

In this section we establish some exponential stability estimates which will be used in the next sections. First we have the following result with a proof from [23].

Theorem 2.1. Under assumptions **(A1)-(A3)** and $u(0) = u_0(x) \in L^2(\Omega)$, the solution u of (1.1) has the property:

$$||u(t)|| \leq e^{-\mu t}||u(0)|| + 2\int_0^t e^{-\mu(t-s)}||f(s)||ds, \quad t \geq 0, \tag{2.1}$$

where $\mu > 0$ is defined in **A3**.

Proof: Since for $t > 0$ the solution $u(t) \in H_0^1(\Omega)$ satisfies

$$\begin{aligned} &(u_t, v) + A(u, v) + \int_0^t K(t-s)B(u(s), v)ds = (f, v), \quad v \in H_0^1(\Omega), \\ &u(0) = u_0(x) \in L^2(\Omega). \end{aligned} \tag{2.2}$$

It follows by letting $u^\mu(t) = e^{\mu t}u(t)$ that for $v \in H_0^1(\Omega)$,

$$\begin{aligned} &(u_t^\mu, v) - \mu(u^\mu, v) + A(u^\mu, v) + \int_0^t K^\mu(t-s)B(u^\mu(s), v)ds = (e^{\mu t}f, v), \\ &u^\mu(0) = u_0(x) \in L^2(\Omega). \end{aligned} \tag{2.3}$$

where $K^\mu = e^{\mu t}K(t)$. It is easy to see from integrating (2.3) from 0 to t with $v = u^\mu(t)$ and **(A1)-(A3)** with $0 < \sigma < 1$ that

$$\begin{aligned} &\frac{1}{2}\left(||u^\mu(t)||^2 - ||u(0)||^2\right) + (\lambda_0\sigma - \mu)\int_0^t ||u^\mu||^2 dt + (1-\sigma)\int_0^t ||u^\mu(s)||_a^2 dt \\ \leq\ & c_0\lambda_0^{\beta/2-1}\int_0^t\int_0^\tau K^\mu(\tau - s)\ \frac{||u^\mu(s)||_a^2 + ||u^\mu(\tau)||_a^2}{2}dsd\tau \\ &+ \int_0^t e^{\mu s}||f(s)||\ ||u^\mu(s)||ds \\ \leq\ & c_0K_\mu\lambda_0^{\beta/2-1}\int_0^t ||u^\mu(s)||_a^2 dt + \int_0^t e^{\mu s}||f(s)||\ ||u^\mu(s)||ds. \end{aligned}$$

Here $\lambda_0||g||^2 \leq ||g||_a^2$ for $g \in H_0^1(\Omega)$ was used. Therefore we find that

$$\begin{aligned} ||u^\mu(t)||^2 &\leq ||u(0)||^2 + 2\int_0^t e^{\mu s}||f(s)||\ ||u^\mu(s)||ds \\ &\leq \max_{0<s<t}||u^\mu(s)||\ \left(||u(0)|| + 2\int_0^t e^{\mu s}||f(s)||ds\right) \end{aligned} \tag{2.4}$$

provided that $\lambda_0\sigma - \mu \geq 0$ and $1 - \sigma \geq c_0K_\mu\lambda_0^{\beta/2-1}$, which is equivalent to $1 - c_0K_\mu\lambda_0^{\beta/2-1} \geq \sigma \geq \mu/\lambda_0$. Thus, Theorem 2.1 follows from taking the maximum on $(0, t)$ on both sides of the above inequality and $u = e^{-\mu t}u^\mu(t)$. ■

Remark: If $K(t) = 0$ then $K_0 = 0$ so that $\mu = \lambda_0$, which reduces (2.1) to the case of the exponential decay of the solutions of parabolic equations. Also if

$$K(t) = \sum_{j=1}^\infty C_j t^{-\alpha_j}e^{-\nu_j t}, \quad t > 0, \quad 0 < \alpha_j < 1, \quad \nu_j > 0, \tag{2.5}$$

then Theorem 2.1 is valid provided that

$$\sum_{j=1}^{\infty} |C_j|\Gamma(\alpha_j, \nu_j) < \frac{\lambda_0^{1-\beta/2}}{c_0}, \quad \Gamma(\alpha_j, \nu_j) = \int_0^{\infty} t^{-\alpha_j} e^{-\nu_j t} dt, \quad j = 1, 2, \cdots.$$

That is, if the memory term in (1.1), even with weakly sigular kernel, is dominated by the diffusion term then the solution u still decay to zero exponentially.

If $C_j > 0$ in (2.5) the kernel $K(t)$ belongs to the class of kernels called positive definite or monotonic [18].

Definition 2.1: The kernel $K(t) \in L^1_{loc}[0, \infty) \cap C(0, \infty)$ is said to be positive definite if

$$\int_0^t \int_0^s K(s-\tau) g(\tau) g(s) d\tau ds \geq 0 \quad \text{for} \quad g \in C(0, \infty) \cap L^1_{loc}(0, \infty). \tag{2.6}$$

It is difficult to verify by this definition if a given kernel $K(t)$ is positive definite. But from the viewpoint of applications to viscoelasticity, it is useful that certain types of sign conditions will guarantee the kernel being positive definite. A common assumption of this type is that the kernel $K(t)$ is a positive convex function [17]:

$$K(t) \geq 0, \quad K'(t) \leq 0, \quad K''(t) \geq 0, \quad \text{and} \quad K'(t) \not\equiv 0.$$

Lemma 2.1. Assume that $K(t)$ is positive definite and B is a symmetric non-negative elliptic operator with $B(\cdot, \cdot)$ being its bilinear form on $H^1_0(\Omega) \times H^1_0(\Omega)$. Then for $g \in C(0, \infty; H^1_0(\Omega)) \cap L^2_{loc}(0, \infty; H^1_0(\Omega))$,

$$\int_0^t \int_0^s K(s-\tau) B(g(\tau), g(s)) d\tau ds \geq 0, \quad t \geq 0. \tag{2.7}$$

Proof: See [5]. ■

We now have the following result for problem (1.1) with a positive definite kernel.

Theorem 2.2 Assume that $K(t)$ and $K^{\mu}(t)$ are positive definite for $0 < \mu \leq \nu_0$, and the operator B is symmetric and non-negative ($\beta = 0$ or $\beta = 2$). Then the solution u of (1.1) has the property:

$$||u(t)|| \leq e^{-\mu t} ||u(0)|| + 2 \int_0^t e^{-\mu(t-s)} ||f(s)|| ds, \quad t \geq 0, \tag{2.8}$$

where $\mu = \min\{\lambda_0, \nu_0\}$.

Proof: As before, we integrate (2.3) from 0 to t with $v = u^{\mu}$ and obtain

$$\begin{aligned} \frac{1}{2} \left(||u^{\mu}(t)||^2 - ||u(0)||^2 \right) + (\lambda_0 - \mu) \int_0^t ||u^{\mu}||^2 dt \\ + \int_0^t \int_0^{\tau} K^{\mu}(\tau - s) B(u^{\mu}(s), u^{\mu}(\tau)) ds d\tau \\ \leq \int_0^t e^{\mu s} ||f(s)|| \, ||u^{\mu}|| ds. \end{aligned}$$

Thus (2.8) follows from the same arguments in Theorem 2.1 since the last term on the left hand side of the above inequality is non-negative because of the positive definiteness of the kernel K and non-negativeness and symmetry of the operator B. ■

Remark: Theorem 2.2 implies that the smallness of the kernel or domination of the diffusion term is not necessary if the kernels K and K^μ are both positive definite.

Now let consider the long time stability of finite element approximations. We see from the above discussions that the following stability results for the semi-discrete finite element approximations are valid.

Theorem 2.3. Let u_h be the solutions of (1.3), and assumptions **A1-A3** be satisfied. Then $u_h(t)$ satisfies

$$||u_h(t)|| \le e^{-\mu t}||u_{0,h}|| + 2\int_0^t e^{-\mu(t-s)}||f(s)||ds, \quad t \ge 0;$$

where $\mu > 0$ is given in **A2**.

Theorem 2.4. Let u_h be the solutions of (1.3). Assume that K and K^μ are positive definite and B is a nonegative symmetric definite elliptic operator ($\beta = 0$ or $\beta = 2$). Then $u_h(t)$ satisfies

$$||u_h(t)|| \le e^{-\mu t}||u_{0,h}|| + 2\int_0^t e^{-\mu(t-s)}||f(s)||ds, \quad t \ge 0;$$

where $\mu = \min\{\lambda_0, \nu_0\}$.

The proofs of the above two theorems are the same as the proofs of Theorem 2.1 and Theorem 2.2 and we omit them.

In the following, two examples with $\beta = 0$ or 2 are constructed to demonstrate that the decay rates given above are sharp. We are not able to construct an example of the sharpness of the decay rates in Theorem 2.1 and Theorem 2.2 when $\beta = 1$, but believe they remain true.

Example 1: Assume that $\beta = 2$. Let $K = \mu exp(-\mu t)$, $\mu > 0$ and $B = -A$ so that $c_0 = 1$ and $c_0 K_0 = \int_0^\infty K(t)ds = 1$. Let u be the solution of

$$\begin{aligned} u_t + Au &= \int_0^t \mu e^{-\mu(t-s)} Au(s)ds, \ \text{in } Q_\infty, \\ u &= 0, \ \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) &= v(x) \ne 0, \ x \in \Omega, \end{aligned} \tag{2.11}$$

Clearly the solution u of (2.11) is given by

$$u(x,t) = \sum_{n=0}^\infty a_n\phi_n(x)\left\{\frac{\mu}{\lambda_n + \mu} + \frac{\lambda_n}{\lambda_n+\mu}e^{-(\lambda_n+\mu)t}\right\}, \tag{2.13}$$

where $\{a_n\}$ is such that $v(x) = \sum_{n=0}^\infty a_n\phi_n(x)$ and $\{\lambda_n, \phi_n\}$ are the eigenpairs in **A3**. We thus have that

$$\lim_{t\to\infty} u(x,t) = \sum_{n=0}^\infty \frac{a_n\mu}{\lambda_n+\mu}\phi_n(x) = u^\infty(x),$$

where $u^\infty(x)$ is the steady-state solution of (2.11). This example indicates that the memory effect takes over the control of the overall asymptotic behavior of the solution as $t \to \infty$.

Example 2. Assume that $\beta = 0$ (B is a zero order operator). Let $B = -c_0 I$ and $K(t) = \mu exp(-\mu t)$, $\mu > 0$, $f = 0$ and let $u(x,t) = \sum_{n=0}^\infty a_n \phi_n(x) T_n(t)$ be the solution of (1.1), where $v(x) = \sum_{n=0}^\infty a_n \phi_n(x)$. Then $T_n(t)$ satisfy

$$T_n'(t) + \lambda_n T_n(t) - c_0 \mu \int_0^t e^{-\mu(t-s)} T_n(s) ds = 0, \quad T_n(0) = 1.$$

where (λ_n, ϕ_n) are the eigenpairs in **A3** or equivalently,

$$T_n''(t) + (\lambda_n + \mu) T_n'(t) + (\lambda_n \mu - c_0 \mu) T_n(t) = 0, \quad T_n(0) = 1, \quad T_n'(0) = -\lambda_n.$$

One finds easily that the characteristic values of the above equation are

$$\lambda_n^\pm = \frac{1}{2}\left\{-(\lambda_n + \mu) \pm \sqrt{(\lambda_n + \mu)^2 - 4\mu(\lambda_n - c_0)}\right\}, \quad n = 0, 1, \cdots. \tag{2.14}$$

Then we see that if $c_0 = \lambda_0$ or $c_0 \lambda_0^{-1} = 1$,

$$T_0(t) = \frac{\mu}{\lambda_0 + \mu} + \frac{\lambda_0}{\lambda_0 + \mu} e^{-(\lambda_0 + \mu)t}, \quad T_n(t) = A_n e^{\lambda_n^+ t} + B_n e^{\lambda_n^- t}, \quad n = 1, 2, \cdots,$$

where

$$A_n = \frac{\lambda_n^- + \lambda_n}{\lambda_n^- - \lambda_n^+}, \qquad B_n = \frac{\lambda_n^+ + \lambda_n}{\lambda_n^+ - \lambda_n^-}, \qquad \lambda_n^+ = -\mu + O(\lambda_n^{-1}), \quad n \to \infty.$$

Therefore, we obtain that $u(x,t) \to \dfrac{a_0 \mu}{\lambda_0 + \mu} \phi_0(x)$ as $t \to \infty$.

In fact if $c_0 > \lambda_0$, the solution u of (2.1) in general goes to ∞ exponentially fast since some of the characteristic values in (2.14) will be positive. However, this behavior also depends upon the initial data. For example, if $v(x) \in L^2(\Omega) \setminus \{\phi_k(x)\}_{k=0}^N$, then the solution u in Example 2 still decays to zero exponentially as long as $c_0 < \lambda_{N+1}$.

3. Error Estimates for Semi-discrete Approximations

In this section the optimal error and long-time stability for semi-discrete approximations are established via the Ritz-Volterra projection techniques [3, 4]. The main difficulty is to show that long-time optimality of the Ritz-Volterra projection $V_h u$ to the solution u.

Theorem 3.1. Let u and u_h be the solutions of (1.1) and (1.3), respectively. Assume that **(A1)-(A3)** are satisfied and $||u_0 - u_{0,h}|| \le Ch^r ||u_0||_r$. Then there exists a constant $C_0 > 0$, independent of h and t, such that

$$||u(t) - u_h(t)|| \le C_0 h^r \left\{ ||u_0||_r \hat{R}(t) + |||u|||_{r,R} + \int_0^t e^{-\mu(t-s)} |||u_t|||_{r,R} ds \right\}, \quad t \ge 0, \tag{3.1}$$

where $\hat{R}(t)$ and $|||u|||_{r,R}$ for $u(\cdot,t) \in H^r(\Omega)$ are defined by

$$\hat{R}(t) = e^{-\mu t} + \int_0^t e^{-\mu(t-s)} \left(R(s) + \int_0^s R(s-\tau)R(\tau)d\tau \right) ds, \quad t \geq 0, \tag{3.2}$$

$$|||u|||_{r,R} = ||u(t)||_r + \int_0^t R(t-s)||u(s)||_r ds + \int_0^t R(t-s) \left(\int_0^s R(s-\tau)||u(\tau)||_r d\tau \right) ds, \tag{3.3}$$

and where μ is defined in **(A3)** and $R(t)$ is the resolvent of the kernel $K(t)$.

Proof: As usual we write the error $e(t) = u(t) - u_h(t)$ as

$$e(t) = (u - V_h u) + (V_h u - u_h) = \rho(t) + \theta(t),$$

where $V_h u$ is the Ritz-Volterra projection of the solution u defined by [3, 4, 15]

$$A(u - V_h u, \chi) + \int_0^t K(t-s)B(u(s) - V_h u(s),\ \chi)ds = 0, \quad \chi \in S_h. \tag{3.4}$$

It follows from Lemma 3.1 (below) that there exists $C_0 > 0$, independent of h and t, such that

$$||\rho(t)|| \leq C_0 h^r |||u(t)|||_{r,R}, \quad t \geq 0, \tag{3.5}$$

$$||\rho_t(t)|| \leq C_0 h^r \left\{ |||u_t(t)|||_{r,R} + ||u_0||_r \left(R(t) + \int_0^t R(t-s)R(s)ds \right) \right\}. \tag{3.6}$$

Thus, we find that only $||\theta(t)||$ needs to be estimated. One finds from (1.1)-(1.3) and (3.4) that $\theta(t)$ satisfies

$$(\theta_t, \chi) + A(\theta, \chi) + \int_0^t K(t-s)B(\theta(s), \chi)ds = -(\rho_t, \chi), \quad \chi \in S_h, \tag{3.7}$$

and then, by the stability estimate of Theorem 2.3 due to $\theta \in S_h$, that

$$||\theta(t)|| \leq e^{-\mu t}||\theta(0)|| + 2\int_0^t e^{-\mu(t-s)}||\rho_t||ds, \quad t \geq 0. \tag{3.8}$$

Hence, the proof is complete by noticing that $||\theta(0)|| \leq ||\rho(0)|| + ||u_0 - u_{0,h}|| \leq C_0 h^r ||u_0||_r$ and using (3.8), (3.5)-(3.6) and the triangle inequality. ■

Theorem 3.2 Let u and u_h be the solutions of (1.1) and (1.2), respectively. Assume that $K(t)$ and $K^\mu(t)$ are positive definite for $0 < \mu \leq \nu_0$, and the operator B is symmetric ($\beta = 0$ or $\beta = 2$). Then

$$||u(t) - u_h(t)|| \leq C_0 h^r \left\{ ||u_0||_r \hat{R}(t) + |||u|||_{r,R} + \int_0^t e^{-\mu(t-s)}|||u_t|||_{r,R} ds \right\}, \quad t \geq 0, \tag{3.9}$$

where $\mu = \min\{\lambda_0, \nu_0\}$ defined in Theorem 2.2 and $C_0 > 0$ is independent of h and t.

Proof: The proof follows from an argument similar to the proofs of Theorem 3.1 and Theorem 2.2 and is omitted. ■

It now remains to prove (3.5)-(3.6) which is the following result.

Lemma 3.1. Under the assumptions **(A1)-(A3)**, there exists a constant $C_0 > 0$, indepedent of h and time t such that for $\rho = u - V_h u, \quad t \geq 0$,

$$\begin{aligned}
(3.10) \qquad ||\rho(t)||_1 &\leq C_0 h^{r-1}\left\{||u||_r + \int_0^t R(t-s)||u||_r ds\right\}, \\
(3.11) \qquad ||\rho(t)|| &\leq C_0 h^r |||u(t)|||_{r,R}, \\
(3.12) \qquad ||\rho_t(t)||_1 &\leq C_0 h^{r-1}\left\{||u_t||_r + R(t)||u_0||_r + \int_0^t R(t-s)||u_t||_r ds\right\}, \\
(3.13) \qquad ||\rho_t(t)|| &\leq C_0 h^r \left\{|||u_t(t)|||_{r,R} + \left(R(t) + \int_0^t R(t-s)R(s)ds\right)||u_0||_r\right\}.
\end{aligned}$$

Proof: Let $\phi \in H^1(\Omega)$ and $\psi \in H_0^1(\Omega) \cap H^2(\Omega)$ such that

$$(3.14) \qquad A\psi = \phi_x \text{ in } \Omega \quad \text{and} \quad \psi = 0 \text{ on } \partial\Omega.$$

Then, it follows from (3.4) that for $x = x_i$, $i = 1, 2, \cdots, n$,

$$\begin{aligned}
-(\rho_x, \phi) &= (\rho, \phi_x) = A(\rho, \psi) + \int_0^t K(t-s)B(\rho(s), \psi)ds \\
&\qquad - \int_0^t K(t-s)B(\rho(s), \psi)ds \\
&= A(\rho, \psi - R_h\psi) + \int_0^t K(t-s)B(\rho(s), \psi - R_h\psi)ds \\
&\qquad - \int_0^t K(t-s)B(\rho(s), \psi)ds \\
&= A(\rho, \psi - R_h\psi) - \int_0^t K(t-s)B(\rho(s), R_h\psi)ds \\
&= A(u - R_h u, \psi - R_h\psi) - \int_0^t K(t-s)B(\rho(s), R_h\psi)ds \\
&\leq C_0\left\{h^{r-1}||u||_r + \int_0^t K(t-s)||\rho(s)||_1 ds\right\}||\psi||_1,
\end{aligned}$$

where R_h is Ritz projection associated with the bilinear form $A(\cdot, \cdot)$ and the stability of R_h in $H_0^1(\Omega)$ [6, 20] were used.

Since $||\psi||_1 \leq C_0||\phi||$ by elliptic regularity , we find from the above inequality that

$$||\rho(t)||_1 \leq C_0\left\{h^{r-1}||u(t)||_r + \int_0^t K(t-s)||\rho(s)||_1 ds\right\}$$

and then by Lemma 1.1 that

$$||\rho(t)||_1 \leq C_0 h^{r-1}\left\{||u(t)||_r + \int_0^t R(t-s)||u(s)||_r ds\right\},$$

where $R(t)$ is the resolvent of $K(t)$. Thus (3.10) has been proved.

In order to estimate $||\rho||$, let $\phi \in L^2(\Omega)$ and $\psi \in H_0^1(\Omega) \cap H^2(\Omega)$ be defined by

$$A\psi = \phi \ \text{ in } \ \Omega \quad \text{ and } \quad \psi = 0 \ \text{ on } \ \partial\Omega. \tag{3.15}$$

we find from an argument similar to the above calculation that

$$\begin{aligned}
(\rho, \phi) &= A(\rho, \psi) + \int_0^t K(t-s)B(\rho(s), \psi)ds - \int_0^t K(t-s)B(\rho(s), \psi)ds \\
&= A(\rho, \psi - R_h\psi) + \int_0^t K(t-s)B(\rho(s), \psi - R_h\psi)ds \\
&\qquad - \int_0^t K(t-s)(\rho(s), B^*\psi)ds \\
&\leq C_0 \left\{ ||\rho||_1 + \int_0^t K(t-s)||\rho(s)||_1 ds \right\} ||\psi - R_h\psi||_1 + C_0 \int_0^t K(t-s)||\rho(s)||ds||\psi||_2 \\
&\leq C_0 h \left\{ ||\rho||_1 + \int_0^t K(t-s)||\rho(s)||_1 ds \right\} ||\psi||_2 + C_0 \int_0^t K(t-s)||\rho(s)||ds||\psi||_2
\end{aligned}$$

where B^* is the adjoint operator of B. Elliptic regularity [6] implies that

$$||\rho|| \ \leq \ C_0 h \left\{ ||\rho||_1 + \int_0^t K(t-s)||\rho(s)||_1 ds \right\} + C_0 \int_0^t K(t-s)||\rho(s)||ds.$$

Now Lemma 1.1 and a simple calculation lead to

$$\begin{aligned}
||\rho|| \ \leq \ C_0 h \Bigg\{ & ||\rho||_1 + \int_0^t K(t-s)||\rho(s)||_1 ds \\
& + \int_0^t R(t-s) \left(||\rho(s)||_1 + \int_0^s K(s-\tau)||\rho(\tau)||_1 d\tau \right) ds \Bigg\}.
\end{aligned}$$

Therefore we have from (3.10), the properties of the resolvent and the above inequality that

$$||\rho|| \leq C_0 h^r \left\{ ||u||_r + \int_0^t R(t-s)||u(s)||_r ds + \int_0^t R(t-s) \int_0^s R(s-\tau)||u(\tau)||_r d\tau ds \right\},$$

which is in fact (3.11).

For (3.12), since (3.4) can be written via $\rho = u(t) - V_h u(t)$ as

$$A(\rho, \chi) + \int_0^t K(t-s)B(\rho(s), \chi)ds = 0, \qquad \chi \in S_h,$$

so that it follows by differention with respect to time t that

$$A(\rho_t, \chi) + K(t)B(\rho(0), \chi) + \int_0^t K(t-s)B(\rho_t(s), \chi)ds = 0, \qquad \chi \in S_h. \tag{3.16}$$

Let ϕ and ψ be given in (3.14), we find by (3.16), the stability of Ritz projection and an argument similar to the proof for $||\rho(t)||_1$ that

$$-(\rho_{tx}, \phi) \ = \ (\rho_t, \phi_x) = A(\rho_t, \psi) + K(t)B(\rho(0), \psi) + \int_0^t K(t-s)B(\rho_t(s), \psi)ds$$

$$
\begin{aligned}
& -K(t)B(\rho(0),\psi) - \int_0^t K(t-s)B(\rho(s),\psi)ds \\
= \;& A(\rho_t,\psi - R_h\psi) - K(t)B(\rho(0),R_h\psi) - \int_0^t K(t-s)B(\rho_t(s),R_h\psi)ds \\
= \;& A(u_t - R_h u_t,\psi - R_h\psi) - K(t)B(\rho(0),R_h\psi) - \int_0^t K(t-s)B(\rho_t(s),R_h\psi)ds \\
\leq \;& C_0\left\{h^{r-1}||u_t||_r + K(t)||\rho(0)||_1 + \int_0^t K(t-s)||\rho_t(s)||_1 ds\right\}||\psi||_1,
\end{aligned}
$$

and then, by the elliptic regularity and Gronwall's inequality that

$$
\begin{aligned}
||\rho_t||_1 \;\leq\; & C_0 h^{r-1}||u_t||_r + C_0 K(t)||\rho(0)||_1 \\
& + C_0\int_0^t R(t-s)\left(h^{r-1}||u_t(s)||_r + K(s)||\rho(0)||_1\right)\Big\} ds \\
\leq\; & C_0 h^{r-1}\left\{||u_t||_r + \int_0^t R(t-s)||u_t(s)||_r ds\right\} + C_0 R(t)||\rho(0)||_1 \\
\leq\; & C_0 h^{r-1}\left\{R(t)||u_0||_r + ||u_t||_r + \int_0^t R(t-s)||u_t(s)||_r ds\right\}.
\end{aligned}
$$

where (1.16) is used again.

Finally, (3.13) can be shown by an argument similar to the proof of (3.11) and we omit it. ■

Remark: If the kernel $K(t) \equiv 0$ then Theorem 1.1, Theorem 2.2, Theorem 3.1 and Theorem 3.2 reduce to the classical stability and long-time error estimates for parabolic equations.

4. Backward Euler Schemes

First, let us recall the backward Euler scheme solution $\{u_h^n\} \subset S_h$ of problem (1.1):

$$
\begin{aligned}
&\left(\frac{u_h^{n+1} - u_h^n}{\Delta t},\ \chi\right) + A(u^{n+1},\chi) + \sum_{j=1}^{n}\omega_{nj}B(u_h^j,\chi) = (f^{n+1},\chi), \qquad \chi \in S_h \\
&u_h^0 = u_{0,h}, \qquad n = 0,1,2,\cdots,
\end{aligned} \tag{4.1}
$$

where the solution u_h^n approximates the solution $u(x,t)$ at the time $t = t_n$.

In particular if the weights can be chosen to be $\omega_{n,j} = \Delta t K(t_{n-j}), j = 0,1,\cdots,n-1$ and $\omega_{n,n} = 0$, notice that $K(0)$ is avoided in the summation which allows for a weak singularity at $t = 0$.

Clearly, if Δt is small the following infinite " sums " is convergent under assumption (**A2**) for any fixed $0 < \mu < \nu_0$.

$$
K_\mu^{\Delta t} = \max\left\{\sup_{n\geq 1}\left(\sum_{j=1}^{n}\omega_{nj}e^{\mu(t_n - t_j)}\right),\quad \sup_{n\geq j\geq 1}\left(\sum_{k=j}^{n}\omega_{kj}e^{\mu(t_k - t_j)}\right)\right\} < \infty. \tag{4.2}
$$

If $\omega_{nj} = \omega_{n-j} = \Delta t K(t_{n+1-j})$, then (4.2) can be verified easily. Thus, for any $0 < \mu_0$ small (μ_0 is smaller than μ defined in (**A3**)) there exists a small $T_0 > 0$ such that $0 < \Delta t \leq T_0$,

$$
1 - c_0\lambda_0^{\beta/2-1}K_{\mu_0}^{\Delta t} \geq \frac{\mu_0}{\lambda_0}e^{\mu_0\Delta t}. \tag{4.3}
$$

Theorem 4.1. Under the assumptions **(A1-A3)** and $\Delta t > 0$ such that (4.2) and (4.3) are satisfied with $\mu = \max\{\mu_0\} > 0$. Then the solution u^n of (4.1) is stable:

$$||u_h^n|| \leq e^{-\mu t_n}||u_h^0|| + 2\Delta t \sum_{j=0}^{n} e^{-\mu t_{n-j}}||f^j||, \quad n \geq 1. \tag{4.4}$$

Proof: Let $U_h^n = e^{\mu t_n}u_h^n$ in (4.1), then it is easy to find that U^n satisfies

$$\begin{aligned}(4.5)\qquad & \left(\frac{U_h^{n+1} - U_h^n}{\Delta t}, \ \chi\right) + \frac{e^{-\mu\Delta t} - 1}{\Delta t}(U_h^{n+1}, \chi) + e^{-\mu\Delta t}A(U_h^{n+1}, \ \chi) \\ & \qquad + e^{-\mu\Delta t}\sum_{j=1}^{n}\omega_{nj}^{\mu}B(U_h^j, \chi) = (e^{\mu t_n}f^n, \chi), \quad \chi \in S_h, \\ & U_h^0 = u_{0,h}, \quad n = 0, 1, 2, \cdots\end{aligned}$$

where $\omega_{nj}^{\mu} = \omega_{nj}e^{\mu(t_n - t_j)}, j = 0, 1, 2, \cdots, n$.

Now taking $\chi = U_h^{n+1} \in S_h$ in (4.5) it follows by the Mean Value Theorem that

$$\begin{aligned} & (2\Delta t)^{-1}(||U_h^{n+1}||^2 - ||U_h^n||^2 + ||U_h^{n+1} - U_h^n||^2) - \mu e^{-\mu\xi}||U_h^{n+1}||^2 + e^{-\mu\Delta t}||U_h^{n+1}||_a^2 \\ (4.6)\quad = & -e^{-\mu\Delta t}\sum_{j=0}^{n}\omega_{nj}^{\mu}B(U_h^j, U_h^{n+1}) + (e^{\mu t_n}f^n, U_h^{n+1}) \\ \leq & \ c_0\lambda_0^{\beta/2-1}e^{-\mu\Delta t}\sum_{j=1}^{n}\omega_{nj}^{\mu}\frac{||U_h^j||^2 + ||U_h^{n+1}||^2}{2} + (e^{\mu t_n}f^n, U_h^{n+1})\end{aligned}$$

for some $0 < \xi < \Delta t$. Then we find by summing on n and using (4.2)-(4.3) that

$$\begin{aligned}(4.7)\qquad & (2\Delta t)^{-1}(||U_h^{N+1}||^2 - ||U_h^0||^2 + \sum_{n=0}^{N}||U_h^{n+1} - U^n||^2) \\ & + (\lambda_0\sigma e^{-\mu\Delta t} - \mu e^{-\mu\xi})\sum_{n=0}^{N}||U_h^{n+1}||^2 + (1-\sigma)e^{-\mu\Delta t}\sum_{n=0}^{N}||U_h^{n+1}||_a^2 \\ \leq & \ c_0\lambda_0^{\beta/2-1}e^{-\mu\Delta t}\sum_{n=0}^{N}\sum_{j=1}^{n}\omega_{nj}^{\mu}\frac{||U_h^j||_a^2 + ||U_h^{n+1}||_a^2}{2} \\ & \quad + \sum_{n=0}^{N}e^{\mu t_n}||f^n|| \ ||U_h^{n+1}|| \\ \leq & \ c_0\lambda_0^{\beta/2-1}K_{\mu}^{\Delta t}e^{-\mu\Delta t}\sum_{n=0}^{N}||U_h^{n+1}||_a^2 + \sum_{n=1}^{N+1}e^{\mu t_n}||f^{n+1}|| \ ||U_h^{n+1}||\end{aligned}$$

where $0 < \sigma < 1$. We have

$$\begin{aligned}||U_h^{N+1}||^2 \ \leq & \ ||U_h^0||^2 + 2\Delta t\sum_{n=0}^{N}e^{\mu t_n}||f^n|| \ ||U_h^{n+1}|| \\ \leq & \max_{1\leq k\leq N+1}||U_h^k||\left(||U_h^0|| + 2\Delta t\sum_{n=0}^{N}e^{\mu t_n}||f^{n+1}||\right)\end{aligned}$$

provided that

$$\lambda_0 \sigma e^{-\mu\Delta t} - \mu e^{-\mu\xi} \geq 0, \quad \text{and} \quad (1-\sigma) \geq c_0 \lambda_0^{\beta/2-1} K_\mu^{\Delta t}, \tag{4.8}$$

which is equivalent to (our assumptions on μ)

$$1 - c_0 \lambda_0^{\beta/2-1} K_\mu^{\Delta t} \geq \sigma \geq \frac{\mu}{\lambda_0} e^{\mu\Delta t}. \tag{4.9}$$

Hence, by taking maximum from 1 to $N+1$, we find that

$$||U_h^{N+1}|| \leq \max_{0\leq k\leq N+1} ||U_h^k|| \leq ||U_h^0|| + 2\Delta t \sum_{n=0}^{N+1} e^{\mu t_n} ||f^n||, \quad N \geq 1.$$

Therefore Theorem 4.1 is proved via $u_h^n = e^{-\mu t_n} U_h^n$. ∎

Now we consider the case of the kernel $K(t)$ being positive. We see from [1] that $\{\omega_{n,j}\} = \{\Delta t K(t_{n-j})\}$ is a positive sequence: ie, for any $\{w^n\}_{n=1}^{\infty}$, there holds

$$\sum_{n=0}^{N} \sum_{j=1}^{n+1} \omega_{n,j} w^j w^{n+1} \geq 0, \quad N \geq 1. \tag{4.10}$$

Lemma 4.1. Assume that $\{\omega_{n,j}\}$ is positive definite and B is a sysmmetric non-negative definite elliptic operator, then for any sequence $\{w^n(x)\} \subset H_0^1(\Omega)$, we have

$$\sum_{n=0}^{N} \sum_{j=1}^{n+1} \omega_{n,j} B(w^j, w^{n+1}) \geq 0, \quad N \geq 1. \tag{4.11}$$

Proof: See [1].

For a positive kernel $K(t)$, the backward Euler scheme $\{u_h^n\}$ is defined by

$$\left(\frac{u_h^{n+1} - u_h^n}{\Delta t}, \chi\right) + A(u_h^{n+1}, \chi) + \sum_{j=1}^{n+1} \omega_{n,j} B(u_h^j, \chi) = (f^{n+1}, \chi), \quad \chi \in S_h, \tag{4.12}$$
$$u_h^0 = u_{0,h}, \quad n = 0, 1, 2, \cdots,$$

where $\omega_{n,j}$ is a positive definite sequence.

Remark: The difference between (4.12) and (4.1) is that (4.12) is a fully implicit scheme, while (4.1) is a semi-implicit scheme since the computation of the present level u_h^{n+1} does not involve in the present level values in the history sum.

Theorem 4.2. Under the assumptions of Theorem 4.1 and Theorem 2.2, the sequences $\{\omega_{n,j}\}$ and $\{\omega_{n,j}^\mu\}$ are both positive definite. If Δt and $\mu > 0$ are such that $\lambda_0 e^{-\mu\Delta t} - \mu \geq 0$, then we have for the solution $\{u_h^n\}$ of (4.12) that

$$||u_h^n|| \leq e^{-\mu t_n} ||u_{0,h}|| + 2\Delta t \sum_{j=0}^{n} e^{-\mu t_{n-j}} ||f^j||, \quad n \geq 1. \tag{4.13}$$

Proof: By taking $\chi = U_h^{n+1} \in S_h$ after rewriting (4.12) using $U_h^n = e^{\mu t_n} u_h^n$, and summing on n, it follows that for some $0 < \xi < \Delta t$,

$$
\begin{aligned}
& (2\Delta t)^{-1}(||U_h^{N+1}||^2 - ||U_h^0||^2 + \sum_{n=0}^{N} ||U_h^{n+1} - U_h^n||^2) \\
& +(\lambda_0 e^{\mu\Delta t} - \mu e^{-\mu\xi}) \sum_{n=0}^{N} ||U_h^{n+1}||^2 + \sum_{n=0}^{N}\sum_{p=1}^{n+1} \omega_{n,j}^{\mu} B(U_h^j, U_h^{n+1}) \\
= \; & \sum_{n=0}^{N} e^{\mu t_n} ||f^n|| \, ||U_h^{n+1}||.
\end{aligned}
$$

Thus, Theorem 4.2 follows from our assumptions and Lemma 4.1. ■

Theorem 4.3. Let u and u_h^n be the solutions of (1.1) and (4.1), respectively. Assume that **(A1)-(A3)** be satisfied and $||u_0 - u_{0,h}|| \leq Ch^r ||u_0||_r$. Then there exists a constant $C_0 > 0$, independent of h and t, such that

$$
\begin{aligned}
||u(t_n) - u_h^n|| \; \leq \; & C_0 h^r \left\{ ||u_0||_r \hat{R}(t_n) + |||u(t_n)|||_{r,R} + \int_0^{t_n} e^{-\mu(t_n - s)} |||u_t|||_{r,R} ds \right\} \\
& + C_0 \Delta t \sum_{j=0}^{n} e^{-\mu(t_n - s)} Q_j(u),
\end{aligned} \tag{4.14}
$$

where $Q_j(u) = Q_{j,q}(u) + Q_{j,T}(u)$, and where $Q_{j,T}(u)$ denotes the time truncation error

$$
Q_{j,T} = \left\| \frac{u(t_j) - u(t_{j-1})}{\Delta t} - u_t(t_j) \right\|
$$

and $Q_{j,q}(u)$ denotes the quadrature error

$$
Q_{j,q}(u) = \left\| \sum_{k=0}^{j-1} \omega_{jk} B u(t_k) - \int_0^{t_j} K(t_j - s) B u(s) ds \right\|.
$$

Proof: The proof of this theorem is the same as the earlier arguments employed in showing stability given above. ■

5. Convergence to the Steady-State u^∞

In this section the asymptotics of the solutions as $t \to \infty$ are considered. We shall first show that the solution $u(t)$ of (1.1) converges to the steady-state solution u^∞, which satisfies

$$
\begin{aligned}
Du^\infty &= f^\infty \;\text{ in }\; \Omega, \\
u^\infty &= 0 \;\text{ on }\; \partial\Omega,
\end{aligned} \tag{5.1}
$$

where the second order elliptic operator $D = A + K_0 B$ with

$$
K_0 = \int_0^\infty K(s) ds \qquad \text{and} \qquad f^\infty = \lim_{t \to \infty} f(t). \tag{5.2}
$$

Then, we show that the semi-discrete finite element approximation $u_h(t)$ of (1.3) and backward Euler solution u_h^n of (1.11) converge to the finite element solution u_h^∞ of (5.1).

First we have the following result:

Lemma 5.1: Assume that **(A1)-(A3)** hold or the kernel $K(t)$ is positive definite and the operator B is non-negative. Then problem (5.1) is well-posed in $H_0^1(\Omega) \cap H^2(\Omega)$ for any $f^\infty \in L^2(\Omega)$ and there exists a positive constant $C > 0$ such that $||u^\infty||_2 \leq C||f^\infty||$.

Proof: First if **(A1)-(A3)** hold then D is a strongly elliptic, i.e., the bilinear form $D(\cdot, \cdot)$ satisfies

$$D(u, u) = A(u, u) + K_0 B(u, u) \geq (1 - c_0 K_0 \lambda_0^{\beta/2-1})||u||_a^2, \quad u \in H_0^1(\Omega).$$

Second, If the kernel $K(t)$ is positive definite and operator B is non-negative, it is easy to see that the operator D is still strongly elliptic. Therefore, Lemma 5.1 follows from the standard elliptic theory [6, 22]. ■

Now let $w = u(t) - u^\infty$, where u and u^∞ are the solutions of (1.1) and (5.1), respectively. Then we see clearly that w satisfies

$$\begin{aligned} & w_t + Aw + \int_0^t K(t-s)Bw(s)ds = F(t) \ \text{ in } \ Q_\infty, \\ & w = 0, \ \text{ on } \ \partial\Omega \times (0, \infty), \\ & w(x, 0) = u_0 - u^\infty, \ \ x \in \Omega, \end{aligned} \tag{5.3}$$

where

$$F(t) = f(t) - f^\infty - \int_t^\infty K(s)ds \; Bu^\infty. \tag{5.4}$$

Theorem 5.1. Under the assumptions of Lemma 5.1, we have the following asymptotic estimate for the solutions u of problem (1.1) and u^∞ of problem (5.1)

$$||u(t) - u^\infty|| \leq ||u_0 - u^\infty||e^{-\mu t} + 2\int_0^t e^{-\mu(t-s)}||F(s)||ds, \quad t \geq 0, \tag{5.5}$$

where μ is defined in Theorem 2.1 or Theorem 2.2 and

$$||F(t)|| \leq ||f(t) - f^\infty|| + \int_t^\infty K(s)ds \; ||Bu^\infty||.$$

Proof: It follows easily from (5.3) and Theorem 2.1 and Theorem 2.2. ■

Remark: (i) Since $||Bu^\infty|| \leq C_0||f^\infty||$ by elliptic regularity, and if $K(t) \leq C_0 e^{-\lambda t}$ and $||f(t) - f^\infty|| \leq C_0 e^{-\lambda t}$ for some $\lambda > 0$ and $t \geq 0$, then one finds from Theorem 5.1 and a simple calculation that there exists $C_0 > 0$, dependent only upon u_0 and f^∞, such that (5.5) is replaced by

$$||u(t) - u^\infty|| \leq C_0 \left(t \, e^{-\mu t} + e^{-\lambda t}\right), \quad t \geq 0. \tag{5.6}$$

(ii) If $K(t) \leq C_0 t^{-\alpha} e^{-\lambda t}$ for $0 < \alpha < 1$, then the above asymptotic estimate (5.6) should be replaced by

$$||u(t) - u^{\infty}|| \leq C_0(\alpha, \lambda) \left(t\, e^{-\mu t} + t^{1-\alpha} e^{-\lambda t}\right), \quad t > 0. \tag{5.7}$$

In fact, since

$$\begin{aligned}\int_t^{\infty} K(s)ds &\leq C_0 \int_t^{\infty} s^{-\alpha} e^{-\lambda s} ds = C_0\, t^{1-\alpha} \int_1^{\infty} \tau^{1-\alpha} e^{-\lambda t \tau} d\tau \\ &\leq C_0 t^{-\alpha} \int_1^{\infty} e^{-\lambda t\tau} t d\tau \leq C_0 \lambda^{-1} t^{-\alpha} e^{-\lambda t}, \quad t > 0,\end{aligned}$$

and then it follows that

$$\begin{aligned}\int_0^t e^{-\mu(t-s)} \left(\int_s^{\infty} K(\tau)d\tau\right) ds &\leq C_0 \lambda^{-1} \int_0^t e^{-\mu(t-s)} s^{-\alpha} e^{-\lambda s} ds \\ &\leq C_0 \lambda^{-1} \int_0^1 e^{-\mu(t-t\tau)} (t\tau)^{-\alpha} e^{-\lambda t \tau} t d\tau \\ &\leq C_0 \lambda^{-1} (1-\alpha)^{-1} t^{1-\alpha} e^{-\lambda t}, \quad t \geq 0.\end{aligned}$$

Thus, (5.7) follows from Theorem 5.1.

(iii) The uniform estimate (5.5) holds true for the singular kernel $K(t)$ only if $Bu^{\infty} = 0$, which implies that the last term on the right hand side of (5.4) vanishes.

Now let $u_h^{\infty} \in S_h$ be a finite element approximation of the solution (5.1) and is defined by

$$D(u_h^{\infty}, \chi) = (f^{\infty}, \chi), \quad \chi \in S_h \tag{5.8}$$

where $D(\cdot, \cdot)$ is the bilinear form associated with the operator $D = A + K_0 B$ on $H_0^1(\Omega) \times H_0^1(\Omega)$. We are now ready to state and prove the following results for semi-discrete and backward Euler approximations.

Theorem 5.2. Assume that **(A1)-(A3)** holds or the kernel $K(t)$ is positive definite and the operator B is non-negative and $u_h(t)$ is the solution of (1.3). Then we have for $t \geq 0$,

$$||u_h(t) - u_h^{\infty}|| \leq ||u_{0,h} - u_h^{\infty}|| e^{-\mu t} + 2 \int_0^t e^{-\mu(t-s)} \left(||f(s) - f^{\infty}|| + ||f^{\infty}|| \int_s^{\infty} K(\tau)d\tau\right) ds, \tag{5.9}$$

where μ is defined in Theorem 2.1 or Theorem 2.2.

Proof: Let the operator $B_h : S_h \to S_h$ be defined by

$$(B_h \phi, \psi) = B(\phi, \psi), \qquad \phi, \psi \in S_h. \tag{5.10}$$

and let $w_h = u_h(t) - u_h^{\infty}$, then we see easily from (1.1), (1.2) and (5.8) that $w_h(t)$ satisfies

$$(w_{h,t}, \chi) + A(w_h, \chi) + \int_0^t K(t-s) B(w_h(s), \chi) ds = (F_h(t), \chi)$$

where

$$F_h(t) = f(t) - f^{\infty} - \int_t^{\infty} K(s)ds\ B_h u_h^{\infty}.$$

Thus we have from Theorem 2.3 and Theorem 2.4 that

$$||u_h(t) - u_h^\infty|| \le ||u_{0,h} - u_h^\infty||e^{-\mu t} + 2\int_0^t e^{-\mu(t-s)}||F_h(s)||ds, \quad t \ge 0, \tag{5.11}$$

with

$$||F_h(t)|| \le ||f(s) - f^\infty|| + \int_t^\infty K(s)ds\, ||B_h u_h^\infty||. \tag{5.12}$$

In order to estimate $B_h u_h^\infty$, let $D_h = A_h + K_0 B_h$, where A_h is defined in a similar way as (5.8) for B_h, we find that D_h^{-1} exists and $||B_h D_h^{-1}\phi||$ is bounded in S_h. In fact, for $\phi, \psi \in S_h$,

$$\begin{aligned}(B_h D_h^{-1}\phi, \psi) &= B(D_h^{-1}\phi - D^{-1}\phi), \psi) + B(D^{-1}\phi, \psi)\\ &\le C_0||(D_h^{-1} - D^{-1})\phi||_1\, ||\psi||_1 + |(BD^{-1}\phi,\ \psi)|\\ &\le C_0 h||\phi||_0 C_0 h^{-1}||\psi|| + C_0||\phi||\, ||\psi|| \le C_0||\phi||\, ||\psi||.\end{aligned}$$

Here the inverse inequality $||\chi||_1 \le Ch^{-1}||\chi||$ for $\chi \in S_h$, the standard finite element estimates for elliptic equations $||(D_h^{-1} - D^{-1})\phi||_1 \le C_0 h||\phi||$ and $||BD^{-1}|| \le C_0$ have been used [6, 21]. Therefore, we find from (5.8) that $D_h u_h^\infty = P_h f$ in S_h and

$$||B_h u_h^\infty|| \le C||B_h D_h^{-1} D_h u_h^\infty|| \le C_0||P_h f^\infty|| \le C_0||f^\infty||, \tag{5.13}$$

where $P_h : L^2(\Omega) \to S_h$ is the L^2-projection. The proof is completed by substituting (5.8) into (5.11)-(5.12). ■

Finally, we have the following estimate for the Euler scheme.

Theorem 5.3. Under the assumptions of Theorem 4.1, we have for the Euler solution u_h^n that

$$||u_h^n(t) - u_h^\infty|| \le ||u_{0,h} - u_h^\infty||e^{-\mu t_n} + 2\Delta t\sum_{j=1}^n e^{-\mu(t_n - t_j)}||F_h^j||, \quad n \ge 0, \tag{5.14}$$

where F_h^n is defined as

$$F_h^n = f^n - f^\infty + \left(\sum_{k=1}^j \omega_{j,k} - K_0\right) B_h u_h^\infty,$$

which is in fact bounded by

$$||F_h^n|| \le ||f^\infty - f^n|| + C_0||f^\infty||\left|\sum_{k=1}^n \omega_{j,k} - K_0\right|.$$

Proof: We find from (4.1) and (5.8) that for $w_h^n = u_h^n - u_h^\infty$

$$\begin{aligned}&\left(\frac{w_h^{n+1} - w_h^n}{\Delta t},\ \chi\right) + A(w^{n+1}, \chi) + \sum_{j=0}^n \omega_{nj} B(w_h^j, \chi) = (F_h^n, \chi), \quad \chi \in S_h,\\ &w_h^0 = u_{0,h} - u_h^\infty, \quad n = 0, 1, 2, \cdots,\end{aligned}$$

where F_h^n is defined above. Thus (5.12) follows from Theorem 4.1. ■

Final Remarks:

(a) If $K(t) = e^{-\mu t}$ and $B = A$ in (1.1), we see that $\mu > 1$ in order to satisfy **(A2)**. It is easy to see from a simple calculation that

$$R(t) = O(e^{-(\mu-1)t}) \quad \text{and} \quad \int_0^t R(t-s)R(s)ds = O(te^{-(\mu-1)t}),$$

so that $\hat{R(t)} = O(te^{-(\mu-1)t})$. This implies that (4.14) may not be the best convergence rate estimates.

(b) Since

$$||u(t) - u_h(t)|| \leq ||u(t) - u^\infty(t)|| + ||u^\infty(t) - u_h^\infty(t)|| + ||u_h^\infty(t) - u_h(t)||$$

so that if the assumptions of Theorem 5.1-5.3, Theorem 3.2 and Theorem 4.3 are satisfied we have

$$||u(t) - u_h(t)|| = O(h^r + te^{-\mu t} + e^{-\lambda t})$$

for the semi-discrete approximation and

$$||u(t_n) - u_h^n|| = O(h^r + \Delta t + te^{-\mu t} + e^{-\lambda t})$$

for the backward Euler scheme. Therefore if $te^{-\mu t_n} + e^{-\lambda t} \leq h^r + \Delta t$ or $t_n \geq t^*$ where t^* is defined by

$$t^* = \begin{cases} \dfrac{1}{\mu} \log\left(\dfrac{1}{\Delta t(h^r + \Delta t)}\right), & \lambda \geq \mu, \\ \dfrac{1}{\mu C^*} \log\left(\dfrac{1}{h^r + \Delta t}\right), & \lambda < \mu, \end{cases}$$

where

$$C^*(\lambda, \mu) = \max\{1 + te^{-(\lambda-\mu)t} \mid t \geq 0\},$$

the numerical scheme is convergent to the steady-state, that is u_h^∞ can be taken as numerical solution for all time $t_n \geq t^*$.

(c) If $K(t) \leq C_0 t^{-\alpha} e^{-\mu t}$ and $||f - f^\infty|| \leq C_0 e^{-\lambda t}$ then we have

$$||u(t_n) - u_h^\infty|| \leq C(\alpha, \lambda)(t_n e^{-\mu t_n} + t_n^{1-\alpha} e^{-\lambda t_n}), \qquad n \geq 1.$$

Thus a similar critical time t^*, dependent upon $h^r + \Delta t$, α, λ and μ, can be obtained after which Computations may be terminated .

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