NUMERICAL WELL MODEL FOR NON-DARCY FLOW

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1. Introduction

When modeling reservoir behavior by numerical simulations, inevitably the dimensions of the grid block containing the well are two order of magnitudes larger that the well-bore radius. The calculated by any numerical method pressure in the well block (or blocks sharing the well as a corner point) will be substantially different from the flowing bottom-hole pressure of the modeled well. Therefore, a fundamental task in the reservoir modeling of wells is to find an accurate correction.

The first comprehensive study of this problem for cell-centered finite difference approximation on square grids was done by Peaceman in [9] for single phase phase Darcy flow in two dimensions. Peaceman's study presented a proper interpretation of the well-block pressure, and showed how it relates to the flowing bottom-hole pressure. The importance of this study is that the computed cell pressure has been associated with the steady-state pressure for the actual well at an equivalent radius r_{eff}. Contrary to the previous studies, which had related the computed cell pressure to the average pressure of the radial flow over the grid cell, Peaceman derived that $r_{eff} \approx 0.2h$ (here h is the cell-size) in three different ways: (a) numerically, by solving the pressure equation on a sequence of grids and producing $r_{eff} = 0.2h$; (b) analytically by assuming that the pressure at the adjacent block is computed exactly by the radial flow model and getting $r_{eff} = 0.208h$; (c) by solving exactly the system of difference equations and using the equation for the pressure drop between injection and producing well in a repeated five-spot pattern given by Muskat [6] and getting $r_{eff} = 0.1987h$.

Peaceman study was extended in various directions (including off center and multiple wells within a wellblock, nonsquare grids, anisortopic permeability, horizontal wells, etc) by a number of numerical analysts and petroleum engineers (see, e.g. [1, 3, 7, 8, 10]). Peaceman himself has extended his study to more general situations including nonsquare grids and anisotropic permeability [10] and more general geometries [11]. For arbitrary location of the well we refer to [1] and for comparative study of numerical simulation of horizontal wells we refer to [7]. To our knowledge, all existing studies are done for cell-centered finite difference approximations of the pressure equation. On the other hand, the finite element approximations have been already successfully used for groundwater flow simulations (see, e.g. [5]). To make use of the finite elements in presence of wells it is necessary to find accurate well models for this important and widely used class of numerical methods.

In this note we extend Peaceman's (b)-approach for well modeling in two different directions: (1) deriving an accurate well models for mixed finite element approximations on triangular grids and Galerkin approximations for bilinear finite elements on squares; (2) deriving well models for cell-centered finite differences on square grids, mixed finite element approximations on triangular grids and Galerkin approximations for bilinear finite elements on squares for flows goverened by by Forchheimer relation between the pressure gradient and the flow velocity.

Our analysis is based on the fumdamental assumption that the flow is radial in a neighborhood of the well. This assumption can be verified for isotropic porous media even in presence of nonlinearities due to the dependence of the viscosity on the pressure and Forchheimer relation between the pressure gradiaent and the flow velocity (see, e.g. [2] and for more general flows [4]). Further, using the technique develpoed in [10] we extend this well models to anisitropic porous media and rectangular gerids. Thus, our analysis can be used for quite general flow models and various numerical methods and techniques.

2. Analytic solution in the neighborhood of the well

The problem of modeling flow from a well with a radius which is substantially smaller than the discretization parameter or mesh size requires the use of analytic formulas. These formulas are only known in the case of simplified flow situations and thus constitute practical limitations in their application. We present analytic formulas for the Forchheimer flow in this section.

The basic assumption is that the flow is radial and that coefficients are constant (at least near the well). Specifically, we assume that

1. The flow is two dimensional in x and y (no gravity term).
2. $\mathbf{K}$ is a constant K times the identity matrix.
3. β is a constant (which we will also denote by β) times the identity matrix.
4. μ and ρ are constant in the neighborhood of the well.
5. The flow is radial in the neighborhood of the well.

We will discuss possible generalizations at the end of this manuscript.

Of the above assumptions, perhaps the most interesting is the last. This implies that the well should be circular or its size so small that the variations in its geometry can be neglected. The decay properties of the Greens function then imply that the flow becomes radial in the limit as one approaches the well (or singularity).

We derive the analytic model as follows. Assume that the well is at the origin. If the flow is radial then the velocity $\vec{u}$ must be of the form

$$\vec{u} = w(r)(\cos\theta, \sin\theta).$$

The function w is a scalar function depending only on the radius r. There are no sources or sinks except at the origin so

$$\nabla \cdot \vec{u} = 0$$

near the well. It follows that

$$w' + r^{-1} w = 0,$$

i.e., $w = cr^{-1}$. The constant c is proportional to the production rate.

The pressure p satisfies the Forchheimer relation (see, e.g. [2]),

$$-\nabla p = (K^{-1}\mu + \rho\beta|\vec{u}|)\vec{u}. \tag{2.1}$$

The pressure will tend to infinity as we approach an idealized (point source well). This results in a positive c above. Dotting (2.1) with the vector $n = (\cos\theta, \sin\theta)$ and integrating from $(r_0, 0)$ to $(r, 0)$ gives

$$p(r) - p(r_0) = F(r) - F(r_0) \tag{2.2}$$

where

$$F(r) = -K^{-1}\mu c \log(r) + \frac{\rho\beta c|c|}{r}.$$

Here we have use the radial flow assumption which implies that $p(x, y) \equiv p(r)$.

Let Q be the injection rate at the well. Then, Q is the mass flux through any small circle B_ϵ centered at the origin, i.e.

$$Q = \int_{B_\epsilon} \vec{u} \cdot n \, ds = 2\pi c.$$

Here n is the outward normal on the circle. Thus,

$$F(r) = -\frac{K^{-1}\mu Q}{2\pi} \log(r) + \frac{\rho\beta Q|Q|}{4\pi^2 r}.$$

The equation (2.2) represents our analytical flow model for flow near the well. As a verification of our codes and the above model we ran several tests. These tests were with physical units. Thus, we considered the equations

$$\begin{aligned} c_1 \nabla \cdot \vec{u} &= c_2 q \\ -\nabla p &= (c_4 K^{-1}\mu + c_5 \rho\beta|\vec{u}|)\vec{u}. \end{aligned} \tag{2.3}$$

We ran both the trilinear finite element code and the triangular mixed method code. We took advantage of symmetry and ran the codes on a square with lower left hand corner at the origin, no flow boundary conditions at $x = 0$ and $y = 0$ and $p = 0$ at $x = 5000$ and $y = 5000$. We set $\rho = \rho(5000) = .178$, $\mu = \mu(5000) = .0256$, $K = 100$, $q = 10^6$, and varied $\beta = 0, 7.6 \times 10^7, 7.6 \times 10^8$.

The absolute magnitude of the pressure cannot be determined from the analytic model since it depends on the placement of the outer boundary and the boundary conditions imposed there. However, we were able to fit the model to the output by aligning them at one point. We did this by choosing some value of r_0 (typically, $r_0 \approx 1000$) and set $P(r_0)$ to be the value computed by the code. The analytical model then predicted the remain values of the pressure near the well with good accuracy. For example, the results from the triangular mixed finite element code for $\beta = 0$ and $\beta = 7.6 \times 10^7$ are given below.

Beta = 0

r	model	computed	
147.313	2351.023	2342.649	
368.284	1740.491	1745.965	
589.255	1437.574	1439.900	
810.226	1231.709	1232.524	
1031.197	1075.481	1075.481	<- Fit point
1252.168	949.587	949.047	
1473.139	844.205	843.215	
1694.109	753.653	752.203	
1915.080	674.355	672.365	
2136.051	603.918	601.255	
2357.022	540.672	537.151	
2577.993	483.411	478.796	
2798.964	431.241	425.243	

Note that there is good agreement between the model and computed values for the pressure. The case of nonzero β is below. At this grid level there is only a modest change in the pressure values compared to $\beta = 0$. However, it turns out that there is a significant difference of one evaluates the models at, e.g., a well of radius 2 ft. The Darcy pressure at 2ft is 5142 while the Forchheimer pressure at 2ft for the run below is 36900. Obviously, the end results of any computation will depend critically on a correctly implemented well model.

Beta = 7.6 x 10^7

r	model	computed	
147.313	2781.126	2762.052	
368.284	1900.631	1906.569	
589.255	1533.394	1535.806	
810.226	1298.192	1299.021	
1031.197	1125.172	1125.172	<- Fit point
1252.168	988.408	987.865	
1473.139	875.418	874.422	
1694.109	779.245	777.783	

1915.080	695.624	693.617
2136.051	621.764	619.075
2357.022	555.743	552.182
2577.993	496.189	491.517
2798.964	442.095	436.018

3. A WELL MODEL FOR CELL CENTERED FINITE DIFFERENCES.

In this section, we derive a well model for cell centered finite differences. This model has built into it the correct behavior resulting from the Forchheimer term.

The basic problem with a numerical approximation on a grid of size much larger than the well-bore is that such a model (without the introduction of singular functions) cannot detect or predict the correct singular behavior of the solution. Such approximations have a tendency of smearing out the singular behavior. In fact, the computed cell block pressure is significantly different than the average of the solution over the cell.

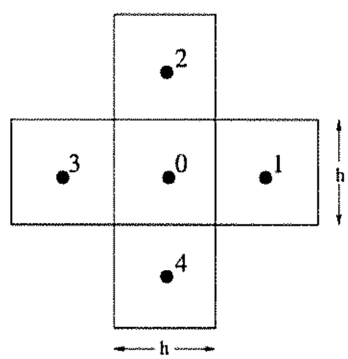

Fig. 1 Block 0 containing a well and its four neighboring blocks.

The goal of a well model is to develop a relationship between the pressure computed in the cell block containing the well and the flow Q. Peaceman developed an empirical model for the case of Darcy flow by examining the ratio

$$\alpha_{ij} = \exp\left(\frac{(P_0 - P_{ij})(2\pi K)}{\mu Q}\right).$$

Here P_{ij} and P_0 are the computed value of the pressures at, respectively, the node ij and the cell containing the well. He found that for cell centered approximations, α_{ij} was proportional to $\sqrt{i^2 + j^2}$. In fact, $\alpha_{ij} \approx 5\sqrt{i^2 + j^2}$. This leads to the Peaceman well model

$$p_w - p_0 = F(r_w) - F(.2h) = -\frac{K^{-1}\mu Q}{2\pi}\log(r_w/(.2h)). \tag{3.1}$$

Here p_w is the bottom-hole pressure for a well bore with radius r_w.

We will derive models similar to [9], which include the Forchheimer effects. We consider the unscaled problem

$$\begin{aligned} \nabla\cdot\vec{u} &= Q \\ -\nabla p &= (K^{-1}\mu + \rho\beta|\vec{u}|)\vec{u}. \end{aligned} \tag{3.2}$$

We consider the case when the well is located in the center of the center square of a square grid (see, Fig 1). We index the cells giving the well cell index 0 and the cell to its right index 1. By summation by parts, the discrete equations which result from cell centered finite difference approximations can be written as

$$A(p,\phi) = Q\phi_0 \tag{3.3}$$

Here p and ϕ are vectors equal to the number of cells. The quadratic form $A(\cdot,\cdot)$ is given by

$$A(v,w) = \sum_{\mathcal{E}_{ij}}(K^{-1}\mu + \rho\beta|\vec{u}(v)_{ij}|)^{-1}(v_i - v_j)(w_i - w_j).$$

Here $\mathcal{E}_{ij}$ is the edge between cells i and j. The quantity $\vec{u}(v)_{ij}$ is the normal component of the velocity associated with the pressure vector v at the edge $\mathcal{E}_{ij}$ and satisfies the Forchheimer relation

$$(K^{-1}\mu + \rho\beta|\vec{u}(v)_{ij}|)\ \vec{u}(v) = -\frac{v_i - v_j}{h}.$$

Taking

$$\phi_i = \begin{cases} 1 \text{ if } i = 0, \\ 0 \text{ otherwise,} \end{cases}$$

in (3.3) and using the symmetry of the solution implies that

$$(K^{-1}\mu + \rho\beta|\vec{u}|)^{-1}(P_0 - P_1) = Q/4. \tag{3.4}$$

Here we have denoted $\vec{u} = \vec{u}(P)_{01}$. It is immediate from (3.4) and the definition of $\vec{u}$ that

$$\vec{u} = -\frac{Q}{4h}.$$

Substituting this back into (3.4) and simplifying gives

$$P_0 - P_1 = \frac{QK^{-1}\mu}{4} + \frac{Q|Q|\rho\beta}{16h}. \tag{3.5}$$

The analytic well model should be a relatively good approximation in cell 1. This means that if we are given a bottom-hole pressure P_w and a well radius r_w,

$$P_1 = P_w + F(r_1) - F(r_w). \tag{3.6}$$

Adding (3.5) and (3.6) gives

$$P_0 = P_w + F(r_1) - F(r_w) + \frac{QK^{-1}\mu}{4} + \frac{Q|Q|\rho\beta}{16h}. \tag{3.7}$$

The above relation suggests that the pressure behavior near the well is significantly more complicated in the Forchheimer case. In particular, the well model depends nonlinearly on Q, ρ, β and the mesh size h.

The above model essentially reproduces the Peaceman result in the case of Darcy flow. Indeed if $\beta = 0$ then (3.7) becomes

$$\begin{aligned} P_0 &= P_w - \frac{QK^{-1}\mu}{2\pi}(\log(h/r_w) - \frac{\pi}{2}) \\ &= P_w - \frac{QK^{-1}\mu}{2\pi}\log(\alpha h/r_w) \end{aligned}$$

where $\alpha = e^{-\pi/2} = .20788\ldots$. This is exactly the value obtained by Peaceman in [9] under the assumption that P_1 is already a very good approximation to the analytic solution. Using different approach (mentioned above as approach (c)) Peaceman [9] computed slightly smaller constant, namely, $\alpha = .1987$.

4. A well model for Galerkin approximations using bilinear finite elements

In a finite element setting using bilinear finite elements on a square grid, we consider an ensemble of four finite elements sharing a common vertex with index 0 (see fig. 2). We assume that the well is placed at the vertex 0 and the flow is radual in its vicinity.

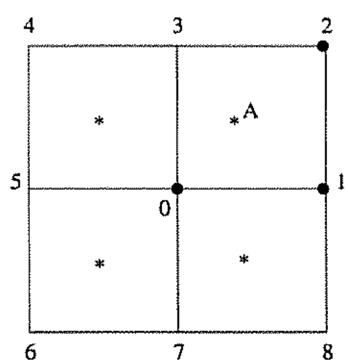

Fig. 2 Four finite elements sharing a common vertex as a well.

We assume that bilinear finite elements have been used and the flow is radial around the well which is located at the point with index 0. For computing the finite element stiffness matrices we employ one-point Gauss quadrature (the quadrature uses the center of the finite element; in the case of element with nodal vertices 0, 1, 2, and 3 this is the point A).

The row in the stiffness matrix corresponding to the unknown P_0 is compiled from the integral

$$\sum_e \int_e \left(K^{-1}\mu + \rho\beta|\vec{u}|\right)^{-1} \nabla P \nabla \phi_0 dx = Q\phi_0, \tag{4.1}$$

where the summation is over the four finite elements sharing the vertex 0. Since P and ϕ_0 are piece-wise linear functions, it is reasonable to evaluate the integral over the finite element by one-point Gauss quadrature and get:

$$\sum_e \left(K^{-1}\mu + \rho\beta|\vec{u}_e|\right)^{-1} \nabla P_e \nabla \phi_{oe} h^2 = Q\phi_0, \tag{4.2}$$

where $\vec{u}_e = \vec{u}(A)$, $\nabla P_e = \nabla P(A)$, and $\nabla \phi_{oe} = \nabla \phi(A)$. As in the previous section $\phi_0 = 1$ at 0 and vanishes at all other grid points 1–8. Taking into account the radial symmetry of the solution P, which result in taking $P_1 = P_3 = P_5 = P_7$ and $P_2 = P_4 = P_6 = P_8$, we get the following finite element equation corresponding to the unknown P_0.

$$\frac{4}{3} \left(K^{-1}\mu + \rho\beta|\vec{u}_e|\right)^{-1} (2P_0 - P_1 - P_2) = Q. \tag{4.3}$$

The element (cell) velocity $\vec{u}_e$ satisfies the Forchheimer law (instead of Darcy's law), which relates the velocity to the pressure gradient. In the case of radial symmetry, one can proceed as follows: first note that approximately we have

$$\left(K^{-1}\mu + \rho\beta|\vec{u}_e|\right) |\vec{u}_e| \approx \frac{P_0 - P_2}{\sqrt{2}h}. \tag{4.4}$$

In order to utilize the equation (4.3) we need an approximate relation of the form

$$\left(K^{-1}\mu + \rho\beta|\vec{u}_e|\right) |\vec{u}_e| \approx \frac{P_0 - P_1}{\gamma h}. \tag{4.5}$$

Obviously, this approximation is not valid for $\gamma = 1$ since this will give the x-derivative of the pressure at the point on a distance $0.5h$ from the well instead of distance $\sqrt{(2)}h$ where the cell-velocity $|\vec{u}_e|$ is computed. To find γ we performed a series of numerical experiments and fitted the parameter. If the ralationships (4.4) and (4.5) were true then the ratio $(P_0 - P_1)/(P_0 - P_2)$ should have a constant value approximately equal to $\gamma/\sqrt{2}$. The results of our computations are given in the following table for a reservoir slightly larger than the one described in section 2. Let us remind that the quarter of the reservoir is 6400×6400 ft with the lowest left corner at the origin and the step-size is given in feet. The last row in the table is computed on a rectangular grid with variable step-sizes in each direction destributed as follows starting from the well: 2 steps of 30 ft, 9 steps of 60 ft and 29 steps of 200 ft.

	Beta = 0	Beta = 7.6 x 10^7	Beta = 7.6 x 10^8
h	computed gamma	computed gamma	computed gamma
800	1.507	1.507	1.505
400	1.507	1.507	1.503
200	1.507	1.507	1.498
100	1.507	1.506	1.488
variable	1.507	1.504	1.461

From the given table we conclude that $\gamma = 1.5$ is a good approximation for this parameter. Our hypothesis is that $\gamma = \sqrt{2}$. Therefore, a reasonable approximation for the Forchheimer relation (2.1) for bilinear elements will be

$$(4.6) \qquad \left(K^{-1}\mu + \rho\beta|\vec{u}_e|\right)|\vec{u}_e|(\gamma+\sqrt{2})h \approx 2P_0 - P_1 - P_2.$$

We rewrite (4.3) and (4.6) in the form

$$\frac{3}{4}Q = \left(K^{-1}\mu + \rho\beta|\vec{u}_e|\right)^{-1}(2P_0 - P_1 - P_2),$$
$$|u_e|(\gamma+\sqrt{2})h = \left(K^{-1}\mu + \rho\beta|\vec{u}_e|\right)^{-1}(2P_0 - P_1 - P_2).$$

The following relationship

$$(4.7) \qquad |\vec{u}_e| = \frac{3}{4}\frac{Q}{(\gamma+\sqrt{2})h} = \frac{0.257Q}{h}.$$

now represents the well model for the Galerkin finite element method using bilinear elements on a square grid.

Here we report some computationally obtained production rates. The table has been completed for a model with a given bottom-hole pressure equal to 1000 [psi], for initial pressure equal to 5000 [psi], and Dercy's law, i.e. $\beta = 0$. We have used three different mesh sizes $h = 500,\ 200,\ 100$ ft. The time is in days and the rest of the parameters are the same as described above. Namely, the domain is square with lower left hand corner at the origin, no flow boundary conditions at $x = 0$ and $y = 0$ and $p = 0$ at $x = 5000$ and $y = 5000$. We set $\rho = \rho(5000) = .178$, $\mu = \mu(5000) = .0256$, $K = 100$.

time	h = 100	h = 200	h = 500
0.0	1.7891e+07	1.4917e+07	1.2229e+07
0.1	1.3967e+07	1.3784e+07	1.2081e+07
0.3	1.1339e+07	1.2285e+07	1.1807e+07
0.5	1.0344e+07	1.1280e+07	1.1553e+07
1.0	9.6154e+06	1.0098e+07	1.1021e+07
1.5	9.2739e+06	9.4977e+06	1.0574e+07
2.0	9.0581e+06	9.1615e+06	1.0198e+07
2.5	8.9010e+06	8.9510e+06	9.8809e+06
3.0	8.7781e+06	8.8044e+06	9.6124e+06
3.5	8.6777e+06	8.6933e+06	9.3844e+06
4.5	8.5281e+06	8.5364e+06	9.0444e+06
5.5	8.4103e+06	8.4158e+06	8.7875e+06
6.5	8.3141e+06	8.3182e+06	8.5904e+06
7.5	8.2332e+06	8.2365e+06	8.4369e+06
8.5	8.1637e+06	8.1664e+06	8.3150e+06
9.5	8.1030e+06	8.1052e+06	8.2166e+06
10.5	8.0491e+06	8.0510e+06	8.1355e+06

As we can see form this table, the results for the production rates are pretty ccurate already for mesh step-size $h = 200$. After the fourth day the differences in the daily production rates between the computations with mesh-size $h = 100$ and $h = 200$ are less than 0.1%.

5. Acknowledgment

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