ASYMPTOTIC ERROR EXPANSION FOR THE LOWEST ORDER RAVIART-THOMAS RECTANGULAR MIXED FINITE ELEMENTS

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Abstract

In this paper, we consider mixed finite element approximations for solving second order elliptic boundary value problems. An asymptotic error expansion is obtained for velocity and Richardson extrapolation schemes are used to reduce the error and to get a higher order approximation.

Keywords elliptic problem, mixed finite element method, rectangular element, asymptotic error expansion, Richardson extrapolation
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1 Introduction

A substantial number of works has been devoted to asymptotic error expansion for the finite element approximations. The research in asymptotic error expansion has been augmented and inter-related with interesting and very important for the computational practice results in superconvergence and postprocessing (see, e.g. [1, 4, 5, 6, 8]). The corresponding results for the asymptotic error expansion of mixed finite element methods can be found in Wang [10], where the lowest order triangular elements have been considered. In this work we concentarte our attention to the error in the Darcy velocity

Let Ω be a rectangle in R^2 with edges parallel to the coordinates axis. We consider the following model problem:

$$\begin{cases} -\Delta p = f & \text{in } \Omega \\ p = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

Let

$$W = L^2(\Omega), \qquad \boldsymbol{V} = \left\{ \boldsymbol{\varphi} \in L^2(\Omega)^2,\ \nabla \cdot \boldsymbol{\varphi} \in L^2(\Omega) \right\},$$

and the Banach space $\boldsymbol{V}$ be equipped with norm $\|\varphi\|_{\boldsymbol{V}}=(\|\varphi\|^2+\|\nabla\cdot\varphi\|^2)^{1/2}$. The inner product and norm in $L^2(\Omega)$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. And for the sake of simplicity, $(\cdot,\cdot)$ and $\|\cdot\|$ are also respectively be used as the inner product and norm in the product space $L^2(\Omega)^2$.

Let $\boldsymbol{u}=-\nabla p$, then pair $(p,\boldsymbol{u})\in W\times\boldsymbol{V}$ satisfies the following mixed variational equation:

$$\begin{cases}(\nabla\cdot\boldsymbol{u},\psi)=(f,\psi), & \forall\,\psi\in W,\\ (\boldsymbol{u},\boldsymbol{\varphi})+(\nabla\cdot\boldsymbol{\varphi},p)=0, & \forall\,\boldsymbol{\varphi}\in\boldsymbol{V},\end{cases}\tag{2}$$

Given finite dimensional spaces $W_h\subset W$ and $\boldsymbol{V}_h\subset\boldsymbol{V}$, $0<h<1$, the so-called mixed finite element approximation $(p_h,\boldsymbol{u}_h)\in W_h\times\boldsymbol{V}_h$ to pair $(p,\boldsymbol{u})\in W\times\boldsymbol{V}$ is the solution of the following problem:

$$\begin{cases}(\nabla\cdot\boldsymbol{u}_h,\psi_h)=(f,\psi_h), & \forall\,\psi_h\in W_h,\\ (\boldsymbol{u}_h,\boldsymbol{\varphi}_h)+(\nabla\cdot\boldsymbol{\varphi}_h,p_h)=0, & \forall\,\boldsymbol{\varphi}_h\in\boldsymbol{V}_h.\end{cases}\tag{3}$$

In this paper, we restrict our analysis to the case of the lowest order rectangular element. Let T_h be a partition of Ω into rectangles and be quasi uniform in the usual sense. For each $e=e_{ij}\in T_h$, h_i and k_j are the lengths of the edges of the rectangle e_{ij} in x- and y-directions, respectively. Set $h=\max\limits_{i,j}\{h_i,k_j\}$. Associated with T_h, define W_h to be the finite dimensional space consisting of piecewise-constant functions. To define $\boldsymbol{V}_h$, let us denote by $Q_{i,j}(e)$ the space of polynomials of degree less than or equal to i in the first variable and to j in the second variable restricted to $e\in T_h$. Define $\boldsymbol{V}_h\subset\boldsymbol{H}^1(\Omega)$ such that its restriction to $e\in T_h$ is $Q_{1,0}(e)\times Q_{0,1}(e)$.

Let $Q_h:\ L^2(\Omega)\to W_h$ denote the L^2 projection and Π_h: $\boldsymbol{H}^1(\Omega)\to\boldsymbol{V}_h$ be defined such that

$$(\nabla\cdot(\boldsymbol{v}-\Pi_h\boldsymbol{v}),\boldsymbol{\varphi}_h)=0,\quad\forall\boldsymbol{\varphi}_h\in\boldsymbol{V}_h.\tag{4}$$

As we know, such a projection can be constructed locally as follows. For each $e\in T_h$, and $\boldsymbol{v}\in\boldsymbol{H}^1(\Omega)$

$$\int_l(\boldsymbol{v}-\Pi_h\boldsymbol{v})\cdot\boldsymbol{n}_l ds=0,\qquad\text{for every side } l \text{ of } e\in T_h,\tag{5}$$

where $\boldsymbol{n}_l$ is the outer normal of side l.

2 Error expansion for the velocity

In this section, we prove an error asymptotic expansion for the velocity.

Theorem 1 *Let $(p, \boldsymbol{u})$ and $(p_h, \boldsymbol{u}_h)$ be the solution of (2) and (3), respectively. If mesh T_h is uniform and $\boldsymbol{u} \in H^4(\Omega)$, there holds the following error asymptotic expansion:*

$$\boldsymbol{u}_h - \mathbf{\Pi}_h \boldsymbol{u} = h_x^2 \mathbf{\Pi}_h \boldsymbol{w}_1 + h_y^2 \mathbf{\Pi}_h \boldsymbol{w}_2 + \boldsymbol{r}, \tag{6}$$

where $\boldsymbol{w}_1, \boldsymbol{w}_2$ are functions independent of h and the remainder $\boldsymbol{r}$ satisfies

$$\| \boldsymbol{r} \| \leq ch^{3+1/2} \| u \|_{\boldsymbol{H}^4(\Omega)} .$$

To prove this theorem we need the following lemmas.

Lemma 1 *For $\boldsymbol{u} \in \boldsymbol{H}$, we have $\mathbf{\Pi}_h \boldsymbol{u} = ((\mathbf{\Pi}_h \boldsymbol{u})_1, (\mathbf{\Pi}_h \boldsymbol{u})_2)$ with*

$$\begin{aligned}(\mathbf{\Pi}_h \boldsymbol{u})_1 &= \frac{1}{2k_j} \int_{y_{j-1}}^{y_j} (u_1(x_{i-1}, y) + u_1(x_i, y))\, dy \\ &+ \frac{x - x_{i-1/2}}{h_i k_j} \int_{e_{ij}} \partial_x u_1(x, y) dx dy,\end{aligned}$$

$$\begin{aligned}(\mathbf{\Pi}_h \boldsymbol{u})_2 &= \frac{1}{2h_i} \int_{x_{i-1}}^{x_i} (u_2(x, y_{j-1}) + u_2(x, y_j))\, dy \\ &+ \frac{y - y_{j-1/2}}{h_i k_j} \int_{e_{ij}} \partial_y u_2(x, y) dx dy.\end{aligned}$$

Lemma 2 *For $e_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ and $g \in W^{4,1}(e_{ij})$, there holds*

$$\begin{aligned}&\frac{1}{2k_j} \int_{y_{j-1}}^{y_j} (g(x_{i-1}, y) + g(x_i, y))\, dy \\ &= g(x_{i-1/2}, y_{j-1/2}) + \frac{h_i^2}{8} \partial_{x^2} g(x_{i-1/2}, y_{j-1/2}) \\ &+ \frac{k_j^2}{24} \partial_{y^2} g(x_{i-1/2}, y_{j-1/2}) + O(h^2) \int_{e_{ij}} \left| \nabla^4 g \right| dx dy,\end{aligned} \tag{7}$$

and

$$\begin{aligned}&\frac{1}{2h_i} \int_{x_{i-1}}^{x_i} (g(x, y_{i-1}) + g(x, y_i))\, dx \\ &= g(x_{i-1/2}, y_{j-1/2}) + \frac{h_j^2}{8} \partial_{y^2} g(x_{i-1/2}, y_{j-1/2}) \\ &+ \frac{h_i^2}{24} \partial_{x^2} g(x_{i-1/2}, y_{j-1/2}) + O(h^2) \int_{e_{ij}} \left| \nabla^4 g \right| dx dy.\end{aligned} \tag{8}$$

Proof : Let g be a polynomial of degree less than or equal to 3. By Taylor's formula, we have

$$\begin{aligned}\int_{-1}^{1} g(-1,y)dy &= \int_{-1}^{1}\left(g(-1,0)+\partial_y g(-1,0)y+\frac{1}{2}\partial_{y^2} g(-1,0)y^2\right)dy \\ &\quad +\frac{1}{6}\int_{-1}^{1}\partial_{y^3} g(-1,0)y^3 dy \\ &= 2g(-1,0)+\frac{1}{3}\partial_{y^2} g(-1,0),\end{aligned}$$

and, similarly,

$$\int_{-1}^{1} g(1,y)dy = 2g(1,0)+\frac{1}{3}\partial_{y^2} g(1,0).$$

Further, in virtue of

$$g(-1,0)=g(0,0)-\partial_x g(0,0)+\frac{1}{2}\partial_{x^2} g(0,0)-\frac{1}{6}\partial_{x^3} g(0,0),$$

$$g(1,0)=g(0,0)+\partial_x g(0,0)+\frac{1}{2}\partial_{x^2} g(0,0)+\frac{1}{6}\partial_{x^3} g(0,0),$$

$$\partial_{y^2} g(-1,0)=\partial_{y^2} g(0,0)-\partial_{xy^2} g(0,0),$$

and

$$\partial_{y^2} g(1,0)=\partial_{y^2} g(0,0)+\partial_{xy^2} g(0,0),$$

it follows that

$$\int_{-1}^{1}\left(g(-1,y)+g(1,y)\right)dy = 4g(0,0)+2\partial_{x^2} g(0,0)+\frac{2}{3}\partial_{y^2} g(0,0).$$

Thus, using Bramble-Hilbert lemma, we complete the proof of (7). The proof of (8) is completely similar. #

Lemma 3 *There holds*
1) *for* $g \in W^{3,1}(e_{ij})$,

$$\begin{aligned}&\frac{1}{h_i k_j}\int_{e_{ij}} g(x,y)dxdy - g(x_{i-1/2},y_{j-1/2}) \\ &= \frac{h_i}{24k_j}\int_{e_{ij}}\partial_{x^2} g(x,y)dxdy+\frac{k_j}{24h_i}\int_{e_{ij}}\partial_{y^2} g(x,y)dxdy \\ &\quad +O\left(h\int_{e_{ij}}\left|\nabla^3 g\right|dxdy\right).\end{aligned}$$

2) *for* $g \in W^{4,1}(e_{ij})$,

$$\begin{aligned}
&\frac{1}{h_i k_j}\int_{e_{ij}} g(x,y)dxdy - g(x_{i-1/2}, y_{j-1/2}) \\
&= \frac{h_i^2}{24}\partial_{x^2} g(x_{i-1/2}, y_{j-1/2}) + \frac{k_j^2}{24}\partial_{y^2} g(x_{i-1/2}, y_{j-1/2}) \\
&+O\left(h^2\int_{e_{ij}}\left|\nabla^4 g\right|dxdy\right).
\end{aligned}$$

Lemma 4 *For* $\boldsymbol{u} \in W^{4,1}(e_{ij})$ *and* $\boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in V_h$*, there holds*

$$\begin{aligned}
\int_{e_{ij}} \left((\mathbf{\Pi}_h \boldsymbol{u})_1 - u_1\right)\varphi_1 dxdy &= \frac{h_i^2}{12}\int_{e_{ij}} \partial_{x^2} u_1 \varphi_1 dxdy + \frac{h_i^4}{720}\int_{e_{ij}} \partial_{x^3} u_1 \partial_x \varphi_1 dxdy \\
&+O\left(h^4 \int_{e_{ij}} \left|\nabla^4 u_1\right| (|\varphi_1| + h\,|\partial_x \varphi_1|)dxdy\right). \qquad (9)
\end{aligned}$$

$$\begin{aligned}
\int_{e_{ij}} \left((\mathbf{\Pi}_h \boldsymbol{u})_2 - u_2\right)\varphi_2 dxdy &= \frac{k_j^2}{12}\int_{e_{ij}} \partial_{y^2} u_2 \varphi_2 dxdy + \frac{k_j^4}{720}\int_{e_{ij}} \partial_{y^3} u_2 \partial_y \varphi_2 dxdy \\
&+O\left(h^4 \int_{e_{ij}} \left|\nabla^4 u_1\right| (|\varphi_1| + h\,|\partial_y \varphi_1|)dxdy\right). \qquad (10)
\end{aligned}$$

Proof : First of all, we assume that u_1 is a polynomial of degree less than or equal to 3. By using Lemma 1 and Taylor's formula, it follows, that

$$\begin{aligned}
(\mathbf{\Pi}_h \boldsymbol{u})_1 - u_1 &= u_1(x_{i-1/2}, y_{j-1/2}) - u_1(x,y) + \frac{h_i^2}{8}\partial_{x^2} u_1(x_{i-1/2}, y_{j-1/2}) \\
&+\frac{x - x_{i-1/2}}{h_i k_j}\int_{e_{ij}} \partial_x u_1 dxdy + \frac{k_j^2}{24}\partial_{y^2} u_1(x_{i-1/2}, y_{j-1/2}) \\
&= -(y - y_{j-1/2})\partial_y u_1(x_{i-1/2}, y_{j-1/2}) \\
&+\left(\frac{h_i^2}{8} - \frac{(x - x_{i-1/2})^2}{2}\right)\partial_{x^2} u_1(x_{i-1/2}, y_{j-1/2}) \\
&+\left(\frac{k_j^2}{24} - \frac{(y - y_{j-1/2})^2}{2}\right)\partial_{y^2} u_1(x_{i-1/2}, y_{j-1/2}) \\
&-(x - x_{i-1/2})(y - y_{j-1/2})\partial_{xy} u_1(x_{i-1/2}, y_{j-1/2}) \\
&\left(\frac{h_i^2}{24} - \frac{(x - x_{i-1/2})^2}{6}\right)(x - x_{i-1/2})\partial_{x^3} u_1(x_{i-1/2}, y_{j-1/2}) \\
&-\frac{(x - x_{i-1/2})^2 (y - y_{j-1/2})}{2}\partial_{x^2 y} u_1(x_{i-1/2}, y_{j-1/2})
\end{aligned}$$

$$\left(\frac{k_j^2}{24} - \frac{(y-y_{j-1/2})^2}{2}\right)(x - x_{i-1/2})\partial_{xy^2}u_1(x_{i-1/2}, y_{j-1/2})$$
$$-\frac{(y-y_{j-1/2})^3}{6}\partial_{y^3}u_1(x_{i-1/2}, y_{j-1/2}). \tag{11}$$

Integrating (11) over e_{ij} we have

$$\begin{aligned}
&\int_{e_{ij}} ((\mathbf{\Pi}_h \boldsymbol{u})_1 - u_1) \cdot \left(\frac{1}{h_i k_j}\int_{e_{ij}} \varphi_1(\xi,\eta)d\xi d\eta\right) dxdy \\
&= \frac{h_i^2}{12}\int_{e_{ij}} \varphi_1 dxdy \partial_{x^2}u_1(x_{i-1/2}, y_{j-1/2}) \\
&= \frac{h_i^2}{12}\int_{e_{ij}} \partial_{x^2}u_1(x,y)\varphi_1(x,y)dxdy. \qquad (12)
\end{aligned}$$

Multiplying (11) by $x - x_{i-1/2}$, and then integrating over e_{ij}, as well as making use of Lemma 3, lead us to

$$\begin{aligned}
&\int_{e_{ij}} ((\mathbf{\Pi}_h \boldsymbol{u})_1 - u_1)(x - x_{i-1/2})dxdy \\
&= \frac{h_i^3 k_j}{12}\left(\frac{h_i}{24k_j}\int_{e_{ij}} \partial_{x^3}u_1 dxdy + \frac{k_j}{24h_i}\int_{e_{ij}} \partial_{xy^2}u_1 dxdy\right) \\
&\quad -\frac{h_i^5 k_j}{480}\partial_{x^3}u_1(x_{i-1/2}, y_{j-1/2}) - \frac{h_i^3 k_j^3}{288}\partial_{xy^2}u_1(x_{i-1/2}, y_{j-1/2}) \\
&= \frac{h_i^4}{720}\int_{e_{ij}} \partial_{x^3}u_1 dxdy \qquad (13)
\end{aligned}$$

Therefore, since φ_1 can be written as follows

$$\varphi_1 = \frac{1}{h_i k_j}\int_{e_{ij}} \varphi_1 dxdy + \left(x - x_{i-1/2}\right)\partial_x \varphi_1,$$

we have shown the desired expansion when u_1 is a polynomial of degree less than or equal to 3. By applying the standard Bramble-Hilbert lemma, we complete the proof. #

Lemma 5 *For $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$, there holds*

$$\|\boldsymbol{u}_h - \mathbf{\Pi}_h \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \le ch^2 \|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)}, \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1.$$

Proof of Theorem 1 For any $\boldsymbol{g} \in L^2(\Omega)^2$, let $q \in H_0^1(\Omega)$ be the solution of $\Delta q = \nabla \cdot \boldsymbol{g}$ and $\boldsymbol{v} = \boldsymbol{g} - \nabla q$. Then $(q, \boldsymbol{v})$ is the the solution of the problem:

$$\begin{cases} (\boldsymbol{v}, \boldsymbol{\varphi}) - (q, \nabla \cdot \boldsymbol{\varphi}) = (\boldsymbol{g}, \boldsymbol{\varphi}), & \forall \boldsymbol{\varphi} \in \boldsymbol{V}, \\ (\nabla \cdot \boldsymbol{v}, \psi) = 0, & \forall \psi \in W. \end{cases} \tag{14}$$

We have $\| q \|_{H^1(\Omega)} + \| \boldsymbol{v} \| \leq c \| \boldsymbol{g} \|$. Let $(q_h, \boldsymbol{v}_h) \in W_h \times \boldsymbol{V}_h$ be the mixed finite element approximation of pair $(q, \boldsymbol{v})$.

$$\begin{cases} (\boldsymbol{v}_h, \boldsymbol{\varphi}_h) - (q_h, \nabla \cdot \boldsymbol{\varphi}_h) = (\boldsymbol{g}, \boldsymbol{\varphi}_h), & \forall \boldsymbol{\varphi}_h \in \boldsymbol{V}_h, \\ (\nabla \cdot \boldsymbol{v}_h, \psi_h) = 0, & \forall \psi_h \in W_h. \end{cases} \tag{15}$$

Then, by definitions, we have

$$\begin{aligned} (\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{g}) &= (\boldsymbol{v}_h, \boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h) - (q_h, \nabla \cdot (\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h)) \\ &= (\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}, \boldsymbol{v}_h) + (p - p_h, \nabla \cdot \boldsymbol{v}_h) = (\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}, \boldsymbol{v}_h). \end{aligned} \tag{16}$$

Applying Lemma 4, we obtain

$$\begin{aligned} (\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{g}) &= \frac{1}{12} \sum_{i,j} \int_{e_{ij}} \left(h_i^2 \partial_{x^2} u_1 v_{1h} + k_j^2 \partial_{y^2} u_2 v_{2h} \right) dxdy \\ &+ \frac{1}{720} \sum_{i,j} \int_{e_{ij}} \left(h_i^4 \partial_{x^3} u_1 \partial_x v_{1h} + k_j^4 \partial_{y^3} u_2 \partial_y v_{2h} \right) dxdy \\ &+ O(h^4) \| \boldsymbol{u} \|_{\boldsymbol{H}^4(\Omega)} \| \boldsymbol{v}_h \| . \end{aligned} \tag{17}$$

Since T_h is uniform, $h_i = h_x, k_j = h_y$. By integration by parts, we have from (17)

$$\begin{aligned} (\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{g}) &= \frac{1}{12} \int_\Omega \left(h_x^2 \partial_{x^2} u_1 v_{1h} + h_y^2 \partial_{y^2} u_2 v_{2h} \right) dxdy \\ &+ \frac{h_x^4}{720} \left(\int_{L_1} + \int_{L_3} \right) \partial_{x^3} u_1 \boldsymbol{v} \cdot \boldsymbol{n} dy + \frac{h_y^4}{720} \left(\int_{L_2} + \int_{L_4} \right) \partial_{y^3} u_2 \boldsymbol{v} \cdot \boldsymbol{n} dx \\ &+ O(h^4) \| \boldsymbol{u} \|_{\boldsymbol{H}^4(\Omega)} \| \boldsymbol{v}_h \| . \end{aligned} \tag{18}$$

Next, let $\boldsymbol{G}_1 = \frac{1}{12}(\partial_{x^2} u_1, 0)$, $\boldsymbol{G}_2 = \frac{1}{12}(0, \partial_{y^2} u_2)$. Denote by $\theta_i \in H_0^1(\Omega)$ the solution of $\Delta \theta = \nabla \cdot \boldsymbol{G}_i, i = 1, 2$. Let $\boldsymbol{w}_i = \boldsymbol{G}_i - \nabla \theta_i$. Then, for $i = 1, 2$, $(\theta_i, \boldsymbol{w}_i)$ satisfies

$$\begin{cases} (\boldsymbol{w}_i, \boldsymbol{\varphi}) - (\theta_i, \nabla \cdot \boldsymbol{\varphi}) = (\boldsymbol{G}_i, \boldsymbol{\varphi}), & \forall \boldsymbol{\varphi} \in \boldsymbol{V}, \\ (\nabla \cdot \boldsymbol{w}_i, \psi) = 0, & \forall \psi \in W. \end{cases} \tag{19}$$

Let $(\theta_{ih}, \boldsymbol{w}_{ih}) \in W_h \times \boldsymbol{V}_h$ be the mixed finite element approximation of pair $(\theta_i, \boldsymbol{w}_i)$, $i = 1, 2$.

$$\begin{cases} (\boldsymbol{w}_{ih}, \boldsymbol{\varphi}_h) - (\theta_{ih}, \nabla \cdot \boldsymbol{\varphi}_h) = (\boldsymbol{G}_i, \boldsymbol{\varphi}_h), & \forall \boldsymbol{\varphi}_h \in \boldsymbol{V}_h, \\ (\nabla \cdot \boldsymbol{w}_{ih}, \psi_h) = 0, & \forall \psi_h \in W_h. \end{cases} \tag{20}$$

We note that $\theta_i \in W^{3,p}(\Omega)$ for any $p \in (1, 2)$ and

$$\| \boldsymbol{w}_i \|_{\boldsymbol{W}^{2,p}(\Omega)} \leq \| \theta_i \|_{W^{3,p}(\Omega)} \leq c_p \| \nabla \cdot \boldsymbol{G}_i \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq c_p \| \boldsymbol{u} \|_{\boldsymbol{H}^4(\Omega)} . \tag{21}$$

According to Lemma 5, (21) and the inverse property, there holds for any $p \in (1,2)$ that

$$\begin{aligned}
&\|\Pi_h \boldsymbol{w}_1 - \boldsymbol{w}_{1h}\| + \|\Pi_h \boldsymbol{w}_2 - \boldsymbol{w}_{2h}\| \\
&\leq ch^{1-2/p}\left(\|\Pi_h \boldsymbol{w}_1 - \boldsymbol{w}_{1h}\|_{\boldsymbol{L}^p(\Omega)} + \|\Pi_h \boldsymbol{w}_2 - \boldsymbol{w}_{2h}\|_{\boldsymbol{L}^p(\Omega)}\right) \\
&\leq ch^{3-2/p}\left(\|\boldsymbol{w}_1\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\boldsymbol{w}_2\|_{\boldsymbol{W}^{2,p}(\Omega)}\right) \\
&\leq ch^{3-2/p}\|\boldsymbol{u}\|_{\boldsymbol{H}^4(\Omega)}. \qquad (22)
\end{aligned}$$

Further, applying the trace theorems, we have

$$\begin{aligned}
&\left|\left(\int_{L_1} + \int_{L_3}\right)\partial_{x^3} u_1 \boldsymbol{v}\cdot \boldsymbol{n}dy\right| + \left|\left(\int_{L_2} + \int_{L_4}\right)\partial_{y^3} u_2 \boldsymbol{v}\cdot \boldsymbol{n}dx\right| \\
&\leq ch^{-1/2}\|\boldsymbol{u}\|_{\boldsymbol{H}^4(\Omega)}\|\boldsymbol{v}_h\|. \qquad (23)
\end{aligned}$$

Thus, from (14), (15), (19), (20) and (21) and Lemma 5,

$$\begin{aligned}
&h_x^2(G_1, \boldsymbol{v}_h) + h_y^2(\boldsymbol{G}_2, \boldsymbol{v}_h) \\
&= h_x^2(\boldsymbol{g}, \boldsymbol{w}_{1h}) + h_y^2(\boldsymbol{g}, \boldsymbol{w}_{2h}) \\
&= h_x^2(\boldsymbol{g}, \Pi_h \boldsymbol{w}_1) + h_y^2(\boldsymbol{g}, \Pi_h \boldsymbol{w}_2) + O\left(h^4\|\boldsymbol{u}\|_{\boldsymbol{H}^4(\Omega)}\right)\|\boldsymbol{g}\| \qquad (24)
\end{aligned}$$

the expansion (18) now can be written as

$$\begin{aligned}
(\Pi_h \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{g}) &= h_x^2(\boldsymbol{g}, \Pi_h \boldsymbol{w}_1) + h_y^2(\boldsymbol{g}, \Pi_h \boldsymbol{w}_2) \\
&\quad + O(h^{3+1/2})\|\boldsymbol{u}\|_{\boldsymbol{H}^4(\Omega)}\|\boldsymbol{v}_h\|. \qquad (25)
\end{aligned}$$

Finally, by letting $\varphi_h = \boldsymbol{v}_h, \psi_h = q_h$ in (15), we have $(\boldsymbol{v}_h, \boldsymbol{v}_h) = (\boldsymbol{g}, \boldsymbol{v}_h)$ which implies $\|\boldsymbol{v}_h\| \leq \|\boldsymbol{g}\|$. Hence, (25) has completed the proof.

3 Neumann boundary value problems

In this section, we consider the Neumann boundary value problem. Namely, we change the boundary condition $p = 0$ on $\partial\Omega$ of (1) into

$$\nabla p \cdot \boldsymbol{n} = 0 \quad \text{or} \quad \boldsymbol{u}\cdot\boldsymbol{n} = 0. \qquad (26)$$

Since this problem is not uniquely solvable, we consider the following problem:

$$\begin{cases} -\Delta p + p = f & \text{in } \Omega \\ \nabla p \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega, \end{cases} \qquad (27)$$

Let $\boldsymbol{V}^0$ be the subspace of $\boldsymbol{V}$ consisting of functions $\boldsymbol{v}$ for which $\boldsymbol{v} \cdot \boldsymbol{n}$ equals zero on $\partial\Omega$ and $\boldsymbol{V}_h^0 = \boldsymbol{V}_h \cap \boldsymbol{V}$. Let $\boldsymbol{u} = -\nabla p$. The mixed variational form of (27) is the following: Find $(p, \boldsymbol{u}) \in W \times \boldsymbol{V}^0$ such that

$$\begin{cases} (\nabla \cdot \boldsymbol{u}, \psi) + (p, \psi) = (f, \psi), & \forall\, \psi \in W, \\ (\boldsymbol{u}, \boldsymbol{\varphi}) + (\nabla \cdot \boldsymbol{\varphi}, p) = 0, & \forall\, \boldsymbol{\varphi} \in \boldsymbol{V}_0, \end{cases} \tag{28}$$

The mixed finite element approximation $(p_h, \boldsymbol{u}_h) \in W_h \times \boldsymbol{V}_h^0$ to pair $(p, \boldsymbol{u}) \in W \times \boldsymbol{V}^0$ is the solution of the following problem:

$$\begin{cases} (\nabla \cdot \boldsymbol{u}_h, \psi_h) + (p_h, \psi_h) = (f, \psi_h), & \forall\, \psi_h \in W_h, \\ (\boldsymbol{u}_h, \boldsymbol{\varphi}_h) + (\nabla \cdot \boldsymbol{\varphi}_h, p_h) = 0, & \forall\, \boldsymbol{\varphi}_h \in \boldsymbol{V}_h^0. \end{cases} \tag{29}$$

The main result of this section is as follows:

Theorem 2 *Let $(p, \boldsymbol{u})$ and $(p_h, \boldsymbol{u}_h)$ be the solution of (28) and (29), respectively. If T_h is uniform and $\boldsymbol{u} \in H^4(\Omega)$, there holds the following error asymptotic expansion:*

$$\boldsymbol{u}_h - \mathbf{\Pi}_h \boldsymbol{u} = h_x^2 \mathbf{\Pi}_h \boldsymbol{w}_1 + h_y^2 \mathbf{\Pi}_h \boldsymbol{w}_2 + \boldsymbol{r}, \tag{30}$$

where $\boldsymbol{w}_1, \boldsymbol{w}_2$ are functions independent of h and the remainder $\boldsymbol{r}$ satisfies

$$\| \boldsymbol{r} \| \leq c h^{4-\varepsilon} \| u \|_{\boldsymbol{H}^4(\Omega)},$$

for any fixed $\varepsilon > 0$.

Proof : For any $\boldsymbol{g} \in C_0^\infty(\Omega)^2$, let $q \in H_0^1(\Omega)$ be the solution of $-\Delta q + q = -\nabla \cdot \boldsymbol{g}$ and $\boldsymbol{v} = \boldsymbol{g} - \nabla q$. As in the proof of Theorem 1, let $(q_h, \boldsymbol{v}_h) \in W_h \times \boldsymbol{V}_h^0$ be the mixed finite element approximation of $(q, \boldsymbol{v})$. Then, we have

$$(\mathbf{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{g}) = (\mathbf{\Pi}_h \boldsymbol{u} - \boldsymbol{u}, \boldsymbol{v}_h). \tag{31}$$

Thus, in virtue of $\boldsymbol{v}_h \cdot \boldsymbol{n} = 0$, (18) shows that

$$\begin{aligned} (\mathbf{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{g}) &= \frac{1}{12} \int_\Omega \left(h_x^2 \partial_{x^2} u_1 v_{1h} + h_y^2 \partial_{y^2} u_2 v_{2h} \right) dx dy \\ &\quad + O(h^4) \| \boldsymbol{u} \|_{\boldsymbol{H}^4(\Omega)} \| \boldsymbol{v}_h \| . \end{aligned} \tag{32}$$

Hence, according to the proof of Theorem 1, we can get the desired expansion (30) easily.

4 Richardson extrapolation

As applications of the error expansions obtained in the previous sections, we employ a Richardson extrapolation scheme to get higher accuracy approximation based on the lowest rectangular element. Let Z_h be the set consisting of centers of all elements in T_h. Let $T_{h/3}$ be the division of Ω obtained by dividing each element of T_h into 9 equal rectangles. Define the semi-norm

$$|||w|||=\left(\sum_{e_{ij}\in T_h} h_x h_y w^2(x_{i-1/2},y_{j-1/2})\right)^{1/2}, \quad |||\boldsymbol{v}|||=|||v_1|||+|||v_2|||.$$

Theorem 3 *Let $(p,\boldsymbol{u})$ and $(p_h,\boldsymbol{u}_h)$ be the solution of* (2) *and* (3), *respectively. If T_h is uniform and $\boldsymbol{u}\in H^4(\Omega)$, then*

$$|||\boldsymbol{u}-\frac{9\boldsymbol{u}_{h/3}-\boldsymbol{u}_h}{8}|||\le ch^{3+1/2}\,\|\boldsymbol{u}\|_{\boldsymbol{H}^4(\Omega)}. \tag{33}$$

Proof : Applying Theorem 1, we have

$$\begin{aligned} &|||9(\boldsymbol{u}_{h/3}-\boldsymbol{\Pi}_{h/3}\boldsymbol{u})-(\boldsymbol{u}_h-\boldsymbol{\Pi}_h\boldsymbol{u})|||\\ =\ &|||h_x^2(\boldsymbol{\Pi}_{h/3}\boldsymbol{w}_1-\boldsymbol{\Pi}_h\boldsymbol{w}_1)+h_y^2(\boldsymbol{\Pi}_{h/3}\boldsymbol{w}_2-\boldsymbol{\Pi}_h\boldsymbol{w}_2)+9\boldsymbol{r}_{h/3}-r_h|||\\ \le\ &h^{3+1/2}\,\|\boldsymbol{u}\|_{\boldsymbol{H}^4(\Omega)}. \end{aligned} \tag{34}$$

Theorem 4 *Let $(p,\boldsymbol{u})$ and $(p_h,\boldsymbol{u}_h)$ be the solution of* (28) *and* (29), *respectively. If T_h is uniform and $\boldsymbol{u}\in H^4(\Omega)$, then*

$$|||\boldsymbol{u}-\frac{9\boldsymbol{u}_{h/3}-\boldsymbol{u}_h}{8}|||\le ch^{4-\varepsilon}\,\|\boldsymbol{u}\|_{\boldsymbol{H}^4(\Omega)}. \tag{35}$$

This error estimate may have various applications ranging from using it as criteria for grid refinement to direct more accurate computations of the velocities especially in ground-water applications. This will be a subject of our further study.

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