PRECONDITIONING TECHNIQUES FOR MIXED AND NONCONFORMING FINITE ELEMENT METHODS*

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1. Introduction. The general goal of this presentation is preconditioning techniques for mixed and nonconforming finite element approximations of elliptic boundary value problems. A special emphasis is placed on problems in three dimensions with possibly large anisotropy in the coefficients of the PDE's along with large jumps in the coefficients across the interfaces separating subregions.

The optimal preconditioners developed exploit the techniques of domain decomposition methods, algebraic substructuring, and multigrid methods. As a result, the proposed iterative processes converge with rates independent of the mesh size, the jumps of the coefficients, and the ratio of anisotropy.

Using an equivalence between nonconforming finite element methods and hybrid-mixed methods [1] the iterative methods constructed for algebraic systems with symmetric positive definite matrices are extended to saddle-point problems which arise from mixed finite element approximations [13].

Let Ω be a convex bounded domain in $\mathbb{R}^d$, $d = 2, 3$, with boundary $\partial\Omega$. Consider an elliptic problem

$$\begin{aligned} -\mathrm{div}\,(K \cdot \nabla u) &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \Gamma_0, \\ (K\nabla u, \mathbf{n}) &= 0 \qquad \text{on } \Gamma_1, \end{aligned} \tag{1.1}$$

where $K(\mathbf{x})$ is a positive definite, uniformly bounded symmetric tensor, $f(\mathbf{x}) \in L^2(\Omega)$, $\overline{\Gamma_0 \cup \Gamma_1} = \partial\Omega$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. We shall consider the case when $\Gamma_0 \equiv \overline{\Gamma_0} \neq \emptyset$. The pure Neumann problem ($\Gamma_0 = \emptyset$) can be treated in a similar way but for the sake of simplicity is not described here.

Let $\mathcal{T}_h$ be a regular partitioning of Ω into simplices τ with mesh-size h and let $V_h(\Omega)$ be the P_1–nonconforming finite element space of functions $v \in L^2(\Omega)$ [1] such that $v|_\tau$ are linear for all $\tau \in \mathcal{T}_h$, v are continuous at the barycenters of $\tau \in \mathcal{T}_h$ and vanish at the barycenters of the boundary faces on Γ_0. Note that the space $V_h(\Omega)$ is not a subspace of $H^1(\Omega)$.

Define the bilinear form on $V_h(\Omega)$ by

$$a^h_\Omega(u, v) = \sum_{\tau \in \mathcal{T}_h} (K\nabla u, \nabla v)_\tau, \qquad \forall\ u, v \in V_h(\Omega), \tag{1.2}$$

where $(\cdot, \cdot)_\tau$ is the inner product in $L^2(\tau)$, $\tau \in \mathcal{T}_h$. Then the P_1–nonconforming finite element discretization of (1.1) has the form: *find* $u_h \in V_h(\Omega)$ *such that*

$$a^h_\Omega(u_h, v) = (f, v), \qquad \forall v \in V_h(\Omega). \tag{1.3}$$

Once a nodal basis $\{\varphi_i(\mathbf{x})\}_{i=1}^N$ for $V_h(\Omega)$ is chosen, (1.3) leads to a system of linear algebraic equations:

$$A\mathbf{u} = \mathbf{f}, \tag{1.4}$$

where $A_{ji} = a^h_\Omega(\varphi_i, \varphi_j)$, $f_j = (f, \varphi_j)$, $i, j = 1, \ldots, N$.

Although the methods of solving (1.4) have been extensively studied in the past few years (see, e.g., [1, 5, 9, 10, 14]), their efficiency depends on the coefficient matrix $K(\mathbf{x})$, and in the case of strong anisotropy in the coefficients the question of constructing effective solution techniques is still open.

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In this work we propose several preconditioners for P_1-nonconforming finite element approximations of anisotropic problems using substructuring ideas [20, 16] and domain decomposition methods [4, 8, 29].

The substructuring approach makes it possible to construct very efficient preconditioning techniques for problems with a high anisotropic ratio in the coefficients in domains of simple form such as a topological parallelepiped [13, 16, 22].

To construct the iterative methods for solving the anisotropic problems approximated by the nonconforming P_1 elements in domains of complex geometric shape we consider numerical methods which involve a solution of analogous problems in domains of relatively simple form. We note that an important difference between nonconforming an conforming cases is that there are no nodes at the vertices (or wire basket) of the subregions. Thus, there is no problem of crosspoints which is an essential part of the domain decomposition method when using conforming finite elements [7, 15, 27, 29]. It makes the construction of the preconditioner for the Schur complement on the interfaces between subdomains very clear and easy. An iterative process whose convergence rate is independent of the mesh step size and the ratio of anisotropy in the coefficients is constructed [24, 26]. It is shown that the number of arithmetic operations required for realization of the method with a given accuracy is proportional to the number of unknowns of the original algebraic problem.

A new construction of iterative methods for mixed finite element approximations of elliptic PDE's on nonmatching grids is considered. The computational domain is considered as a union of nonintersecting subdomains. In each subdomain the grid is constructed in accordance with its own coordinate system (for example, using the main directions of anisotropy) [25]. The original elliptic problem is posed as a problem with Lagrange multipliers at the interface between the subdomains, which ensure the continuity conditions of the solution (in a weak sense) [21, 23]. A mortar finite element subspace is constructed in the space of Lagrange multipliers, which results in algebraic systems of a saddle-point type.

Based on the technique of domain decomposition and fictitious components methods a construction of block diagonal preconditioners for the algebraic systems arising in the mortar finite element method is developed [25, 26]. The fictitious components method is used to precondition subdomain problems, while the interface problems are preconditioned by an inner Chebyshev iterative procedure. It is shown that the developed preconditioner is spectrally equivalent to the original saddle-point matrix.

Applications of the newly developed iterative methods and preconditioning techniques are considered. In particularly, these methods are applied in the simulator of fluid flow in porous media [12].

2. Substructuring preconditioner for nonconforming approximations of elliptic problems with anisotropy.

2.1. Description of the approach. In this part of the talk we consider a method of constructing the preconditioner for (1.4) using an idea of algebraic substructuring which can be described as follows [22, 26].

Let us partition the domain Ω into subdomains Ω_s, $s = 1, \ldots, n$, such that each Ω_s is a union of simplices $\tau \in \mathcal{T}_h$,

$$\Omega = \bigcup_{s=1}^{n} \Omega_s, \qquad \Omega_s = \bigcup_{l=1}^{n_s} \{\tau_l \in \mathcal{T}_h : \tau_l \subset \Omega_s\}.$$

Below these subdomains Ω_s are called superelements.

Let us introduce local stiffness matrices A_s on each superelement Ω_s as follows:

$$(A_s \mathbf{u}_s, \mathbf{v}_s) = \sum_{\tau_l \subset \Omega_s} (K(\mathbf{x}) \nabla u_h, \nabla v_h)_{\tau_l}, \qquad \forall u_h, v_h \in V_h(\Omega_s).$$

All these matrices are at least positive semidefinite, and the global stiffness matrix is determined

by assembling the local stiffness matrices over all the superelements:

$$(A\mathbf{u},\mathbf{v}) = \sum_{s=1}^{n} (A_s \mathbf{u}_s, \mathbf{v}_s), \qquad \forall \mathbf{u},\mathbf{v} \in \mathbb{R}^N.$$

We can symbolically write $A = \{A_s\}_{s=1}^n$, where $\{\cdot\}_{s=1}^n$ denotes assembling with respect to the partitioning $\{\Omega_s\}_{s=1}^n$ of Ω.

In the above notation each superelement matrix A_s can be represented in terms of local stiffness matrices over simplices τ_l from Ω_s, i.e. $A_s = \{A_{sl}\}_{\tau_l \subset \Omega_s}$. Note that matrices A_{sl} are also at least positive semidefinite.

Following [20, 22], let us introduce on each simplex $\tau \in \mathcal{T}_T$ another matrix $\hat{A}_{sl}$ which has the same kernel as A_{sl} (i.e. Ker A_{sl} = Ker $\hat{A}_{sl}$). Define the matrix $\hat{A}_s$ on each superelement Ω_s by assembling $\hat{A}_{sl}$:

$$\hat{A}_s = \left\{ \hat{A}_{sl} \right\}_{\tau_l \subset \Omega_s}.$$

Then it can easily be shown that Ker A_s = Ker $\hat{A}_s$ and the matrices $\hat{A}_s$ are also at least positive semidefinite.

Now let us define an $N \times N$ matrix $\hat{A}$ by assembling $\hat{A}_s$ over all the superelements

$$\hat{A} = \left\{ \hat{A}_s \right\}_{s=1}^{n}.$$

It can be easily shown that to estimate the extreme eigenvalues of $\hat{A}^{-1}A$ it is sufficient to consider the local problems

$$A_s \mathbf{u}_s = \mu^{(s)} \hat{A}_s \mathbf{u}_s, \qquad \mathbf{u}_s \perp \text{Ker } \hat{A}_s,$$

on all the superelements Ω_s, $s = 1, \ldots, n$.

Thus, if the superelement matrices A_s and $\hat{A}_s$ are spectrally equivalent with respect to Ker A_s, i.e. there exist constants $c_{0,s}$ and $c_{1,s}$ such that

$$c_{0,s}(\hat{A}_s \mathbf{u}_s, \mathbf{u}_s) \le (A_s \mathbf{u}_s, \mathbf{u}_s) \le c_{1,s}(\hat{A}_s \mathbf{u}_s, \mathbf{u}_s), \qquad \forall \mathbf{u}_s \in \mathbb{R}^{N_s}, \qquad N_s = \dim \Omega_s,$$

where constants $c_{0,s}$, $c_{1,s}$ do not depend on mesh-size parameter h, then matrices $\hat{A}$ and A are also spectrally equivalent, i.e.

$$c_0(\hat{A}\mathbf{u},\mathbf{u}) \le (A\mathbf{u},\mathbf{u}) \le c_1(\hat{A}\mathbf{u},\mathbf{u}), \qquad \forall \mathbf{u} \in \mathbb{R}^N,$$

with $c_0 = \min\limits_s c_{0,s}$ and $c_1 = \max\limits_s c_{1,s}$.

Now let us partition all the unknowns in (1.4) into two groups:

$$\mathbf{u} = (\mathbf{u}_1^T, \mathbf{u}_2^T)^T, \qquad \dim \mathbf{u}_1 = N_1, \qquad \dim \mathbf{u}_2 = N - N_1,$$

so that matrix $\hat{A}$ is represented in a block form:

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \tag{2.1}$$

such that block $\hat{A}_{22}$ is easily invertible. Then introducing the Schur complement $S = \hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21}$, we can rewrite matrix $\hat{A}$ as

$$\hat{A} = \begin{bmatrix} S + \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}. \tag{2.2}$$

Following [6, 19], we construct a matrix $\tilde{S}$ which is spectrally equivalent to S, i.e.

$$d_0(\tilde{S}\mathbf{v},\mathbf{v}) \le (S\mathbf{v},\mathbf{v}) \le d_1(\tilde{S}\mathbf{v},\mathbf{v}), \qquad \forall \mathbf{v} \in \mathbb{R}^{N_1},$$

where constants $0 < d_0 \le d_1$ are independent of mesh-size parameter h. Then the matrix

$$B = \begin{bmatrix} \tilde{S} + \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \tag{2.3}$$

is spectrally equivalent to matrix A, i.e.

$$r_0(B\mathbf{u},\mathbf{u}) \le (A\mathbf{u},\mathbf{u}) \le r_1(B\mathbf{u},\mathbf{u}), \qquad \forall \mathbf{u} \in \mathbb{R}^{N},$$

where $r_0 = c_0 \min\{1; d_0\}$, $r_1 = c_1 \max\{1; d_1\}$. To construct such a matrix $\tilde{S}$, again, we can use the idea of the algebraic substructuring described above.

Concluding this overview, we can say that the algebraic substructuring procedure consists of the following main steps:

(A) the reconstruction of the directed graph of matrix A from (1.4) in such a way that the resulting matrix $\hat{A}$ has the same kernel and is still positive definite (or positive semidefinite if matrix A is singular);

(B) the representation of matrix $\hat{A}$ in 2×2 block form (2.1) in such a way that one of the blocks, $\hat{A}_{11}$ or $\hat{A}_{22}$, is easily invertible;

(C) the replacement of the Schur complement S in (2.2) by a spectrally equivalent matrix $\tilde{S}$; we can use steps (A) and (B) to construct such a matrix $\tilde{S}$.

Note that we can first represent matrix A in 2×2 block form (2.2) and then use steps (A)–(C) to construct a preconditioner for the Schur complement $S = A_{11} - A_{12}A_{22}^{-1}A_{21}$. Implementing a finite number of these steps, we can get matrix B which is spectrally equivalent to the given matrix A.

Because of the algebraic nature of such a procedure this approach strongly depends on the structure of the graph of matrix A and consequently on the type of the nonconforming finite element space V_h.

2.2. Model three-dimensional problem. To explain this approach we consider the model problem when Ω is a unit cube in $\mathbb{R}^3$, Γ_0 is a union of some faces of Ω, the boundary conditions are homogeneous, and $K(\mathbf{x})$ satisfies the following assumption.

ASSUMPTION 2.1. *Assume that the coefficient matrix of equation* (1.1) *is a diagonal tensor* $K(\mathbf{x}) = diag\{k_1, k_2, k_3\}$, *where* k_i, $i = 1, 2, 3$, *are constants over the cube* Ω *such that a coefficient* k_* *in some direction is not less then the coefficients in the other directions. For the sake of definiteness we assume that this is the "z-direction". That is* $\kappa = \min\{k_3/k_1, k_3/k_2\} \ge 1$.

The extension of the method to the case in which Ω is a union of topological parallelepipeds and the tensor $K(\mathbf{x})$ is a full symmetric matrix satisfying some requirements can be found in [22, 24, 25, 26].

Let $\mathcal{C}_h = \{C^{(i,j,k)}\}$ be a partition of Ω into uniform cubes with edge length $h = 1/n$; here (x_i, y_j, z_k) is the right back upper corner of cube $C^{(i,j,k)}$. Next, we divide each cube $C^{(i,j,k)}$ into 5 tetrahedra as shown in Figure 2.1. We denote this partitioning of Ω into tetrahedra by $\mathcal{T}_h$. Note that we have two types of partitioning of cubes $C^{(i,j,k)}$ into tetrahedra, the cube with one type of partitioning having all the adjacent cubes of another type.

We introduce the set of barycenters of all the faces of the tetrahedral partition of Ω and the set Q_h of those barycenters that do not belong to Γ_0. The Crouzeix-Raviart P_1–nonconforming finite element space V_h is defined by

$$V_h = \Big\{ v \in L^2(\Omega) : \quad v|_T \in P_1(T),\ \forall T \in \mathcal{T}_h;\ v \text{ is continuous at the barycenters from } Q_h \text{ and vanishes at the barycenters of faces on } \Gamma_0 \Big\}. \tag{2.4}$$

Let its dimension be N. Note that $N \approx 10n^3$.

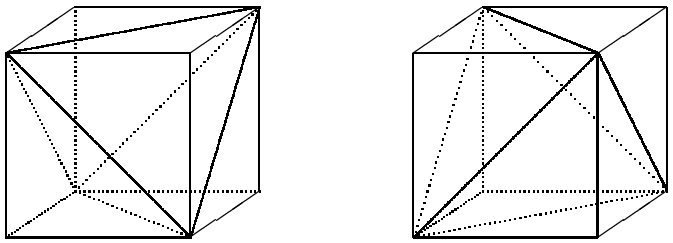


FIG. 2.1. *Partition of cubes $C^{(i,j,k)}$ into 5 tetrahedra.*

2.3. Algebraic substructuring preconditioner. Here we outline the construction of the algebraic substructuring preconditioner. We divide all the unknowns in the system into two groups:

1. The first group consists of the unknowns corresponding to the faces of the tetrahedra that are internal for each cube.
2. The second group consists of all the unknowns corresponding to the faces of the cubes in partition $\mathcal{C}_h$, without the faces on Γ_0.

Splitting the space $\mathbb{R}^N$ into two groups induces a vector presentation: $\mathbf{v}^T = (\mathbf{v}_1^T, \mathbf{v}_2^T)$, where $\mathbf{v}_1 \in \mathbb{R}^{N_1}$ and $\mathbf{v}_2 \in \mathbb{R}^{N_2}$; here $\mathbf{v}_2$ corresponds to the unknowns of the 2-nd group. Obviously, $N_1 = 4n^3$ and $N_2 = N - 4n^3$. Then stiffness matrix A is represented in the following block form:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{2.5}$$

where $A_{11} : \mathbb{R}^{N_1} \to \mathbb{R}^{N_1}$ corresponds to the unknowns on the faces of tetrahedra which are internal for each cube and matrix A_{22} is diagonal. It can be easily verified that A_{11} has the form:

$$A_{11} = k_1 A_x + k_2 A_y + k_3 A_z,$$

where the matrices A_x, A_y, and A_z do not depend on the coefficients of the problem (1.1).

Along with matrix A we introduce the matrix B

$$B = A + \frac{3h}{2} \begin{bmatrix} \tilde{B}_{11} & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.6}$$

where $\tilde{B}_{11} = (k_1 + k_2)A_z$. Thus, matrix B can be represented in the form:

$$B = \frac{3h}{2} \begin{bmatrix} B_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $B_{11} = k_1(A_x + A_z) + k_2(A_y + A_z) + k_3 A_z$.

Using superelement analysis it is easy to verify that the eigenvalues μ of the problem

$$A\mathbf{u} = \mu B\mathbf{u} \tag{2.7}$$

belong to the interval $[1/3, 1]$ provided Assumption 2.1.

Let us denote by $\hat{B}_{11} = B_{11} - A_{12}A_{22}^{-1}A_{21}$ the Schur complement of B obtained by elimination of the vector $\mathbf{v}_2$. Then $B_{11} = \hat{B}_{11} + A_{12}A_{22}^{-1}A_{21}$ and hence matrix B has the form:

$$B = \begin{bmatrix} \hat{B}_{11} + A_{12}A_{22}^{-1}A_{21} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{2.8}$$

Now we need to develop a preconditioner for matrix $\hat{B}_{11}$. Again, using algebraic substructuring we can construct a sparse separable matrix $\tilde{B}_{11}$ spectrally equivalent to $\hat{B}_{11}$ so that the resulting matrix

$$\tilde{B} = \begin{bmatrix} \tilde{B}_{11} + A_{12}A_{22}^{-1}A_{21} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{2.9}$$

is spectrally equivalent to the initial matrix A. In this case we shall use the method of separation of variables and discrete fast Fourier algorithm in order to solve the system of linear equations with matrix $\tilde{B}_{11}$.

The use of discrete fast Fourier transform yields a total number of arithmetic operations proportional to $N_1 \ln(N_1)$ or $N \ln(0.4N)$, where the constants of proportionality do not depend on the number of unknowns N and on coefficients k_1, k_2, and k_3.

REMARK 2.1. In the method outlined above, we note that the condition number depends neither on mesh-size h nor on the value of the coefficients when $k_3 \geq \max\{k_1, k_2\}$. Because the condition number of matrix $\tilde{B}^{-1}A$ depends on the value of parameter κ it is very important to choose the "z-direction" in the proper way. If, for example, we have the problem in which coefficient k_1 is greater than coefficients k_2 and k_3, we rearrange the variables so that the new variable z coincides with the old variable x. It means that we simply rename the axes of the coordinate system.

2.4. Results of numerical experiments. The method of preconditioning is tested on the model problem (1.1) with diagonal matrix coefficient $K = \text{diag}\{k_1, k_2, k_3\}$.

The domain is divided into n^3 cubes (n in each direction). Each cube is partitioned into 5 tetrahedra. The dimension of the original algebraic system is $N = 10n^3 - 6n^2$. The problem (1.4) is solved by preconditioned conjugate method (PCG) with accuracy $\varepsilon = 10^{-6}$. Coefficients k_i, $i = 1, 2, 3$, are constants on each cube. The results are summarized in Table 2.1.

TABLE 2.1
Dependency on parameters κ_i.

			$16 \times 16 \times 16$ $N = 39424$		$20 \times 20 \times 20$ $N = 77600$		$30 \times 30 \times 30$ $N = 264600$	
k_1	k_2	k_3	Iter	Cond	Iter	Cond	Iter	Cond
1	1	1	14	4.87	14	4.93	14	5.03
1	1	10	12	3.72	12	3.94	12	4.28
1	1	100	9	2.28	10	2.55	10	3.00
1	1	1000	8	1.55	8	1.58	8	1.73
1	1	10000	8	1.48	8	1.49	8	1.51
1	1	0.1	31	19.4	31	19.6	31	19.8
1	1	0.01	62	133.	71	149.	82	168.
10	1	1	24	12.0	25	12.1	25	12.1
1	10	1	24	12.1	24	12.1	24	12.0
100	1	1	58	99.3	63	100.	62	100.
1	100	1	62	100.	60	100.	60	99.5
1	10	10	14	4.72	14	4.81	14	4.94
1	10	100	12	3.62	12	3.85	12	4.25
1	10	1000	9	2.14	10	2.42	10	2.92
1	100	10000	9	2.20	10	2.42	10	2.92

From Table 2.1 we see that the condition number depends on the maximal ratio

$$\kappa = \max\left\{\frac{k_1}{k_3}, \frac{k_2}{k_3}\right\}.$$

The numerical results are in full agreement with the theoretical estimates. One can see that the proposed preconditioner is optimal if $\kappa \leq 1$. In the case of $\kappa < 1$ the method has a better convergence than in the case of the Poisson equation (i.e. $k_1 = k_2 = k_3 = 1$). If $\kappa > 1$, the preconditioner looses its optimal order and the corresponding relative condition numbers increased strongly with κ. It is a rather predictable result since we defined preconditioning matrix B in (2.6), taking some "additional

positiveness" from the direction with the dominated anisotropy (z-direction) to other directions. Experiments show that this procedure is "well behaved" if the coefficient in the z-direction (k_3) is greater than coefficients k_1 and k_2. And the method loses its effectiveness if we choose the wrong direction, i.e. coefficient k_3 is small compared with coefficients k_1 and k_2.

Remember that in the method described here we need only an assumption that coefficient k_* in some direction is not less then the coefficients in the other directions. Thus, if, for example, we have the problem where coefficient k_1 is not less than coefficients k_2 and k_3 we can simply rename variables in such a way that a new z variable corresponds to the old x variable. The results will be the same.

3. Block bordering method for anisotropic problems. In this section we present a construction of the domain decomposition method for solving systems of grid equations approximating boundary value problems for second-order elliptic problems with anisotropic coefficients. We consider problems for which the computational domain Ω can be represented as a union of nonoverlapping subdomains $\Omega = \bigcup_{i=1}^{m} \Omega_i$ inside which the equation coefficients vary insignificantly.

A variant of the block bordering method [27] for the anisotropic problem is considered. This algorithm uses the well defined multigrid or substructuring preconditioners for problems in subdomains (e.g., the preconditioner outlined in Section 2). For the problem at the interfaces we construct a preconditioner in the form of the inner Chebyshev iterative procedure. More precisely, this is a preconditioner for the Schur complement of the original symmetric positive definite matrix, which results after eliminating the block corresponding to the unknowns in the subdomains.

This approach combines the ideas of domain decomposition methods [4, 8, 29, 30] and the algorithms of multilevel and algebraic multigrid methods [2, 9, 19] with the bordering method for solving systems of mesh equations.

3.1. Problem formulation. Let Ω be a bounded domain on a plane $\mathbb{R}^2$, which is composed of open rectangles Ω_i whose sides are parallel to the coordinate axes $\Omega = \bigcup_{i=1}^{m} \Omega_i$. Consider an elliptic problem (1.1) with a positive definite symmetric coefficient matrix $K(\mathbf{x})$.

Assume that the interior of each side of the rectangles Ω_i either entirely belongs to Γ_0 or Γ_1, or lies inside Ω. Also assume that $\overline{\Omega}_i$, $i = 1, \ldots, m$, can have either a common side or only a common vertex, or they do not overlap. It is obvious that any domain composed of rectangles can be partitioned by additional lines into subdomains Ω_i satisfying this assumption.

Again we consider a regular partitioning $\mathcal{T}_h$ of Ω into simplices τ with mesh-size h and P_1 nonconforming finite element space of functions $V_h(\Omega)$ as described in Section 1.

ASSUMPTION 3.1. *We assume that there exist a diagonal coefficient matrix* $\tilde{K}(\mathbf{x}) = \mathrm{diag}\,\{k_x(\mathbf{x}), k_y(\mathbf{x})\}$ *such that* $k_x(\mathbf{x}) = k_{x,i}$, $k_y(\mathbf{x}) = k_{y,i}$, $\mathbf{x} \in \Omega_i$, $i = 1, \ldots, m$, *with constants* $k_{x,i} > 0$, $k_{y,i} > 0$, *such that the bilinear form* $a^h_\Omega(\cdot,\cdot)$ *defined by* (1.2) *and bilinear form*

$$\tilde{a}^h_\Omega(u, v) = \sum_{\tau \in \mathcal{T}_h} (\tilde{K} \nabla u, \nabla v)_\tau, \qquad \forall u, v \in V_h(\Omega),$$

satisfy inequalities

$$\alpha_0 \cdot \tilde{a}^h_\Omega(u, u) \le a^h_\Omega(u, u) \le \alpha_1 \cdot \tilde{a}^h_\Omega(u, u), \quad \forall u \in V_h(\Omega), \tag{3.1}$$

with some positive constants α_0, α_1.

Inequalities (3.1) suggest considering form $\tilde{a}^h_\Omega(\cdot,\cdot)$ as a preconditioning form to $a^h_\Omega(\cdot,\cdot)$. Thus below we consider the problem (1.1) with diagonal coefficient matrix K.

Let $\mathbf{u}^{(i)}$ and $\mathbf{v}^{(i)}$ denote the vectors corresponding to the finite element functions u and v from $V_h(\Omega_i)$. Let $A^{(i)}$ denote the local stiffness matrix arising from $a^h_{\Omega_i}(\cdot,\cdot)$:

$$(A^{(i)}\mathbf{u}^{(i)}, \mathbf{v}^{(i)}) = a^h_{\Omega_i}(u, v), \qquad \forall u, v \in V_h(\Omega_i). \tag{3.2}$$

For each subdomain Ω_i, $i = 1, \ldots, m$, we can partition the degrees of freedom $\mathbf{u}^{(i)}$ into two sets. The first set includes the degrees of freedom at the nodes in the interior of subdomain Ω_i, denoted $\mathbf{u}^{(i)}_I$,

and the second set corresponds to the degrees of freedom at the nodes on the boundary $\partial\Omega_i \setminus \Gamma_0$, denoted $\mathbf{u}_\Gamma^{(i)}$. Such a partitioning induces the partitioning of $A^{(i)}$ given by

$$(A^{(i)}\mathbf{u}^{(i)}, \mathbf{v}^{(i)}) = \left(\begin{bmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(i)} \\ \mathbf{u}_\Gamma^{(i)} \end{bmatrix}, \begin{bmatrix} \mathbf{v}_I^{(i)} \\ \mathbf{v}_\Gamma^{(i)} \end{bmatrix} \right). \tag{3.3}$$

Finite element system (1.4) has the obvious algebraic representation:

$$\begin{bmatrix} A_{II}^{(1)} & & \mathbf{0} & A_{I\Gamma}^{(1)} \\ & \ddots & & \vdots \\ \mathbf{0} & & A_{II}^{(m)} & A_{I\Gamma}^{(m)} \\ A_{\Gamma I}^{(1)} & \cdots & A_{\Gamma I}^{(m)} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{v}_I^{(1)} \\ \vdots \\ \mathbf{v}_I^{(m)} \\ \mathbf{v}_\Gamma \end{bmatrix} = \begin{bmatrix} \mathbf{g}_I^{(1)} \\ \vdots \\ \mathbf{g}_I^{(m)} \\ \mathbf{g}_\Gamma \end{bmatrix}, \tag{3.4}$$

with block $A_{\Gamma\Gamma}$ defined by

$$(A_{\Gamma\Gamma}\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) = \sum_{i=1}^{m} (A_{\Gamma\Gamma}^{(i)}\mathbf{u}_\Gamma^{(i)}, \mathbf{v}_\Gamma^{(i)}). \tag{3.5}$$

Note that blocks $A_{II}^{(i)}$, $i = 1, \ldots, m$, correspond to the boundary value problems in rectangles Ω_i

$$a_{\Omega_i}^h(u_h, v) = G(v), \qquad \forall v \in V_h(\Omega_i), \quad i = 1, \ldots, m, \tag{3.6}$$

with homogeneous Dirichlet boundary conditions imposed on the boundaries $\partial\Omega_i$. Denote the number of degrees of freedom in $\bigcup_{i=1}^{m} \Omega_i$ and $\bigcup_{i=1}^{m} \partial\Omega_i \setminus \Gamma_0$ by N_I and N_Γ, respectively.

Eliminating the unknowns $\mathbf{v}_I^{(i)}$, $i = 1, \ldots, m$, in (3.4), we obtain the following Schur system:

$$\Lambda_\Gamma \mathbf{v}_\Gamma = \mathbf{G}_\Gamma, \tag{3.7}$$

where

$$\Lambda_\Gamma = A_{\Gamma\Gamma} - \sum_{i=1}^{m} A_{\Gamma I}^{(i)} \left[A_{II}^{(i)}\right]^{-1} A_{I\Gamma}^{(i)}, \qquad \mathbf{G}_\Gamma = \mathbf{g}_\Gamma - \sum_{i=1}^{m} A_{\Gamma I}^{(i)} \left[A_{II}^{(i)}\right]^{-1} \mathbf{g}_I^{(i)}. \tag{3.8}$$

Thus, the solution to system (3.4) can be reduced to the construction of an efficient algorithm for solving systems (3.6) in subdomains and system (3.7).

The algorithm for solving subdomain problems with matrices $A_{II}^{(i)}$ (for 3D problem) is considered in Section 2. It can be shown that these problems can be solved very efficiently.

The main goal of this section is to construct an easily invertible matrix B which is spectrally equivalent to matrix Λ_Γ:

$$c_0(B_\Gamma \mathbf{v}_\Gamma, \mathbf{v}_\Gamma) \le (\Lambda_\Gamma \mathbf{v}_\Gamma, \mathbf{v}_\Gamma) \le c_1(B_\Gamma \mathbf{v}_\Gamma, \mathbf{v}_\Gamma), \qquad \forall \mathbf{v}_\Gamma \in \mathbb{R}^{N_\Gamma},$$

where constants c_0 and c_1 are independent of mesh size parameter h, the subdomain diameters, and value of the coefficients.

3.2. Preconditioner for interface problems. In this subsection we construct a preconditioner for the problem at the interface in the form of an inner iterative procedure. More precisely, we construct a preconditioner for the Schur complement of the original matrix.

The method is based on the following statement [24, 26].

LEMMA 3.1. *There exists an h-independent constant α such that*

$$\alpha \cdot h\, (A_{\Gamma\Gamma}\mathbf{u}_\Gamma, \mathbf{u}_\Gamma) \le (\Lambda_\Gamma \mathbf{u}_\Gamma, \mathbf{u}_\Gamma) \le (A_{\Gamma\Gamma}\mathbf{u}_\Gamma, \mathbf{u}_\Gamma), \qquad \forall \mathbf{u}_\Gamma \in \mathbb{R}^{N_\Gamma} \setminus \mathrm{Ker}\ \Lambda_\Gamma. \tag{3.9}$$

The preconditioner B_Γ is defined in the form of an inner Chebyshev iterative procedure [2, 8, 19]. From Lemma 3.1 we know that the eigenvalues of matrix $A_{\Gamma\Gamma}^{-1}\Lambda_\Gamma$ belong to segment $[\alpha h, 1]$. Let $P_L(y)$ be the polynomial of least deviation from zero on this segment and that satisfies the condition $P_L(0) = 1$. Denote by β_l, $l = 1, \ldots, L$, the inverses of the roots of the polynomial $P_L(y)$. The formulae for $P_L(y)$ and its roots $1/\beta_l$, $l = 1, \ldots, L$, can be found, e.g., in [18]. Then preconditioner B_Γ for matrix Λ_Γ is determined by:

$$B_\Gamma^{-1} = \left\{ I_\Gamma - \prod_{l=1}^{L} \left(I_\Gamma - \beta_l A_{\Gamma\Gamma}^{-1} \Lambda_\Gamma \right) \right\} \Lambda_\Gamma^{+}. \tag{3.10}$$

Lemma 3.1 and the theory of Chebyshev iterative methods imply the following result [26].

STATEMENT 3.1. *Let $L \geq (5/h)^{1/2}$. Then matrix B_Γ in (3.10) is spectrally equivalent to matrix Λ_Γ with constants of equivalence independent of mesh size parameter h and the value of coefficients $k_{x,i}$, $k_{y,i}$, $i = 1, \ldots, m$, in the subdomains.*

Using the partial solution technique [3, 28] one can show that the procedure of multiplying a vector by matrix B_Γ^{-1} can be implemented for $O(h^{-2} + h^{-3/2} \ln^2 (h^{-1}))$.

Now assume that we use the algebraic multigrid method (AMG) to solve the subdomain problems. Then the arithmetical complexity of solving the problem (1.4) is estimated by $O(N + N^{3/4} \ln^2 (N))$.

3.3. Results of numerical experiments. In this subsection the domain decomposition method is tested on the model problem in the unit square $\Omega = [0,1]^2$ with diagonal coefficient matrix $K(\mathbf{x}) = \mathrm{diag}\{k_x, k_y\}$. Domain Ω is composed of 4 subdomains as shown in Figure 3.1. Coefficients k_x and k_y are constants in each subdomain.

Ω_3 $k_x = 1$ $k_y = k$	Ω_4 $k_x = k$ $k_y = 1$
Ω_1 $k_x = k$ $k_y = 1$	Ω_2 $k_x = 1$ $k_y = k$

FIG. 3.1. *Coefficients in the subdomains for a model problem.*

The domain is divided into n^2 squares (n in each direction) and each square is partitioned into 2 triangles. The dimension of the original algebraic system is $N = 3n^2 - 2n$ and the dimension of the Schur complement after elimination of the subdomain problems is $N_\Gamma = 4n$. The problem (1.4) is solved by preconditioned conjugate method (PCG) with accuracy $\varepsilon = 10^{-6}$. The degree of matrix polynomial (3.10) equals $L = [\sqrt{2.5n}] + 1$, where $[\eta]$ is an integer part of η. The condition number of matrix $A_{\Gamma\Gamma}^{-1}\Lambda_\Gamma$ is calculated by the relation between the conjugate gradient and the Lanczos algorithm [18]. The results are summarized in Table 3.1.

4. Domain decomposition method on nonmatching grids. In this section we describe an algorithm for solving systems of linear algebraic equations arising from nonconforming finite element approximations of the anisotropic diffusion equations on nonmatching grids.

First, the original differential problem (1.1) is represented in the hybrid-mixed form using the Arnold-Brezzi formulation [1] via nonoverlapping domain decomposition using additional Lagrange multipliers to enforce the necessary continuity of the solution on the interfaces between subdomains [11, 17, 23]. Next, using the equivalence between hybrid-mixed and nonconforming finite element methods we replace the original three-field formulation in each subdomain with the simple nonconforming one. The original elliptic problem is thus imposed as a nonconforming discrete problem with Lagrange multipliers at the interfaces between the subdomains, into which the original domain is decomposed. At these interfaces certain continuity conditions on the solution are imposed.

TABLE 3.1
Results of experiments with bordering method.

	100×100 $N = 29800$		200×200 $N = 119600$		400×400 $N = 479200$	
k	Iter	Cond	Iter	Cond	Iter	Cond
1	23	10.7	25	10.9	26	10.9
10	23	9.2	24	9.8	26	10.2
100	20	8.3	19	7.9	20	8.1
1000	12	4.2	14	6.2	14	6.4
10000	6	1.5	7	2.0	7	2.1
100000	3	1.1	4	1.1	4	1.1

This construction is done to inherit the properties of the Lagrange multiplier space defined on the interfaces between the subdomains. The Dirichlet boundary conditions, if any, are also given by the Lagrange multipliers. We should note that the corresponding matrix is symmetric but indefinite. The iterative method to be considered involves a block diagonal preconditioner with the inner Chebyshev iterative procedure and the preconditioned Lanczos method as an outer iterative procedure.

In each subdomain we introduce its own grid, namely, a triangular one in two dimensions and a tetrahedral one in three dimensions and corresponding P_1-nonconforming finite element space. A mortar finite element space is constructed in the space of the Lagrange multipliers.

Using a variant of fictitious domain method [26] it is shown that for subdomains we can choose the substructuring preconditioners constructed in Section 2. For the problem on the interfaces the preconditioner is introduced in the form of the inner Chebyshev procedure for the matrix which is spectrally equivalent to the Schur complement. The construction and the analysis of this preconditioner is based on the new approach recently developed in [25, 26] for solving finite element problems on nonmatching grids with Lagrange multipliers on the interfaces between the subdomains.

It is shown that the proposed block-diagonal preconditioner is spectrally equivalent to the original saddle-point matrix with constants independent of mesh-size parameter and coefficients of the problem. We remind that the symmetric matrix $\mathcal{A}$ and the symmetric positive definite matrix $\mathcal{B}$ are said to be spectrally equivalent if the spectrum of matrix $\mathcal{B}^{-1}\mathcal{A}$ belongs to the set $[d_1, d_2] \cup [d_3, d_4]$, $d_1 \leq d_2 < 0 < d_3 \leq d_4$, with the boundaries of the segments independent of the mesh size parameter h.

Using the partial solution technique [3, 28] it can be shown that the arithmetical complexity of the proposed algorithm for solving problem (1.1) in two-dimensional domains is estimated from above by $O(N \ln^2 (N/m))$, where N is the dimension of the problem and m is a number of subdomains. This estimate is independent of mesh size parameter h, size of subdomains, and coefficients of the problem $K(\mathbf{x})$.

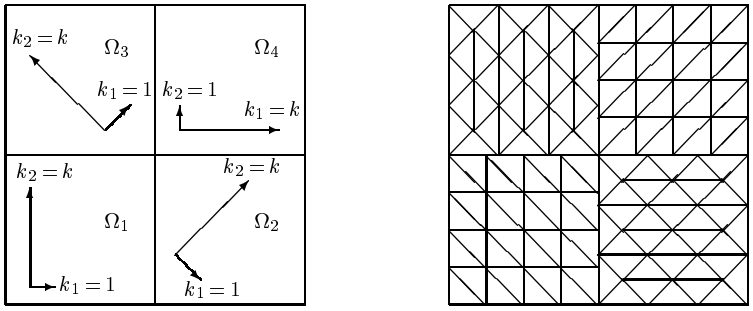
4.1. Results of numerical experiments. In this subsection we present the results of numerical experiments with the domain decomposition method on nonmatching grids for the model problem in unit square $\Omega = [0,1]^2$:

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(k_{ij}\frac{\partial u}{\partial x_j}) = f, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega. \tag{4.1}$$

Domain Ω is composed of 4 subdomains. The coefficients k_{ij}, $i, j = 1, 2$, are constants in each subdomain:

$$K_1 = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}, \; K_2 = \frac{1}{2}\begin{bmatrix} k+1 & k-1 \\ k-1 & k+1 \end{bmatrix}, \; K_3 = \frac{1}{2}\begin{bmatrix} 1+k & 1-k \\ 1-k & 1+k \end{bmatrix}, \; K_4 = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix},$$

with some parameter k. The main directions of the anisotropy in each subdomain and an example of the grids used in experiments are shown in Figures 4.1a and 4.1b, respectively.



(a) Main directions of the anisotropy. *(b) Nonmatching grids.*

FIG. 4.1. *Main directions of the anisotropy and grids in the subdomains.*

Each subdomain Ω_k is embedded in a rectangle Π_k constructed in the local coordinate system as described in [26]. Each rectangle Π_k is divided into n^2 squares (n in each direction) and each square is partitioned into 2 triangles. The dimension of the original algebraic system is N.

The results are summarized in Table 4.1. Here *Iter* denotes the number of iterations of the generalized Lanczos method and *Cond* denotes the condition number of preconditioned interface system.

TABLE 4.1
Results of experiments with nonmatching grids.

	$n = 100$ $N = 90\,600$		$n = 200$ $N = 361\,200$		$n = 400$ $N = 1\,442\,400$	
k	Iter	Cond	Iter	Cond	Iter	Cond
1	25	12.3	28	13.9	29	14.3
10	23	10.2	25	11.8	27	11.2
100	20	9.4	22	9.9	22	10.1
1000	17	6.2	19	7.2	19	7.4
10000	16	3.5	18	4.0	18	4.1
100000	16	2.1	18	2.1	17	2.1

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