SUBSTRUCTURING DOMAIN DECOMPOSITION METHOD FOR NONCONFORMING FINITE ELEMENT APPROXIMATIONS OF ELLIPTIC PROBLEMS WITH ANISOTROPY*

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Abstract. An optimal iterative method for solving systems of linear algebraic equations arising from nonconforming finite element discretizations of second order elliptic boundary value problems with anisotropic coefficients is constructed and studied. The technique suggested is based on decomposition of the original domain into nonoverlapping subdomains. It is assumed that the equation coefficients in each subdomain vary inconsiderably. This approach combines the ideas of domain decomposition methods, the algorithms of multilevel and algebraic multigrid methods with the bordering method. An iterative process whose convergence rate is independent of the mesh step size and the ratio of anisotropy in the coefficients is constructed. It is shown that the number of arithmetic operations required for realization of the method with a given accuracy is proportional to the number of unknowns of the original algebraic problem.

Key words. Second order elliptic problem, nonconforming finite element method, domain decomposition method, algebraic substructuring preconditioner, separable matrix, condition number.

AMS subject classifications. 65N30, 65N22, 65F10.

In this paper we propose an iterative method for solving systems of grid equations approximating boundary value problems for second order elliptic equations with anisotropic coefficients. The technique suggested is based on decomposition of the original domain composed of rectangles into nonoverlapping subdomains inside which the equation coefficients vary inconsiderably.

At present, there are numerous results on non-overlapping domain decomposition (see, e.g., [6, 7, 14, 26, 28]). Almost all of them can be viewed as so-called additive Schwarz method [15]. Algorithms of this kind, which have satisfactory convergence properties for the case of many regions, have one feature in common. In addition to subspaces and subproblems directly related to individual subdomains, there is a global, coarse subspace. The absence of such a subspace in general results in slow convergence [15, 26]. All such methods are based on decomposing the domain into subdomains of size d and involve the solution of related problems on the subdomains and lower-order coupling systems on the subdomain boundaries. The best condition number for the preconditioned system is shown to be on the order of $(1+\ln(d/h))^\alpha$, $\alpha = 1, 2$, ([8, 15, 25]), where h is the mesh-size parameter.

Another approaches consist in developing the bordering method [21, 22] or capacitance matrix method [5, 12, 14]. The main idea is to use well known techniques to solve or precondition the problems in the subdomains. For the problem at the interfaces a preconditioner is constructed in the form of an inner Chebyshev iterative procedure. More precisely, the preconditioner is constructed for the Schur complement of the original symmetric positive definite matrix, defined after eliminating the blocks corresponding to the unknowns in the subdomains [9, 21]. This approach for solving systems of mesh equations combines the ideas of domain decomposition methods [6, 9, 14, 26, 28], the algorithms of multilevel and algebraic multigrid methods [2, 10, 19], with the bordering method [21, 22].

The work in this direction has been done in [9, 5, 12, 13, 14, 21, 22]. Preconditioning the Schur complement on the interface between subdomains makes it able to construct such methods that the condition numbers for the resulting preconditioned systems are independent of mesh-size parameter h and size of the subdomains d. Although the results obtained in all these references include the cases of large jumps in the coefficients of the problem, none of them deals with the case of anisotropic coefficient tensor.

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The theory developed in this paper provides an approach which is applicable to second order elliptic boundary value problems with large anisotropy ratio in the coefficients. The main idea we follow here is close to the construction considered in [9, 21]. In these works along with the Schur complement on the interfaces S_Γ authors introduced some auxiliary interface operator $\tilde{B}_\Gamma$ such that the computation of its inverse is very cheap and the condition number of the preconditioned operator $\mathrm{Cond}(\tilde{B}_\Gamma^{-1} S_\Gamma)$ is bounded from above by $C \cdot h^{-1}$, with a positive constant C and h is a mesh-size parameter. Then the preconditioner B_Γ for operator S_Γ is defined in the form of matrix polynomial of degree m:

$$B_\Gamma^{-1} = P_m(\tilde{B}_\Gamma^{-1} S_\Gamma) S_\Gamma^{-1}.$$

Using Chebyshev polynomials it is shown that the preconditioner B_Γ can be constructed to be spectrally equivalent to the Schur complement S_Γ and condition number for the resulting preconditioned system is independent of the diameters of the subdomains d and mesh-size parameter h with only a slight increase in computational effort. The arithmetical cost of solving the problem with matrix B_Γ is proportional to the dimension of the initial discrete problem, i.e. $O(h^{-2})$.

The purpose of this paper is to develop a variant of the bordering method for second order elliptic partial differential equations with anisotropy in the coefficients approximated by the nonconforming P_1 elements. We note that an important difference between nonconforming an conforming cases is that there are no nodes at the vertices (or wire basket) of the subregions. Thus, there is no problem of crosspoints which is an essential part of the domain decomposition method when using conforming finite elements [8, 15, 21, 22, 26]. It makes the construction of the preconditioner for the Schur complement on the interfaces between subdomains very clear and easy. The construction of this interface preconditioner is the most interesting part of the work. The main result of this paper is that the condition number of the preconditioned operator is bounded by a constant independent of the mesh-size parameter and the coefficients of the problem.

The outline of this paper is as follows. In Section 1 we pose the problem, give its nonconforming finite element discretization, the matrix formulation and outline the construction of a block diagonal preconditioner to the algebraic system. In Section 2 we consider a model problem in a unit square with constant coefficients and construct the preconditioners for the problems in subdomains. Section 3 contains the main result of the paper. It is subdivided into three subsections. In the first subsection we construct a preconditioner for the problem at the interface in the form of an inner iterative procedure considering the union of two rectangular subdomains. The second subsection describes an algorithm for implementing the interface preconditioner. Then, in the third subsection we construct the interface preconditioner for a general domain composed of rectangles. The arithmetic cost of solving the system with the preconditioner proposed is proportional to the number of unknowns of the original algebraic system, i.e. the preconditioner constructed is of the optimal order of arithmetical complexity. In Section 4 we provide the results of numerical experiments. These computations show that the theoretical estimates are fully realized in practice.

1. Problem Formulation. Let Ω be a bounded domain on a plane $\mathbb{R}^2$, which is composed of open rectangles Ω_i whose sides are parallel to the coordinate axes, $\Omega = \bigcup_{i=1}^{m} \Omega_i$. Consider an elliptic problem

$$\begin{aligned} -\nabla \cdot (\tilde{K} \cdot \nabla u) + \tilde{c}_0 \cdot u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ (\tilde{K} \nabla u, \mathbf{n}) &= 0 && \text{on } \Gamma_1, \end{aligned} \tag{1.1}$$

where $\tilde{K}(\mathbf{x})$ is a positive definite symmetric coefficient matrix, $\tilde{c}_0(\mathbf{x})$ is a nonnegative bounded function, $f(\mathbf{x}) \in L^2(\Omega)$ is a given function, $\overline{\Gamma_0 \cup \Gamma_1} = \partial\Omega$, and $\Gamma_0 \cap \Gamma_1 = \emptyset$. We consider the case of $\Gamma_0 \equiv \overline{\Gamma_0} \neq \emptyset$. The pure Neumann problem ($\Gamma_0 = \emptyset$) can be treated in a similar way but for the sake of simplicity is not described here.

Assume that the interior of each side of rectangles Ω_i either entirely belongs to Γ_0 or Γ_1, or lies inside Ω. Also assume that $\overline{\Omega}_i$, $i = 1,\dots,m$, can have either a common side or only a common vertex, or they do not overlap. It is obvious that any domain composed of rectangles can be partitioned by additional lines into subdomains Ω_i satisfying this assumption.

Let the bilinear form $\tilde{a}(\cdot,\cdot)$ be defined by

$$\tilde{a}(u,v) = (\tilde{K}\nabla u, \nabla v) + (\tilde{c}_0 \cdot u, v), \qquad u,v \in V_0(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0\},$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. Assume that there a diagonal coefficient matrix $K(\mathbf{x}) = \operatorname{diag}\{k_x(\mathbf{x}), k_y(\mathbf{x})\}$ and a piecewise constant function $c_0(\mathbf{x})$, such that

$$k_x(\mathbf{x}) = k_{x,i}, \quad k_y(\mathbf{x}) = k_{y,i}, \quad c_0(\mathbf{x}) = c_{0,i}, \qquad \mathbf{x} \in \Omega_i, \qquad i == 1,\dots,m,$$

with constants $k_{x,i} > 0$, $k_{y,i} > 0$, $c_{0,i} \geq 0$, satisfying inequalities

$$\alpha_0\ \left((K\nabla u, \nabla u) + (c_0 \cdot u, u)\right) \leq a(u,u) \leq \alpha_1\ \left((K\nabla u, \nabla u) + (c_0 \cdot u, u)\right), \tag{1.2}$$

for any $u \in V_0(\Omega)$ with some positive constants α_0, α_1.

The standard weak form of (1.1) is: find $u \in V_0(\Omega)$ such that

$$\tilde{a}(u,v) = (f,v), \qquad \forall v \in V_0(\Omega). \tag{1.3}$$

Let $\mathcal{C}_h$ be a rectangular mesh in Ω. Assume that in each rectangle Ω_i the mesh steps $h_{x,i}$, $h_{y,i}$, $i = 1,\dots,m$, are constant in each direction, and the boundaries $\partial\Omega_i$ of the rectangles belong to the mesh lines. Also assume that there exist constants c_0 and c_1 independent of h such that

$$c_0 h \leq \min_{i=1,\dots,m}\{h_{x,i}, h_{y,i}\} \leq \max_{i=1,\dots,m}\{h_{x,i}, h_{y,i}\} \leq c_1 h.$$

Here $h = 1/\sqrt{M}$, where M is the number of mesh nodes belonging to $\overline{\Omega} \setminus \Gamma_0$.

Let $\mathcal{T}_h$ be a regular partitioning of $\mathcal{C}_h$ into triangles τ [11] and let $V_h(\Omega)$ be the P_1–nonconforming finite element space of functions $v \in L^2(\Omega)$ [1]: that is $v|_\tau$ are linear for all $\tau \in \mathcal{T}_h$, v are continuous at the middle points of the sides of $\tau \in \mathcal{T}_h$, and vanish at the middle points of the sides of triangles on Γ_0. Note that the space $V_h(\Omega)$ is not a subspace of $H^1(\Omega)$.

Define the bilinear forms on $V_h(\Omega)$ by

$$\begin{aligned} \tilde{a}^h_\Omega(u,v) &= \textstyle\sum_{\tau\in\mathcal{T}_h} (\tilde{K}\nabla u, \nabla v)_\tau + (\tilde{c}_0 \cdot u, v)_\tau, \qquad \forall\ u,v \in V_h(\Omega), \\ a^h_\Omega(u,v) &= \textstyle\sum_{\tau\in\mathcal{T}_h} (K\nabla u, \nabla v)_\tau + (c_0 \cdot u, v)_\tau, \qquad \forall\ u,v \in V_h(\Omega), \end{aligned} \tag{1.4}$$

where $(\cdot,\cdot)_\tau$ is the inner product in $L^2(\tau)$, $\tau \in \mathcal{T}_h$. Then the P_1–nonconforming finite element discretization of (1.1) is: find $u_h \in V_h$ such that

$$\tilde{a}^h_\Omega(u_h, v) = (f,v), \qquad \forall v \in V_h(\Omega). \tag{1.5}$$

Once a basis $\{\varphi_i(\mathbf{x})\}_{i=1}^N$ for $V_h(\Omega)$ is chosen, where $N = \dim V_h(\Omega)$, then (1.5) leads to a system of linear algebraic equations. Write $u(\mathbf{x}) = \sum_{i=1}^N\ u_i\varphi_i(\mathbf{x})$. Then (1.3) becomes

$$\sum_{i=1}^N u_i \tilde{a}^h_\Omega(\varphi_i, \varphi_j) = (f, \varphi_j), \qquad j = 1,\dots,N,$$

or in matrix representation

$$\tilde{A}\mathbf{u} = \mathbf{f}, \tag{1.6}$$

where $\tilde{A}_{ji} = \tilde{a}^h_\Omega(\varphi_i,\varphi_j)$, $f_j = (f,\varphi_j)$, $i,j = 1,\dots,N$.

In the same way we define matrix A by $A_{ji} = a^h_\Omega(\varphi_i, \varphi_j)$, $i,j = 1,\dots,N$. Then (1.2) implies that

$$\alpha_0(A\mathbf{u},\mathbf{u}) \le (\tilde{A}\mathbf{u},\mathbf{u}) \le \alpha_1(A\mathbf{u},\mathbf{u}), \qquad \forall \mathbf{u} \in \mathbb{R}^N, \tag{1.7}$$

i.e. matrices $\tilde{A}$ and A are spectrally equivalent.

The underlying method to solve (1.6) is a preconditioned iterative method. Inequalities (1.7) suggest considering matrix A as a preconditioner to $\tilde{A}$. Therefore, we need to find an efficient method for solving the problem

$$A\mathbf{v} = \mathbf{g}. \tag{1.8}$$

Let $\mathbf{u}^{(i)}$ and $\mathbf{v}^{(i)}$ denote the vectors corresponding to the finite element functions u and v from $V_h(\Omega_i)$, respectively. Let $A^{(i)}$ denote the local stiffness matrix arising from $a^h_{\Omega_i}(\cdot,\cdot)$:

$$(\tilde{A}^{(i)}\mathbf{u}^{(i)}, \mathbf{v}^{(i)}) = \tilde{a}^h_{\Omega_i}(u,v), \qquad \forall u,v \in V_h(\Omega_i). \tag{1.9}$$

For each subdomain Ω_i, $i = 1,\dots,m$, we can partition the degrees of freedom $\mathbf{u}^{(i)}$ into two sets. The first set includes the degrees of freedom at the nodes in the interior of subdomain Ω_i, denoted $\mathbf{u}^{(i)}_I$, and the second set corresponds to the degrees of freedom at the nodes on the boundary $\partial\Omega_i \setminus \Gamma_0$, denoted $\mathbf{u}^{(i)}_\Gamma$. Such a partitioning induces the partitioning of $A^{(i)}$ given by

$$(A^{(i)}\mathbf{u}^{(i)}, \mathbf{v}^{(i)}) = \left(\begin{bmatrix} A^{(i)}_{II} & A^{(i)}_{I\Gamma} \\ A^{(i)}_{\Gamma I} & A^{(i)}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{u}^{(i)}_I \\ \mathbf{u}^{(i)}_\Gamma \end{bmatrix}, \begin{bmatrix} \mathbf{v}^{(i)}_I \\ \mathbf{v}^{(i)}_\Gamma \end{bmatrix} \right). \tag{1.10}$$

Finite element system (1.8) has the obvious algebraic representation:

$$\begin{bmatrix} A^{(1)}_{II} & & \mathbf{0} & A^{(1)}_{I\Gamma} \\ & \ddots & & \vdots \\ \mathbf{0} & & A^{(m)}_{II} & A^{(m)}_{I\Gamma} \\ A^{(1)}_{\Gamma I} & \dots & A^{(m)}_{\Gamma I} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{v}^{(1)}_I \\ \vdots \\ \mathbf{v}^{(m)}_I \\ \mathbf{v}_\Gamma \end{bmatrix} = \begin{bmatrix} \mathbf{g}^{(1)}_I \\ \vdots \\ \mathbf{g}^{(m)}_I \\ \mathbf{g}_\Gamma \end{bmatrix}, \tag{1.11}$$

with block $A_{\Gamma\Gamma}$ defined by

$$(A_{\Gamma\Gamma}\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) = \sum_{i=1}^m (A^{(i)}_{\Gamma\Gamma}\mathbf{u}^{(i)}_\Gamma, \mathbf{v}^{(i)}_\Gamma). \tag{1.12}$$

Note that blocks $A^{(i)}_{II}$, $i = 1,\dots,m$, correspond to the boundary value problems in the rectangles Ω_i

$$a^h_{\Omega_i}(u_h, v) = G(v), \qquad \forall v \in V_h(\Omega_i), \quad i = 1,\dots,m, \tag{1.13}$$

with homogeneous Dirichlet boundary conditions imposed on the boundaries $\partial\Omega_i$. Denote the number of degrees of freedom in Ω_i, $\partial\Omega_i \setminus \Gamma_0$, $\bigcup_{i=1}^m \Omega_i$ and $\bigcup_{i=1}^m \partial\Omega_i \setminus \Gamma_0$ by $N^{(i)}_I$, $N^{(i)}_\Gamma$, N_I and N_Γ, $i = 1,\dots,m$, respectively.

Eliminating the unknowns $\mathbf{v}^{(i)}_I$, $i = 1,\dots,m$, in (1.11), we obtain the following Schur system:

$$\Lambda_\Gamma \mathbf{v}_\Gamma = \mathbf{G}_\Gamma, \tag{1.14}$$

where

$$\Lambda_\Gamma = A_{\Gamma\Gamma} - \sum_{i=1}^m A^{(i)}_{\Gamma I} \left[A^{(i)}_{II}\right]^{-1} A^{(i)}_{I\Gamma}, \qquad \mathbf{G}_\Gamma = \mathbf{g}_\Gamma - \sum_{i=1}^m A^{(i)}_{\Gamma I} \left[A^{(i)}_{II}\right]^{-1} \mathbf{g}^{(i)}_I. \tag{1.15}$$

Thus, The solution to system (1.11) can be reduced to the construction of an efficient algorithm for solving systems (1.13) in subdomains and the Schur complement system (1.14).

The algorithm for solving subdomain problems with matrices $A_{II}^{(i)}$ is considered in Section 2. It is shown that these problems can be solved very efficiently.

The main goal of this paper is to construct an easily invertible matrix B spectrally equivalent to matrix Λ_Γ:

$$c_0(B_\Gamma \mathbf{v}_\Gamma, \mathbf{v}_\Gamma) \le (\Lambda_\Gamma \mathbf{v}_\Gamma, \mathbf{v}_\Gamma) \le c_1(B_\Gamma \mathbf{v}_\Gamma, \mathbf{v}_\Gamma), \qquad \forall \mathbf{v}_\Gamma \in \mathrm{I\!R}^{N_\Gamma},$$

where constants c_0 and c_1 are independent of the mesh size parameter h, the subdomain diameters, and value of the coefficients. This issue is discussed in detail in Section 3.

2. Model problem in a unit square. Here we discuss an algorithm to solve the subdomain systems. Consider a model problem on a unit square:

$$(2.1)\qquad \begin{aligned} -k_x \frac{\partial^2 u}{\partial x^2} - k_y \frac{\partial^2 u}{\partial y^2} + c_0 u &= f, &\quad &\text{in } \Omega \equiv [0,1]^2,\\ u &= 0, &\quad &\text{on } \partial\Omega, \end{aligned}$$

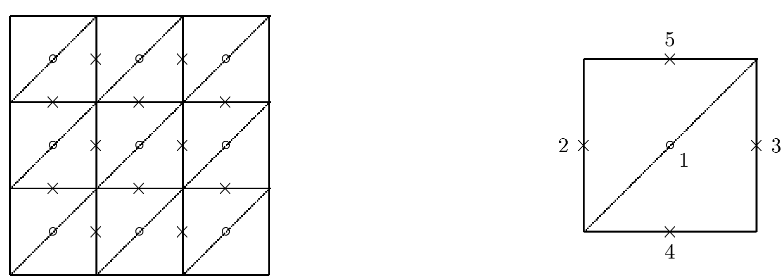
where coefficients $k_x > 0$, $k_y > 0$, $c_0 \ge 0$, are constants in Ω. It is clear that a method developed for this model problem can be easily generalized for the case of rectangular domain and mixed boundary conditions.

Let $\mathcal{C}_h = \{C^{(i,j)}\}$ be a partition of Ω into uniform squares with the length of the side $h = 1/n$, where (x_i, y_j) is the lower left corner of the square $C^{(i,j)}$. We enumerate the squares in a lexicographical order, first, in the y-direction, then in the x-direction. Next, we divide each square $C^{(i,j)}$ into 2 triangles as shown in Figure 1a. The partitioning of Ω into triangles is denoted by $\mathcal{T}_h$.

We introduce the set of centers of all edges of the triangulation of Ω, and the set Q_h of those centers that are not on the Dirichlet boundary $\Gamma_0 = \partial\Omega$ (see Fig. 1a). The Crouzeix-Raviart P_1–nonconforming finite element space V_h is defined by

$$(2.2)\qquad \begin{aligned} V_h = \{v \in L^2(\Omega) :\ & v|_\tau \in P_1(\tau),\ \forall \tau \in \mathcal{T}_h;\ v \text{ is continuous at the points}\\ & \text{from } Q_h \text{ and vanishes at the middle points of edges on } \Gamma_0\}. \end{aligned}$$

Let the dimension of V_h be N. Obviously, $N \approx 3n^2$.



(a) Triangilation of the domain Ω. *(b) Local enumeration of the degrees of freedom.*

FIGURE 1. *Triangilation and partition of the degrees of freedom.*

Now we define the bilinear form on V_h by

$$(2.3)\qquad a_h(u,v) = \sum_{\tau \in \mathcal{T}_h} \int_\tau \left(k_x \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + k_y \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + c_0 uv \right) d\mathbf{x}, \qquad \forall\, u, v \in V_h.$$

Thus the nonconforming discretization of problem (2.1) is given by seeking $u_h \in V_h$ such that

$$(2.4)\qquad a_h(u_h, v) = (f, v), \qquad \forall\, v \in V_h.$$

For any function $v \in V_h$ we denote by $\mathbf{v} \in \mathbb{R}^N$ its representation with respect to the basis in V_h.

Let $(\mathbf{u}, \mathbf{v})_N$ be a standard bilinear form defined on $\mathbb{R}^N$ by $(\mathbf{u}, \mathbf{v})_N = \sum_{\mathbf{x} \in Q_h} u(\mathbf{x}) v(\mathbf{x})$, $\forall u, v \in V_h$. Then define symmetric and positive definite operator $A : \mathbb{R}^N \to \mathbb{R}^N$ by

$$(A\mathbf{u}, \mathbf{v})_N = a_h(u, v), \qquad u, v \in V_h. \tag{2.5}$$

For each square $C = C^{(i,j)} \in \mathcal{C}_h$, denote by V_h^C the subspace of the restriction of the functions from V_h into C. For each $v \in V_h^C$, we indicate by $\mathbf{v}_c$ the corresponding vector. The dimension of V_h^C is denoted by N_c. Obviously, for a square without faces on Γ_0 we have $N_c = 5$.

The local stiffness matrix A^C on a square $C \in \mathcal{C}_h$ is given by

$$(A^C \mathbf{u}_c, \mathbf{v}_c)_{N_c} = \sum_{\tau \subset C} \left(k_x (\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x})_\tau + k_y (\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y})_\tau + c_0 (u, v)_\tau \right), \quad \forall u, v \in V_h^C. \tag{2.6}$$

Note that matrices A^C are positive definite when $C \cap \Gamma_0 \neq 0$ and at least semidefinite otherwise (if $c_0 \neq 0$ then all the matrices A^C are positive definite). The global stiffness matrix is determined by assembling the local stiffness matrices:

$$(A\mathbf{u}, \mathbf{v})_N = \sum_{C \in \mathcal{C}_h} (A^C \mathbf{u}_c, \mathbf{v}_c)_{N_c}, \qquad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^N. \tag{2.7}$$

To define the solution procedure we divide all the unknowns in the system into two groups:

1. The first group consists of the unknowns corresponding to the edges of the triangles that are internal for each square (these are unknowns corresponding to the nodes marked by "$\circ$" in Fig. 1). We denote these unknowns by $vc_{(i,j)}$, $i, j = 1, \ldots, n$.
2. The second group consists of all unknowns corresponding to the edges of the squares in the partition $\mathcal{C}_h$, without the faces on Γ_0 (Fig. 1, the nodes marked by "$\times$").
   (a) First, we enumerate the unknowns on the edges perpendicular to the x-axis (nodes 2 and 3 in Fig. 1b). We denote these unknowns by $vx_{(i,j)}$, $i = 1, \ldots, n-1$, $j = 1, \ldots, n$.
   (b) Second, we enumerate the unknowns on the edges perpendicular to the y-axis (nodes 4 and 5 in Fig. 1b). We denote these unknowns by $vy_{(i,j)}$, $i = 1, \ldots, n$, $j = 1, \ldots, n-1$.

Now we consider a square C that has no face on the boundary $\partial\Omega$ and enumerate the edges s_j, $j = 1, \ldots, 5$, of the triangles in this square in correspondence with the partitioning introduced above as is shown in Figure 1b. Then the local stiffness matrix of this square has the following form:

$$A^C = \begin{bmatrix} A_{11,c} & A_{12,c} \\ A_{21,c} & A_{22,c} \end{bmatrix}. \tag{2.8}$$

Introducing parameter $c = c_0 h^2/12$ we can write

$$A_{11,c} = 4\left[k_x + k_y + c\right], \qquad A_{12,c} = A_{21,c}^T = \left[-2k_x, -2k_x, -2k_y, -2k_y\right], \tag{2.9}$$

$$A_{22,c} = \begin{bmatrix} 2k_x & & & \\ & 2k_x & & \\ & & 2k_y & \\ & & & 2k_y \end{bmatrix} + 2c \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

The splitting of space $\mathbb{R}^N$ induces the presentation of the vectors: $\mathbf{v}^T = (\mathbf{v}_1^T, \mathbf{v}_2^T)$, where $\mathbf{v}_1 \in \mathbb{R}^{N_1}$, $\mathbf{v}_2 \in \mathbb{R}^{N_2}$, and $\mathbf{v}_2$ corresponds to the unknowns of the 2-nd group. Obviously, $N_1 = n^2$ and $N_2 = N - n^2$. Then matrix A can be presented in the following block form:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{2.10}$$

where $A_{22} : \mathbb{R}^{N_2} \to \mathbb{R}^{N_2}$ is a diagonal matrix.

Now denote by $\hat{A}_{11} = A_{11} - A_{12}A_{22}^{-1}A_{21}$ the Schur complement of A obtained by elimination of the vector $\mathbf{v}_2$. Then $A_{11} = \hat{A}_{11} + A_{12}A_{22}^{-1}A_{21}$, so matrix A has the form

$$A = \begin{bmatrix} \hat{A}_{11} + A_{12}A_{22}^{-1}A_{21} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{2.11}$$

To understand the structure of the Schur complement $\hat{A}_{11}$ let us write explicitly the matrix equation

$$A\mathbf{v} = \mathbf{g}$$

in terms of the unknowns $vc_{i,j}$, $vx_{i,j}$, and $vy_{i,j}$. For any square $C^{(i,j)} \cap \partial\Omega \neq 0$ we have:

$$4(k_x + k_y + c)vc_{i,j} - 2k_x(vx_{i,j} + vx_{i+1,j}) - 2k_y(vy_{i,j} + vy_{i,j+1}) = gc_{i,j}, \qquad i,j = 1,\dots,n,$$

$$4(k_x + c)vx_{i,j} - 2k_x(vc_{i-1,j} + vc_{i,j}) = gx_{i,j}, \qquad i = 2,\dots,n,\ j = 1,\dots,n,$$

$$4(k_y + c)vy_{i,j} - 2k_y(vc_{i,j-1} + vc_{i,j}) = gy_{i,j}, \qquad i = 1,\dots,n,\ j = 2,\dots,n.$$

After eliminating the unknowns $vx_{i,j}$ and $vy_{i,j}$ we have a 5-point computational scheme for the unknowns $vc_{i,j}$:

$$(2a_x + 2a_y + b)vc_{i,j} - a_x(vc_{i-1,j} + vc_{i+1,j}) - a_y(vc_{i,j-1} + vc_{i,j+1}) = \tilde{gc}_{i,j}, \tag{2.12}$$

where

$$a_x = \frac{k_x}{1 + c/k_x}, \quad a_y = \frac{k_y}{1 + c/k_y}, \qquad b = 4c\left(1 + \frac{1}{1 + c/k_x} + \frac{1}{1 + c/k_y}\right). \tag{2.13}$$

It is easy to see matrix $\hat{A}_{11}$ can be represented in a tensor product form (according to the enumeration introduced earlier in this section):

$$\hat{A}_{11} = a_x(A_x \otimes I_y) + a_y(I_x \otimes A_y) + b(I_x \otimes I_y), \tag{2.14}$$

where the matrices $I_x, I_y : \mathbb{R}^n \to \mathbb{R}^n$ are identity ones, and the matrices $A_x, A_y : \mathbb{R}^n \to \mathbb{R}^n$ are tridiagonal:

$$A_x = A_y = \begin{bmatrix} 3 & -1 & & & \mathbf{0} \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ \mathbf{0} & & & -1 & 3 \end{bmatrix}. \tag{2.15}$$

To solve the problem with separable matrix $\hat{A}_{11}$ we can use either the discrete fast Fourier transform [24] or an algebraic multigrid method (AMG) [2, 10, 19, 20, 28]. When an implementation cost of the first method is estimated by $O(h^{-2}\ln(h^{-1}))$, the AMG methods have the optimal order of arithmetic complexity $O(h^{-2})$. Since these methods are well described in the literature we are not going to discuss them in greater detail.

3. Preconditioner for the interface problem. In this section we construct a preconditioner for the problem at the interface in the form of an inner iterative procedure. More precisely, we construct here a preconditioner for the Schur complement of the original matrix.

3.1. Model problem. For the sake of simplicity, let us consider the problem (2.1) for the case of Ω being a rectangle composed of two squares $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$, $\Gamma = \bar{\Omega}_1 \cap \bar{\Omega}_2$ (see Fig. 2):

$$\begin{aligned} \Omega &= \{(x, y) : 0 < x < 2, 0 < y < 1\} \\ \Omega_i &= \{(x, y) : i-1 < x < i, 0 < y < 1\}, \qquad i = 1, 2. \end{aligned} \tag{3.1}$$

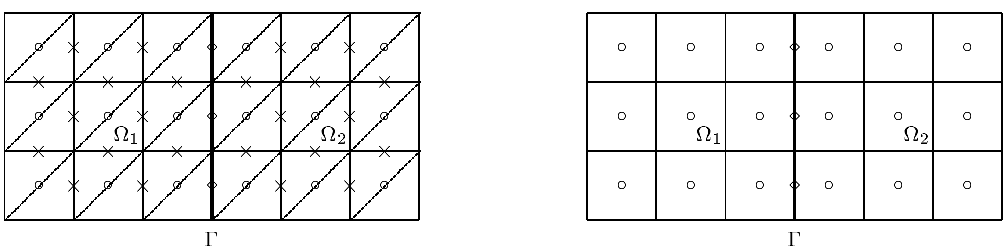
Assume that coefficients k_x, k_y, c_0 are constants in Ω_i, $i = 1, 2$, i.e.

$$k_x \equiv k_{x,i} = const > 0, \qquad k_y \equiv k_{y,i} = const > 0, \qquad c_0 \equiv c_{0,i} = const \geq 0.$$

Let $\mathcal{T}_h$ be a regular triangulation of domain Ω with mesh-size h (as described in Section 1). In this case (1.11) has the form:

$$\begin{bmatrix} A_{II}^{(1)} & \mathbf{0} & A_{I\Gamma}^{(1)} \\ \mathbf{0} & A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma I}^{(2)} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{v}_I^{(1)} \\ \mathbf{v}_I^{(2)} \\ \mathbf{v}_\Gamma \end{bmatrix} = \begin{bmatrix} \mathbf{g}_I^{(1)} \\ \mathbf{g}_I^{(2)} \\ \mathbf{g}_\Gamma \end{bmatrix}. \tag{3.2}$$

Note that blocks $A_{II}^{(i)}$, $i = 1, 2$, correspond to the boundary value problems in subdomains Ω_i and block $A_{\Gamma\Gamma}$ is a diagonal matrix.



(a) Triangulation of the domain Ω. *(b) The degrees of freedom of the reduced problem.*

FIGURE 2. *Degrees of freedom of real and reduced problems.*

Eliminating the unknowns $vx_{i,j}$ and $vy_{i,j}$ in each subdomain as is discussed in Section 2 we get the problem

$$\begin{bmatrix} \hat{A}_{II}^{(1)} & \mathbf{0} & \hat{A}_{I\Gamma}^{(1)} \\ \mathbf{0} & \hat{A}_{II}^{(2)} & \hat{A}_{I\Gamma}^{(2)} \\ \hat{A}_{\Gamma I}^{(1)} & \hat{A}_{\Gamma I}^{(2)} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{v}_c^{(1)} \\ \mathbf{v}_c^{(2)} \\ \mathbf{v}_\Gamma \end{bmatrix} = \begin{bmatrix} \mathbf{g}_c^{(1)} \\ \mathbf{g}_c^{(2)} \\ \mathbf{g}_\Gamma \end{bmatrix}, \tag{3.3}$$

where blocks $\hat{A}_{II}^{(i)}$, $i = 1, 2$, are separable matrices, and vectors $\mathbf{v}_c^{(i)}$, $i = 1, 2$, consist of the unknowns $vc_{i,j}$ in each subdomain. In Figure 2b, the nodes corresponding to these unknowns are marked by "$\circ$".

Matrix $A_{\Gamma\Gamma}$ is defined by equality (1.12)

$$(A_{\Gamma\Gamma}\mathbf{u}_\Gamma, \mathbf{v}_\Gamma) = \sum_{i=1}^{2} (A_{\Gamma\Gamma}^{(i)}\mathbf{u}_\Gamma^{(i)}, \mathbf{v}_\Gamma^{(i)}).$$

Introducing the subdomain Schur complements

$$\Lambda_\Gamma^{(i)} = A_{\Gamma\Gamma}^{(i)} - \hat{A}_{\Gamma I}^{(i)} \left[\hat{A}_{II}^{(i)}\right]^{-1} \hat{A}_{I\Gamma}^{(i)}, \qquad i = 1, 2, \tag{3.4}$$

the Schur complement (1.15) can be rewritten in the form:

$$\Lambda_\Gamma = A_{\Gamma\Gamma} - \sum_{i=1}^{2} \hat{A}^{(i)}_{\Gamma I} \left[\hat{A}^{(i)}_{II}\right]^{-1} \hat{A}^{(i)}_{I\Gamma} = \Lambda^{(1)}_\Gamma + \Lambda^{(2)}_\Gamma. \tag{3.5}$$

We construct a preconditioner B_Γ for Λ_Γ by defining preconditioners $B^{(1)}_\Gamma$ and $B^{(2)}_\Gamma$ for $\Lambda^{(1)}_\Gamma$ and $\Lambda^{(2)}_\Gamma$, respectively, and setting

$$B_\Gamma = B^{(1)}_\Gamma + B^{(2)}_\Gamma. \tag{3.6}$$

Let us consider subdomain Ω_1, matrices $\Lambda^{(1)}_\Gamma$ and $A^{(1)}_{\Gamma\Gamma}$, and omit the index "(1)" for simplicity. Boundary nodes belonging to Γ (marked by "$\diamond$") and internal nodes (marked by "$\circ$") are schematically shown in Figure 3.

The following lemma is valid.

LEMMA 3.1. *There exists an h-independent constant α such that*

$$\alpha \cdot h\,(A_{\Gamma\Gamma}\mathbf{u}_\Gamma, \mathbf{u}_\Gamma) \le (\Lambda_\Gamma \mathbf{u}_\Gamma, \mathbf{u}_\Gamma) \le (A_{\Gamma\Gamma}\mathbf{u}_\Gamma, \mathbf{u}_\Gamma), \qquad \forall \mathbf{u}_\Gamma \in \mathbb{R}^{N_\Gamma}. \tag{3.7}$$

Proof. Consider an eigenvalue problem

$$\Lambda_\Gamma \mathbf{u}_\Gamma = \mu A_{\Gamma\Gamma}\mathbf{u}_\Gamma, \qquad \mathbf{u}_\Gamma \in \mathbb{R}^{N_\Gamma}. \tag{3.8}$$

Since the symmetric matrices Λ_Γ and $A_{\Gamma\Gamma}$ are positive definite problem (3.8) has N_Γ positive eigenvalues.

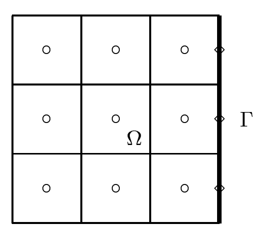


FIGURE 3. *Degrees of freedom of the model subdomain problem.*

It is obvious that eigenvalues μ_i, $i = 1, \dots, N_\Gamma$, of problem (3.8) can be found from the system of equations

$$\begin{aligned} &\hat{A}_{II}\mathbf{u}_I + \hat{A}_{I\Gamma}\mathbf{u}_\Gamma = 0, \\ &\hat{A}_{\Gamma I}\mathbf{u}_I + A_{\Gamma\Gamma}\mathbf{u}_\Gamma = \mu A_{\Gamma\Gamma} u_\Gamma. \end{aligned} \tag{3.9}$$

Here matrix $\hat{A}_{II}$ is defined by (2.14). The $N_I \times N_\Gamma$ matrix $\hat{A}_{I\Gamma}$ has a form

$$\hat{A}_{I\Gamma} = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ -2k_x I_y \end{bmatrix} \equiv \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -2k_x \end{bmatrix} \otimes I_y,$$

and the diagonal $N_\Gamma \times N_\Gamma$ matrix $A_{\Gamma\Gamma}$ is defined by $A_{\Gamma\Gamma} = 2(k_x + c)I_y$.

Define by $\boldsymbol{\xi}_l \in \mathbb{R}^n$ an eigenvector of $n \times n$ matrix A_y (2.15) corresponding to the eigenvalue λ_l:

$$A_y \boldsymbol{\xi}_l = \lambda_l \boldsymbol{\xi}_l, \qquad l = 1, \dots, n. \tag{3.10}$$

Note that we explicitly know the eigenvalues and eigenvectors of matrix A_y:

$$\lambda_l = 4\sin^2\frac{\pi l}{2n}, \qquad \boldsymbol{\xi}_l = \left\{\sin\frac{\pi l(2i-1)}{2n}\right\}_{i=1}^{n}, \qquad l = 1,\dots,n. \tag{3.11}$$

Fix some $l \in [1,n]$ and define a vector

$$\begin{bmatrix} \mathbf{u}_I \\ \mathbf{u}_\Gamma \end{bmatrix} = \begin{bmatrix} \mathbf{v}\otimes\boldsymbol{\xi}_l \\ \boldsymbol{\xi}_l \end{bmatrix},$$

with some vector $\mathbf{v} \in \mathbb{R}^n$ and substitute it in system (3.9). From the second equation of (3.9) it follows that eigenvalues of problem (3.8) are defined by the expressions

$$\mu_l = 1 - \frac{k_x}{k_x + c}\, v_n^{(l)}, \qquad l = 1,\dots,n, \tag{3.12}$$

where a parameter $v_n^{(l)}$ is the n-th component of vector $\mathbf{v}^{(l)}$ from the system

$$\tilde{A}_{II}(\mathbf{v}^{(l)}\otimes\boldsymbol{\xi}_l) = -\tilde{A}_{I\Gamma}\boldsymbol{\xi}_l$$

or, using expressions (2.13),

$$(a_x A_x + (\lambda_l a_y + b)I_x)\,\mathbf{v}^{(l)} = 2k_x \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}^T. \tag{3.13}$$

Introducing notations:

$$d_l = \lambda_l\frac{a_y}{a_x} + \frac{b}{a_x}, \qquad e = \frac{2k_x}{a_x} \equiv 2\left(1 + \frac{c}{k_x}\right), \tag{3.14}$$

system (3.13) can be rewritten in the form:

$$\begin{bmatrix} 3+d_l & -1 & & & \\ -1 & 2+d_l & -1 & & \mathbf{0} \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2+d_l & -1 \\ \mathbf{0} & & & -1 & 3+d_l \end{bmatrix} \cdot \begin{bmatrix} v_1^{(l)} \\ v_2^{(l)} \\ \vdots \\ v_{n-1}^{(l)} \\ v_n^{(l)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ e \end{bmatrix}. \tag{3.15}$$

Set $x_l = 1 + \frac{1}{2}d_l$ and consider the following recurrent sequence:

$$\begin{aligned} \alpha_0 &= 1 \\ \alpha_1 &= 2x_l + 1 \\ \alpha_{i+1} &= 2x_l\alpha_i - \alpha_{i-1}, \qquad i = 1,\dots,n-1. \end{aligned} \tag{3.16}$$

It is easy to see that

$$\alpha_i = U_i(x_l) + U_{i-1}(x_l), \qquad i = 1,\dots,n,$$

where $U_m(x)$ is the Chebyshev polynomial of the 2-nd kind of degree m:

$$U_m(x) = \frac{1}{2\sqrt{x^2-1}}\left(\left(x + \sqrt{x^2-1}\right)^{m+1} - \left(x + \sqrt{x^2-1}\right)^{-(m+1)}\right).$$

By induction it can be shown that

$$v_n^{(l)} = e\cdot\frac{\alpha_{n-1}}{\alpha_{n-1} + \alpha_n} = e\cdot\frac{U_{n-1}(x_l) + U_{n-2}(x_l)}{U_n(x_l) + 2U_{n-1}(x_l) + U_{n-2}(x_l)}. \tag{3.17}$$

Since $x_l \geq 1$ then $v_n^{(l)} \geq 1$ for any $l = 1, \ldots, n$. From (3.12) it follows that the maximal eigenvalue of problem (3.8) is bounded from above by $\mu_{max} \leq 1$.

Now we have to estimate the minimal eigenvalue μ_{min} of problem (3.8) from below. Taking into account that $U_n(x_l) = 2x_l U_{n-1}(x_l) - U_{n-2}(x_l)$ we get

$$v_n^{(l)} = e \cdot \frac{(2x_l + 1)U_{n-1}(x_l) - U_n(x_l)}{(2x_l + 2)U_{n-1}(x_l)} = \frac{e}{2} \cdot \left(\frac{2x_l + 1}{x_l + 1} - \frac{U_n(x_l)}{(x_l + 1)U_{n-1}(x_l)} \right). \tag{3.18}$$

Since

$$\begin{aligned} \frac{U_n(x)}{U_{n-1}(x)} &= \frac{(x + \sqrt{x^2 - 1})^{n+1} - (x + \sqrt{x^2 - 1})^{-(n+1)}}{(x + \sqrt{x^2 - 1})^{n} - (x + \sqrt{x^2 - 1})^{-n}} = \\ &= x + \sqrt{x^2 - 1} \left(1 + \frac{2}{(x + \sqrt{x^2 - 1})^{2n} - 1} \right), \end{aligned}$$

from (3.12), (3.14), and (3.18) it follows that

$$\begin{aligned} \mu_l &= \sqrt{\frac{x_l - 1}{x_l + 1}} \left(1 + \frac{2}{\left(x_l + \sqrt{x_l^2 - 1} \right)^{2n} - 1} \right) = \\ &= \sqrt{\frac{d_l}{4 + d_l}} \left(1 + \frac{2}{\left(1 + \frac{1}{2}d_l + \sqrt{d_l + \frac{1}{4}d_l^2} \right)^{2n} - 1} \right). \end{aligned} \tag{3.19}$$

To estimate expression (3.19) from below we consider two cases:

1. $y \equiv \left(\frac{1}{2}d_l + \sqrt{d_l + \frac{1}{4}d_l^2} \right) \geq 1/2n$. It means that $d_l + \sqrt{d_l(4 + d_l)} \geq 1/n$, or $d_l \geq \frac{1}{2n(2n+1)}$. Then we have

$$\mu_l \geq \sqrt{\frac{d_l}{4 + d_l}} \geq \frac{1}{\sqrt{1 + 8n(2n + 1)}} = \frac{1}{4n + 1} \geq \frac{h}{5}.$$

2. $y < 1/2n$. In this case $d_l < \frac{1}{2n(2n+1)}$. So, (3.19) is estimated from below as follows

$$\begin{aligned} \mu_l &= \sqrt{\frac{d_l}{4 + d_l}} \left(1 + \frac{2}{(1 + y)^{2n} - 1} \right) \geq \sqrt{\frac{d_l}{4 + d_l}} \left(1 + \frac{2}{e^{2ny} - 1} \right) \geq \\ &\geq \sqrt{\frac{d_l}{4 + d_l}} \left(1 + \frac{2}{(e - 1)2ny} \right) \geq \sqrt{\frac{d_l}{4 + d_l}} \cdot \frac{1 + ny}{2ny} = \\ &= \sqrt{\frac{d_l}{4 + d_l}} \left(\frac{\frac{1}{n} + \frac{1}{2}\left(d_l + \sqrt{d_l(4 + d_l)} \right)}{d_l + \sqrt{d_l(4 + d_l)}} \right) = \frac{\frac{1}{n} + \frac{1}{2}\left(d_l + \sqrt{d_l(4 + d_l)} \right)}{4 + \left(d_l + \sqrt{d_l(4 + d_l)} \right)} \geq \\ &\geq \min\left\{ \frac{1}{2}; \frac{1}{4n} \right\} \geq \frac{h}{4}. \end{aligned}$$

From these estimates it follows that $\mu_l \geq h/5$ for any $l = 1, \ldots, n$. Thus, the minimal eigenvalue of problem (3.8) is bounded from below by $\mu_{min} \geq h/5$ and we have

$$\frac{h}{5} \, (A_{\Gamma\Gamma}^{(1)} \mathbf{u}_\Gamma, \mathbf{u}_\Gamma) \leq (\Lambda_\Gamma^{(1)} \mathbf{u}_\Gamma, \mathbf{u}_\Gamma) \leq (A_{\Gamma\Gamma}^{(1)} \mathbf{u}_\Gamma, \mathbf{u}_\Gamma), \qquad \forall \mathbf{u}_\Gamma \in \mathbb{R}^{N_\Gamma}.$$

Analogously, we have the same estimates for matrices $A^{(2)}_{\Gamma\Gamma}$ and $\Lambda^{(2)}_{\Gamma}$, which completes the proof. □

REMARK 3.1. Lemma 3.1 holds true if on some of the other edges of the rectangular subdomain Ω_1 a homogeneous Neumann boundary condition is imposed.

REMARK 3.2. If a homogeneous Neumann boundary condition is imposed on the whole remaining part of the boundary of subdomain $\partial\Omega_1 \setminus \Gamma$ then inequalities (3.7) of Lemma 3.1 should be replaced by:

$$\begin{aligned} \alpha \cdot h\,(A_{\Gamma\Gamma}\mathbf{u}_\Gamma, \mathbf{u}_\Gamma) \le (\Lambda_\Gamma \mathbf{u}_\Gamma, \mathbf{u}_\Gamma) \le (A_{\Gamma\Gamma}\mathbf{u}_\Gamma, \mathbf{u}_\Gamma), \\ \forall \mathbf{u}_\Gamma \in \mathbb{R}^{N_\Gamma} \setminus \ker(\Lambda_\Gamma), \end{aligned} \tag{3.20}$$

where constant α does not depend on mesh size parameter h and coefficients of the subdomain problems.

We proceed with the construction of preconditioner B_Γ. We define it in the form of an inner Chebyshev iterative procedure [2, 9, 19]. From Lemma 3.1 we know that the eigenvalues of matrix $A^{-1}_{\Gamma\Gamma}\Lambda_\Gamma$ belong to segment $[h/5, 1]$. Let $P_L(y)$ be the polynomial of least deviation from zero on this segment that satisfies the condition $P_L(0) = 1$. Denote by β_l, $l = 1, \ldots, L$, the inverses of the roots of polynomial $P_L(y)$. The formulae for $P_L(y)$ and its roots $1/\beta_l$, $l = 1, \ldots, L$, can be found, e.g. in [27]. Then preconditioner B_Γ for matrix Λ_Γ is determined by:

$$B^{-1}_\Gamma = \left\{ I_\Gamma - \prod_{l=1}^{L} \left(I_\Gamma - \beta_l A^{-1}_{\Gamma\Gamma}\Lambda_\Gamma \right) \right\} \Lambda^{-1}_\Gamma. \tag{3.21}$$

The procedure for calculating the vector $\mathbf{w}_\Gamma = B^{-1}_\Gamma \mathbf{g}_\Gamma$ for given $\mathbf{g}_\Gamma \in \mathbb{R}^{N_\Gamma}$ has the form:

$$\begin{aligned} \mathbf{w}^{(0)}_\Gamma &= \mathbf{0}, \\ \mathbf{w}^{(l)}_\Gamma &= \mathbf{w}^{(l-1)}_\Gamma - \beta_l A^{-1}_{\Gamma\Gamma}\left(\Lambda_\Gamma \mathbf{w}^{(l-1)}_\Gamma - \mathbf{g}_\Gamma\right), \qquad l = 1, \ldots, L, \\ \mathbf{w}_\Gamma &= \mathbf{w}^{(L)}_\Gamma. \end{aligned} \tag{3.22}$$

For computational stability, instead of (3.22), we can use the three-term recurrence relation [27]. We return to the realization of the iterative procedure (3.22) in the next subsection.

Lemma 3.1 and the theory of Chebyshev iterative methods imply the following basic result.

THEOREM 3.2. *Let $L \ge (5/h)^{1/2}$. Then matrix B_Γ in* (3.21) *is spectrally equivalent to matrix Λ_Γ with constants of equivalence independent of mesh size parameter h and the value of coefficients $k_{x,i}$, $k_{y,i}$, $c_{0,i}$, $i = 1, 2$, in the subdomains.*

REMARK 3.3. Clearly, in the theory L is chosen to be of the order $(5/h)^{1/2}$. In practice it is calculated explicitly after the boundaries of the spectrum of matrix $A^{-1}_{\Gamma\Gamma}\Lambda_\Gamma$ have been calculated by an appropriate iterative procedure [17].

3.2. Arithmetical complexity of the interface preconditioner. In order to estimate the arithmetic complexity of multiplying a vector by matrix B^{-1}_Γ it is sufficient to estimate the arithmetic complexity of multiplying a vector by matrix Λ_Γ because we know that matrix $A_{\Gamma\Gamma}$ is diagonal and the number of iterations L in the inner Chebyshev iterative procedure (3.22) is of the order $(5n)^{1/2}$.

The procedure of finding the product $\Lambda_\Gamma \mathbf{u}_G$ is based on a partial solution technique [3, 4, 18], which we outline here. Let us consider the terms $\Lambda^{(1)}_\Gamma \mathbf{u}_\Gamma$ and $\Lambda^{(2)}_\Gamma \mathbf{u}_\Gamma$ in the expression $\mathbf{v}_\Gamma = \Lambda^{(1)}_\Gamma \mathbf{u}_\Gamma + \Lambda^{(2)}_\Gamma \mathbf{u}_\Gamma$ separately. Below we define the multiplication procedure only for the first term $\Lambda^{(1)}_\Gamma \mathbf{u}_\Gamma$. The second term is treated in the same manner. Again, we omit the index "(1)" for simplicity.

First, vector $\mathbf{v}_\Gamma = \Lambda_\Gamma \mathbf{u}_\Gamma = A_{\Gamma\Gamma}\mathbf{u}_\Gamma - \tilde{A}_{\Gamma I}\tilde{A}_{II}^{-1}\tilde{A}_{I\Gamma}\mathbf{u}_\Gamma$ is rewritten as

$$\mathbf{v}_\Gamma = A_{\Gamma\Gamma}\mathbf{u}_\Gamma + \tilde{A}_{\Gamma I}\mathbf{u}_I, \tag{3.23}$$

where vector $\mathbf{u}_I$ is defined from the system of equations

$$\tilde{A}_{II}\mathbf{u}_I = -\tilde{A}_{I\Gamma}\mathbf{u}_\Gamma \equiv \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ 2k_x\mathbf{u}_\Gamma \end{bmatrix}, \tag{3.24}$$

with matrix $\tilde{A}_{II}$ defined by (2.14).

Let us denote by W_y an orthogonal $n \times n$ matrix of eigenvectors of problem (3.10) and by L_y a diagonal $n \times n$ matrix of the eigenvalues of matrix A_y (3.11):

$$W_y = \{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n\}, \qquad L_y = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}.$$

Then we define an $N_I \times N_I$ orthogonal matrix $Q = I_x \otimes W_y$. Introducing a vector $\mathbf{v}_I = Q^T\mathbf{u}_\Gamma$ and multiplying both parts of equation (3.24) by matrix Q^T we get the following matrix equation:

$$Q^T\tilde{A}_{II}Q\mathbf{v}_I \equiv \left(a_x(A_x \otimes I_y) + a_y(I_x \otimes L_y) + b(I_x \otimes I_y)\right)\mathbf{v}_I = 2k_x \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ W_y^T\mathbf{u}_\Gamma \end{bmatrix}. \tag{3.25}$$

Note that this system can be decoupled into n independent linear systems:

$$(a_xA_x + (b+\lambda_l a_y)I_x) \begin{bmatrix} v_1^{(l)} \\ \vdots \\ v_{n-1}^{(l)} \\ v_n^{(l)} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 2k_x w_l \end{bmatrix}, \qquad l = 1, \ldots, n, \tag{3.26}$$

where the component w_l is the l-th component of the backward Fourier transform $\mathbf{w} = W_y^T\mathbf{u}_\Gamma$ of vector $\mathbf{u}_\Gamma$.

As soon as we find vector $\mathbf{v}_I$ from systems (3.26) we compute the product

$$\tilde{A}_{\Gamma I}\mathbf{u}_I = \tilde{A}_{\Gamma I}Q\mathbf{v}_I = -2k_xW_y \begin{bmatrix} v_n^{(1)} \\ \vdots \\ v_n^{(n)} \end{bmatrix}. \tag{3.27}$$

From (3.27) it follows that to define product $\Lambda_\Gamma\mathbf{u}_\Gamma$ we need to know only the last components $v_n^{(l)}$, $l = 1, \ldots, n$, of the solution vectors $\mathbf{v}^{(l)}$ from systems (3.26).

Now we define the partial solution algorithm:

0. On the initial step we solve n systems with three-diagonal $n \times n$ matrices:

$$(a_xA_x + (b+\lambda_l a_y)I_x) \begin{bmatrix} x_1^{(l)} \\ \vdots \\ x_{n-1}^{(l)} \\ x_n^{(l)} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

   and store the last components of the vectors $\mathbf{x}^{(l)}$, i.e. parameters $x_n^{(l)}$, $l = 1, \ldots, n$. It requires $O(n^2)$ operations and $O(n)$ elements of memory.

1. Given vector $\mathbf{u}_\Gamma$ we use the discrete fast Fourier transform algorithm to compute a vector $\mathbf{w} = 2k_x W_y^T \mathbf{u}_\Gamma$ for only $O(n \ln n)$ operations.
2. Then, we compute a vector $\mathbf{v} = \left[v_n^{(1)}, \dots, v_n^{(n)}\right]$ from (3.27) by the formulae $v_n^{(l)} = w_l \cdot x_n^{(l)}$, $l = 1, \dots, n$.
3. Again, we use the discrete fast Fourier transform algorithm to compute a vector $\mathbf{p} = -2k_x W_y \mathbf{v}$.
4. Finally, we compute the vector $\mathbf{v}_\Gamma = A_{\Gamma\Gamma}\mathbf{u}_\Gamma + \mathbf{p}$.

As a result of this algorithm the procedure of multiplying a vector by matrix B_Γ^{-1} can be implemented for $O(n^2 + n^{3/2} \ln n)$ operations. Note that this estimate does not depend on the coefficients of the problem.

REMARK 3.4. We provided computations of the complexity for so-called "parallel" partial solutions, that is, when the grid line where we need to find a solution is parallel to the grid line where the right-hand side is nonzero. For the case of so-called "perpendicular" partial solutions, when the grid line where we need the solution is perpendicular to the grid line with the nonzero right-hand side we can use an algorithm of the approximate partial solution [3, 18, 23]. Instead of computing the exact value of $\mathbf{v}_\Gamma = \Lambda_\Gamma \mathbf{u}_\Gamma$, we calculate the approximation $\mathbf{v}_\Gamma^{(\varepsilon)} = \Lambda_\Gamma^{(\varepsilon)} \mathbf{u}_\Gamma$. It can be shown [3, 23] that taking the accuracy parameter $\varepsilon \sim h^p$, $p \geq 2$, the matrix $\Lambda_\Gamma^{(\varepsilon)}$ is spectrally equivalent to Λ_Γ.

As a consequence one can develop the partial solution algorithm for the general case when we need to find the partial solution on the entire boundary of the rectangular subdomain. Using the results of [3, 18] the partial solution on the boundary $\partial\Omega_i$ of the rectangular subdomain Ω_i can be found for $O(n^2 + pn \ln^2(n))$, where the first term of this estimate corresponds to the initial step and can be obtained before starting the iterative process. If we use the algorithm of the approximate partial solution developed in [23] we have an estimate of computational cost $O(n^2 + pn^{3/2} \ln^2(n))$.

Therefore, the algorithm of multiplying a vector by matrix B_Γ^{-1} can be implemented for $O(n^2 + pn^{3/2} \ln^2(n))$ operations. This estimate does not depend on the coefficients of the problem.

3.3. Interface preconditioner for general problem. Now let us consider problem (1.1) in the domain $\Omega = \bigcup_{i=1}^{m} \Omega_i$, being the union of m rectangles Ω_i, $i = 1, \dots, m$, as is described in Section 1.

Using the same arguments as in Subsection 3.1 it is easy to show that the statement of Lemma 3.1 holds true even for a general domain Ω composed of rectangles. Since preconditioner B_Γ is constructed by defining subdomain preconditioners $B_\Gamma^{(i)}$ it is sufficient to consider the model problem in the rectangular subdomain Ω_i with homogeneous Neumann boundary condition on the whole boundary $\partial\Omega \equiv \Gamma = \bigcup_{i=1}^{4} \Gamma_i$. Boundary nodes belonging to Γ ("$\diamond$") and internal nodes ("$\circ$") are schematically shown in Figure 4.

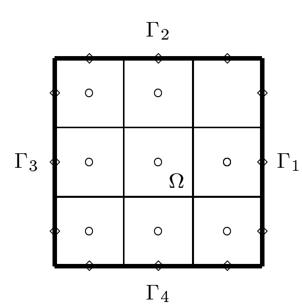


FIGURE 4. *Degrees of freedom of the Neumann subdomain problem.*

Denote by $\mathbf{u}_\Gamma$ and by $\mathbf{u}_{\Gamma_i}$, $i = 1, \dots, 4$, the vectors of the degrees of freedom on the boundaries Γ and Γ_i, $i = 1, \dots, 4$, respectively. Following [21], one can show that for any vector $\mathbf{v}_\Gamma \in \mathbb{R}^{N_\Gamma}$ such that $\mathbf{v}_\Gamma \perp \ker(\Lambda_\Gamma)$ the following is valid:

$$(3.28)\quad (\Lambda_\Gamma \mathbf{v}_\Gamma, \mathbf{v}_\Gamma) = \inf_{\substack{v^h \in V_h(\Omega) \\ v^h|_\Gamma = v^h_\Gamma}} a^h_\Omega(v^h, v^h) \geq \inf_{\substack{w^h_i \in V_h(\Omega) \\ w^h_i|_{\Gamma_i} = v^h_{\Gamma_i}}} a^h_\Omega(w^h_i, w^h_i) = (\Lambda_{\Gamma_i} \mathbf{v}_{\Gamma_i}, \mathbf{v}_{\Gamma_i}),\ i = 1, \dots, 4,$$

where u^h_Γ, $u^h_{\Gamma_i}$, $i = 1, \dots, 4$, are piecewise constant functions defined on Γ, Γ_i, $i = 1, \dots, 4$, and generated by vectors $\mathbf{u}_\Gamma$, $\mathbf{u}_{\Gamma_i}$, $i = 1, \dots, 4$, respectively.

From (3.7), (3.20), and (3.28) it follows that for any vector $\mathbf{v}_\Gamma \in \mathbb{R}^{N_\Gamma}$ such that $\mathbf{v}_\Gamma \perp \ker(\Lambda_\Gamma)$ we have

$$(3.29)\quad \begin{aligned} (A_{\Gamma\Gamma}\mathbf{u}_\Gamma, \mathbf{u}_\Gamma) &\geq (\Lambda_\Gamma \mathbf{u}_\Gamma, \mathbf{u}_\Gamma) \geq \tfrac{1}{4}\sum_{i=1}^{4} (\Lambda_{\Gamma_i}\mathbf{u}_{\Gamma_i}, \mathbf{u}_{\Gamma_i}) \geq \\ &\geq \alpha h \cdot \sum_{i=1}^{4} (\hat{A}_{\Gamma_i\Gamma_i}\mathbf{u}_{\Gamma_i}, \mathbf{u}_{\Gamma_i}) \geq \alpha h \cdot (A_{\Gamma\Gamma}\mathbf{u}_\Gamma, \mathbf{u}_\Gamma). \end{aligned}$$

It means that we can define preconditioner B_Γ for the Schur complement (1.15) in the form (3.21). By Theorem 3.2 matrix B_Γ is spectrally equivalent to matrix Λ_Γ provided that the degree L of the matrix polynomial (3.21) is chosen to be $O(h^{-1/2})$.

Using the partial solution technique described in Subsection 3.2 and Remark 3.4 one can shown that the procedure of multiplying a vector by matrix B_Γ^{-1} can be implemented for $O(h^{-2} + h^{-3/2}\ln^2 h^{-1})$.

Now assume that we use the AMG methods to solve the subdomain problems. Then problem (1.8) with matrix $\hat{A}$ can be solved for $O(h^{-2} + h^{-3/2}\ln^2 h^{-1})$ operations. As was mentioned in Section 2 we use the matrix A as the preconditioner in a preconditioned iterative method to solve the problem (1.6).

Summarizing the results of Sections 2 and 3 we can formulate the following statement.

STATEMENT 3.1. *Under the above assumptions the arithmetic complexity of the proposed algorithm for solving problem* (1.6) *is estimated from above by*

$$C \cdot (h^{-2} + h^{-3/2}\ln^2 h^{-1}),$$

where constant C is independent of mesh size parameter h and coefficients of the problem $\tilde{K}(\mathbf{x})$ and $\tilde{a}_0(\mathbf{x})$.

4. Results of the numerical experiments. In this section the method of preconditioning being presented is tested on the model problem in the unit square:

$$\begin{aligned} -k_x\frac{\partial^2 u}{\partial x^2} - k_y\frac{\partial^2 u}{\partial y^2} &= f, \qquad && \text{in } \Omega \equiv [0,1]^2, \\ u &= 0, && \text{on } \partial\Omega. \end{aligned}$$

Domain Ω is composed of 4 subdomains as shown in Figure 5. Coefficients k_x and k_y are constants in each subdomain.

Ω_3 $k_x = 1$ $k_y = k$	Ω_4 $k_x = k$ $k_y = 1$
Ω_1 $k_x = k$ $k_y = 1$	Ω_2 $k_x = 1$ $k_y = k$

FIGURE 5. *Coefficients in the subdomains for a model problem.*

The domain is divided into n^2 squares (n in each direction) and each square is partitioned into 2 triangles. The dimension of the original algebraic system is $N = 3n^2 - 2n$ and the dimension of the Schur complement after elimination of the subdomain problems is $N_\Gamma = 4n$. The right-hand side is generated randomly, and the accuracy parameter is taken as $\varepsilon = 10^{-6}$. The degree of matrix polynomial (3.21) equals $L = \left[\sqrt{2.5n}\right] + 1$, where $[\eta]$ is an integer part of η. The condition number of matrix $A_{\Gamma\Gamma}^{-1}\Lambda_\Gamma$ is calculated by the relation between the conjugate gradient and the Lanczos algorithm [16].

The results are summarized in Table 1, where $n_{\rm iter}$ and $Cond$ denote the iteration number and condition number, respectively. All experiments are carried out on Sun Workstation.

TABLE 1.

	100×100 $N = 29800$		200×200 $N = 119600$		400×400 $N = 479200$	
k	$n_{\rm iter}$	$Cond$	$n_{\rm iter}$	$Cond$	$n_{\rm iter}$	$Cond$
1	23	10.7	25	10.9	26	10.9
10	23	9.2	24	9.8	26	10.2
100	20	8.3	19	7.9	20	8.1
1000	12	4.2	14	6.2	14	6.4
10000	6	1.5	7	2.0	7	2.1
100000	3	1.1	4	1.1	4	1.1

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