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MULTIGRID AND MULTILEVEL METHODS FOR NONCONFORMING ROTATED Q1 ELEMENTS

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ABSTRACT. In this paper we systematically study multigrid algorithms and multilevel preconditioners for discretizations of second-order elliptic problems using nonconforming rotated Q_1 finite elements. We first derive optimal results for the $\mathcal{W}$-cycle and variable $\mathcal{V}$-cycle multigrid algorithms; we prove that the $\mathcal{W}$-cycle algorithm with a sufficiently large number of smoothing steps converges in the energy norm at a rate which is independent of grid number levels, and that the variable $\mathcal{V}$-cycle algorithm provides a preconditioner with a condition number which is bounded independently of the number of grid levels. In the case of constant coefficients, the optimal convergence property of the $\mathcal{W}$-cycle algorithm is shown with any number of smoothing steps. Then we obtain suboptimal results for multilevel additive and multiplicative Schwarz methods and their related $\mathcal{V}$-cycle multigrid algorithms; we show that these methods generate preconditioners with a condition number which can be bounded at least by the number of grid levels. Also, we consider the problem of switching the present discretizations to spectrally equivalent discretizations for which optimal preconditioners already exist. Finally, the numerical experiments carried out here complement these theories.

1. INTRODUCTION

In recent years there has been analyses and applications of the nonconforming rotated (NR) Q_1 finite elements for the numerical solution of partial differential problems. These nonconforming rectangular elements were first proposed and analyzed in [23] for numerically solving the Stokes problem; they are the simplest divergence-free nonconforming elements on rectangles (respectively, rectangular parallelepipeds). Then they were used to simulate the deformation of martensitic crystals with microstructure [17] due to their simplicity. Conforming finite element methods can be used to approximate the microstructure with layers which are oriented with respect to meshes, while nonconforming finite element methods allow the microstructure to be approximated on meshes which are not aligned with the microstructure (see, e.g., [17] for the references).

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Independently, the NR Q_1 elements have been derived within the framework of mixed finite element methods [11, 1]. It has been shown that the nonconforming method using these elements is equivalent to the mixed method exploiting the lowest-order Raviart-Thomas mixed elements on rectangles (respectively, rectangular parallelepipeds) [24]. Based on this equivalence theory, both the NR Q_1 and the Raviart-Thomas mixed methods have been applied to model semiconductor devices [11]; they have been effectively employed to compute the electric potential equation with a doping profile which has a sharp junction.

Error estimates of the NR Q_1 elements can be derived by the classical finite element analysis [23, 16]. They can be also obtained from the known results on the mixed method based on the equivalence between these two methods [1]. It has been shown that the so-called "nonparametric" rotated Q_1 elements produce optimal-order error estimates. As a special case of the nonparametric families, the optimal-order errors can be obtained for partitions into rectangles (respectively, rectangular parallelepipeds) oriented along the coordinate axes. Finally, in the case of cubic triangulations, superconvergence results can be obtained [1, 16].

Unlike the simplest triangular nonconforming elements, i.e., the nonconforming P_1 elements, the NR Q_1 elements do not have any reasonable conforming subspace. Consequently, there are differences between these two types of nonconforming elements. The NR Q_1 elements can be defined on rectangles (respectively, rectangular parallelepipeds) with degrees of freedom given by the values at the midpoints of edges of the rectangles (respectively, the centers of faces of the rectangular parallelepipeds), or by the averages over the edges of the rectangles (respectively, the faces of the rectangular parallelepipeds). While these two versions lead to the same definition for the nonconforming P_1 elements, they can produce very different results in terms of implementation for the NR Q_1 elements. With the second version of the NR Q_1 elements, we are able to prove all the theoretical results for the multigrid algorithms and multilevel additive and multiplicative Schwarz methods considered in this paper. However, we are unable to obtain these results with their first version. In particular, as numerical tests in [22] indicate, the energy norm of the iterates of the usual intergrid transfer operators, which enters both upper and lower bounds for the condition number of preconditioned systems, deteriorates with the number of grid levels for the first version. But it is bounded independently of the number of grid levels for the second version, as shown here.

The other major difference between the nonconforming P_1 and the NR Q_1 elements is that the former contains the conforming P_1 elements, while the latter does not contain any reasonable conforming subspace, as mentioned above. As a result of this, the convergence of the standard $\mathcal{V}$-cycle algorithm for the nonconforming P_1 elements can be shown when the coarse-grid correction steps of this algorithm are established on the conforming P_1 spaces [18, 12]. But this is not the case for the NR Q_1 elements. On the other hand, within the context of the nonconforming methods, i.e., when the coarse-grid correction steps are defined on the nonconforming P_1 spaces themselves, the convergence of the $\mathcal{V}$-cycle algorithm has not been shown, and the $\mathcal{W}$-cycle algorithm has been proven to converge only under the assumption that the number of smoothing steps is sufficiently large [7, 8, 3, 4, 25, 1, 12, 14]. However, we are here able to show the convergence of the $\mathcal{W}$-cycle algorithm with any number of smoothing steps for the Laplace equation using the NR Q_1 elements. This optimal property cannot be proven for the nonconforming P_1 elements using the present techniques.

The multigrid algorithms for the NR Q_1 elements were first developed and analyzed in [1], and further discussed in [12] and [9]. The second version of these elements was used in [1] and [12], while their first version was exploited in [9]. Moreover, the analysis in [9] was given for elliptic boundary value problems which are not required to have full elliptic regularity. However, in all these three papers, only the $\mathcal{W}$-cycle algorithm with a sufficiently large number of smoothing steps was shown to converge using the standard proof of convergence of multigrid algorithms for conforming finite element methods [2]. We finally mention that the study of the NR Q_1 elements in the context of domain decomposition methods has been given in [13].

In this paper we systematically study multigrid algorithms and multilevel preconditioners for discretizations of second-order elliptic problems using the NR Q_1 elements. We first consider the convergence of the $\mathcal{W}$-cycle and variable $\mathcal{V}$-cycle algorithms for these nonconforming elements. We prove that the $\mathcal{W}$-cycle algorithm with a sufficiently large number of smoothing steps converges in the energy norm at a rate which is independent of grid number levels, and that the variable $\mathcal{V}$-cycle algorithm provides a preconditioner with a condition number which is bounded independently of the number of grid levels. A main observation in this paper is that the optimal convergence property for the $\mathcal{W}$-cycle algorithm holds with any number of smoothing steps, when the coefficient of the differential problems is constant. Explicit bounds for the convergence rate and condition number are given. The NR Q_1 elements has so far been the first type of nonconforming elements which are shown to possess this feature for the $\mathcal{W}$-cycle algorithm with any number of smoothing steps.

We then study multilevel preconditioners of hierarchical basis and BPX type [5] for the NR Q_1 elements. We develop a convergence theory for the multilevel additive and multiplicative Schwarz methods and their related $\mathcal{V}$-cycle algorithms. We follow the general theory introduced in [22] where the analysis of the hierarchical basis and BPX type for nonconforming discretizations of partial differential equations was carried out. A key ingredient in the analysis is to control the energy norm growth of the iterated coarse-to-fine grid operators, which enters both upper and lower bounds for the condition number of preconditioned systems as outlined above. So far, the energy norm of the iterated intergrid transfer operators has been shown to be bounded independently of grid levels solely for the nonconforming P_1 elements [19]. In this paper we prove this property for the NR Q_1 elements. Based on the present theory, we derive a suboptimal result for the multilevel preconditioners of hierarchical basis and BPX type for the NR Q_1 elements.

Finally, we study the problem of switching the NR Q_1 discretization system to a spectrally equivalent discretization system for which optimal preconditioners are already available. This switching strategy has been used in the setting of the multilevel additive Schwarz method; see [21] for the references. After we find a spectrally equivalent reference discretization for the NR Q_1 system, we are able to obtain optimal preconditioner results for the NR Q_1 elements.

Thanks to the equivalence between the rotated Q_1 nonconforming method and the lowest-order Raviart-Thomas mixed rectangular method, all the results derived here carry over directly to the latter method [1, 12].

For technical reasons, all the results in this paper are shown for partitions into uniform squares (respectively, cubes). They can be extended to triangulations in which the finest triangulation can be mapped to a square (respectively, cubic) triangulation in an affine-invariant fashion. Also, the analysis is given for the two-dimensional domain; an extension to three space dimensions is straightforward for most of the results below.

The rest of the paper is organized as follows. In the next section we prove some preliminary results for the intergrid transfer operators. Then the multigrid algorithms and multilevel preconditioners are discussed in §3 and §4, respectively. The problem of switching to a spectrally equivalent discretization is considered in §5. Finally, the numerical results presented in §6 complement the present theories.

2. PRELIMINARY RESULTS

For expositional convenience, let $\Omega = (0,1)^2$ be the unit square, and let $H^s(\Omega)$ and $L^2(\Omega) = H^0(\Omega)$ be the usual Sobolev spaces with the norm

$$||v||_s = \left(\int_\Omega \sum_{|\alpha| \leq s} |D^\alpha v|^2 \mathrm{d}x \right)^{1/2},$$

where s is a nonnegative integer. Also, let $(\cdot, \cdot)$ denote the $L^2(\Omega)$ or $(L^2(\Omega))^2$ inner product, as appropriate. The $L^2(\Omega)$ norm is indicated by $||\cdot||$. Finally,

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_\Gamma = 0\},$$

where $\Gamma = \partial\Omega$.

Let h_1 and $\mathcal{E}_{h_1} = \mathcal{E}_1$ be given, where $\mathcal{E}_{h_1}$ is a partition of Ω into uniform squares with length h_0 and oriented along the coordinate axes. For each integer $2 \leq k \leq K$, let $h_k = 2^{1-k} h_1$ and $\mathcal{E}_{h_k} = \mathcal{E}_k$ be constructed by connecting the midpoints of the edges of the squares in $\mathcal{E}_{k-1}$, and let $\mathcal{E}_h = \mathcal{E}_K$ be the finest grid. Also, let $\partial\mathcal{E}_k$ be the set of all interior edges in $\mathcal{E}_k$. In this and the following sections, we replace subscript h_k simply by subscript k.

For each k, we introduce the rotated Q_1 nonconforming space

$$V_k = \Big\{ v \in L^2(\Omega) : v|_E = a_E^1 + a_E^2 x + a_E^3 y + a_E^4 (x^2 - y^2),\ a_E^i \in \mathbb{R},\ \forall E \in \mathcal{E}_k;$$
$$\text{if } E_1 \text{ and } E_2 \text{ share an edge } e, \text{ then } \int_e \xi|_{\partial E_1} \mathrm{d}s = \int_e \xi|_{\partial E_2} \mathrm{d}s;$$
$$\text{and } \int_{\partial E \cap \Gamma} \xi|_\Gamma \mathrm{d}s = 0 \Big\}.$$

Note that $V_k \not\subset H_0^1(\Omega)$ and $V_{k-1} \not\subset V_k$, $k \geq 2$.

We introduce the space

$$\hat{V}_k = \sum_{l=1}^k V_l \supset V_k,$$

the discrete energy scalar product on $\hat{V}_k \oplus H_0^1(\Omega)$ by

$$(v,w)_{\mathcal{E},k} = \sum_{E \in \mathcal{E}_k} (\nabla v, \nabla w)_E, \qquad v,\, w \in \hat{V}_k \oplus H_0^1(\Omega),$$

and the discrete norm on $\hat{V}_k \oplus H_0^1(\Omega)$ by

$$||v||_{\mathcal{E},k} = \sqrt{(v,v)_{\mathcal{E},k}}, \qquad v \in \hat{V}_k \oplus H_0^1(\Omega).$$

We introduce two sets of intergrid transfer operators $I_k : V_{k-1} \to V_k$ and $P_{k-1} : V_k \to V_{k-1}$ as follows. Following [1, 12], if $v \in V_{k-1}$ and e is an edge of a square in $\mathcal{E}_k$, then $I_k v \in V_k$ is defined by

$$\frac{1}{|e|}\int_e I_k v \mathrm{d}s = \begin{cases} 0 & \text{if } e \subset \partial\Omega, \\ \frac{1}{|e|}\int_e v\mathrm{d}s & \text{if } e \not\subset \partial E \text{ for any } E \in \mathcal{E}_{k-1}, \\ \frac{1}{2|e|}\int_e (v|_{E_1} + v|_{E_2})\mathrm{d}s & \text{if } e \subset \partial E_1 \cap \partial E_2 \text{ for some } E_1, E_2 \in \mathcal{E}_{k-1}. \end{cases}$$

If $v \in V_k$ and e is an edge of an element in $\partial\mathcal{E}_{k-1}$, then $P_{k-1}v \in V_{k-1}$ is given by

$$\frac{1}{|e|}\int_e P_{k-1}v\mathrm{d}s = \frac{1}{2}\left\{\frac{1}{|e_1|}\int_{e_1} v\mathrm{d}s + \frac{1}{|e_2|}\int_{e_2} v\mathrm{d}s\right\},$$

where e_1 and e_2 in $\partial\mathcal{E}_k$ form the edge $e \in \partial\mathcal{E}_{k-1}$. Note that the definition of P_{k-1} automatically preserves the zero average values on boundary edges. Also, it can be seen that

$$P_{k-1}I_k v = v, \qquad v \in V_{k-1},\ k \geq 1. \tag{2.1}$$

That is, $P_{k-1}I_k$ is the identity operator Id_{k-1} on V_{k-1}. This relation is not satisfied when the NR Q_1 elements are defined with degrees of freedom given by the values at the midpoints of edges of elements.

We also define the iterates of I_k and P_{k-1} by

$$R_k^K = I_K \cdots I_{k+1} : V_k \to V_K,$$
$$Q_k^K = P_k \cdots P_{K-1} : V_K \to V_k.$$

Finally, we make the convention on the discrete energy scalar product on the space $\hat{V}_K$:

$$(v, w)_{\mathcal{E}} = (v, w)_{\mathcal{E},K}, \quad v,\, w \in \hat{V}_K.$$

Obviously, we have the inverse inequality

$$||v||_{\mathcal{E}} \leq C2^k||v||, \quad v \in \hat{V}_k,\, 1 \leq k \leq K, \tag{2.2}$$

(here and later, by C, c,... we denote generic positive constants which are independent of k).

In this section we collect some basic properties of the intergrid transfer operators P_{k-1} (respectively, I_k) and their iterates Q_k^K (respectively, R_k^K). The crucial results are the boundedness of the operators I_k with constant $\sqrt{2}$ and the uniform boundedness of the operators R_k^K with respect to the discrete energy norm $||\cdot||_{\mathcal{E}}$.

Lemma 2.1. *It holds that P_{k-1} $(2 \leq k \leq K)$ is an orthogonal projection with respect to the energy scalar product; i.e., for any $v \in V_k$,*

$$\begin{aligned}(v - P_{k-1}v, w)_{\mathcal{E}} &= 0, \qquad \forall w \in V_{k-1}, \\ ||v||_{\mathcal{E}}^2 &= ||v - P_{k-1}v||_{\mathcal{E}}^2 + ||P_{k-1}v||_{\mathcal{E}}^2.\end{aligned} \tag{2.3}$$

Moreover, there are constants C and c, independent of v, such that the difference $\hat{v} = v - P_{k-1}v \in \hat{V}_k$ satisfies

$$c2^k||\hat{v}|| \leq ||\hat{v}||_{\mathcal{E}} \leq C2^k||\hat{v}||. \tag{2.4}$$

PROOF. For any $E \in \mathcal{E}_{k-1}$ with the four subsquares $E_i \in \mathcal{E}_k$ ($i = 1, \ldots, 4$, see Figure 1), an application of Green's formula implies that

$$\begin{aligned}(\nabla[v - P_{k-1}v], \nabla w)_E &= \sum_{i=1}^4 (\nabla[v - P_{k-1}v], \nabla w)_{E_i} \\ &= \sum_{i=1}^4 \sum_{j=1}^4 \frac{\partial w}{\partial \nu_{E_i}^j}\Big|_{e_{E_i}^j} \int_{e_{E_i}^j} (v - P_{k-1}v)|_{E_i} \mathrm{d}s,\end{aligned} \tag{2.5}$$

where $e_{E_i}^j$ are the four edges of E_i with the outer unit normals $\nu_{E_i}^j$, $i = 1, \ldots, 4$. Note that in (2.5) the line integrals over edges interior to $E \in \mathcal{E}_{k-1}$ cancel by continuity of $P_{k-1}v$ in the interior of E. Also, if $e_{E_i}^j$ and $e_{E_{\hat{i}}}^{\hat{j}}$ form an edge of E, it follows by the definition of P_{k-1} that

$$\int_{e_{E_i}^j} (v - P_{k-1}v)|_{E_i} \mathrm{d}s + \int_{e_{E_{\hat{i}}}^{\hat{j}}} (v - P_{k-1}v)|_{E_{\hat{i}}} \mathrm{d}s = 0,$$

and that

$$\frac{\partial w}{\partial \nu_{E_i}^j}\Big|_{e_{E_i}^j} = \frac{\partial w}{\partial \nu_{E_{\hat{i}}}^{\hat{j}}}\Big|_{e_{E_{\hat{i}}}^{\hat{j}}}$$

is constant. Then, by (2.5), we see that

$$(\nabla[v - P_{k-1}v], \nabla w)_E = 0.$$

Now, sum on all $E \in \mathcal{E}_{k-1}$ to derive the orthogonality relations in (2.3).

The upper estimate in (2.4) directly follows from (2.2). The lower bound can be easily obtained from a direct calculation of the energy norms of $v - P_{K-1}$ on all $E \in \mathcal{E}_{k-1}$. This completes the proof. #

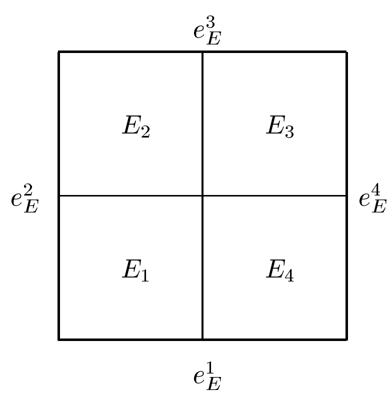

FIGURE 1. Edges and subsquares of $E \in \mathcal{E}_{k-1}$.

Before we start with the investigation of the prolongations I_k, it will be useful to collect some formulas. For $E \in \mathcal{E}_{k-1}$ and any $v \in V_{k-1}$, define

$$\frac{1}{|e_E^i|} \int_{e_E^i} v \mathrm{d}s = b_E^i,$$

(see Figure 1 for the notation), and set

$$\begin{aligned} s_E &= b_E^1 + b_E^2 + b_E^3 + b_E^4, \qquad & \triangle_E^1 &= b_E^3 - b_E^1, \\ \theta_E^0 &= b_E^1 + b_E^3 - b_E^2 - b_E^4, \qquad & \triangle_E^2 &= b_E^4 - b_E^2. \end{aligned}$$

Then, with the subscript E omitted, we have the next lemma.

Lemma 2.2. *It holds that*

$$\begin{aligned} ||v||^2_{L^2(E)} &= h^2_{k-1}\left(\tfrac{1}{16}s^2 + \tfrac{1}{12}\{(\triangle^1)^2 + (\triangle^2)^2\} + \tfrac{1}{40}(\theta^0)^2\right), \\ ||\nabla v||^2_{L^2(E)} &= (\triangle^1)^2 + (\triangle^2)^2 + \tfrac{3}{2}(\theta^0)^2, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \tfrac{h^2_{k-1}}{10}\left\{(b^1)^2 + (b^2)^2 + (b^3)^2 + (b^4)^2\right\} &\le ||v||^2_{L^2(E)} \\ &\le \tfrac{h^2_{k-1}}{4}\left\{(b^1)^2 + (b^2)^2 + (b^3)^2 + (b^4)^2\right\}. \end{aligned} \tag{2.7}$$

Proof. Using the affine invariance of the local interpolation problem connecting v with its edge averages b^i, it suffices to prove (2.6) and (2.7) for the master square $E = (-1, 1)^2$. A straightforward calculation gives

$$v = v(x, y) = \frac{1}{4}s + \frac{\triangle^2}{2}x + \frac{\triangle^1}{2}y - \frac{3}{8}\theta^0(x^2 - y^2). \tag{2.8}$$

Now direct integration yields the desired results in (2.6). Also, (2.7) follows from the first equation of (2.6) by computing the eigenvalues of the symmetric 4×4 matrix $\mathcal{T}^t D\mathcal{T}$, where D =diag$(1/16, 1/12, 1/12, 1/40)$, $\mathcal{T}$ stands for the transformation matrix from the vector (b^1, b^3, b^2, b^4) to $(s, \triangle^1, \triangle^2, \theta^0)$, and $\mathcal{T}^t$ is the transpose of $\mathcal{T}$. These eigenvalues are $1/10$, $1/6$, $1/6$, and $1/4$, which implies (2.7). #

Lemma 2.2 is the basis for computing all the discrete energy and L^2 norms needed in the sequel. The formula (2.8) valid for the master square can be used to derive explicit expressions for the edge averages of $I_k v$ and $I_k v - v$. Toward this end, we first compute the corresponding values for the master square, and then use the invariance of the local interpolation problem for v under affine transformations (for the square triangulations under consideration, these transformations are just dilation and translation) to return to the notation on each $E \in \mathcal{E}_{k-1}$.

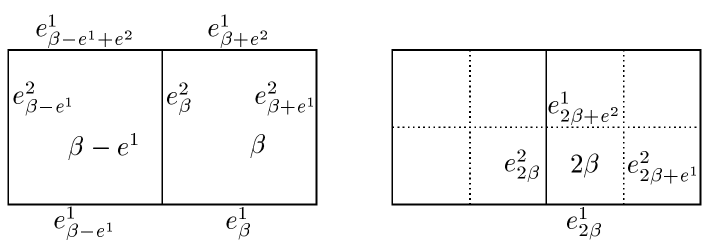

Figure 2. An illustration for Lemma 2.3.

Note that, by the definition of the triangulation $\mathcal{E}_{k-1}$, to each $E \in \mathcal{E}_{k-1}$ is uniquely assigned a $\beta = (\beta_1, \beta_2)$ such that $0 \le \beta_1,\ \beta_2 \le 2^{k-1}$. For notational convenience, let b^1_β and b^2_β denote the averages of $v \in V_{k-1}$ over the horizontal and vertical edges e^1_β and e^2_β, respectively, in $\partial\mathcal{E}_{k-1}$ (see Figure 2, where $e^2_{2\beta}$, $e^2_{2\beta+e^2}$, $e^1_{2\beta+e^2}$, $e^1_{2\beta+e^2-e^1} \in \partial\mathcal{E}_k$, $e^1 = (1, 0)$, and $e^2 = (0, 1)$). The corresponding quantities for $I_k v \in V_k$ are indicated by a^j_α, $j = 1, 2$. Now, introduce the notation

$$\begin{aligned} \hat\theta^1_\beta &= b^1_\beta + b^1_{\beta-e^1} - b^1_{\beta+e^2} - b^1_{\beta+e^2-e^1}, \\ \hat\theta^2_\beta &= b^2_\beta + b^2_{\beta-e^2} - b^2_{\beta+e^1} - b^2_{\beta+e^1-e^2}. \end{aligned}$$

With these notation, it follows from the definition of I_k that the edge averages of I_kv can be written as follows:

$$
\begin{aligned}
a^1_{2\beta} &= b^1_\beta + \tfrac{1}{8}\hat\theta^2_\beta,\\
a^1_{2\beta+e^1} &= b^1_\beta - \tfrac{1}{8}\hat\theta^2_\beta,\\
a^1_{2\beta+e^2} &= \tfrac{5}{8}b^2_\beta + \tfrac{1}{8}b^2_{\beta+e^1} + \tfrac{1}{8}b^1_\beta + \tfrac{1}{8}b^1_{\beta+e^2},\\
a^1_{2\beta+e^2+e^1} &= \tfrac{5}{8}b^2_{\beta+e^1} + \tfrac{1}{8}b^2_\beta + \tfrac{1}{8}b^1_\beta + \tfrac{1}{8}b^1_{\beta+e^2},
\end{aligned} \tag{2.9}
$$

and

$$
\begin{aligned}
a^2_{2\beta} &= b^2_\beta + \tfrac{1}{8}\hat\theta^1_\beta,\\
a^2_{2\beta+e^2} &= b^2_\beta - \tfrac{1}{8}\hat\theta^1_\beta,\\
a^2_{2\beta+e^1} &= \tfrac{5}{8}b^1_\beta + \tfrac{1}{8}b^1_{\beta+e^2} + \tfrac{1}{8}b^2_\beta + \tfrac{1}{8}b^2_{\beta+e^1},\\
a^2_{2\beta+e^2+e^1} &= \tfrac{5}{8}b^1_{\beta+e^2} + \tfrac{1}{8}b^1_\beta + \tfrac{1}{8}b^2_\beta + \tfrac{1}{8}b^2_{\beta+e^1},
\end{aligned} \tag{2.10}
$$

when the edge average a^j_α is associated with an interior edge in $\partial\mathcal{E}_k$; for boundary edges, this value is set to be zero.

Note that I_k (as well as P_{k-1}) can be extended to the larger spaces $\hat V_k$ in a natural way. In order to define the extension $\hat I_j : \hat V_k \to V_k$, observe that any $v \in \hat V_k$ coincides on each $E \in \mathcal{E}_k$ with a polynomial from $V_k|_E$, so the form of the previous definition for I_k remains the same for $\hat I_j$. Clearly, $\hat I_k|_{V_{k-1}} = I_k$ and $\hat I_k|_{V_k} = \mathrm{Id}_k$.

To express the edge averages of $I_kv - v$, set

$$
\begin{aligned}
\theta^1_\beta &= b^1_{\beta+e^2} - b^1_\beta + b^1_{\beta-e^1} - b^1_{\beta+e^2-e^1},\\
\theta^2_\beta &= b^2_{\beta+e^1} - b^2_\beta + b^2_{\beta-e^2} - b^2_{\beta+e^1-e^2},
\end{aligned}
$$

if e^1_β and e^2_β are interior edges in $\partial\mathcal{E}_{k-1}$. For boundary edges, they need to be modified to give the correct expressions for $I_kv - v$. If e^1_β is a boundary edge, for example, we define

$$\theta^2_\beta = 2(b^2_{\beta+e^1} - b^2_\beta).$$

With these, we see that

$$
\begin{aligned}
&\int_{e^1_{2\beta+e^2-e^1}} (I_kv - v)|_{\beta-e^1}\mathrm{d}s = \int_{e^1_{2\beta+e^2}} (I_kv - v)|_\beta \mathrm{d}s = 0,\\
&\frac{1}{|e^2_{2\beta}|}\int_{e^2_{2\beta}} (I_kv - v)|_{\beta-e^1}\mathrm{d}s = \frac{1}{|e^2_{2\beta+e^2}|}\int_{e^2_{2\beta+e^2}} (I_kv - v)|_\beta \mathrm{d}s = -\tfrac{1}{8}\theta^1_\beta,\\
&\frac{1}{|e^2_{2\beta}|}\int_{e^2_{2\beta}} (I_kv - v)|_\beta \mathrm{d}s = \frac{1}{|e^2_{2\beta+e^2}|}\int_{e^2_{2\beta+e^2}} (I_kv - v)|_{\beta-e^1} \mathrm{d}s = \tfrac{1}{8}\theta^1_\beta,
\end{aligned} \tag{2.11}
$$

where $(I_kv - v)|_\beta$ denotes the restriction of $I_kv - v$ to the element associated with β. The averages of $I_kv - v$ on other edges are given similarly. Then, by Green's formula and (2.11), we see that

$$||I_kv||^2_{\mathcal{E}} - ||v||^2_{\mathcal{E}} = ||I_kv - v||^2_{\mathcal{E}}, \qquad v \in V_{k-1}. \tag{2.12}$$

From (2.9)–(2.12) and Lemma 2.2, we immediately have the next lemma. Below the notation $\approx$ stands for two-sided inequalities with constants independent of k.

Lemma 2.3. *It holds that*

$$
\begin{aligned}
||\hat I_kv|| &\le \sqrt{\tfrac{5}{2}}||v||, \qquad \forall v \in \hat V_k,\\
||I_kv||_{\mathcal{E}} &\le \sqrt{2}||v||_{\mathcal{E}}, \qquad \forall v \in V_{k-1},
\end{aligned} \tag{2.13}
$$

and

$$2^k||I_kv - v|| \approx ||I_kv - v||_{\mathcal{E}} \approx \sqrt{\sum_\beta \{(\theta^1_\beta)^2 + (\theta^2_\beta)^2\}}, \quad \forall v \in V_{k-1}. \tag{2.14}$$

We now prove the following property of the iterated coarse-fine intergrid transfer operators R^K_k.

Lemma 2.4. *It holds that*

$$||R^K_k v||_{\mathcal{E}} \leq C||v||_{\mathcal{E}}, \qquad \forall v \in V_k,\ 1 \leq k \leq K. \tag{2.15}$$

PROOF. The proof is technical; it follows the idea of the proof of an analogous statement for the P_1 nonconforming elements [19]. First, we consider the case of $\Omega = \mathbb{R}^2$. That is, we assume that all our definitions are extended to infinite square partitions of $\mathbb{R}^2$; due to the local character of all constructions, this is easy to do. We keep the same notation for the extended partitions $\mathcal{E}_k$, edges $e^j_\alpha \in \partial\mathcal{E}_k$, squares $E \in \mathcal{E}_k$, etc. In order to guarantee the finiteness of all norm expressions, we restrict our attention to functions $v \in V_k$ with finite support. By the construction of I_k, this property is preserved when applying the operators I_k and R^K_k.

After the extension to the shift-invariant setting of $\mathbb{R}^2$, it is clear that it suffices to consider the case of $k = 1$. Set, for simplicity, $\tilde{R}^k = R^k_1$, $k = 1, \dots, K$. Our main observation from numerical experiments [21] was that the sequence

$$\{||\tilde{R}^k v - \tilde{R}^{k-1}v||^2_{\mathcal{E}},\ k = 2, \dots, K\}$$

decays geometrically. What we want to prove next is the mathematical counterpart to this observation. To formulate the technical result, introduce

$$\sigma_j = \sum_{\alpha \in \mathbf{Z}^2} (\theta^j_\alpha)^2, \qquad j = 0, 1, 2,$$

where the quantities θ^j_α are determined from the edge averages of $v \in V_1$ by the same formulas as above. The corresponding quantities computed for $\tilde{v} = I_2 v \in V_2$ are denoted by $\tilde{\theta}^j_\alpha$ and $\tilde{\sigma}_j$, $j = 0, 1, 2$. From (2.14) in Lemma 2.3, we see that

$$\sigma_1 + \sigma_2 \approx ||\tilde{R}^2 v - v||^2_{\mathcal{E}} \quad \text{and} \quad \tilde{\sigma}_1 + \tilde{\sigma}_2 \approx ||\tilde{R}^3 v - \tilde{R}^2 v||^2_{\mathcal{E}};$$

moreover, we can iterate this construction. Thus, if we can prove that

$$\tilde{\sigma} \equiv c^*\tilde{\sigma}_0 + \tilde{\sigma}_1 + \tilde{\sigma}_2 \leq \gamma^*\sigma \equiv \gamma^*(c^*\sigma_0 + \sigma_1 + \sigma_2), \tag{2.16}$$

where $0 < \gamma^* < 1$ and $c^* > 0$ are constants independent of v, then, by Lemmas 2.2 and 2.3,

$$\begin{aligned} ||R^K_1 v||_{\mathcal{E}} &\leq ||v||_{\mathcal{E}} + \textstyle\sum_{k=2}^K ||\tilde{R}^k v - \tilde{R}^{k-1}v||_{\mathcal{E}} \\ &\leq ||v||_{\mathcal{E}} + C \textstyle\sum_{k=1}^{K-1} \sqrt{(\gamma^*)^k}\sqrt{\sigma} \\ &\leq C||v||_{\mathcal{E}}. \end{aligned} \tag{2.17}$$

Since this gives the desired boundedness of R^K_k (for $\mathbb{R}^2$) via dilation, we concentrate on (2.16).

From (2.9) and (2.10) we find the following formulas for $\tilde\theta^j_\alpha$:

$$\begin{aligned}
&\tilde\theta^0_{2\beta+e^1} = \tfrac18\theta^1_{\beta+e^1} - \tfrac18\theta^2_\beta + \tfrac14\theta^0_\beta,\\
&\tilde\theta^0_{2\beta} = -\tfrac18\theta^1_\beta + \tfrac18\theta^2_\beta + \tfrac14\theta^0_\beta,\\
&\tilde\theta^0_{2\beta+e^2} = \tfrac18\theta^1_\beta - \tfrac18\theta^2_{\beta+e^2} + \tfrac14\theta^0_\beta,\\
&\tilde\theta^0_{2\beta+e^1+e^2} = -\tfrac18\theta^1_{\beta+e^1} + \tfrac18\theta^2_{\beta+e^2} + \tfrac14\theta^0_\beta,\\
&\tilde\theta^1_{2\beta} = \tfrac12\theta^1_\beta - \tfrac18(\theta^2_\beta + \theta^2_{\beta-e^1}) - \tfrac38(\theta^0_\beta - \theta^0_{\beta-e^1}),\\
&\tilde\theta^1_{2\beta+e^1} = \tfrac14\theta^2_\beta,\\
&\tilde\theta^1_{2\beta+e^2} = \tfrac12\theta^1_\beta - \tfrac18(\theta^2_{\beta+e^2} + \theta^2_{\beta-e^1+e^2}) + \tfrac38(\theta^0_\beta - \theta^0_{\beta-e^1}),\\
&\tilde\theta^1_{2\beta+e^1+e^2} = -\tfrac14\theta^2_{\beta+e^2},\\
&\tilde\theta^2_{2\beta} = \tfrac12\theta^2_\beta - \tfrac18(\theta^1_\beta + \theta^1_{\beta-e^2}) + \tfrac38(\theta^0_\beta - \theta^0_{\beta-e^2}),\\
&\tilde\theta^2_{2\beta+e^2} = \tfrac14\theta^1_\beta,\\
&\tilde\theta^2_{2\beta+e^1} = \tfrac12\theta^2_\beta - \tfrac18(\theta^1_{\beta+e^1} + \theta^1_{\beta-e^2+e^1}) - \tfrac38(\theta^0_\beta - \theta^0_{\beta-e^2}),\\
&\tilde\theta^2_{2\beta+e^1+e^2} = -\tfrac14\theta^1_{\beta+e^1}.
\end{aligned}$$

These formulas are used to compute the quantities $\tilde\sigma_j$. In order to write them in reasonably short form, we introduce the notation

$$\sigma_j^{\beta^*} = \sum_{\beta\in\mathbf{Z}^2}\theta^j_\beta\theta^j_{\beta+\beta^*}, \quad \sigma_{jl}^{\beta^*} = \sum_{\beta\in\mathbf{Z}^2}\theta^j_\beta\theta^l_{\beta+\beta^*}, \quad k,l=0,1,2\ (j\neq l);$$

if $\beta^*\in\mathbf{Z}^2$ is the null vector, it is omitted in this notation. With them, we see, by carefully evaluating all squares, that

$$\begin{aligned}
\tilde\sigma_0 &= \textstyle\sum_\alpha(\tilde\theta^0_\alpha)^2 = \sum_\beta\Big((\tilde\theta^0_{2\beta})^2 + (\tilde\theta^0_{2\beta+e^1})^2 + (\tilde\theta^0_{2\beta+e^2})^2 + (\tilde\theta^0_{2\beta+e^1+e^2})^2\Big)\\
&= \tfrac14\sigma_0 + \tfrac1{16}(\sigma_1+\sigma_2) - \tfrac1{32}\underbrace{(\sigma_{12} + \sigma_{12}^{e^2} + \sigma_{12}^{-e^1} + \sigma_{12}^{e^2-e^1})}_{\equiv\sigma^*},\\
\tilde\sigma_1 &= \tfrac9{16}\sigma_0 + \tfrac12\sigma_1 + \tfrac3{16}\sigma_2 - \tfrac9{16}\sigma_0^{e^1} + \tfrac1{16}\sigma_2^{e^1}\\
&\quad -\tfrac18\underbrace{(\sigma_{12} + \sigma_{12}^{e^2} + \sigma_{12}^{-e^1} + \sigma_{12}^{e^2-e^1})}_{\equiv\sigma^*} - \tfrac3{32}(\sigma_{02}^{e^1} + \sigma_{02}^{-e^1+e^2} - \sigma_{02}^{-e^1} - \sigma_{02}^{e^1+e^2}),\\
\tilde\sigma_2 &= \tfrac9{16}\sigma_0 + \tfrac3{16}\sigma_1 + \tfrac12\sigma_2 - \tfrac9{16}\sigma_0^{e^2} + \tfrac1{16}\sigma_1^{e^2}\\
&\quad -\tfrac18\underbrace{(\sigma_{12} + \sigma_{12}^{e^2} + \sigma_{12}^{-e^1} + \sigma_{12}^{e^2-e^1})}_{\equiv\sigma^*} - \tfrac3{32}(\sigma_{01}^{-e^2} + \sigma_{01}^{e^1+e^2} - \sigma_{01}^{e^2} - \sigma_{01}^{e^1-e^2}).
\end{aligned}$$

Thus, introducing $\mathcal{A} = \sigma_1 + \sigma_2$ and $\tilde{\mathcal{A}} = \tilde\sigma_1 + \tilde\sigma_2$, we have

$$\begin{aligned}
\tilde\sigma_0 &= \tfrac14\sigma_0 + \tfrac1{16}\mathcal{A} - \tfrac1{32}\sigma^*,\\
\tilde{\mathcal{A}} &= \tfrac98\sigma_0 + \tfrac{11}{16}\mathcal{A} - \tfrac9{16}(\sigma_0^{e^1} + \sigma_0^{e^2}) + \tfrac1{16}(\sigma_1^{e^2} + \sigma_2^{e^1}) - \tfrac14\sigma_1^* - \tfrac3{32}\sigma^{**},
\end{aligned}\tag{2.18}$$

where

$$\sigma^{**} = \sigma_{01}^{-e^2} + \sigma_{01}^{e^1+e^2} + \sigma_{02}^{e^1} + \sigma_{02}^{-e^1+e^2} - \sigma_{01}^{e^2} - \sigma_{01}^{e^1-e^2} - \sigma_{02}^{-e^1} - \sigma_{02}^{e^1+e^2}.$$

Next, we simplify σ^* and σ^{**}. Note that

$$\begin{aligned}
\sigma^* - 2\sigma_1^{e^2} &= \textstyle\sum_\beta \theta_\beta^1(\theta_\beta^2 + \theta_{\beta+e^2}^2 + \theta_{\beta-e^1+e^2}^2 + \theta_{\beta-e^1}^2 - \theta_{\beta+e^2}^1 - \theta_{\beta-e^2}^1) \\
&= \textstyle\sum_\beta \theta_\beta^1(\theta_{\beta+e^2-e^1}^0 + \theta_{\beta-e^2}^0 - \theta_{\beta-e^1-e^2}^0 - \theta_{\beta+e^2}^0 + 2\theta_\beta^1), \\
\sigma^* - 2\sigma_2^{e^1} &= \textstyle\sum_\beta \theta_\beta^2(\theta_\beta^1 + \theta_{\beta-e^2}^1 + \theta_{\beta+e^1-e^2}^1 + \theta_{\beta+e^1}^1 - \theta_{\beta+e^1}^2 - \theta_{\beta-e^1}^2) \\
&= \textstyle\sum_\beta \theta_\beta^2(\theta_{\beta-e^2-e^1}^0 + \theta_{\beta+e^1}^0 - \theta_{\beta+e^1-e^2}^0 - \theta_{\beta-e^1}^0 + 2\theta_\beta^2),
\end{aligned}$$

so that

$$\sigma^* = \sigma_1^{e^2} + \sigma_2^{e^1} + 2\mathcal{A} - \frac{1}{2}\sigma^{**}.$$

Analogously, we can simplify σ^{**} as follows:

$$\begin{aligned}
\sigma^{**} &= \textstyle\sum_\beta \theta_\beta^0\big((\theta_{\beta+e^1+e^2}^1 + \theta_{\beta-e^2}^1 + \theta_{\beta+e^1}^2 + \theta_{\beta-e^1+e^2}^2) \\
&\qquad -(\theta_{\beta+e^2}^1 + \theta_{\beta+e^1-e^2}^1 + \theta_{\beta-e^1}^2 + \theta_{\beta+e^1+e^2}^2)\big) \\
&= \textstyle\sum_\beta \theta_\beta^0\big((\theta_{\beta+e^1+e^2}^0 + \theta_{\beta+e^1-e^2}^0 + \theta_{\beta-e^1-e^2}^0 + \theta_{\beta-e^1+e^2}^0) \\
&\qquad -2(\theta_{\beta+e^1}^0 + \theta_{\beta+e^2}^0 + \theta_{\beta-e^1}^0 + \theta_{\beta-e^2}^0) + 4\theta_\beta^0)\big) \\
&= 2(\sigma_0^{e^1+e^2} + \sigma_0^{e^1-e^2}) - 4(\sigma_0^{e^1} + \sigma_0^{e^2}) + 4\sigma_0.
\end{aligned}$$

Substitution of σ^* and σ^{**} into (2.18) leads to

$$\begin{aligned}
\tilde\sigma_0 &= \tfrac{1}{4}\sigma_0 + \tfrac{1}{32}\mathcal{A} - \tfrac{1}{32}(\sigma_1^{e^2} + \sigma_2^{e^2}) + \tfrac{1}{64}\sigma^{**} \\
&= \tfrac{5}{16}\sigma_0 + \tfrac{1}{32}\mathcal{A} + \tfrac{1}{32}\left(\sigma_0^{e^1-e^2} + \sigma_0^{e^1+e^2} - 2\sigma_0^{e^1} - 2\sigma_0^{e^2}\right) \\
&\quad -\tfrac{1}{32}\left(\sigma_1^{e^2} + \sigma_2^{e^1}\right) \\
&\le \tfrac{1}{2}\sigma_0 + \tfrac{1}{16}\mathcal{A},
\end{aligned} \tag{2.19}$$

where we have used the fact that $|\sigma_j^{\beta^*}| \le \sigma_j,\ j = 0,1,2$, which is valid for arbitrary β^*. With the same argument, we see that

$$\begin{aligned}
\tilde{\mathcal{A}} &= \tfrac{9}{8}\sigma_0 + \tfrac{7}{16}\mathcal{A} - \tfrac{9}{16}\left(\sigma_0^{e^1} + \sigma_0^{e^2}\right) - \tfrac{3}{16}\left(\sigma_1^{e^2} + \sigma_2^{e^2}\right) + \tfrac{1}{32}\sigma^{**} \\
&= \tfrac{5}{4}\sigma_0 + \tfrac{7}{16}\mathcal{A} + \tfrac{1}{16}\left(\sigma_0^{e^1-e^2} + \sigma_0^{e^1+e^2}\right) \\
&\quad -\tfrac{11}{16}\left(\sigma_0^{e^1} + \sigma_0^{e^2}\right) - \tfrac{3}{16}\left(\sigma_1^{e^2} + \sigma_2^{e^1}\right) \\
&\le \tfrac{11}{4}\sigma_0 + \tfrac{5}{8}\mathcal{A}.
\end{aligned} \tag{2.20}$$

Now, set $\mathcal{B} = c\sigma_0$ and $\tilde{\mathcal{B}} = c\tilde\sigma_0$. Then it follows from (2.18) and (2.19) that

$$\tilde{\mathcal{A}} \le \frac{5}{8} + \frac{11}{4c}\mathcal{B}, \qquad \tilde{\mathcal{B}} \le \frac{c}{16}\mathcal{A} + \frac{1}{2}\mathcal{B},$$

and

$$(\tilde{\mathcal{A}} + \tilde{\mathcal{B}}) \le \max\left(\frac{5}{8} + \frac{c}{16}, \frac{11}{4c} + \frac{1}{2}\right)(\mathcal{A} + \mathcal{B}).$$

Let $c = c^* \equiv 3\sqrt{5} - 1$, so we see that (2.16) holds with

$$\gamma^* = \frac{5}{8} + \frac{c^*}{16} = \frac{11}{4c^*} + \frac{1}{2} = \frac{3\sqrt{5}+9}{16} < 1.$$

It remains to reduce the assertion of Lemma 2.4 to the shift-invariant situation just considered. To this end, starting with any $v \in V_k$ on the unit square, we repeatedly use an odd extension. Namely, set $\hat{v} = v$ on $[0,1]^2$ and

$$\hat{v}(x,y) = -\hat{v}(-x,y), \qquad (x,y) \in [-1,0) \times [0,1];$$

after this, define

$$\hat{v}(x,y) = -\hat{v}(x,-y), \qquad (x,y) \in [-1,1] \times [-1,0),$$

and continue this extension process with the unit square replaced by $[-1,1]^2$ such that after the next two steps $\hat{v}$ is defined on $[-1,3]^2$. Outside this larger square we continue by zero. Clearly, $||\hat{v}||_{\mathcal{E}}^2 = 16||v||_{\mathcal{E}}^2$, where the norms for $\hat{v}$ and v are taken with respect to $\mathbb{R}^2$ and the unit square, respectively.

It is not difficult to check by induction that on $[0,1]^2$ the functions $R_k^K\hat{v}$ (obtained by the repeated application of the prolongations defined on $\mathbb{R}^2$) and $R_k^K v$ (as defined above with respect to $[0,1]^2$) coincide. Also, the values of $I_{k+1}\hat{v}$ on $[-2^{-(k+1)}, 1+2^{-(k+1)}]^2$ depend solely on the values of $\hat{v}$ on the square $[-2^{-k}, 1+2^{-k}]^2$, and on this enlarged square $I_{k+1}\hat{v}$ coincides with its odd extension from $[0,1]^2$. Finally, the zero edge averages are automatically reproduced along the boundary of $[0,1]^2$ from the above extension procedure. Therefore, by (2.17) and the dilation argument, we obtain

$$||R_k^K v||_{\mathcal{E}}^2 \le ||R_k^K \hat{v}||_{\mathcal{E}}^2 \le C||\hat{v}||_{\mathcal{E}}^2 = 16C||v||_{\mathcal{E}}^2,$$

which finishes the proof of Lemma 2.4. #

The second inequality in (2.13) is critical for the convergence results of multigrid algorithms developed in the next section, while (2.15) is crucial for the multilevel preconditioner results in §4.

3. MULTIGRID ALGORITHMS

In this section and the next section we consider multigrid algorithms and multilevel preconditioners for the numerical solution of the second-order elliptic problem

$$\begin{aligned} -\nabla \cdot (\mathbf{A}\nabla u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^2$ is a simply connected bounded polygonal domain with the boundary Γ, $f \in L^2(\Omega)$, and the coefficient $\mathbf{A} \in (L^\infty(\Omega))^{2\times 2}$ satisfies

$$\alpha_1 \xi^t \xi \ge \xi^t \mathbf{A}(x,y)\xi \ge \alpha_0 \xi^t \xi, \qquad (x,y) \in \Omega,\ \xi \in \mathbb{R}^2, \tag{3.2}$$

with fixed constants $\alpha_1,\ \alpha_0 > 0$. The condition number of preconditioned linear systems to be analyzed later depends on the ratio α_1/α_0.

Problem (3.1) is recast in weak form as follows. The bilinear form $a(\cdot,\cdot)$ is defined as follows:

$$a(v,w) = (\mathbf{A}\nabla v, \nabla w), \qquad v,\ w \in H^1(\Omega).$$

Then the weak form of (3.1) for the solution $u \in H_0^1(\Omega)$ is

$$a(u,v) = (f,v), \qquad \forall\ v \in H_0^1(\Omega). \tag{3.3}$$

Associated with each V_k, we introduce a bilinear form on $V_k \oplus H_0^1(\Omega)$ by

$$a_k(v,w) = \sum_{E \in \mathcal{E}_k} (\mathbf{A}\nabla v, \nabla w)_E, \qquad v,\ w \in V_k \oplus H_0^1(\Omega).$$

The NR Q_1 finite element discretization of (3.1) is to find $u_K \in V_K$ such that

$$a_K(u_K, v) = (f, v), \qquad \forall\, v \in V_K. \tag{3.4}$$

Let $A_k : V_k \to V_k$ be the discretization operator on level k given by

$$(A_k v, w) = a_k(v, w), \qquad \forall\, w \in V_k. \tag{3.5}$$

The operator A_k is clearly symmetric (in both the $a_k(\cdot,\cdot)$ and $(\cdot,\cdot)$ inner products) and positive definite. Also, we define the operators $R_{k-1} : V_k \to V_{k-1}$ and $R^0_{k-1} : V_k \to V_{k-1}$ by

$$a_{k-1}(R_{k-1}v, w) = a_k(v, I_k w), \qquad \forall\, w \in V_{k-1},$$

and

$$\left(R^0_{k-1}v, w\right) = (v, I_k w), \qquad \forall\, w \in V_{k-1}.$$

It is easy to see that $I_k R_{k-1}$ is a symmetric operator wtih respect to the a_k form. Note that neither R^0_k nor R_k is a projection in the nonconforming case. Finally, let Λ_k dominate the spectral radius of A_k.

The multigrid processes below result in a linear iterative scheme with a reduction operator equal to $I - B_K A_K$, where $B_K : V_K \to V_K$ is the multigrid operator to be defined below.

Multigrid Algorithm 3.1. Let $2 \le k \le K$ and p be a positive integer. Set $B_1 = A_1^{-1}$. Assume that B_{k-1} has been defined and define $B_k g$ for $g \in V_k$ as follows:

1. Set $x^0 = 0$ and $q^0 = 0$.
2. Define x^l for $l = 1, \dots, m(k)$ by
$$x^l = x^{l-1} + S_k(g - A_k x^{l-1}).$$
3. Define $y^{m(k)} = x^{m(k)} + I_k q^p$, where q^i for $i = 1, \dots, p$ is defined by
$$q^i = q^{i-1} + B_{k-1}\left[R^0_{k-1}\left(g - A_k x^{m(k)}\right) - A_{k-1}q^{i-1}\right].$$
4. Define y^l for $l = m(k)+1, \dots, 2m(k)$ by
$$y^l = y^{l-1} + S_k\left(g - A_k y^{l-1}\right).$$
5. Set $B_k g = y^{2m(k)}$.

In Algorithm 3.1, $m(k)$ gives the number of pre- and post-smoothing iterations and can vary as a function of k. In this section, we set $S_k = (\Lambda_k)^{-1}\mathrm{Id}_k$ in the pre- and post-smoothing steps. If $p = 1$, we have a $\mathcal{V}$-cycle multigrid algorithm. If $p = 2$, we have a $\mathcal{W}$-cycle algorithm. A variable $\mathcal{V}$-cycle algorithm is one in which the number of smoothings $m(k)$ increase exponentially as k decreases (i.e., $p = 1$ and $m(k) = 2^{K-k}$).

We now follow the methodology developed in [6] to state convergence results for Algorithm 3.1. The two ingredients in their analysis are the regularity and approximation property and the boundedness of the intergrid transfer operator:

$$\left|a_k\left(v - I_k R_{k-1}v, v\right)\right| \le C\frac{\|A_k v\|}{\sqrt{\lambda_k}}\sqrt{a_k(v,v)}, \qquad \forall\, v \in V_k, \tag{3.6}$$

and

$$a_k(I_k v, I_k v) \le C a_{k-1}(v, v), \qquad \forall\, v \in V_{k-1}, \tag{3.7}$$

for $k = 2, \dots, K$, where λ_k is the largest eigenvalue of A_k. The proof of (3.6) is standard; see the proof of a similar result for the P_1 nonconforming elements in [14]. Inequality (3.7) has been shown in [1] using the approximation property of the operator I_k. However, here we see that if $\mathbf{A} = \alpha_0 \mathbf{I}$ is a scalar multiple of the two-by-two identity matrix $\mathbf{I}$, by the second inequality in (2.13) in Lemma 2.3, we actually have

$$a_k(I_k v, I_k v) \le 2a_{k-1}(v,v), \qquad \forall\, v \in V_{k-1}. \tag{3.8}$$

This leads to the following main result of this section. Let the convergence rate for Algorithm 3.1 on the kth level be measured by the convergence factor δ_k satisfying

$$|a_k\,(v - B_k A_k v, v)| \le \delta_k a_k(v,v), \qquad \forall\, v \in V_k.$$

Theorem 3.1. (i) *Define B_k by $p = 1$ and $m(k) = 2^{K-k}$ for $k = 2, \dots, K$ in Algorithm 3.1. Then there are η_0, $\eta_1 > 0$, independent of k, such that*

$$\eta_0 a_k(v,v) \le a_k(B_k A_k v, v) \le \eta_1 a_k(v,v), \qquad \forall v \in V_k,$$

with

$$\eta_0 \ge \sqrt{m(k)}/(C + \sqrt{m(k)}) \quad \text{and} \quad \eta_1 \le (C + \sqrt{m(k)})/\sqrt{m(k)}.$$

(ii) *Define B_k by $p = 2$ and $m(k) = m$ for all k in Algorithm 3.1. Then if $\mathbf{A} = \alpha_0 \mathbf{I}$ is constant, there exists $C > 0$, independent of k, such that*

$$\delta_k \le \delta \equiv \frac{C}{C + \sqrt{m}}.$$

The same conclusion holds if the assumption that $\mathbf{A} = \alpha_0 \mathbf{I}$ is replaced by requiring that $m \ge m_0$, where m_0 is sufficiently large, but independent of k.

The proof of this theorem follows from (3.6)–(3.8) and Theorems 6 and 7 in [6]. From Theorem 3.1, we have an optimal convergence property of the $\mathcal{W}$-cycle and a uniform condition number estimate for the variable $\mathcal{V}$-cycle preconditioner.

4. MULTILEVEL PRECONDITIONERS

In this section we discuss multilevel preconditioners of hierarchical basis and BPX [5] type for (3.4). More precisely, we derive the condition numbers of the additive subspace splittings

$$\{V_K; (\cdot,\cdot)_{\mathcal{E}}\} = R_1^K \{V_1; (\cdot,\cdot)_{\mathcal{E}}\} + \sum_{k=2}^{K} R_k^K \{V_k; 2^{2k}(\cdot,\cdot)\}, \tag{4.1}$$

and

$$\{V_K; (\cdot,\cdot)_{\mathcal{E}}\} = R_1^K \{V_1; (\cdot,\cdot)_{\mathcal{E}}\} + \sum_{k=2}^{K} R_k^K \{(\mathrm{Id}_k - I_k P_{k-1}) V_k; 2^{2k}(\cdot,\cdot)\}. \tag{4.2}$$

The condition number of (4.1) is given by [20]

$$\kappa = \frac{\lambda_{\max}}{\lambda_{\min}}, \quad \lambda_{\max} = \sup_{v \in V_K} \frac{||v||_{\mathcal{E}}^2}{|||v|||^2}, \quad \lambda_{\min} = \inf_{v \in V_K} \frac{||v||_{\mathcal{E}}^2}{|||v|||^2}, \tag{4.3}$$

where

$$|||v|||^2 = \inf_{v_k \in V_k \;:\; v = \sum_k R_k^K v_k} \left\{ ||v_1||_{\mathcal{E}}^2 + \sum_{k=2}^{K} 2^{2k} ||v_k||^2 \right\}.$$

A similar definition can be given for (4.2).

Theorem 4.1. *There are positive constants c and C, independent of K, such that*

$$c \le \frac{||v||_{\mathcal{E}}^2}{|||v|||^2} \le CK, \qquad \forall v \in V_K, \tag{4.4}$$

and

$$c \le \frac{||v||_{\mathcal{E}}^2}{||||v||||^2} \le CK, \qquad \forall v \in V_K, \tag{4.5}$$

where

$$||||v||||^2 = ||Q_1^K v||_{\mathcal{E}}^2 + \sum_{k=2}^{K} 2^{2k} ||(\mathrm{Id}_k - I_k P_{k-1}) Q_k^K v||^2.$$

That is, the condition numbers of the additive subspace splittings (4.1) *and* (4.2) *are bounded by* $\mathrm{O}(K)$ *as* $K \to \infty$.

Proof. For $k = 2, \dots, K$, it follows from the definitions of I_k, $\hat{I}_k$, and Q_k^K, (2.4), and the first inequality of (2.13) that

$$\begin{aligned} 2^{2k}||(\mathrm{Id}_k - I_k P_{k-1}) Q_k^K v||^2 &= 2^{2k} ||\hat{I}_k (\mathrm{Id}_k - P_{k-1}) Q_k^K v||^2 \\ &\le \tfrac{5}{2}\, 2^{2k} ||(\mathrm{Id}_k - P_{k-1}) Q_k^K v||^2 \\ &\le C||(\mathrm{Id}_k - I_k P_{k-1}) Q_k^K v||_{\mathcal{E}}^2 \\ &= C||Q_k^K v - Q_{k-1}^K v||^2. \end{aligned}$$

Summing on j and using the orthogonality relations in (2.3), we see that

$$\begin{aligned} &\inf_{v_k \in V_k \,:\, v = \sum_k R_k^K v_k} \left\{ ||v_1||_{\mathcal{E}}^2 + \textstyle\sum_{k=2}^K 2^{2k} ||v_k||^2 \right\} \\ &\quad \le ||Q_1^K v||_{\mathcal{E}}^2 + \textstyle\sum_{k=2}^K 2^{2k} ||(\mathrm{Id}_k - I_k P_{k-1}) Q_k^K v||^2 \\ &\quad \le C||v||_{\mathcal{E}}^2, \end{aligned}$$

which implies the lower bounds in (4.4) and (4.5).

For the upper bounds, we consider an arbitrary decomposition $v = \sum_{k=1}^K R_k^K v_k$ with $v_k \in V_k$. Then we see, by Lemma 2.4, that

$$||v||_{\mathcal{E}}^2 \le \left(\sum_{k=1}^K ||R_k^K v_k||_{\mathcal{E}} \right)^2 \le K \sum_{k=1}^K ||R_k^K v_k||_{\mathcal{E}}^2 \le CK \sum_{k=1}^K ||v_k||_{\mathcal{E}}^2.$$

Consequently, by (2.2), we have

$$||v||_{\mathcal{E}}^2 \le CK \left(||v_1||_{\mathcal{E}}^2 + \sum_{k=2}^K 2^{2k} ||v_k||^2 \right).$$

Now, taking the infimum with respect to all decompositions, we obtain

$$\begin{aligned} ||v||_{\mathcal{E}}^2 &\le CK \inf_{v_k \in V_k \,:\, v = \sum_k R_k^K v_k} \left\{ ||v_1||_{\mathcal{E}}^2 + \textstyle\sum_{k=2}^K 2^{2k} ||v_k||^2 \right\} \\ &\le CK \left(||Q_1^K v||_{\mathcal{E}}^2 + \textstyle\sum_{k=2}^K 2^{2k} ||(\mathrm{Id}_k - I_k P_{k-1}) Q_k^K v||^2 \right), \end{aligned}$$

which finishes the proof of the theorem. #

We now discuss the algorithmical consequences for the splittings (4.1) and (4.2). Theoretically, Theorem 4.1 already produces suitable preconditioners for the matrix A_K using (4.1) and (4.2). However, they are still complicated since they involve L^2-projections onto V_k, $1 < k < K$, which means to solve large linear systems within each preconditioning step. To get more practicable algorithms, we replace the L^2 norms in V_k and $W_k = (\mathrm{Id}_k - I_k P_{k-1})V_k \subset V_k$, $k = 2, \dots, K$, by their suitable discrete counterparts. We first consider the splitting (4.1); (4.2) will be discussed later.

Let $\{\phi^j_{\alpha,k}\}$ be the basis functions of V_k such that the edge average of $\phi^j_{\alpha,k}$ equals one at $e^j_{\alpha,k}$ and zero at all other edges. Then each $v \in V_k$ has the representation

$$v = \sum_{j=1}^{2} \sum_{\alpha} a^j_\alpha \phi^j_{\alpha,k}.$$

Thus, by the uniform L^2-stability of the bases, which follows from (2.7) in Lemma 2.2, we see that

$$\frac{1}{5} 2^{-2k} \sum_{j=1}^{2} \sum_{\alpha} (a^j_\alpha)^2 \le ||v||^2 \le \frac{1}{2} 2^{-2k} \sum_{j=1}^{2} \sum_{\alpha} (a^j_\alpha)^2. \tag{4.6}$$

Note that (with the same argument as in Lemma 2.2)

$$2^{2k} ||\phi^j_{\alpha,k}||^2 = \frac{41}{120}, \quad a_k(\phi^j_{\alpha,k}, \phi^j_{\alpha,k}) \approx ||\phi^j_{\alpha,k}||^2_{\mathcal{E}} = 5, \tag{4.7}$$

so (4.6) can be interpreted as the two-sided inequality associated with the stability of any of the splittings

$$\{V_k; 2^{2k}(\cdot,\cdot)\} = \sum_{j=1}^{2} \sum_{\alpha} \{V^j_{\alpha,k}; 2^{2k}(\cdot,\cdot)\}, \tag{4.8}$$

$$\{V_k; 2^{2k}(\cdot,\cdot)\} = \sum_{j=1}^{2} \sum_{\alpha} \{V^j_{\alpha,k}; (\cdot,\cdot)_{\mathcal{E}}\}, \tag{4.9}$$

and

$$\{V_k; 2^{2k}(\cdot,\cdot)\} = \sum_{j=1}^{2} \sum_{\alpha} \{V^j_{\alpha,k}; a_k(\cdot,\cdot)\}, \tag{4.10}$$

into the direct sum of one-dimensional subspaces $V^j_{\alpha,k}$ spanned by the basis functions $\phi^j_{\alpha,k}$. Any of the splittings (4.8)–(4.10) can be used to refine (4.1). As we will see below, the difference is just in a diagonal scaling (i.e., a multiplication by a diagonal matrix) in the final algorithms. As example, we consider the splitting (4.10) in detail; the other two cases can be analyzed in the same fashion.

With (4.1) and (4.10), we have the splitting

$$\{V_K; a_K(\cdot,\cdot)\} = R^K_1 \{V_1; a_1(\cdot,\cdot)\} + \sum_{k=2}^{K} \sum_{j=1}^{2} \sum_{\alpha} R^K_k \{V^j_{\alpha,k}; a_k(\cdot,\cdot)\}. \tag{4.11}$$

It follows from (4.4), (4.6), and (4.7) that the condition number κ for (4.11) still behaves like $\mathrm{O}(K)$. Now, associated with this splitting we can explicitly state the additive Schwarz operator

$$\mathcal{P}_K = R^K_1 T_1 + \sum_{k=2}^{K} \sum_{j=1}^{2} \sum_{\alpha} R^K_k T^j_{\alpha,k}, \tag{4.12}$$

where

$$T^j_{\alpha,k}v = \frac{a_K(v, R^K_k \phi^j_{\alpha,k})}{a_k(\phi^j_{\alpha,k}, \phi^j_{\alpha,k})}\phi^j_{\alpha,k},$$

and $T_1 v \in V_1$ solves the elliptic problem

$$a_1(T_1 v, w) = a_K(v, R^K_1 w), \qquad \forall\, w \in V_1.$$

Thus the matrix representations of all operators with respect to the bases of the respective V_k are

$$T_k = \sum_{j=1}^{2}\sum_{\alpha} T^j_{\alpha,k} = S_k (R^K_k)^t A_K, \qquad S_k = \operatorname{diag}(a_j(\phi^j_{\alpha,k}, \phi^j_{\alpha,k})^{-1}),$$

for $2 \le k \le K$, and

$$T_1 = A_1^{-1}(R^K_1)^t A_K,$$

where for convenience the same notation is used for operators and matrices. Hence it follows from (4.12) that

$$\mathcal{P}_K = \left(R^K_1 A_1^{-1}(R^K_1)^t + \sum_{k=2}^{K} R^K_k S_k (R^K_k)^t \right) A_K \equiv C_K A_K,$$

which, together with the definition of $R^K_k = I_K \cdots I_{k+1}$, leads to the typical recursive structure for the preconditioner C_K

$$C_k = I_k C_{k-1} I_k^t + S_k, \quad k = K, \dots, 2, \quad S_1 = C_1 \equiv A_1^{-1}. \tag{4.13}$$

Note that with these choices for S_k, the multiplication of a vector by C_K is formally a special case of Algorithm 3.1 if one sets $m(k) = 1$, $p = 1$, removes the post-smoothing step, and replaces A_k by a zero matrix for all $k \ge 2$.

From (4.13) and the definitions of I_k and S_k, we see that a multiplication by C_K only involves $\mathrm{O}(n_K + \ldots + n_2 + n_1^3) = \mathrm{O}(n_K)$ arithmetical operations, where $n_k \approx 2^{2k}$ is the dimension of V_k. This, together with (4.4), yields suboptimal work estimates for a preconditioned conjugate gradient method for (3.4) with the preconditioner C_K. That is, an error reduction by a factor ϵ in the preconditioned conjugate gradient algorithm can be achieved by $\mathrm{O}(n_K\sqrt{\log n_K}\log(\epsilon^{-1}))$ operations.

We now turn to the discussion of the algorithmical consequences for the splitting (4.2). To do this, we need to construct basis functions in W_k, $k = 2, \dots, K$. Starting with the bases $\{\phi^j_{\alpha,k}\}$ in V_k, to each interior edge $e^j_{\beta,k-1} \in \partial\mathcal{E}_{k-1}$, we replace the two associated basis functions $\phi^j_{2\beta,k}, \phi^j_{2\beta+e^j,k}$ with their linear combinations

$$\psi^j_{2\beta,k} = \phi^j_{2\beta,k} + \phi^j_{2\beta+e^j,k}, \quad \psi^j_{2\beta+e^j,k} = \phi^j_{2\beta,k} - \phi^j_{2\beta+e^j,k}, \quad j = 1, 2,$$

where $e^j_{2\beta,k}$ and $e^j_{2\beta+e^j,k} \in \partial\mathcal{E}_k$ form the edge $e^j_{\beta,k-1}$. For all other interior edges $e^j_{\alpha,k}$, which do not belong to any edge in $\partial\mathcal{E}_{k-1}$, we set

$$\psi^j_{\alpha,k} = \phi^j_{\alpha,k}.$$

The new bases $\{\psi^j_{\alpha,k}\}$ in V_k is still L^2-stable; i.e., they satisfy an analogous inequality to (4.6). Moreover, if

$$v = \sum_{j=1}^{2}\sum_{\alpha} b^j_\alpha \psi^j_{\alpha,k},$$

we have

$$P_{k-1}v = \sum_{j=1}^{2}\sum_{\beta} b_{2\beta}^{j}\phi_{\beta,k-1}^{j},$$

and

$$(\mathrm{Id}_k - I_k P_{k-1})v = \sum_{j=1}^{2}\sum_{\alpha\neq 2\beta} c_{\alpha}^{j}\psi_{\alpha,k}^{j},$$

since $\psi_{2\beta,k}^{j} - I_k\phi_{\beta,k-1}^{j}$ can be completely expressed by the functions $\psi_{\alpha,k}^{l}$ with $\alpha \neq 2\beta$ only. More precisely, we have

$$\begin{aligned}
c_{2\beta+e^1}^{1} &= b_{2\beta+e^1}^{1} - \tfrac{1}{8}(b_{2\beta}^{2} + b_{2(\beta-e^2)}^{2} - b_{2(\beta+e^1)}^{2} - b_{2(\beta+e^1-e^2)}^{2}),\\
c_{2\beta+e^2}^{1} &= b_{2\beta+e^2}^{1} - \tfrac{1}{8}(5b_{2\beta}^{2} + b_{2\beta}^{1} + b_{2(\beta+e^1)}^{2} + b_{2(\beta+e^2)}^{1}),\\
c_{2\beta+e^1+e^2}^{1} &= b_{2\beta+e^2}^{1} - \tfrac{1}{8}(5b_{2(\beta+e^1)}^{2} + b_{2\beta}^{1} + b_{2\beta}^{2} + b_{2(\beta+e^2)}^{1}),
\end{aligned}$$

and similar relations hold for $j = 2$. Hence any function from W_k has a unique representation by linear combinations of $\{\psi_{\alpha,k}^{j} : \alpha \neq 2\beta\}$, and this basis system is L^2-stable. With this basis system, as in (4.11), we have the corresponding splitting

$$\{V_K; a_K(\cdot,\cdot)\} = R_1^K\{V_1; a_1(\cdot,\cdot)\} + \sum_{k=2}^{K}\sum_{j=1}^{2}\sum_{\alpha\neq 2\beta} R_k^K\{W_{\alpha,k}^{j}; a_k(\cdot,\cdot)\} \tag{4.14}$$

into a direct sum of $R_1^K V_1$ and one-dimensional spaces $R_k^K W_{\alpha,k}^{j}$ spanned by the basis $\psi_{\alpha,k}^{j}$. Then, with the same argument as for (4.13), we derive an additive preconditioner $\hat{C}_K$ for A_K recursively defined by

$$\hat{C}_k = I_k\hat{C}_{k-1}I_k^t + \hat{I}_k\hat{S}_k\hat{I}_k^t, \quad k = K,\dots,2, \quad \hat{C}_1 = \hat{S}_1 \equiv A_1^{-1}, \tag{4.15}$$

where

$$\hat{S}_k = \mathrm{diag}\left(a_k(\psi_{\alpha,k}^{j}, \psi_{\alpha,k}^{j})^{-1},\, \alpha \neq 2\beta,\, j = 1,2\right)$$

are diagonal matrices and $\hat{I}_k$ is the rectangular matrix corresponding to the natural embedding $W_k \subset V_k$ with respect to the bases $\{\psi_{\alpha,k}^{j}\}$ in W_k and $\{\phi_{\alpha,k}^{j}\}$ in V_k (one may use the bases $\{\psi_{\alpha,k}^{j}\}$ for all V_k, which would change the I_k representations, but keep $\hat{I}_k$ maximally simple). (4.15) has the same arithmetical complexity as before.

We now summarize the results in Theorem 4.1 and the above discussion in the next theorem.

Theorem 4.2. *The symmetric preconditioners C_K and $\hat{C}_K$ defined in* (4.13) *and* (4.15) *and associated with the multilevel splittings* (4.11) *and* (4.14)*, respectively, have an* $\mathrm{O(n_K)}$ *operation count per matrix-vector multiplication and produce the following the condition numbers*:

$$\kappa(C_K A_K) \le CK, \quad \kappa(\hat{C}_K A_K) \le CK, \quad K \ge 1. \tag{4.16}$$

The splitting (4.11) can be viewed as the nodal basis preconditioner of BPX type [5], while the splitting (4.14) is analogous to the hierarchical basis preconditioner.

We now consider multiplicative algorithms for (3.4). One iteration step of a multiplicative algorithm corresponding to the splitting (4.11) takes the form

$$
\begin{aligned}
&y^0 = x_K^j,\\
&y^{l+1} = y^l - \omega R_{K-l}^K S_{K-l} (R_{K-l}^K)^t (A_K y^l - f_K), \quad l = 0, \dots, K-1,\\
&x_K^{j+1} = y^K,
\end{aligned}
\tag{4.17}
$$

where ω is a suitable relaxation parameter (the range of relaxation parameters for which the algorithm in (4.17) converges is determined mainly by the constant in the inverse inequality (2.2) [27, 26, 15]. The method (4.17) corresponds to a $\mathcal{V}$-cycle algorithm in Algorithm 3.1 with A_k replaced by $\tilde{A}_k = (R_k^K)^t A_K R_k^K$, one pre-smoothing and no post-smoothing steps.

The iteration matrix $M_{K,\omega}$ in (4.17) is given by

$$
M_{K,\omega} = (\mathrm{Id}_K - \omega E_1) \cdots (\mathrm{Id}_K - \omega E_{K-1})(\mathrm{Id}_K - \omega E_K)\ , \quad E_k \equiv R_k^K S_k (R_k^K)^t A_K.
$$

An analogous multiplicative algorithm for (3.4) corresponding to the splitting (4.14) can be defined.

From the general theory on multiplicative algorithms [27] and by the same argument as for Theorem 4.2, we can show the following result.

Theorem 4.3. *For properly chosen relaxation parameter ω the multiplicative schemes corresponding to the splittings* (4.11) *and* (4.14) *possess the following upper bounds for the convergence rate:*

$$
\inf_{\omega} ||M_{K,\omega}||_{\mathcal{E}} \le 1 - \frac{C}{K}, \quad \inf_{\omega} ||\hat{M}_{K,\omega}||_{\mathcal{E}} \le 1 - \frac{C}{K}, \quad K \to \infty,
\tag{4.18}
$$

where $M_{K,\omega}$ and $\hat{M}_{K,\omega}$ denote the iteration matrices associated with (4.11) *and* (4.14)*, respectively.*

We end with two remarks. First, one example for the choice of ω is that $\omega \approx K^{-1}$, which leads to the upper bounds in (4.18). Second, the diagonal matrices S_k and $\hat{S}_k$ in (4.13) and (4.15) can be replaced by any other spectrally equivalent symmetric matrices of their respective dimension.

5. EQUIVALENT DISCRETIZATIONS

To improve the estimates in Theorems 4.2 and 4.3, we now consider the problem of switching the NR Q_1 discretization system (3.4) to a spectrally equivalent discretization system for which optimal preconditioners are already available. This switching strategy, as mentioned in the introduction, has been used in the context of the multilevel additive Schwarz method; see [21] for the references.

The most natural candidate for a switching procedure is the space of conforming bilinear elements

$$
U_K = \{\xi \in C^0(\overline{\Omega}) : \xi|_E \in Q_1(E),\ \forall E \in \mathcal{E}_k \text{ and } \xi|_\Gamma = 0\},
$$

on the same partition. We introduce two linear operators $Y_K : U_K \to V_K$ and $\hat{Y}_K : V_K \to U_K$ as follows. If $\xi \in U_K$ and e is an edge of an element in $\mathcal{E}_K$, then $Y_K \xi \in V_K$ is given by

$$
\int_e Y_K \xi \mathrm{d}s = \int_e \xi \mathrm{d}s,
\tag{5.1}
$$

which preserves the zero average values on the boundary edges. If $v \in V_K$, we define $\hat{Y}_K v \in U_K$ by

$$
\begin{aligned}
&(\hat{Y}_K v)(z) = 0 \quad \text{for all boundary vertices } z \text{ in } \mathcal{E}_K, \\
&(\hat{Y}_K v)(z) = \text{average of } v_j(z) \quad \text{for all internal vertices } z \text{ in } \mathcal{E}_K,
\end{aligned}
\tag{5.2}
$$

where $v_j = v|_{E_j}$ and $E_j \in \mathcal{E}_K$ contains z as a vertex.

Another choice for U_K is the space of conforming P_1 elements

$$U_K = \left\{\xi \in C^0(\overline{\Omega}) : \xi|_E \in P_1(E),\ \forall E \in \tilde{\mathcal{E}}_K \text{ and } \xi|_\Gamma = 0\right\},$$

where $\tilde{\mathcal{E}}_K$ is the triangulation of Ω generated by connecting the two opposite vertices of the squares in $\mathcal{E}_K$. The two linear operators $Y_K : U_K \to V_K$ and $\hat{Y}_K : V_K \to U_K$ are defined as in (5.1) and (5.2), respectively. Moreover, for both the conforming bilinear elements and the conforming P_1 elements, it can be easily shown that there is a constant C, independent of K, such that

$$
\begin{aligned}
2^K||\xi - Y_K\xi|| &\le C||\xi||_{\mathcal{E}}, \qquad \forall \xi \in U_K, \\
2^K||v - \hat{Y}_K v|| &\le C||v||_{\mathcal{E}}, \qquad \forall v \in V_K.
\end{aligned}
\tag{5.3}
$$

Since optimal preconditioners exist for the discretization system $\overline{A}_K$ generated by the conforming bilinear elements (respectively, the conforming P_1 elements), the next result follows from (5.3) and the general switching theory in [21].

Theorem 5.1. *Let $\overline{C}_K$ be any optimal symmetric preconditioner for $\overline{A}_K$; i.e., we assume that a matrix-vector multiplication by $\overline{C}_K$ can be performed in $\mathrm{O}(n_K)$ arithmetical operations, and that $\kappa(\overline{C}_K\overline{A}_K) \le C$, with constant independent of K. Then*

$$\overline{C}^*_K = S_K + Y_K\overline{C}_K(Y_K)^t \tag{5.4}$$

is an optimal symmetric preconditioner for A_K.

6. NUMERICAL EXPERIMENTS

In this section we present the results of numerical examples to illustrate the theories developed in the earlier sections. These numerical examples deal with the Laplace equation on the unit square:

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega = (0,1)^2, \\
u &= 0 \quad \text{on } \Gamma,
\end{aligned}
\tag{6.1}
$$

where $f \in L^2$. The NR Q_1 finite element method (3.4) is used to solve (6.1) with $\{\mathcal{E}_k\}_{k=1}^K$ being a sequence of dyadically, uniformly refined partitions of Ω into squares. The coarsest grid is of size $h_1 = 1/2$.

The first test concerns the convergence of Algorithm 3.1. The analysis of the third section guarantees the convergence of the $\mathcal{W}$-cycle algorithm with any number of smoothing steps and the uniform condition number property for the variable $\mathcal{V}$-cycle algorithm, but does not give any indication for the convergence of the standard $\mathcal{V}$-cycle algorithm, i.e., Algorithm 3.1 with $p = 1$ and $m(k) = 1$ for all k. The first two rows of Table 1 show the results for levels $K = 3, \dots, 7$ for this symmetric V-cycle, where (κ_v, δ_v) denote the condition number for the system B_KA_K and the reduction factor for the system $\mathrm{Id}_K - B_KA_K$ as a function of the mesh size on the finest grid h_K. While there is no complete theory for this $\mathcal{V}$-cycle algorithm, it is of practical interest that the condition numbers for this cycle remain relatively small.

$1/h_K$	8	16	32	64	128
κ_v	1.54	1.70	1.84	1.96	2.06
δ_v	0.23	0.27	0.32	0.33	0.35
κ_m	1.75	1.81	1.84	1.85	1.85

Table 1. Numerical results for the multiplicative $\mathcal{V}$-cycles.

For comparison, we run the same example by a symmetrized multilevel multiplicative Schwarz method corresponding to (4.17). One step of the symmetric version consists of two substeps, the first coinciding with (4.17) and the second repeating (4.17) in reverse order. The condition numbers κ_m for $\tilde{M}_{K,\omega}A_K$ with $\omega \approx K^{-1}$ are presented in the third row of Table 1, where $\tilde{M}_{K,\omega} = M^t_{K,\omega}M_{K,\omega}$ is now symmetric. The results are better than expected from the upper bounds of Theorem 4.3 which seem to be only suboptimal.

In the second test we treat the above multigrid algorithm and symmetrized multilevel multiplicative method as preconditioners for the conjugate gradient method. In this test the problem (6.1) is assumed to have the exact solution

$$u(x,y) = x(1-x)y(1-y)e^{xy}.$$

Table 2 shows the number of iterations required to achieve the error reduction 10^{-6}, where the starting vector for the iteration is zero. The iteration numbers $(\text{iter}_v, \text{iter}_m)$ correspond to Algorithm 3.1 with $p = 1$ and $m(k) = 1$ for all k and the symmetrized multiplicative algorithm (4.17), respectively. Note that iter_v and iter_m remain almost constant when the step size increases.

$1/h_K$	8	16	32	64	128
iter_v	8	8	9	9	10
iter_m	9	9	9	10	10

Table 2. Iteration numbers for the pcg-iteration.

In the final test we report analogous numerical results (condition numbers and pcg-iteration count) for the additive preconditioner C_K associated with the splitting (4.11) (subscript a), and the preconditioner $\overline{C}^*_K$ (subscript s) which uses the switch from the system arising from (3.4) to the spectrally equivalent system generated by the conforming bilinear elements via the operators in (5.1) and (5.2). We have implemented the standard BPX-preconditioner [5], with diagonal scaling, as $\overline{C}_K$.

These results are shown in Table 3. The numbers show the slight growth, which is typical for most of the additive preconditioners and level numbers $K < 10$. The condition numbers κ_s for the switching procedure are practically identical to the condition numbers for $\overline{C}_K \overline{A}_K$ characterizing the BPX-preconditioner [5] in the conforming bilinear case. The switching procedure is clearly favorable as can be expected from the theoretical bounds of Theorems 4.2 and 5.1; however, the computations do not indicate whether the upper bound (4.16) is sharp or could be further improved.

$1/h_K$	8	16	32	64	128	256	512
κ_a	9.6	12.3	14.4	16.1	17.4	18.3	19.3
iter_a	18	22	24	26	27	28	28
κ_s	3.37	3.87	4.24	4.54	4.80	5.05	-
iter_s	10	11	13	13	14	15	-

Table 3. Results for the preconditioners C_K and $\overline{C}_K^*$.

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