MULTILEVEL PRECONDITIONERS FOR MIXED METHODS FOR SECOND ORDER ELLIPTIC PROBLEMS

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Abstract. A new approach of constructing algebraic multilevel preconditioners for mixed finite element methods for second order elliptic problems with tensor coefficients on general geometry is proposed. The linear system arising from the mixed methods is first algebraically condensed to a symmetric, positive definite system for Lagrange multipliers, which corresponds to a linear system generated by standard nonconforming finite element methods. Algebraic multilevel preconditioners are then constructed for this system based on a triangulation of parallelepipeds into tetrahedral substructures. Explicit estimates of condition numbers and simple computational schemes are established for the constructed preconditioners. Finally, numerical results for the mixed finite element methods are presented to illustrate the present theory.

Key words. mixed method, nonconforming method, multilevel preconditioner, condition number, second order elliptic problem

AMS(MOS) subject classifications. 65N30, 65N22, 65F10

1. Introduction.

Let Ω be a bounded domain in $\mathbb{R}^d$, $d = 2$ or 3, with the polygonal boundary $\partial\Omega$. We consider the elliptic problem

$$\begin{aligned} &-\nabla\cdot(a\nabla u) = f \qquad &&\text{in } \Omega,\\ &u = 0 &&\text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $a(x)$ is a uniformly positive definite, bounded, symmetric tensor and $f(x) \in L^2(\Omega)$ ($H^k(\Omega) = W^{k,2}(\Omega)$ is the Sobolev space of k differentiable functions in $L^2(\Omega)$). Let $(\,\cdot\,,\,\cdot\,)_S$ denote the $L^2(S)$ inner product (we omit S if $S = \Omega$), and let

$$\begin{aligned} V &= H(\text{div};\Omega) = \left\{v \in \left(L^2(\Omega)\right)^d : \nabla\cdot v \in L^2(\Omega)\right\},\\ W &= L^2(\Omega). \end{aligned}$$

Then (1.1) is formulated in the following mixed form for the pair $(\sigma, u) \in V \times W$:

$$\begin{aligned} &(\nabla\cdot\sigma, w) = (f, w), \qquad &&\forall w \in W,\\ &(a^{-1}\sigma, v) - (u, \nabla\cdot v) = 0, &&\forall v \in V. \end{aligned} \tag{1.2}$$

It can be easily seen that (1.1) is equivalent to (1.2) through the relation

$$\sigma = -a\nabla u.$$

To define a finite element method, we need a partition $\mathcal{T}_h$ of Ω into elements T, say, simplexes, rectangular parallelepipeds, and/or prisms. In $\mathcal{T}_h$, we also need that adjacent elements completely share their common edge or face; let $\partial\mathcal{T}_h$ denote the set of all *interior* edges ($d = 2$) or faces ($d = 3$) e of $\mathcal{T}_h$.

Let $V_h \times W_h \subset V \times W$ denote some standard mixed finite element space for second order elliptic problems defined over $\mathcal{T}_h$ (see, e.g., [8], [9], [10], [14], [16], [29], [30], and

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[31]). This space is finite dimensional and defined locally on each element $T \in \mathcal{T}_h$, so let $V_h(T) = V_h|_T$ and $W_h(T) = W_h|_T$. Then the mixed finite element method for (1.1) is to find $(\sigma_h, u_h) \in V_h \times W_h$:

$$
\begin{aligned}
&(\nabla \cdot \sigma_h, w) = (f, w), && \forall w \in W_h, \\
&(a^{-1}\sigma_h, v) - (u_h, \nabla \cdot v) = 0, && \forall v \in V_h.
\end{aligned} \tag{1.3}
$$

It has been known that, due to its saddle point property, it is difficult to solve the linear system arising from (1.3). In particular, it is hard to construct efficient preconditioners for the mixed method system. While some preconditioning algorithms for solving the saddle point system have been proposed and studied (see, e.g., [3], [6], [17], [20], [32], [33]), their efficiency strongly depends on the geometry of the domain, the coefficient matrix $a(x)$, and the type of finite elements exploited. Especially, the preconditioners in these papers can be very expensive to obtain. An alternate approach was suggested by means of a nonmixed formulation. Namely, it is shown that the mixed finite element method is equivalent to a modification of the nonconforming Galerkin method [1], [2], [11], [28]. The nonconforming method yields a symmetric and positive definite problem (i.e., a minimization problem). This is explained below and in the next section in detail.

The constraint $V_h \subset V$ says that the normal components of the members of V_h are continuous across the interior boundaries in $\partial\mathcal{T}_h$. Following [2], we relax this constraint on V_h by defining

$$\tilde{V}_h = \{v \in L^2(\Omega) : v|_T \in V_h(T) \text{ for each } T \in \mathcal{T}_h\}.$$

We then need to introduce Lagrange multipliers to enforce the required continuity on $\tilde{V}_h$, so define

$$L_h = \left\{\mu \in L^2\left(\bigcup_{e \in \partial\mathcal{T}_h} e\right) : \mu|_e \in V_h \cdot \nu|_e \text{ for each } e \in \partial\mathcal{T}_h\right\},$$

where ν is the unit normal to e. Also, to establish a relationship between the mixed method and the nonconforming Galerkin method and to construct efficient preconditioners, following [11] we introduce the projection of the coefficient, i.e., $\alpha_h = P_h a^{-1}$, where P_h is the L^2-projection onto W_h. Then the hybrid form of the mixed method for (1.1) is to find $(\sigma_h, u_h, \lambda_h) \in \tilde{V}_h \times W_h \times L_h$ such that

$$
\begin{aligned}
&\textstyle\sum_{T\in\mathcal{T}_h} (\nabla \cdot \sigma_h, w)_T = (f, w), && \forall w \in W_h, \\
&(\alpha_h \sigma_h, v) - \textstyle\sum_{T\in\mathcal{T}_h} \left[(u_h, \nabla \cdot v)_T - (\lambda_h, v \cdot \nu_T)_{\partial T\setminus\partial\Omega}\right] = 0, && \forall v \in \tilde{V}_h, \\
&\textstyle\sum_{T\in\mathcal{T}_h} (\sigma_h \cdot \nu_T, \mu)_{\partial T\setminus\partial\Omega} = 0, && \forall \mu \in L_h.
\end{aligned} \tag{1.4}
$$

Note that the last equation in (1.4) enforces the continuity requirement mentioned above, so in fact $\sigma_h \in V_h$. In [1], [2], [11], and [28], it was shown that the solution to (1.4) can be obtained from a certain modified nonconforming Galerkin method by means of augmenting the latter with bubble functions. In this paper, following [12] and [13], we show that the linear system generated by (1.4) can be algebraically condensed to a symmetric, positive definite system for the Lagrange multiplier λ_h. It is then shown that this linear system can be obtained from the standard nonconforming Galerkin method without using any bubbles.

The main objective of this paper is to construct algebraic multilevel preconditioners for the mixed finite element method. We first use the above equivalence to construct multilevel preconditioners for the linear system for the Lagrange multipliers. Then the mixed method solutions σ_h and u_h are recovered via these multipliers.

The construction of multilevel preconditioners for the mixed methods is inspired by the fundamental work [7], [22], where new systematic representations for preconditioners in the Neumann-Dirichlet domain decomposition methods for conforming finite elements were suggested. The multilevel domain decomposition versions of these methods were outlined in detail in [23], [24] and their multigrid versions were described in [25], [26] for model elliptic equations. In addition, the superelement approach used here to estimate condition numbers for two level methods is based on that used in [4], [5], [18], [25], and [27].

This paper unifies and refines the results previously announced in [18] and [19]. Briefly, the approach used to construct preconditioners includes two main stages. First, using the idea of partitioning (decomposing) a parallelepiped grid into tetrahedral substructures a two-level preconditioner is constructed for a block "7-point" algebraic system with 2×2 blocks on the coarse level, and the condition number of the preconditioned matrix is estimated. A detailed description of procedures to construct such preconditioners for parallelepiped grids can be found in [18], [23], [24], and [26]. The explicit bounds of spectrum of the preconditioned matrix are obtained with help of the superelement approach [18], [25].

On the second stage, introducing a special rotation we reduce the above block 7-point algebraic system to a series of plane problems and an exact 7-point-scheme problem with one unknown per parallelepiped. The constructed preconditioners are spectrally equivalent to the original stiffness matrix and their arithmetic cost depends on the method of solving the 7-point problem on the coarsest level. In our implementation we use a method of separation of variables although some variants of multigrid methods or domain decomposition techniques can be used. Explicit estimates of condition numbers are obtained for these multilevel preconditioners. A computational scheme for implementing these preconditioners is also considered, and a three-step preconditioned conjugate gradient method using the present technique is described as well.

The case where $a(x)$ is a diagonal tensor and Ω is a regular parallelepiped is first analyzed in detail for the multilevel preconditioners. Then the analysis is extended to the case in which $a(x)$ is a full tensor and Ω is a rather general domain. Throughout this paper, only the three-dimensional problem is discussed; the two-dimensional case is treated as a special case of the present technique.

The rest of the paper is organized as follows. In the next section we consider an elimination procedure for (1.4). Then, in §3 we develop multilevel preconditioners for the resulting linear system. In §4, we consider arbitrary tetrahedral meshes and full tensors. Finally, in §5 extensive numerical results are presented for both regular and logical parallelepipeds to illustrate the present theory.

2. The mixed finite element method.

We now consider the most useful partition $\mathcal{T}_h$ of Ω into tetrahedra. The lowest-order Raviart-Thomas-Nedelec space [31], [29] defined over $T \in \mathcal{T}_h$ is given by

$$\begin{aligned}
&V_h(T) = \big(P_0(T)\big)^3 \oplus \big((x,y,z)P_0(T)\big),\\
&W_h(T) = P_0(T),\\
&L_h(e) = P_0(e),
\end{aligned}$$

where $P_i(T)$ is the restriction of the set of all polynomials of total degree not bigger than $i \geq 0$ to the set $T \in \mathcal{T}_h$. For each T in $\mathcal{T}_h$, let

$$\overline{f}_T = \frac{1}{|T|}(f,1)_T,$$

where $|T|$ denotes the volume of T. Also, set $\alpha_h = (\alpha_{ij})$ and $\sigma_h|_T = (\sigma_{T1}, \sigma_{T2}, \sigma_{T3}) = (q_T^1 + t_T x, q_T^2 + t_T y, q_T^3 + t_T z)$. Then, by the first equation of (1.4), it follows that

$$t_T = \overline{f}_T/3. \tag{2.1}$$

Now, take $v = (1,0,0)$ in T and $v = 0$ elsewhere, $v = (0,1,0)$ in T and $v = 0$ elsewhere, and $v = (0,0,1)$ in T and $v = 0$ elsewhere, respectively, in the second equation of (1.4) to obtain

$$(\textstyle\sum_{i=1}^3 \alpha_{ji}\sigma_{Ti}, 1)_T + \sum_{i=1}^4 |e_T^i| \nu_T^{i(j)} \lambda_h|_{e_T^i} = 0, \quad j = 1,2,3, \tag{2.2}$$

where $|e_T^i|$ is the area of the face e_T^i, and $\nu_T^i = (\nu_T^{i(1)}, \nu_T^{i(2)}, \nu_T^{i(3)})$. Let $\beta^T = (\beta_{ij}^T) = ((\alpha_{ij},1)_T)^{-1}$. Then (2.2) can be solved for q_T^j:

$$\begin{aligned} q_T^j = \quad & -\textstyle\sum_{i=1}^4 |e_T^i|(\beta_{j1}^T \nu_T^{i(1)} + \beta_{j2}^T \nu_T^{i(2)} + \beta_{j3}^T \nu_T^{i(3)})\lambda_h|_{e_T^i} \\ & -\tfrac{\overline{f}_T}{3}(\textstyle\sum_{i=1}^3 \beta_{ji}^T(\alpha_{i1}x + \alpha_{i2}y + \alpha_{i3}z), 1)_T, \quad j = 1,2,3. \end{aligned} \tag{2.3}$$

Let the basis in L_h be chosen as usual. Namely, take $\mu = 1$ on one face and $\mu = 0$ elsewhere in the last equation of (1.4). Then, apply (2.1) and (2.3) to see that the contributions of the tetrahedron T to the stiffness matrix and the right-hand side are

$$m_{ij}^T = \overline{\nu}_T^i \beta^T \overline{\nu}_T^j, \quad F_i^T = -\frac{(J_T^f, \overline{\nu}_T^i)_T}{|T|} + (J_T^f, \nu_T^i)_{e_T^i}, \quad T \in \mathcal{T}_h,$$

where $\overline{\nu}_T^i = |e_T^i|\nu_T^i$ and $J_T^f = \overline{f}_T(x,y,z)/3$. Hence we obtain the system for λ_h:

$$M\lambda = F. \tag{2.4}$$

After the computation of λ_h, we can recover σ_h via (2.1) and (2.3). Also, if u_h is required, it follows from the second equation of (1.4) that

$$u_T = \frac{1}{3|T|}\left((\alpha\sigma_h, (x,y,z))_T + \sum_{i=1}^4 \lambda_h|_{e_T^i}((x,y,z), \nu_T^i)_{e_T^i}\right), \quad T \in \mathcal{T}_h.$$

The above result is summarized in the following lemma.

Lemma 1. *Let*

$$M_h(\chi,\mu) = \textstyle\sum_{T\in\mathcal{T}_h}(\chi,\nu_T)_{\partial T}\beta^T(\mu,\nu_T)_{\partial T}, \qquad \chi,\, \mu \in L_h,$$

$$F_h(\mu) = -\textstyle\sum_{T\in\mathcal{T}_h}\frac{1}{|T|}(J^f,1)_T\cdot(\mu,\nu_T)_{\partial T} + \sum_{T\in\mathcal{T}_h}(\mu J^f, \nu_T)_{\partial T}, \qquad \mu \in L_h,$$

where J^f is such that $J^f|_T = J_T^f$. Then $\lambda_h \in \mathcal{L}_h$ satisfies

$$M_h(\lambda_h, \mu) = F_h(\mu), \quad \forall \mu \in \mathcal{L}_h, \tag{2.5}$$

where

$$\mathcal{L}_h = \{\mu \in L_h : \mu|_e = 0 \text{ for each } e \subset \partial\Omega\}.$$

Note that there are at most seven nonzero entries per row in the stiffness matrix M. Also, it is easy to see that the matrix M is a symmetric and positive definite matrix; moreover, if the angles of every T in $\mathcal{T}_h$ are not bigger than $\pi/2$, then it is an M-matrix. Finally, (2.4) corresponds to the P_1 nonconforming finite element method system, as described below. This equivalence is used to construct our multilevel preconditioners later.

Let

$$\mathcal{N}_h = \{v \in L^2(\Omega) : \begin{array}{l} v|_T \in P_1(T),\ \forall T \in \mathcal{T}_h;\ v \text{ is continuous} \\ \text{at the barycenters of interior faces and} \\ \text{vanishes at the barycenters of faces on } \partial\Omega\}. \end{array} \tag{2.6}$$

Proposition 2. *Let $f_h = P_h f$. Then* (2.4) *corresponds to the linear system produced by the problem: Find $\psi_h \in \mathcal{N}_h$ such that*

$$a_h(\psi_h, \varphi) = (f_h, \varphi), \quad \forall \varphi \in \mathcal{N}_h, \tag{2.7}$$

where $a_h(\psi_h, \varphi) = \sum_{T\in\mathcal{T}_h} (\alpha_h^{-1}\nabla\psi_h, \nabla\varphi)_T$.

PROOF. From the definition of the basis $\{\psi_i^h\}$ of $\mathcal{N}_h$, for each $T \in \mathcal{T}_h$ we have

$$\psi_i^h|_T = \frac{1}{|T|}\overline{\nu}_T^i \cdot ((x,y,z) - p_l), \quad i \neq l,$$

for some barycenter p_l. Then, we see that

$$(\alpha_h^{-1}\nabla\psi_i^h, \nabla\psi_j^h)_T = \overline{\nu}_T^i \beta^T \overline{\nu}_T^j,$$

which is m_{ij}^T. Also, note that for any linear functions ψ and ϕ on a tetrahedron $T \in \mathcal{T}_h$

$$(\psi, \phi)_T = \tfrac{1}{4}|T| \textstyle\sum_{i=1}^4 \psi(p_i)\phi(p_i), \tag{2.8}$$

where the p_i's are the barycenters of the faces of T, so that

$$\begin{aligned} F_i^T &= -\tfrac{(J_T^f, \overline{\nu}_T^i)_T}{|T|} + (J_T^f, \nu_T^i)_{e_T^i} \\ &= -\tfrac{\overline{f}_T}{3}(1, \psi_i^h)_T + \tfrac{|T|\overline{f}_T}{3|e_T^i|}(\psi_i^h, 1)_{e_T^i} \\ &= \overline{f}_T \left(1, \psi_i^h\right)_T, \end{aligned}$$

which is $(f_h, \psi_i^h)_T$. □

3. Multilevel preconditioners over a cube. In this section we consider multilevel preconditioners for (2.4) based on partitioning regular parallelepipeds into tetrahedral substructures, following the ideas in [18] and [19]. Here we treat the case where Ω is a unit cube and $a(x)$ is a diagonal tensor. A general case is considered in the next section.

3.1. Two level preconditioners. Let $\mathcal{C}_h = \{C^{(i,j,k)}\}$ be a partition of Ω into uniform cubes with the length $h = 1/n$, where (x_i, y_j, z_k) is the right back upper corner of the cube $C^{(i,j,k)}$. Next, each cube $C^{(i,j,k)}$ is divided into two prisms $P_1 = P_1^{(i,j,k)}$ and $P_2 = P_2^{(i,j,k)}$ as shown in Figure 1. The resulting partition of Ω is denoted by $\mathcal{P}_h$. Finally, we divide each prism into three tetrahedra as illustrated in Figure 1 and denote this partition of Ω into tetrahedra by $\mathcal{T}_h$.

Let $W_{c,h}$ be the space of piecewise constants associated with $\mathcal{C}_h$, and $P_{c,h}$ be the L^2-projection onto $W_{c,h}$. To define our preconditioners, we introduce $\alpha_h = P_{c,h}a^{-1}$ in the hybrid form (1.4) instead of $\alpha_h = P_h a^{-1}$. Obviously, Lemma 1 and Proposition 2 are still valid for this modification since $\mathcal{T}_h$ is a refinement of $\mathcal{C}_h$. With this modification, α_h^{-1} is a constant on each cube. For notational convenience, we drop the subscript h and simply write α_h^{-1} =diag(a_1, a_2, a_3).

Let $\mathcal{N}_h$ be the nonconforming finite element space associated with $\mathcal{T}_h$ as defined in (2.6), and let its dimension be N. All the unknowns on the faces of $\partial\Omega$ are excluded. For any function $v_h \in \mathcal{N}_h$, we denote by $v \in \mathbb{R}^N$ the corresponding vector of its degrees of freedom. Introduce the inner product

$$(u, v)_N = h^3 \sum_{p_i \in \partial\mathcal{T}_h} u_h(p_i) v_h(p_i), \quad u_h, v_h \in \mathcal{N}_h, \tag{3.1}$$

where the p_i's are the barycenters of the interior faces. By (2.8) the norm induced by (3.1) is equivalent to the L^2-norm on Ω.

For each prism $P = P^{(i,j,k)} \in \mathcal{P}_h$, denote by $\mathcal{N}_h^P$ the subspace of the restriction of the functions in $\mathcal{N}_h$ onto P. For each $v \in \mathcal{N}_h^P$, we indicate by v_P its corresponding vector. The dimension of $\mathcal{N}_h^P$ is denoted by N^P. Obviously, for a prism without faces on $\partial\Omega$ its dimension is 10, i.e., $N^P = 10$.

The local stiffness matrix M^P on prism $P \in \mathcal{P}_h$ is given by

$$(M^P u_P, v_P)_{N^P} = \sum_{T \subset P} (\alpha_h^{-1} \nabla u_h, \nabla v_h)_T. \tag{3.2}$$

Then the global stiffness matrix is determined by assembling the local stiffness matrices:

$$(Mu, v)_N = \sum_{P \in \mathcal{P}_h} (M^P u_P, v_P)_{N^P}. \tag{3.3}$$

Now we consider a prism P of a cube that has no face on the boundary $\partial\Omega$ and enumerate the faces s_j, $j = 1, \ldots, 10$ of the tetrahedra in this prism as shown in Figure 2. Then the local stiffness matrix of this prism has the following form:

$$\mathrm{M}^{\mathrm{P}} = \frac{3\mathrm{h}}{2} \left[\begin{array}{cccccc|cccc}
a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_2 & 0 \\
0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_1 \\
0 & 0 & a_1 & 0 & 0 & 0 & -a_1 & 0 & 0 & 0 \\
0 & 0 & 0 & a_2 & 0 & 0 & 0 & -a_2 & 0 & 0 \\
0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & -a_3 & 0 \\
0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & -a_3 \\
\hline
0 & 0 & -a_1 & 0 & 0 & 0 & a_1 + a_2 & 0 & -a_2 & 0 \\
0 & 0 & 0 & -a_2 & 0 & 0 & 0 & a_1 + a_2 & 0 & -a_1 \\
-a_2 & 0 & 0 & 0 & -a_3 & 0 & -a_2 & 0 & 2(a_2 + a_3) & -a_3 \\
0 & -a_1 & 0 & 0 & 0 & -a_3 & 0 & -a_1 & -a_3 & 2(a_1 + a_3)
\end{array} \right],$$

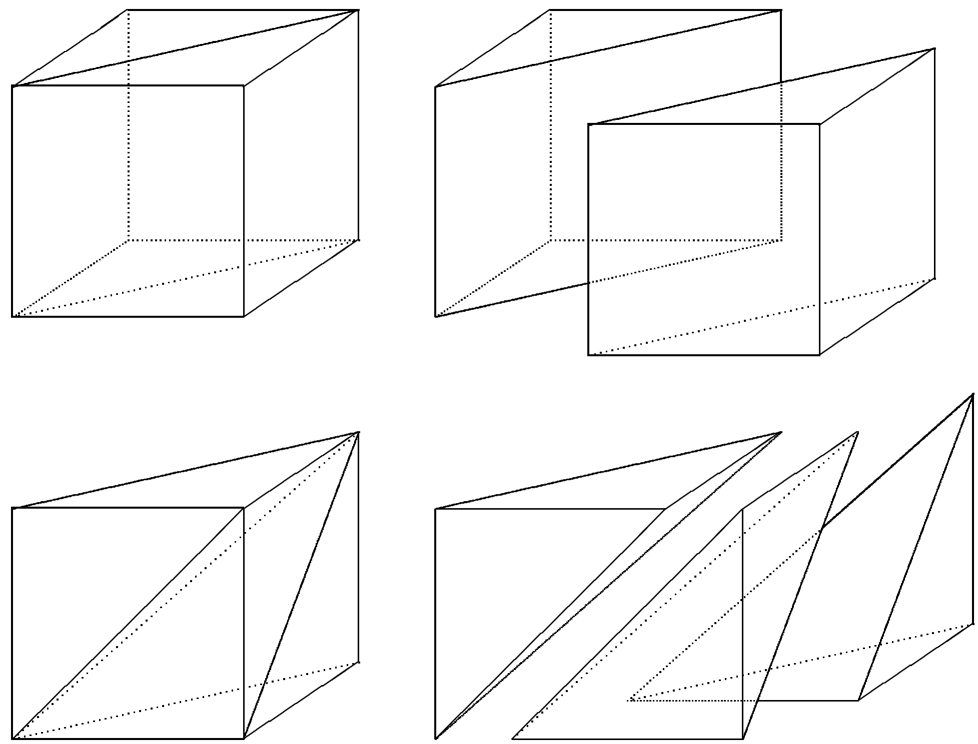


FIGURE 1. *The partition of a cube into prisms and tetrahedra.*

which we write as

$$M^P = \frac{3h}{2} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}. \tag{3.4}$$

Along with matrix M^P we also introduce the matrix B^P defined on the same space $\mathcal{N}_h^P$:

$$B^P = \frac{3h}{2} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & B_{22} \end{bmatrix}, \tag{3.5}$$

where

$$B_{22} = \begin{bmatrix} a_1+a_2+b & -b & -a_2 & 0 \\ -b & a_1+a_2+b & 0 & -a_1 \\ -a_2 & 0 & 2a_2+a_3 & 0 \\ 0 & -a_1 & 0 & 2a_1+a_3 \end{bmatrix},$$

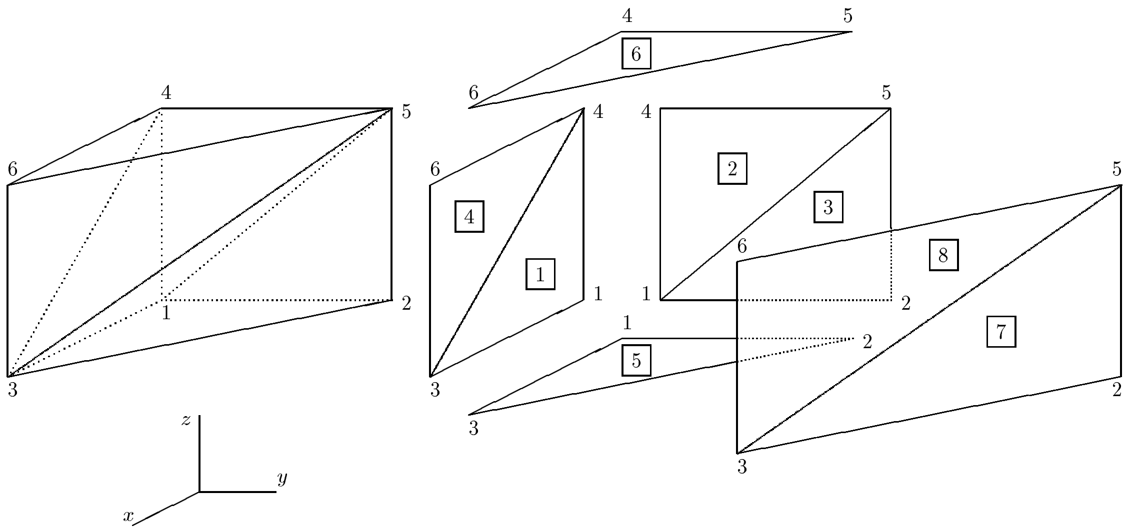
with some parameter b to be specified later.

Proposition 3. *It holds that*

$$\ker M^P = \ker B^P.$$

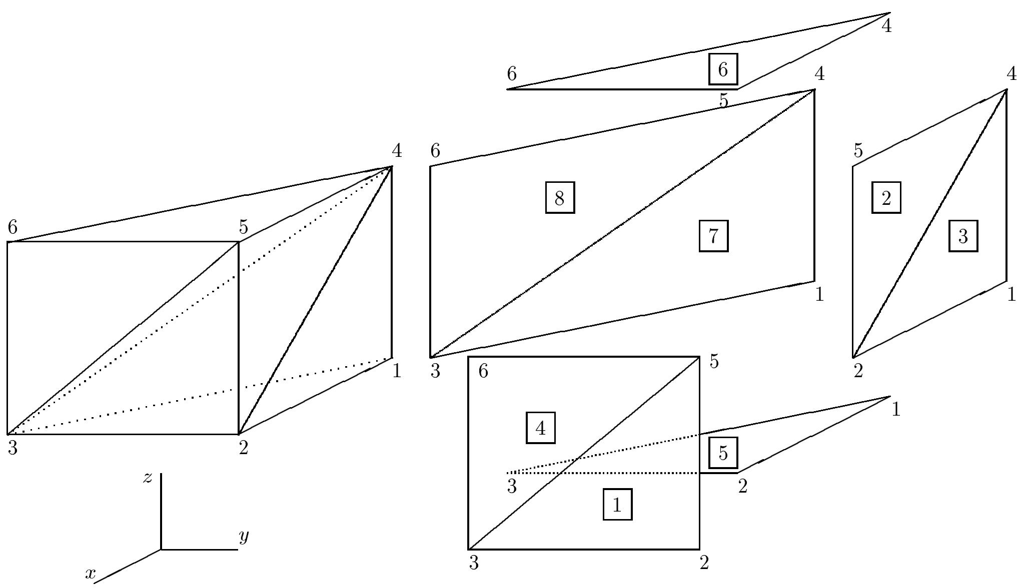
PROOF. It is easy to see from the definitions of M^P and B^P that

$$\ker M^P = \ker B^P = \{v = (v_1, v_2, \cdots, v_{10})^T \in \mathbb{R}^{10} : v_i = v_1,\ i = 2, \cdots, 10\}. \quad \square$$



(a) P_1

$s_1 = (1,4,3)$ $s_3 = (1,2,5)$ $s_5 = (1,2,3)$ $s_7 = (2,5,3)$ $s_9 = (1,5,3)$
$s_2 = (1,4,5)$ $s_4 = (3,4,6)$ $s_6 = (4,5,6)$ $s_8 = (3,5,6)$ $s_{10} = (3,4,5)$



(b) P_2

$s_1 = (2,3,5)$ $s_3 = (1,2,4)$ $s_5 = (1,2,3)$ $s_7 = (1,3,4)$ $s_9 = (2,3,4)$
$s_2 = (2,4,5)$ $s_4 = (3,5,6)$ $s_6 = (4,5,6)$ $s_8 = (3,4,6)$ $s_{10} = (3,4,5)$

FIGURE 2. *A local enumeration of faces in a prism.*

Remark. If the prism $P \in \mathcal{P}_h$ has a face on $\partial\Omega$, then the matrix M^P does not have the rows and columns which correspond to the nodes on that face, and the modification of B_{22} is obvious.

We now define the $N \times N$ matrix B by the following equality:

$$(Bu, v)_N = \sum_{P \in \mathcal{P}_h} (B^P u_P, v_P)_{N^P}, \qquad \forall u, v \in \mathbb{R}^N.$$

Since the matrix B is used for preconditioning the original problem (2.4), it is important to estimate the condition number of $B^{-1}M$:

$$Mu = \mu Bu. \tag{3.6}$$

Lemma 4. *Let μ_P satisfy the equality*

$$M^P u_P = \mu_P B^P u_P, \qquad P \in \mathcal{P}_h. \tag{3.7}$$

Then we have

$$\max_{(Bu,u)_N \neq 0} \frac{(Mu,u)_N}{(Bu,u)_N} \leq \max_{P \in \mathcal{P}_h} \mu_P \quad \textit{and} \quad \min_{(Bu,u)_N \neq 0} \frac{(Mu,u)_N}{(Bu,u)_N} \geq \min_{P \in \mathcal{P}_h} \mu_P. \tag{3.8}$$

Proof. For each $P \in \mathcal{P}_h$, it follows from (3.7) that

$$(M^P u_P, u_P)_{N^P} = \mu_P (B^P u_P, u_P)_{N^P}.$$

It then follows from the fact that all local stiffness matrices are nonnegative that

$$\begin{aligned} \sum_{P \in \mathcal{P}_h} (M^P u_P, u_P)_{N^P} &= \sum_{P \in \mathcal{P}_h} \mu_P (B^P u_P, u_P)_{N^P} \\ &\leq \max_{P \in \mathcal{P}_h} \mu_P \sum_{P \in \mathcal{P}_h} (B^P u_P, u_P)_{N^P}. \end{aligned}$$

Hence from the definitions of M and B, we see that

$$(Mu, u)_N \leq \max_{P \in \mathcal{P}_h} \mu_P (Bu, u)_N.$$

Consequently, the first inequality in (3.8) is true. The same argument can be used to show the second inequality. □

From Lemma 4, we see that, to estimate the condition number of $B^{-1}M$, it suffices to consider the local problem (3.7). Using a superelement analysis [23], to estimate $\max_{P \in \mathcal{P}_h} \mu_P$ and $\min_{P \in \mathcal{P}_h} \mu_P$, it suffices to treat the worst case where the prism $P \in \mathcal{P}_h$ has no face on the boundary $\partial\Omega$. From (3.4) and (3.5), a direct calculation shows that the eigenvalue μ_P is within the interval $[\mu_P^-, \mu_P^+]$, where

$$\mu_P^{\pm} = \frac{1}{2}\left(1 + \frac{a_3}{a_1} + \frac{a_3}{a_2} + \frac{a_3}{b}\right)\left(1 \pm \sqrt{1 - \frac{4a_3/b}{(1 + a_3/a_1 + a_3/a_2 + a_3/b)^2}}\right). \tag{3.9}$$

We now consider two useful choices of b. The κ below is given by

$$\max_{P \in \mathcal{P}_h} \left\{ \frac{a_3}{a_1}, \frac{a_3}{a_2} \right\} \leq \kappa.$$

Theorem 5. *The eigenvalues of problem* (3.6) *with the parameter $b = a_3$ belong to the interval*

$$\left[1 + \kappa - \sqrt{\kappa^2 + 2\kappa},\, 1 + \kappa + \sqrt{\kappa^2 + 2\kappa}\right],$$

and the condition number is thus estimated by

$$\mathrm{cond}(B^{-1}M) \leq 3 + 8\kappa + 4\kappa^2.$$

If the parameter b *is chosen by* $b^{-1} = a_1^{-1} + a_2^{-1} + a_3^{-1}$, *then the eigenvalues of problem* (3.6) *are within the interval*

$$\left[(1+2\kappa)\left(1-\sqrt{\frac{2\kappa}{1+2\kappa}}\right), (1+2\kappa)\left(1+\sqrt{\frac{2\kappa}{1+2\kappa}}\right)\right],$$

and the condition number is then estimated as follows:

$$\mathrm{cond}(B^{-1}M) \leq 3 + 8\kappa.$$

PROOF. We only prove the case where $b = a_3$; the other choice can be shown with the same argument. When $b = a_3$, the $\mu_P^{\pm}$ can be written as

$$\mu_P^{\pm} = \frac{1}{2}\left(2 + \frac{a_3}{a_1} + \frac{a_3}{a_2} \pm \sqrt{\left(2 + \frac{a_3}{a_1} + \frac{a_3}{a_2}\right)^2 - 4}\right).$$

Then we consider the functions

$$f_{\pm}(x) = \frac{1}{2}\left(x \pm \sqrt{x^2 - 4}\right), \quad x \geq 2.$$

Note that f_+ is a nondecreasing function and f_- is a nonincreasing function. Hence the desired result follows from the definition of κ. □

We stress that the condition number of the matrix $B^{-1}M$ is bounded by a constant independent of the step size of the mesh h. Furthermore, the second choice of b has a much better estimate of the condition number than the first choice. Hence only the second choice will be considered in the following sections. Since we introduced a two level subdivision, the matrix B can be referred to as a two level preconditioner.

3.2. Three level preconditioners. While the preconditioner B has good properties, it is not economical to invert it. In this section we propose a modification of the matrix B and consider its properties and computational scheme. Toward that end, we divide all unknowns in the system into two groups:

1. The first group consists of all unknowns corresponding to faces of the prisms in the partition $\mathcal{P}_h$, excluding the faces on $\partial\Omega$ (see Figure 2).
2. The second group consists of the unknowns corresponding to the faces of the tetrahedra that are internal for each prism (these are faces s_9 and s_{10} on Figure 2).

This splitting of the space IR^N induces the presentation of the vectors: $v^T = (v_1, v_2)$, where $v_1 \in \mathrm{IR}^{N_1}$ and $v_2 \in \mathrm{IR}^{N_2}$. Obviously, $N_1 = N - 4n^3$. Then the matrix B can be presented in the following block form:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \qquad \mathrm{dim} B_{11} = N_1. \tag{3.10}$$

Denote now by $\hat{B}_{11} = B_{11} - B_{12}B_{22}^{-1}B_{21}$ the Schur complement of B obtained by elimination of the vector v_2. Then $B_{11} = \hat{B}_{11} + B_{12}B_{22}^{-1}B_{21}$, so the matrix B has the form

$$B = \begin{bmatrix} \hat{B}_{11} + B_{12}B_{22}^{-1}B_{21} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}. \tag{3.11}$$

Note that for each prism $P \in \mathcal{P}_h$ the unknowns on the faces s_9 and s_{10} (see Figure 2) are connected only with the unknowns associated with this prism and therefore can be eliminated locally; that is, the matrix B_{22} is block diagonal with 2×2 blocks and can be inverted locally (prism by prism). Thus matrix $\hat{B}_{11}$ is easily computable. The proposed modification of the matrix B in (3.11) is of the form

$$\tilde{B} = \begin{bmatrix} \tilde{B}_0 + B_{12}B_{22}^{-1}B_{21} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where $\tilde{B}_0$ is to be defined later.

3.2.1. Group partitioning of grid points. For the sake of simplicity of representation of matrices and computational schemes we introduce the partitioning of all nodes in $\partial\mathcal{T}_h$ into the following three groups. Denote by $s_{r,l,m}^{(i,j,k)}$ the face of the cube $C^{(i,j,k)}$ with vertices r, l, m (see Figure 3).

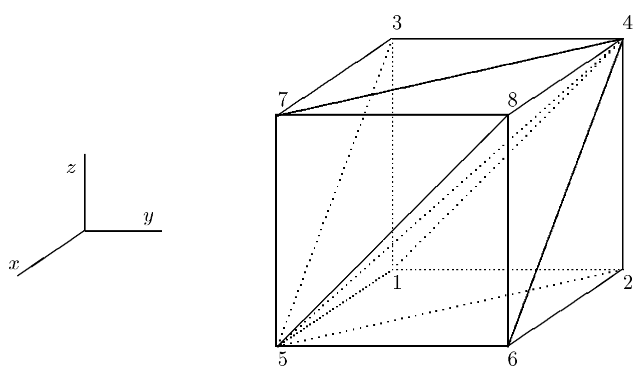


FIGURE 3. *A cube* $C^{(i,j,k)}$.

1. First, we group the nodes on the faces

$$s_{2,4,5}^{(i,j,k)} \quad \text{and} \quad s_{4,5,7}^{(i,j,k)}, \qquad i,j,k = \overline{1,n};$$

we denote the unknowns at these nodes by $VI_\ell^{(i,j,k)}$, $\ell = 1,2,\ i,j,k = \overline{1,n}$.

2. Second, we take the nodes on the faces perpendicular to the x, y, and z axes:

$$\text{(i)} \quad s_{1,2,4}^{(i,j,k)}, \quad s_{1,3,4}^{(i,j,k)}, \quad i = \overline{2,n}, \quad j,k = \overline{1,n};$$

we denote the unknowns at these nodes by $Vx_\ell^{(i,j,k)}$, $\ell = 1,2,\ i = \overline{2,n}$, $j,k = \overline{1,n}$.

$$\text{(ii)} \quad s_{1,3,5}^{(i,j,k)}, \quad s_{5,3,7}^{(i,j,k)}, \quad j = \overline{2,n}, \quad i,k = \overline{1,n};$$

we denote the unknowns at these nodes by $Vy_\ell^{(i,j,k)}$, $\ell = 1,2,\ j = \overline{2,n}$, $i,k = \overline{1,n}$.

$$\text{(iii)} \quad s_{1,2,5}^{(i,j,k)}, \quad s_{2,5,6}^{(i,j,k)}, \quad i,j = \overline{1,n}, \quad k = \overline{2,n};$$

we denote the unknowns at these nodes by $Vz_\ell^{(i,j,k)}$, $\ell = 1,2,\ i,j = \overline{1,n}$, $k = \overline{2,n}$.

3. Finally, we take the remaining nodes on the faces

$$s_{1,4,5}^{(i,j,k)}, \quad s_{3,4,5}^{(i,j,k)}, \quad s_{4,5,6}^{(i,j,k)}, \quad s_{4,5,8}^{(i,j,k)}, \quad i,j,k=\overline{1,n};$$

we denote the unknowns at these nodes by $VA_\ell^{(i,j,k)}$, $\ell=\overline{1,4}$, $i,j,k=\overline{1,n}$.

3.2.2. Three level preconditioners. We partition each cube $C^{(i,j,k)}$ into the left and right prisms $P_p^{(i,j,k)}$, $p=1,2$; see Figure 1. Below we skip the indices '(i,j,k)' and the superscript 'P' when no ambiguity occurs.

In the local numeration (Figure 2) the matrices B_1 and B_2 corresponding to the left and right prisms have the form (3.5). We rewrite these matrices in the above group partitioning:

$$B_1=\frac{3h}{2}\left[\begin{array}{cc|cccccc|cc}
a_1+a_2+b & -b & -a_1 & 0 & 0 & 0 & 0 & 0 & -a_2 & 0\\
-b & a_1+a_2+b & 0 & 0 & 0 & -a_2 & 0 & 0 & 0 & -a_1\\
\hline
-a_1 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & -a_1\\
0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & -a_2 & 0\\
0 & -a_2 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 & -a_3 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 & -a_3\\
\hline
-a_2 & 0 & 0 & 0 & -a_2 & 0 & -a_3 & 0 & 2a_2+a_3 & 0\\
0 & -a_1 & 0 & -a_1 & 0 & 0 & 0 & -a_3 & 0 & 2a_1+a_3
\end{array}\right],$$

$$B_2=\frac{3h}{2}\left[\begin{array}{cc|cccccc|cc}
a_1+a_2+b & -b & 0 & 0 & -a_2 & 0 & 0 & 0 & -a_1 & 0\\
-b & a_1+a_2+b & 0 & -a_1 & 0 & 0 & 0 & 0 & 0 & -a_2\\
\hline
0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & -a_1 & 0\\
0 & -a_1 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0\\
-a_2 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & -a_2\\
0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 & -a_3 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 & -a_3\\
\hline
-a_1 & 0 & -a_1 & 0 & 0 & 0 & -a_3 & 0 & 2a_1+a_3 & 0\\
0 & -a_2 & 0 & 0 & 0 & -a_2 & 0 & -a_3 & 0 & 2a_2+a_3
\end{array}\right].$$

The partitioning of nodes into the above three groups induces the following block forms of the matrices B_p, $p=1,2$:

$$B_p=\begin{bmatrix} B_{11,p} & B_{12,p}\\ B_{21,p} & B_{22,p}\end{bmatrix}, \qquad p=1,2, \tag{3.12}$$

where the blocks $B_{22,p}$ correspond to the unknowns of the last group and the blocks $B_{11,p}$ correspond to the unknowns of the first and second groups.

We eliminate the unknowns of the last group from each matrix B_p, $p=1,2$, which is done locally on each prism. Then we get the matrices

$$\hat{B}_{11,p}=B_{11,p}-B_{12,p}B_{22,p}^{-1}B_{21,p}, \qquad p=1,2,$$

where

$$B_{12,1}B_{22,1}^{-1}B_{21,1} = \frac{3h}{2}\left[\begin{array}{cc|cccccc}
\frac{a_2^2}{2a_2+a_3} & 0 & 0 & 0 & \frac{a_2^2}{2a_2+a_3} & 0 & \frac{a_2a_3}{2a_2+a_3} & 0 \\
0 & \frac{a_1^2}{2a_1+a_3} & 0 & \frac{a_1^2}{2a_1+a_3} & 0 & 0 & 0 & \frac{a_1a_3}{2a_1+a_3} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{a_1^2}{2a_1+a_3} & 0 & \frac{a_1^2}{2a_1+a_3} & 0 & 0 & 0 & \frac{a_1a_3}{2a_1+a_3} \\
\frac{a_2^2}{2a_2+a_3} & 0 & 0 & 0 & \frac{a_2^2}{2a_2+a_3} & 0 & \frac{a_2a_3}{2a_2+a_3} & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\
\frac{a_2a_3}{2a_2+a_3} & 0 & 0 & 0 & \frac{a_2a_3}{2a_2+a_3} & 0 & \frac{a_3^2}{2a_2+a_3} & 0 \\
0 & \frac{a_1a_3}{2a_1+a_3} & 0 & \frac{a_1a_3}{2a_1+a_3} & 0 & 0 & 0 & \frac{a_3^2}{2a_1+a_3}
\end{array}\right],$$

and a similar form holds for $B_{12,2}B_{22,2}^{-1}B_{21,2}$.

Following [18], we introduce on each prism the following modification of the matrices $\hat{B}_{11,p}$:

$$\mathrm{B}_0 = \frac{3\mathrm{h}}{2}\left[\begin{array}{cc|cccccc}
a_1+a_2+b+s_2 & -b & -a_1 & 0 & -a_2 & 0 & -s_2/2 & -s_2/2 \\
-b & a_1+a_2+b+s_1 & 0 & -a_1 & 0 & -a_2 & -s_1/2 & -s_1/2 \\
\hline
-a_1 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 \\
0 & -a_1 & 0 & a_1 & 0 & 0 & 0 & 0 \\
-a_2 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 \\
0 & -a_2 & 0 & 0 & 0 & a_2 & 0 & 0 \\
-s_2/2 & -s_1/2 & 0 & 0 & 0 & 0 & \frac{s_1+s_2}{2} & 0 \\
-s_2/2 & -s_1/2 & 0 & 0 & 0 & 0 & 0 & \frac{s_1+s_2}{2}
\end{array}\right],$$

with some parameters s_1 and s_2.

Proposition 6. *The matrices $\hat{B}_{11,p}$, $p = 1,2$, and B_0 have the same kernel, i.e.,*

$$\ker\hat{B}_{11,p} = \ker B_0.$$

Proof. It can be easily checked that

$$\ker\hat{B}_{11,p} = \ker B_0 = \{v = (v_1, v_2, \cdots, v_8)^T \in \mathbb{R}^8 : v_i = v_1,\ i = 2, \cdots, 8\}, \qquad p = 1,2. \quad \square$$

We now consider the eigenvalue problem

$$\hat{B}_{11,p}u = \mu B_0 u, \quad u \in \mathbb{R}^8, \quad p = 1,2, \tag{3.13}$$

with the following choices of s_1 and s_2.

Proposition 7. *For the case where $s_i = 2a_ia_3/(2a_i + a_3)$, $i = 1,2$, the eigenvalues of problem* (3.13) *belong to the interval*

$$\left[\frac{3+2\kappa}{4+2\kappa}(1-\frac{1}{\sqrt{3}}), \frac{3+2\kappa}{4+2\kappa}(1+\frac{1}{\sqrt{3}})\right].$$

If we choose $s_i = \max\{a_i, a_3\}$, $i = 1,2$, the eigenvalues of problem (3.13) *are within the interval*

$$\left[\frac{3+\kappa}{4+2\kappa}(1-\frac{1}{\sqrt{3}}), \frac{3+\kappa}{4+2\kappa}(1+\frac{1}{\sqrt{3}})\right].$$

Both cases have the same estimate of the condition number

$$\mathrm{cond}(B_0^{-1}\hat{B}_{11,p}) \le 2+\sqrt{3}.$$

PROOF. A direct calculation shows that $\mu \in [\mu^-, \mu^+]$ where

$$\mu^- = \min\left\{ \frac{a_i}{4a_i+2a_3}\left(1+\frac{a_3}{a_i}+\frac{2a_3}{s_i}\right)\left(1-\sqrt{1-\frac{a_3^2/(a_i s_i)+2a_3/s_i}{1+a_3/a_i+2a_3/s_i}}\right) : i=1,2 \right\},$$

and

$$\mu^+ = \max\left\{ \frac{a_i}{4a_i+2a_3}\left(1+\frac{a_3}{a_i}+\frac{2a_3}{s_i}\right)\left(1+\sqrt{1-\frac{a_3^2/(a_i s_i)+2a_3/s_i}{1+a_3/a_i+2a_3/s_i}}\right) : i=1,2 \right\}.$$

With $s_i = 2a_i a_3/(2a_i+a_3)$, $i=1,2$, and the definition of κ, it can be seen as in Theorem 5 that

$$\mu_- \ge \frac{3+2\kappa}{4+2\kappa}\left(1-\sqrt{1-\frac{2+3\kappa/2+\kappa^2/2}{3+2\kappa}}\right),$$

and

$$\mu_+ \ge \frac{3+2\kappa}{4+2\kappa}\left(1+\sqrt{1-\frac{2+3\kappa/2+\kappa^2/2}{3+2\kappa}}\right).$$

Note that

$$1-\frac{2+3\kappa/2+\kappa^2/2}{3+2\kappa} \le \frac{1}{3},$$

so that the first case follows. The same argument applies to the second case. □

Now we define a new matrix on each prism:

$$\tilde{B}_p = \begin{bmatrix} B_0 + B_{12,p}B_{22,p}^{-1}B_{21,p} & B_{12,p} \\ B_{21,p} & B_{22,p} \end{bmatrix}, \qquad p=1,2. \tag{3.14}$$

As remarked before, in the case where a cube C has nonempty intersection with $\partial\Omega$, the matrices B_0, $B_{12,p}$, and $B_{21,p}$, $p=1,2$, do not have the rows and columns corresponding to the nodes on the boundary.

For each prism $P \in \mathcal{P}_h$ we now consider the eigenvalue problem:

$$B^P u = \mu \tilde{B}^P u, \tag{3.15}$$

where $B^P = B_p^P$ is defined in (3.12) and $\tilde{B}^P = \tilde{B}_p^P$ in (3.14), $p=1,2$. Below we only consider the simpler choice: $s_i = \max\{a_i, a_3\}$, $i=1,2$.

Proposition 8. *The eigenvalues of problem* (3.15) *belong to the interval*

$$\left[\frac{3+\kappa}{4+2\kappa}\left(1-\frac{1}{\sqrt{3}}\right), \frac{3+\kappa}{4+2\kappa}\left(1+\frac{1}{\sqrt{3}}\right)\right].$$

Moreover, on each prism $P \in \mathcal{P}_h$ the eigenvalues of the problem

$$M^P u = \mu \tilde{B}^P u, \tag{3.16}$$

are within the interval $[\mu_-, \mu_+]$, *where*

$$\mu_\pm = (1+2\kappa)\left(1 \pm \sqrt{\frac{2\kappa}{1+2\kappa}}\right)\frac{3+\kappa}{4+2\kappa}\left(1 \pm \frac{1}{\sqrt{3}}\right).$$

PROOF. The first statement follows directly from Proposition 7, and the second one then follows from Theorem 5. □

Now we define the symmetric positive-definite $N_1 \times N_1$ matrix $\tilde{B}_0$ by

$$(\tilde{B}_0 u_1, v_1) = \sum_{P \in \mathcal{P}_h} (B_0 u_{1,P}, v_{1,P}),$$

where $v_1, u_1 \in \mathbb{R}^{N_1}$, and $u_{1,P}$ and $v_{1,P}$ are their respective restrictions on the prism P. As in (3.11), we introduce the matrix

$$\tilde{B} = \begin{bmatrix} \tilde{B}_0 + B_{12}B_{22}^{-1}B_{21} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}. \tag{3.17}$$

Using Proposition 8 and the same proof as in Theorem 5, we have the following theorem.

Theorem 9. *The matrix* $\tilde{B}$ *defined in* (3.17) *is spectrally equivalent to the matrix* M, *i.e.,*

$$\mu_* \tilde{B} \le M \le \mu^* \tilde{B}.$$

Moreover,

$$\text{cond}(\tilde{B}^{-1}M) \le \overline{\mu} \equiv \mu^*/\mu_* \le (3+8k)(2+\sqrt{3}). \tag{3.18}$$

Instead of the matrix B in the form (3.11) we take the matrix $\tilde{B}$ in (3.17) as a preconditioner for the matrix M. Because we have introduced a two-level subdivision of the matrix $\tilde{B}_0$, the matrix $\tilde{B}$ can be considered as a three-level preconditioner.

As we noted earlier, the matrix B_{22} is block-diagonal and can be inverted locally on prisms. So we concentrate on the linear system

$$\tilde{B}_0 u = G. \tag{3.19}$$

In terms of the group partitioning in §3.2.1, the matrix $\tilde{B}_0$ has the block form

$$\tilde{B}_0 = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \tag{3.20}$$

where the block C_{22} corresponds to the nodes from the second group, which are on the faces of tetrahedra perpendicular to the coordinate axes. From the definition of B_0, it can be seen that the matrix C_{22} is diagonal. In the above partitioning, we present u and G in (3.19) in the form

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}. \tag{3.21}$$

Then, after elimination of the second group of unknowns:

$$u_2 = C_{22}^{-1}(G_2 - C_{21}u_1),$$

we get the system of linear equations

$$(C_{11} - C_{12}C_{22}^{-1}C_{21})u_1 = G_1 - C_{12}C_{22}^{-1}G_2 \equiv \tilde{G}_1, \tag{3.22}$$

where the vector u_1 and the block C_{11} correspond to the unknowns from the first group, which have only two unknowns per each cube. The dimension of vectors u_1 and G_1 is equal to

$$\dim(u_1) = 2n^3. \tag{3.23}$$

The above simplification of (3.19) is carried out in detail in the next section.

3.2.3. Computational scheme. We now consider the computational scheme for (3.19). In terms of the unknowns introduced in §3.2.1:

$$\begin{aligned}
&uI_\ell^{(i,j,k)}, \quad GI_\ell^{(i,j,k)}, \quad \ell = 1,2, \quad i,j,k = \overline{1,n},\\
&ux_\ell^{(i,j,k)}, \quad Gx_\ell^{(i,j,k)}, \quad \ell = 1,2, \quad i = \overline{2,n}, \qquad j,k = \overline{1,n},\\
&uy_\ell^{(i,j,k)}, \quad Gy_\ell^{(i,j,k)}, \quad \ell = 1,2, \quad j = \overline{2,n}, \qquad i,k = \overline{1,n},\\
&uz_\ell^{(i,j,k)}, \quad Gz_\ell^{(i,j,k)}, \quad \ell = 1,2, \quad k = \overline{2,n}, \qquad i,j = \overline{1,n},
\end{aligned}$$

the system (3.19) with $a(x) \equiv 1$ can be written as

$$\begin{aligned}
&\tfrac{1}{3}\begin{bmatrix} 20 & -2 \\ -2 & 20 \end{bmatrix} uI^{(i,j,k)} - (1-\delta_{i1})ux^{(i-1,j,k)} - (1-\delta_{in})ux^{(i,j,k)}\\
&-(1-\delta_{j1})uy^{(i,j-1,k)} - (1-\delta_{jn})uy^{(i,j,k)}\\
&-(1-\delta_{k1})\tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} uz^{(i,j,k-1)} - (1-\delta_{kn})\tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} uz^{(i,j,k)}\\
&\qquad = \tfrac{2}{3h}GI^{(i,j,k)}, \quad i,j,k = \overline{1,n},
\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
&2ux^{(i,j,k)} - uI^{(i+1,j,k)} - uI^{(i,j,k)} = \tfrac{2}{3h}Gx^{(i,j,k)}, \quad i = \overline{1,n-1},\ j,k = \overline{1,n},\\
&2uy^{(i,j,k)} - uI^{(i,j+1,k)} - uI^{(i,j,k)} = \tfrac{2}{3h}Gy^{(i,j,k)}, \quad j = \overline{1,n-1},\ i,k = \overline{1,n},\\
&2uz^{(i,j,k)} - \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} uI^{(i,j,k+1)} - \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} uI^{(i,j,k)} = \tfrac{2}{3h}Gz^{(i,j,k)},\\
&\qquad k = \overline{1,n-1}, \quad i,j = \overline{1,n},
\end{aligned} \tag{3.25}$$

where δ_{ij} (the Kronecker symbol) is introduced to take into account the Dirichlet boundary conditions. Eliminating unknowns $ux_\ell^{(i,j,k)}$, $uy_\ell^{(i,j,k)}$, $uz_\ell^{(i,j,k)}$, $\ell = 1,2$, from equations (3.24), we obtain the block "seven-point" scheme with 2×2-blocks for the unknowns $uI_\ell^{(i,j,k)}$, $\ell = 1,2,\ i,j,k = \overline{1,n}$. From (3.25) we have

$$
\begin{aligned}
ux^{(i,j,k)} &= \frac{1}{2}\left(\frac{2}{3h}\right)Gx^{(i,j,k)} + \frac{1}{2}uI^{(i+1,j,k)} + \frac{1}{2}uI^{(i,j,k)}, \\
&\qquad\qquad i = \overline{1,n-1}, \quad j,k = \overline{1,n}, \\
uy^{(i,j,k)} &= \frac{1}{2}\left(\frac{2}{3h}\right)Gy^{(i,j,k)} + \frac{1}{2}uI^{(i,j+1,k)} + \frac{1}{2}uI^{(i,j,k)}, \\
&\qquad\qquad j = \overline{1,n-1}, \quad i,k = \overline{1,n}, \\
uz^{(i,j,k)} &= \frac{1}{2}\left(\frac{2}{3h}\right)Gz^{(i,j,k)} + \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} uI^{(i,j,k+1)} \\
&\quad + \tfrac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} uI^{(i,j,k)}, \quad k = \overline{1,n-1}, \quad i,j = \overline{1,n}.
\end{aligned}
\tag{3.26}
$$

Substituting (3.26) into (3.24), we see that

$$
\begin{aligned}
&\tfrac{1}{3}\begin{bmatrix} 20 & -2 \\ -2 & 20 \end{bmatrix} uI^{(i,j,k)} - (1-\delta_{i1})\tfrac{1}{2}\left(uI^{(i-1,j,k)} + uI^{(i,j,k)}\right) \\
&-(1-\delta_{in})\tfrac{1}{2}\left(uI^{(i+1,j,k)} + uI^{(i,j,k)}\right) \\
&-(1-\delta_{j1})\tfrac{1}{2}\left(uI^{(i,j-1,k)} + uI^{(i,j,k)}\right) \\
&-(1-\delta_{jn})\tfrac{1}{2}\left(uI^{(i,j+1,k)} + uI^{(i,j,k)}\right) \\
&-(1-\delta_{k1})\tfrac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\left(uI^{(i,j,k-1)} + uI^{(i,j,k)}\right) \\
&-(1-\delta_{kn})\tfrac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\left(uI^{(i,j,k+1)} + uI^{(i,j,k)}\right) = g^{(i,j,k)}, \quad i,j,k = \overline{1,n},
\end{aligned}
\tag{3.27}
$$

where

$$
\begin{aligned}
g^{(i,j,k)} = \tfrac{2}{3h}\Big\{ \quad & GI^{(i,j,k)} + (1-\delta_{i1})\tfrac{1}{2}Gx^{(i-1,j,k)} + (1-\delta_{in})\tfrac{1}{2}Gx^{(i,j,k)} \\
&+(1-\delta_{j1})\tfrac{1}{2}Gy^{(i,j-1,k)} + (1-\delta_{jn})\tfrac{1}{2}Gy^{(i,j,k)} \\
&+(1-\delta_{k1})\tfrac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} Gz^{(i,j,k-1)} \\
&+(1-\delta_{kn})\tfrac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} Gz^{(i,j,k)}\Big\}.
\end{aligned}
\tag{3.28}
$$

To solve system (3.27) we introduce the rotation matrix

$$
Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},
$$

and the new vectors $v^{(i,j,k)} = (v_1^{(i,j,k)},\ v_2^{(i,j,k)})^T$, $i,j,k = \overline{1,n}$, given by

$$v^{(i,j,k)} = Q \cdot uI^{(i,j,k)}, \qquad i,j,k = \overline{1,n}. \tag{3.29}$$

Then multiplying both sides of the matrix equation (3.27) by the matrix Q and using the relation

$$uI^{(i,j,k)} = Q^T \cdot v^{(i,j,k)}, \qquad i,j,k = \overline{1,n}, \tag{3.30}$$

we obtain the following problem for the unknowns $v^{(i,j,k)}$:

$$\begin{aligned}
&\begin{bmatrix} 6 & 0 \\ 0 & 22/3 \end{bmatrix} v^{(i,j,k)} - (1-\delta_{i1})\tfrac{1}{2}\left(v^{(i-1,j,k)} + v^{(i,j,k)}\right) \\
&-(1-\delta_{in})\tfrac{1}{2}\left(v^{(i+1,j,k)} + v^{(i,j,k)}\right) \\
&-(1-\delta_{j1})\tfrac{1}{2}\left(v^{(i,j-1,k)} + v^{(i,j,k)}\right) \\
&-(1-\delta_{jn})\tfrac{1}{2}\left(v^{(i,j+1,k)} + v^{(i,j,k)}\right) \\
&-(1-\delta_{k1})\tfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\left(v^{(i,j,k-1)} + v^{(i,j,k)}\right) \\
&-(1-\delta_{kn})\tfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\left(v^{(i,j,k+1)} + v^{(i,j,k)}\right) \\
&\qquad = Q \cdot g^{(i,j,k)} \equiv \tilde{g}^{(i,j,k)}, \quad i,j,k = \overline{1,n}.
\end{aligned} \tag{3.31}$$

It is easy to see that problem (3.31) can be decomposed into the following two independent problems:

$$\begin{aligned}
&6v_1^{(i,j,k)} - (1-\delta_{i1})\tfrac{1}{2}\left(v_1^{(i-1,j,k)} + v_1^{(i,j,k)}\right) \\
&-(1-\delta_{in})\tfrac{1}{2}\left(v_1^{(i+1,j,k)} + v_1^{(i,j,k)}\right) \\
&-(1-\delta_{j1})\tfrac{1}{2}\left(v_1^{(i,j-1,k)} + v_1^{(i,j,k)}\right) \\
&-(1-\delta_{jn})\tfrac{1}{2}\left(v_1^{(i,j+1,k)} + v_1^{(i,j,k)}\right) \\
&-(1-\delta_{k1})\tfrac{1}{2}\left(v_1^{(i,j,k-1)} + v_1^{(i,j,k)}\right) \\
&-(1-\delta_{kn})\tfrac{1}{2}\left(v_1^{(i,j,k+1)} + v_1^{(i,j,k)}\right) = \tilde{g}_1^{(i,j,k)}, \\
&\qquad\qquad i,j,k = \overline{1,n},
\end{aligned} \tag{3.32}$$

and

$$
\begin{aligned}
&\tfrac{22}{3} v_2^{(i,j,k)} - (1-\delta_{i1})\tfrac{1}{2}\left(v_2^{(i-1,j,k)} + v_2^{(i,j,k)}\right) \\
&-(1-\delta_{in})\tfrac{1}{2}\left(v_2^{(i+1,j,k)} + v_2^{(i,j,k)}\right) \\
&-(1-\delta_{j1})\tfrac{1}{2}\left(v_2^{(i,j-1,k)} + v_2^{(i,j,k)}\right) \\
&-(1-\delta_{jn})\tfrac{1}{2}\left(v_2^{(i,j+1,k)} + v_2^{(i,j,k)}\right) = \tilde{g}_2^{(i,j,k)}, \\
&\qquad\qquad i,j = \overline{1,n}, \quad \forall\ k = \overline{1,n}.
\end{aligned}
\tag{3.33}
$$

That is, we reduced the linear system (3.31) of dimension $(2n^3)$ to one linear system of equations (3.32) of dimension n^3 and n linear systems of equations (3.33) of dimension n^2. For all these problems the method of separation of variables can be used, as shown in (3.36) and (3.38) below.

After we find the solution of these problems we easily retrieve vectors $uI^{(i,j,k)}$ by using the relations (3.30).

3.2.4. A method of separation of variables. In this section we consider a method of separation of variables for solving problems (3.32) and (3.33). Problem (3.32) can be represented in the form

$$C^{(3)} v_1 = \tilde{g}_1, \qquad v_1, \tilde{g}_1 \in \mathbb{R}^{n^3}, \tag{3.34}$$

where

$$C^{(3)} = C_0 \otimes I_0 \otimes I_0 + I_0 \otimes C_0 \otimes I_0 + I_0 \otimes I_0 \otimes C_0,$$

I_0 is the $(n \times n)$-identity matrix, $\otimes$ denotes the tensor product of matrices, and C_0 takes the form

$$C_0 = \frac{1}{2}\begin{bmatrix} 3 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 3 \end{bmatrix}. \tag{3.35}$$

If C_0 is factorized by

$$C_0 = Q_0 \Lambda_0 Q_0^T,$$

where Λ_0 is an $(n \times n)$-diagonal matrix and Q_0 is an $(n \times n)$-orthogonal matrix $(Q_0^{-1} = Q_0^T)$, then the matrix $C^{(3)}$ can be rewritten as follows:

$$C^{(3)} = Q^{(3)} \Lambda^{(3)} Q^{(3)},$$

where

$$
\begin{aligned}
Q^{(3)} &= Q_0 \otimes Q_0 \otimes Q_0, \\
\Lambda^{(3)} &= \Lambda_0 \otimes I_0 \otimes I_0 + I_0 \otimes \Lambda_0 \otimes I_0 + I_0 \otimes I_0 \otimes \Lambda_0.
\end{aligned}
$$

Note that $Q^{(3)}$ is an $(n^3 \times n^3)$-orthogonal matrix and $\Lambda^{(3)}$ is an $(n^3 \times n^3)$-diagonal matrix. We can now use the following method to solve the system (3.32):

$$\begin{aligned} &(1) \qquad \tilde{f}_1 = \left(Q^{(3)}\right)^T \tilde{g}_1, \\ &(2) \qquad \Lambda^{(3)} w = \tilde{f}_1, \\ &(3) \qquad v_1 = Q^{(3)} w. \end{aligned} \tag{3.36}$$

The same argument can be exploited to solve (3.33). Problem (3.33) can be rewritten as

$$C^{(2)} v_2 = \tilde{g}_2, \qquad v_2,\, \tilde{g}_2 \in \mathbb{R}^{n^2}, \tag{3.37}$$

where

$$C^{(2)} = K_0 \otimes I_0 + I_0 \otimes K_0,$$

and the $(n \times n)$-matrix K_0 is given by

$$K_0 = \frac{1}{6} \begin{bmatrix} 19 & -3 & & & \\ -3 & 16 & -3 & & \\ & \ddots & \ddots & \ddots & \\ & & -3 & 16 & -3 \\ & & & -3 & 19 \end{bmatrix}.$$

Again, if we write K_0 as

$$K_0 = R_0 D_0 R_0^T,$$

where D_0 is an $(n \times n)$-diagonal matrix and R_0 is an $(n \times n)$-orthogonal matrix, we can rewrite the matrix $C^{(2)}$ as follows:

$$C^{(2)} = Q^{(2)} \Lambda^{(2)} Q^{(2)^T},$$

where $Q^{(2)} = R_0 \otimes R_0$ and $\Lambda^{(2)} = D_0 \otimes I_0 + I_0 \otimes D_0$. Then the system (3.33) can be solved with the following method:

$$\begin{aligned} &(1) \qquad \tilde{f}_2 = \left(Q^{(2)}\right)^T \tilde{g}_2, \\ &(2) \qquad \Lambda^{(2)} w = \tilde{f}_2, \\ &(3) \qquad v_2 = Q^{(2)} w. \end{aligned} \tag{3.38}$$

3.2.5. Preconditioned conjugate gradient method. We now solve system (2.4) by a three-step preconditioned conjugate gradient method in the following form: Given λ^{-1}, $\lambda^0 \in \mathbb{R}^N$, find λ^k, $k = 1, 2, \cdots, K_\epsilon$, such that

$$\lambda^{(k+1)} = \lambda^k - \frac{1}{q_k} \left[\tilde{B}^{-1} \xi^k - d_k (\lambda^k - \lambda^{k-1}) \right], \tag{3.39}$$

where

$$\begin{aligned} \xi^k &= M \lambda^k - F, \\ q_k &= \frac{\|B^{-1} \xi^{k-1}\|_M^2}{\|\xi^k\|_{B^{-1}}^2} - d_{k-1}, \\ d_k &= q_k \frac{\|\xi^k\|_{B^{-1}}^2}{\|\xi^{k-1}\|_{B^{-1}}^2}. \end{aligned}$$

Theorem 10. *The number of operations for solving system* (2.4) *by method* (3.39) *with the matrix* $\tilde{B}$ *defined in* (3.17) *and with accuracy* ϵ *in the sense*

$$\|\lambda^{K_\epsilon+1} - \lambda^*\|_M \le \epsilon\|\lambda^0 - \lambda^*\|_M, \quad 0 < \epsilon \ll 1, \tag{3.40}$$

is estimated by $\mathcal{Q}N^{4/3}\ln\left(\frac{\epsilon}{2}\right)$*, where* $\lambda^* = M^{-1}F$*,* $\lambda^0 \in \mathrm{I\!R}^N$ *is any initial vector, and the constant* $\mathcal{Q} > 0$ *does not depend on* N*.*

Proof. It is known [21] that for a given $\epsilon > 0$, to achieve the accuracy (3.40) the number of iterations K_ϵ can be estimated by the inequality

$$K_\epsilon \le \frac{\ln(\epsilon/2)}{\ln q_\mu},$$

where $q_\mu = (\sqrt{\overline{\mu}} - 1)/(\sqrt{\overline{\mu}} + 1)$ and $\overline{\mu}$ is defined in (3.18). Thus it is easy to see that the arithmetical cost of the procedure (3.39) for solving (2.4) is approximately $(\ln(\epsilon/2)/\ln q_\mu)$ times the cost per iteration. The cost per iteration is $O(N^{4/3})$ by the method of separation of variables (3.36) and (3.38), so the desired result follows from Theorem 9. □

4. Preconditioners for a general case. In this section we consider the case where the coefficient a is a full tensor and the domain Ω satisfies the assumption that there is an orientation-preserving smooth mapping $\mathcal{L}$ from the unit cube $\hat{\Omega}$ onto Ω and there are positive constants r and $\mathcal{Q}$ such that

$$r^{-1}\|\mathcal{J}(x)\| \le \mathcal{Q}, \qquad \forall x \in \hat{\Omega}, \tag{4.1}$$

and

$$r\|\mathcal{J}^{-1}(x)\| \le \mathcal{Q}, \qquad \forall x \in \Omega, \tag{4.2}$$

where $\mathcal{J}(x)$ is the Jacobian matrix of $\mathcal{L}$ at x and $\|\cdot\|$ denotes the matrix 2-norm. Note that the domain Ω is of size r.

Next, we consider the definition of the nonconforming finite element space. Let $\mathcal{C}_{\hat{h}}$, $\mathcal{P}_{\hat{h}}$, and $\mathcal{T}_{\hat{h}}$ be the partitions of $\hat{\Omega}$ into cubes, prisms, and tetrahedra, respectively, associated with the mesh size $\hat{h}$, as defined in §3.1, and let $\mathcal{N}_{\hat{h}}$ be the P_1 nonconforming space associated with $\mathcal{T}_{\hat{h}}$, as given in (2.6). Set $h = r\hat{h}$ and define

$$\mathcal{N}_h = \left\{\varphi = \psi \circ \mathcal{L}^{-1} \, : \psi \in \mathcal{N}_{\hat{h}}\right\}.$$

Also, we introduce the mapping $\mathcal{I} : \mathcal{N}_h \to \mathcal{N}_{\hat{h}}$ defined by $\mathcal{I}v = v \circ \mathcal{L}$.

We now define the stiffness matrix M on the domain Ω by

$$(Mu, v)_N = a_h(u_h, v_h), \qquad \forall u_h, v_h \in \mathcal{N}_h, \tag{4.3}$$

where

$$\begin{aligned} a_h(u_h, v_h) &= \textstyle\sum_{T\in\mathcal{T}_h}(\alpha_h^{-1}\nabla u_h, \nabla v_h)_T \\ &= \textstyle\sum_{T\in\mathcal{T}_{\hat{h}}}(|\det(\mathcal{J})|\alpha_h^{-1}\mathcal{J}^{-1}\nabla\mathcal{I}u_h, \mathcal{J}^{-1}\nabla\mathcal{I}v_h)_T, \end{aligned} \tag{4.4}$$

where $|\det(\mathcal{J})|$ is the Jacobian of the mapping.

For each cube $C \in \mathcal{C}_{\hat{h}}$, we introduce the diagonal matrix α_C^{-1} =diag$\{a_{1,C}, a_{2,C}, a_{3,C}\}$ with some as yet unspecified constants $a_{i,C}$, $i = 1, 2, 3$. Then we define

$$b_h(u_h, v_h) = \sum_{C\in\mathcal{C}_h} \delta_C \left(\sum_{T\in C} (\alpha_C^{-1}\nabla u_h, \nabla v_h)_T \right), \qquad \forall u_h, v_h \in \mathcal{N}_{\hat{h}}, \tag{4.5}$$

where the constants δ_C are scaling factors. One reasonable choice is to take $\delta_C = (\lambda_{1,C} + \lambda_{0,C})/2$, where $\lambda_{1,C}$ and $\lambda_{0,C}$ are the largest and smallest eigenvalues of the eigenvalue problem

$$\tilde{\alpha}_h^{-1}(x_0)v = \lambda_C \alpha_C v, \qquad v \in \mathbb{R}^3, \tag{4.6}$$

where $\tilde{\alpha}_h^{-1} = |\det(\mathcal{J})| \left(\mathcal{J}^{-1}\right)^T \alpha_h^{-1}\mathcal{J}^{-1}$ and $x_0 \in \mathcal{L}(C) \subset \Omega$ is some point. Note that the assumptions (4.1) and (4.2) imply that there are two constants $\mathcal{Q}_0$ and $\mathcal{Q}_1$ independent of r and h such that

$$\mathcal{Q}_0 a_h(u_h, u_h) \le r b_h(\mathcal{I}u_h, \mathcal{I}u_h) \le \mathcal{Q}_1 a_h(u_h, u_h), \qquad \forall u_h \in \mathcal{N}_h. \tag{4.7}$$

We consider two useful choices of the matrix α_C^{-1}:

1. $\alpha_C^{-1} = I$, $\forall C \in \mathcal{C}_h$, i.e., the matrix α_C^{-1} is the identity matrix.
2. α_C^{-1} =diag$\{\tilde{\alpha}_h^{-1}(x_0)\}$, $\forall C \in \mathcal{C}_h$, i.e., the matrix α_C^{-1} is the diagonal part of $\tilde{\alpha}_h^{-1}(x_0)$ at some point $x_0 \in \mathcal{L}(C)$.

Note that in both cases the constants $\mathcal{Q}_0$ and $\mathcal{Q}_1$ in (4.7) only depend on the local variation of the coefficients $\left\{\left(\tilde{\alpha}_h^{-1}\right)_{kl}\right\}$. Hence the problem of defining a preconditioner for $a_h(\cdot,\cdot)$ is reduced to the problem of finding a preconditioner for $rb_h(\cdot,\cdot)$, which has a diagonal coefficient tensor and is defined on the unit cube $\hat{\Omega}$. Namely, all the analyses in §3 can be carried out here.

5. Results of the numerical experiments. In this section the method (3.39) is tested on the model problem

$$\begin{aligned} -\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(a_i \frac{\partial u}{\partial x_i}\right) &= f \qquad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

We present two numerical examples. In the first example, the domain Ω is the unit cube: $\Omega = (0,1)^3$. The domain is divided into n^3 cubes (n in each direction) and each cube is partitioned into 6 tetrahedra. The dimension of the original algebraic system is $N = 12n^3 - 6n^2$. The right hand side is generated randomly, and the accuracy parameter is taken as $\epsilon = 10^{-6}$. The condition number of the matrix $B^{-1}M$ is calculated by the relation between the conjugate gradient and Lanczos algorithms [21]. The coefficients a_i, $i = 1, 2, 3$, are constants on each cube. The results are summarized in Table 1, where Iter and Cond denote the iteration number and condition number, respectively.

From Table 1 we see that the condition number depends on the maximal ratio $\kappa = \max_{C\in\mathcal{C}_h}\left\{\frac{a_3}{a_1}, \frac{a_3}{a_2}\right\}$. The case of $\kappa < 1$ has a better convergence than the case of the Poisson equation (i.e., $a_1 = a_2 = a_3 = 1$).

In the second example we treat the Poisson equation on the domain Ω as shown in Figure 4. The domain is subdivided into $90 \times 90 \times 10$ cubes and the number of unknowns is then $N = 955440$. This problem is solved with accuracy $\epsilon = 10^{-6}$. Twenty iterations are needed to achieve the desired accuracy, and the computed condition number of the matrix $B^{-1}M$ is equal to ten.

Table 1

a_1	a_2	a_3	$N = 47,616$ Iter	Cond	$N = 387,072$ Iter	Cond
1	1	1	18	7.5	17	7.6
1	1	0.1	13	3.7	13	3.8
1	1	0.01	10	2.8	11	3.0
10	1	1	16	6	16	6.2
1	10	1				
100	1	1	14	4.7	14	5.2
1	100	1				
1	1	10	34	41	34	42
1	1	100	75	315	80	328
0.1	1	1	32	30	31	29
1	0.1	1				
0.01	1	1	68	198	72	203
1	0.01	1				

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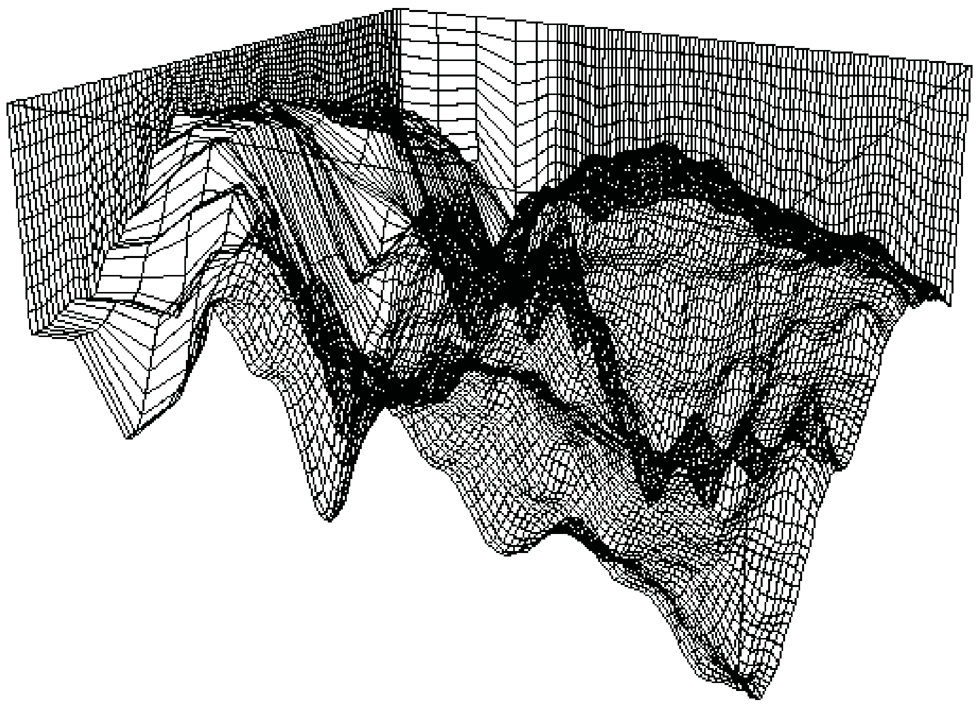

FIGURE 4. An example of the grid domain Ω.

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