PRECONDITIONING OF NONCONFORMING FINITE ELEMENT APPROXIMATIONS OF SECOND ORDER ELLIPTIC PROBLEMS

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Abstract

We consider the finite element approximation of Poisson equation in a parallelepiped using linear tetrahedral nonconforming Crouzeix-Raviart elements. Using the idea of substructuring we eliminate most of the unknowns and precondition the obtained Schur complement by a spectrally equivalent very sparse matrix. The numerical experiments show that this solution procedure is very efficient, robust and is a good preconditioner for approximations of general elliptic equations of second order on domains, topologically equivalent to a parallelepiped.

1. Introduction

Let Ω be a convex polyhedral domain in $\mathbb{R}^3$, $f(x) \in L^2(\Omega)$ and $K(x)$ be a three by three symmetric matrix-valued function on $\bar{\Omega}$ satisfying the uniform positive definiteness condition: there exists $\alpha > 0$ such that

$$\alpha^{-1}\xi^T\xi \leq \xi^T K(x)\xi \leq \alpha\xi^T\xi\,, \qquad \forall x \in \bar{\Omega}\,, \forall\xi \in \mathbb{R}^3\,. \tag{1.1}$$

We consider the Dirichlet boundary value problem:

$$\mathbf{q} + K\nabla u = 0 \quad \text{in} \;\; \Omega\,, \qquad \nabla\cdot\mathbf{q} = f \quad \text{in} \;\; \Omega\,, \qquad u|_{\partial\Omega} = 0, \qquad (1.2),$$

where $\partial\Omega$ is the boundary of Ω. In applications of fluid flow in porous media, $u(x)$ is refered to as pressure and $\mathbf{q}$ a to Darcy velocity vector.

This problem can be discretized in various ways. Among the most popular and frequently used methods of approximation are the finite volume method, the Galerkin finite element method and the mixed finite element method. Each of these methods has its advantages and disadvantages when applied to particular engineering problem. For example, for reservoir problems in geometrically simple domains and heterogeneous media the finite volume method has proven to be reliable, accurate and mass conserving cell-by-cell. Many engineering problems, e.g. petroleum recovery, ground-water contamination, seismoic exploration etc. need very accurate velocity (flux) determination in the presence of heterogeneities, anisotropy and large jumps in the coefficient martix $A(x)$. More accurate approximation of the velocity can be achieved through the use of mixed finite element method. As shown by Wiser and Wheeler in [21] the lowest order mixed finite element method with special quadratures on rectangular grids is equivalent to the finite volume method. Based on that equivalence Bramble et al. in [3] have developed efficient multigrid solution procedure for structured grids. However, in general the mixed finite element method leads to an algebraic saddle point problem that is more difficult and more expensive to solve. Although some reliable preconditioning algorithms for these saddle point problems have been proposed and studied (see, e.g. [4, 17, 19]), their efficiency depends strongly on the geometry of the domain, on the coefficient matrix $A(x)$ and on the type of the finite elements used.

An alternative approach can be taken by developing of hybrid methods. This approach has been studied by Arnold and Brezzi [2] where the continuity of the velocity vector normal to the boundary of each element is enforced by Lagrange multipliers. This idea can be explained in the following way: first, define the spaces

$$\mathbf{V} = H(div;\Omega) = \left\{\mathbf{q} \in \left(L^2(\Omega)\right)^3, \nabla\cdot\mathbf{q} \in L^2(\Omega)\right\}, \qquad W = L^2(\Omega);$$

then the weak formulation of the system (1.2) is: find a pair $(\mathbf{q}, u) \in \mathbf{V} \times W$ such that

$$(\nabla\cdot\mathbf{q}, w) + (K^{-1}\mathbf{q}, \mathbf{p}) - (u, \nabla\cdot\mathbf{p}) = (f, w), \qquad \forall\, w \in W, \quad \forall\, \mathbf{p} \in \mathbf{V}. \tag{1.3}$$

The standard mixed finite element approximation to (1.5) reads as follows: let $\bar{\mathbf{V}}_h \times W_h \subset \mathbf{V} \times W$ be a finite element space over the partition $\mathcal{T}_T$ of Ω into tetrahedra (or over the partition $\mathcal{T}_C$ into cubes) (see [6]). The requirement $\bar{\mathbf{V}}_h \subset \mathbf{V}$ implies that the normal component of the vector $\mathbf{q}$ is continious accross the interelement boundaries $\partial\mathcal{T}_T$. The construction of Arnold and Brezzi [2] is based on the idea of backing off this continuity requirement and defining the space $\mathbf{V}_h = \left\{\mathbf{q} \in (L_h^2(\Omega))^3 : \mathbf{q}|_T \in \bar{\mathbf{V}}_h,\ T \in \mathcal{T}_T\right\}$. In order to introduce the interelement continuity of the normal component of $\mathbf{q}$ we introduce the space of the Lagrange multipliers

$$L_h = \left\{\lambda \in L^2(\partial\mathcal{T}_T) : \lambda|_{\partial T} \in \bar{\mathbf{V}}_h \cdot \nu \ \text{ for each } T \in \mathcal{T}_T\right\},$$

where ν is the normal to ∂T vector.

Now the approximation to (1.3) using Lagrange multipliers is formulated for the unknown triple $(\mathbf{q}_h, u_h, \lambda_h) \in \mathbf{V}_h \times W_h \times L_h$. We skip the details of the weak formulation of (1.3) over $\mathbf{V}_h \times W_h \times L_h$ refering to [1, 5, 7]. If the vectors $\mathbf{Q}$, $\mathbf{U}$ and $\mathbf{\Lambda}$ correspond to the representation of $\mathbf{q}_h$, u_h and λ_h with respect to the bases in $\mathbf{V}_h$, W_h and L_h, respectively, the algebraic form of this approximation is (see Brezzi and Fortin [6])

$$\begin{pmatrix} M & B & C \\ B^T & 0 & 0 \\ C^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{Q} \\ \mathbf{U} \\ \mathbf{\Lambda} \end{pmatrix} = \mathbf{F}, \tag{1.4}$$

where M is a symmetric and positive definite matrix. Important feature of the matrices M and B is that they are block diagonal since the unknown nodal values of $\mathbf{q}_h$ and u_h over a given finite element T are related to the nodal values on the adjacent element only through the Lagrange multipliers. Therefore, using element-by-element elimination we can reduce this system to the form

$$S\,\mathbf{\Lambda} = \mathbf{\Phi}. \tag{1.5}$$

For the description of the structure of the Shur complement S for particular finite element spaces we refer to [1, 5, 7].

The important discovery of Arnold and Brezzi [2] is that the system (1.5) can be obtained also from application to (1.2) the Galerkin method using nonconforming elements. Namely, in [2] is shown that the lowest-order Raviart-Thomas mixed element approximations are equivalent to the usual P_1-nonconforming finite element approximations when the classical P_1-nonconforming space is augmented with P_3-bubbles. Such a relationship has been studied recently for a large variety of mixed finite element spaces [1, 5, 7].

This equivalence between the hybrid mixed and the nonconforming finite element methods establishes a framework for preconditioning and/or solving the algebraic problem and for postprocessing of the finite element solution. This framework includes the following three steps: (1) forming the reduced algebraic problem for the Lagrange multipliers, which is equivalent to the nonconforming problem; (2) construction and study of efficient methods, based on multigrid, multilevel or domain decomposition, for solving or preconditioning of the reduced problem; (3) recovery of the solution $u(x)$ and the velocity $\mathbf{q}$ from the Lagrange multipliers that were already found.

The recent progress in each of these steps described above (see, e.g. [10, 18, 20]) gives us an indication that the mixed finite element method can be used as an accurate and efficient method for solving general elliptic problems of second order in domains with complicated geometry.

The goal of this paper is to construct, study and implement efficient preconditioners for the nonconforming finite element approximations of problem (1.2) on tetrahedral meshes.

2. Problem Formulation

To explain our approach we consider the model case when Ω is a unit cube in $\mathbb{R}^3$ and $K(x) = a(x)\, I$. Let $\mathcal{T}_T$ be a regular partitioning of Ω on tetrahedra T with a characteristic size $h = diam(T)$ (see Fig. 1).

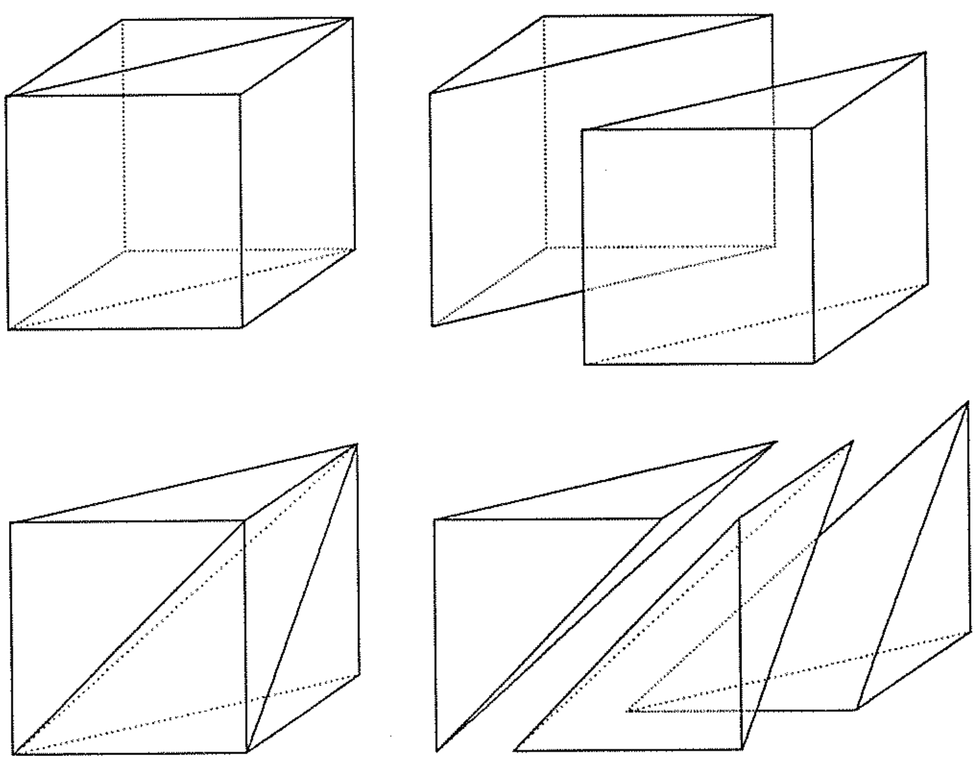

FIGURE 1 *Partition of cube into prisms and tetrahedra.*

We introduce the set of barycenters of all faces of the tetrahedral partition of Ω, and the set Q_h of those barycenters that are strictly inside Ω. The Crouzeix-Raviart nonconforming finite element space V_h consists of all piecewise linear functions on $\mathcal{T}_T$ that vanish at barycenters of the boundary faces and are continuous at the barycenters of Q_h. Note that the space V_h is not a subspace of $H_0^1(\Omega)$.

Now we define the bilinear form on V_h by

$$a_h(u,v) = \sum_{T \in \mathcal{T}_T} \int_T a(x) \nabla u \cdot \nabla v dx, \qquad \forall\, u, v \in V_h. \tag{2.1}$$

Thus the nonconfirming discretization of problem (1.2) is given by seeking $u_h \in V_h$ such that

$$a_h(u_h, v) = (f, v), \qquad \forall\, v \in V_h. \tag{2.2}$$

The natural degrees of freedom of Crouzeix-Raviart nonconforming elements are the values at the barycenters of the faces of the tetrahedral elements. Denote the vector of the unknown values corresponding to a function $v_h \in V_h$ by $\mathbf{v}$ and assume that its dimension is N, i.e., $\mathbf{v} \in \mathbb{R}^N$. Note, that all unknowns on faces on the boundary with Dirichlet data are excluded.

Let $(\mathbf{u}, \mathbf{v})$ and $\|\mathbf{v}\|$ be a bilinear form and the corresponding norm, defined on $\mathbb{R}^N$ by

$$(\mathbf{u}, \mathbf{v})_N = h^3 \sum_{x \in Q_h} u(x) v(x), \qquad \|\mathbf{v}\| = (\mathbf{v}, \mathbf{v})_N^{1/2}, \qquad u, v \in V_h. \tag{2.3}$$

Then the discretization operator $A : \mathbb{R}^N \to \mathbb{R}^N$, which is symmetric and positive definite is defined by

$$(A\mathbf{v}, \mathbf{w})_N = a_h(u, w), \qquad u, v \in V_h. \tag{2.4}$$

Similarly, we introduce the vector $\mathbf{F}$ as $(f, v) = (\mathbf{F}, \mathbf{v})_N \quad \forall\, v \in V_h$. Now, the problem (2.2) can be rewritten in a matrix form

$$A\mathbf{u} = \mathbf{F}. \tag{2.5}$$

3. Matrix Formulation and Its Properties

Our goal is to introduce an algebraic formulation of the approximate problem using a type of static condensation that eliminates some of the unknowns. In this way we can reduce substantially the size of the problem. For this approach we need a special partitioning of the domain into tetrahedra that have some regularity and preserve the simplicity of the algebraic problem.

First, we partition Ω into cubes with size of the edges $h = 1/n$ and denote them by $C = C^{(i,j,k)}$ where (x_{1i}, x_{2j}, x_{3k}) is the right back upper corner of the cube. This partitioning is denoted by $\mathcal{T}_C$. Next, we divide each cube $C = C^{(i,j,k)}$ into two prisms $P_1 = P_1^{(i,j,k)}$ and $P_2 = P_2^{(i,j,k)}$ as shown in Fig. 1 and denote this partitioning of Ω by $\mathcal{T}_P$. Finally, we divide each prism into three tetrahedra (see Fig. 1) and denote this partitioning of Ω into tetrahedra by $\mathcal{T}_T$.

Let $P = P^{(i,j,k)} \in \mathcal{T}_P$ be a particular prism of the partition $\mathcal{T}_P$. Denote by V_h^P the subspace of restrictions of the functions in V_h onto P. These restrictions define vectors $\mathbf{u}_P$ that are restrictions of a vector $\mathbf{u} \in \mathbb{R}^N$. The dimension of V_h^P we denote by N^P. Obviously, for prisms with no faces on $\partial\Omega$ the dimension $N^P = 10$.

Local stiffness matrices A^P on prisms $P \in \mathcal{T}_P$ is defined by

$$(A^P \mathbf{u}_P, \mathbf{v}_P)_N = \sum_{T \subset P} \int_T a(x) \nabla u_h \cdot \nabla v_h \, dx\,, \tag{3.1}$$

for any $P \in \mathcal{T}_P$. Then the global stiffness matrix A is determined by assembling the local stiffness matrices.

Now we consider a prism P of an arbitrary cube that has no face on the boundary $\partial\Omega$ and enumerate its faces s_j, $j = 1, \ldots, 10$. Then the local stiffness matrix of this prism for the case $a(x) \equiv 1$ has the following form:

$$A^P = \frac{3h}{2}\left[\begin{array}{cccccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 4 & -1 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 4 \end{array}\right] \equiv \frac{3h}{2}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad (3.2)$$

where

$$A_{22} = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}.$$

Along with matrix A^P we introduce the following matrix B^P defined on the same space V_h^P:

$$B^P = \frac{3h}{2}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & B_{22} \end{bmatrix}, \qquad B_{22} = \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & 0 & -1 \\ -1 & 0 & 3 & 0 \\ 0 & -1 & 0 & 3 \end{bmatrix}. \qquad (3.3)$$

It easy to show that $ker A^P = ker B^P$.

Then we define the $N \times N$ matrix B by the following equality:

$$(B\mathbf{u}, \mathbf{v})_N = \sum_{P \in \mathcal{T}_P} (B^P \mathbf{u}_P, \mathbf{v}_P)_N \qquad \forall\, \mathbf{u}, \mathbf{v} \in \mathbb{R}^N, \qquad (3.4)$$

and prove that $\text{cond}(B^{-1}A) \leq (2+\sqrt{3})^2$.

Now we divide all unknowns in the system into two groups: (1) the first group consists of all unknowns corresponding to faces of the prisms in the partition $\mathcal{T}_P$, excluding, of course, the faces on $\partial\Omega$: (2) the second group consists of the unknowns corresponding to the faces of the tetrahedra that are internal for each prism.

This splitting of the space $\mathbb{R}^N$ induces the following presentation of the vectors $\mathbf{v}^T = (\mathbf{v}_1^T, \mathbf{v}_2^T)$, where $\mathbf{v}_1 \in \mathbb{R}^{N_1}$ and $\mathbf{v}_2 \in \mathbb{R}^{N_2}$. Obviously, $N_2 = 4n^3$. Then matrix B can be presented in the following block form:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \qquad \text{dim} B_{11} = N_1. \qquad (3.5)$$

Denote now by $\hat{B}_{11} = B_{11} - B_{12}B_{22}^{-1}B_{21}$ the Schur complement of B obtained by elimination of the vector $\mathbf{v}_2$. Then $B_{11} = \hat{B}_{11} + B_{12}B_{22}^{-1}B_{21}$, so the matrix B has the form

$$B = \begin{bmatrix} \hat{B}_{11} + B_{12}B_{22}^{-1}B_{21} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}. \qquad (3.6)$$

Note that for each prism $P \in \mathcal{T}_P$ the unknowns on the internal for P faces are connected through the equation $B\mathbf{v} = \mathbf{F}$, only with the unknowns associated with this prism, i.e., the matrix B_{22} is block diagonal with 2×2 blocks and can be inverted locally (prism by prism). Thus matrix $\hat{B}_{11}$ is easily computable.

4. Three-Level Preconditioner

Next, we replace the matrix B of (3.6) by

$$\tilde{B} = \left[\begin{array}{cc} \tilde{B}_1 + B_{12} B_{22}^{-1} B_{21} & B_{12} \\ B_{21} & B_{22} \end{array} \right]. \tag{4.1}$$

This is done in the following way: take the stiffness matrix of any prism and eliminate the unknowns associated with the internal for that prism faces of the three tertahedra. Using certain ordering of the unknowns we get the matrix:

$$\hat{B}_{1,P} \;=\; \left(\frac{3h}{2}\right) \left[\begin{array}{rr|rrrrrr} 8/3 & -1 & -1 & 0 & -1/3 & 0 & -1/3 & 0 \\ -1 & 8/3 & 0 & -1/3 & 0 & -1 & 0 & -1/3 \\ \hline -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/3 & 0 & 2/3 & 0 & 0 & 0 & -1/3 \\ -1/3 & 0 & 0 & 0 & 2/3 & 0 & -1/3 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1/3 & 0 & 0 & 0 & -1/3 & 0 & 2/3 & 0 \\ 0 & -1/3 & 0 & -1/3 & 0 & 0 & 0 & 2/3 \end{array} \right]. \tag{4.2}$$

Then, replace this prism element matrix by the following spectrally equivalent matrix (they have the same kernel):

$$\tilde{B}_{1,P} = \left(\frac{3h}{2}\right) \left[\begin{array}{rr|rrrrrr} 8/3 & -1 & -2/3 & 0 & -2/3 & 0 & -1/6 & -1/6 \\ -1 & 8/3 & 0 & -2/3 & 0 & -2/3 & -1/6 & -1/6 \\ \hline -2/3 & 0 & 2/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2/3 & 0 & 2/3 & 0 & 0 & 0 & 0 \\ -2/3 & 0 & 0 & 0 & 2/3 & 0 & 0 & 0 \\ 0 & -2/3 & 0 & 0 & & 2/3 & 0 & 0 \\ -1/6 & -1/6 & 0 & 0 & 0 & 0 & 1/3 & 0 \\ -1/6 & -1/6 & 0 & 0 & 0 & 0 & 0 & 1/3 \end{array} \right]. \tag{4.3}$$

The matrix $\tilde{B}$ is obtained by assembling these prismatic element stiffness matrices.

Again, using superelement analysis, we prove that $\text{cond}(\tilde{B}^{-1}A) \leq 3(2 + \sqrt{3})^2$.

Now consider the linear system with matrix $\tilde{B}_1$: $\tilde{B}_1 \mathbf{u} = \mathbf{f}$. Matrix $\tilde{B}_1$ can be represented in the block form

$$\tilde{B}_1 = \left[\begin{array}{cc} C_{11} & C_{12} \\ C_{21} & C_{22} \end{array} \right], \tag{4.4}$$

where the matrix C_{22} is diagonal. After eliminating the unknowns of the second block we split the reduced system to one linear system of dimension n^3 and n linear systems each of dimension n^2. Thus, we have constructed a spectrally equivalent sparse preconritioner for the Schur compliment after the elimination of more than 80 % of the original unknowns.

5. Results of the Numerical Experiments

The method of preconditioning on the basis of multilevel substructuring as discussed above was tested on the model problem with diagonal matrix K:

$$Lu \equiv -div\, K\, grad\, u = f \quad \text{in } \Omega = (0,1)^3 \subset \mathbb{R}^3, \qquad u|_{\partial\Omega} = 0\,,$$

using described above Crouzeix-Raviart nonconformal finite elements.

The domain was divided into n^3 cubes (n in each direction) and each cube was partitioned into 6 tetrahedra. The total dimension of the original algebraic system was $N = 12n^3 - 6n^2$.

The right hand side was generated randomly. The original algebraic problem has been solved by the conjugate gradient method with the preconditioner in the form of (4.1) with accuracy $\varepsilon = 10^{-6}$. For comparison that problem has been solved by the same method without preconditioning. The condition number of matrix $B^{-1}A$ was calculated from the relation between conjugate gradients and Lanczos algorithm.

Two cases were considered:

1. $K = I$, where I is unity matrix; the results are shown in Table 1;
2. $K = \mathrm{diag}\{a_1, a_2, a_3\}$, where a_i, $i = 1, 2, 3$, are piece-wise constant finctions; results are shown in Table 2.

Table 1

		CG Without Preconditioning			CG With Preconditioning		
n	N	n_{iter}	cond	time (sec)	n_{iter}	cond	time (sec)
4	672	40	66	0.18	22	9.84	0.22
8	5760	73	265	2.18	24	10.7	1.27
16	47616	130	1062	49.2	24	11.94	15.7
32	387072	200<	—	1248	25	12.2	163
40	758400				25	12.26	376
50	1485000				25	12.33	771

Table 2

			$N = 47,616$		$N = 387,072$	
a_1	a_2	a_3	Iter	Cond	Iter	Cond
1	1	1	21	11.	21	11.1
1	1	0.1	16	5.3	16	5.4
1	1	0.01	14	4.5	14	4.5
10	1	1	18	7	18	7.1
1	10	1				
100	1	1	16	5.7	16	5.8
1	100	1				
1	1	10	54	125	55	130
1	1	100	100	1266	102	1270
0.1	1	1	43	50	43	50.5
0.01	1	1	88	330	88	335

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