

Superconvergence of the mixed finite element approximations to parabolic equations

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Abstract

Semidiscrete mixed finite element approximation to parabolic initial-boundary value problems is introduced and analyzed. Superconvergence estimates for both pressure and velocity are obtained. The estimates for the errors in pressure and velocity depend on the smoothness of the initial data including the limiting cases of data in L^2 and data in H^r , for r sufficiently large. Because of the smoothing properties of the parabolic operator these estimates for large time levels essentially coincide with the estimates obtained earlier for smooth solutions. However, for small time intervals we obtain the correct convergence orders for nonsmooth data.

1 Introduction

Since the pioneering work of Raviart and Thomas [14] the mixed finite element approximations to second order elliptic problems have drawn the attention of many specialists on numerical partial differential equations. This method provides direct approximation of the physical quantities such as fluxes and velocities and leads to schemes that are locally conservative. Due largely to Babuška[1] and Brezzi [2], it is now well understood that the finite

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element spaces approximating different physical quantities (pressure and velocity, temperature and flux, etc.) cannot be chosen independently. Then the so-called *inf-sup* condition of Babuška–Brezzi is essential if one wants to construct unconditionally stable schemes with optimal convergence rates.

Superconvergence results are important from an application point of view since under reasonable assumptions on the grid and with additional smoothness of the solution, they provide higher order accuracy. To our knowledge the first superconvergence results in the mixed method for second order elliptic problems were proven by Douglas and Milner in [5] and by Nakata, Weiser and Wheeler in [12]. Since then a wide variety of results has been obtained by many other authors, see, e.g., Duran [7], Douglas and Wang [6], Wang [18], and Ewing, Lazarov and Wang [9].

The corresponding error analysis of mixed finite element methods applied to time-dependent problems was developed by Johnson and Thomée in [10]. Then, Squeff in [15] using the quasi-projections of Douglas, Dupont and Wheeler, and the averaging method of Bramble and Schatz obtained superconvergence rates for parabolic mixed finite element solutions in R^2 .

All known superconvergence results for mixed parabolic finite element equations require that the solutions be sufficiently smooth, uniformly in time. However, many practical problems involve irregularities, such as, in the initial data. This may result in a break-down of the uniform regularity of the solutions and, therefore, a nonavailability of the error estimates obtained before. Fortunately, the linear parabolic operators have the so-called *smoothing property*. Namely, if the initial data given at $t = 0$ is nonsmooth or does not satisfy the compatibility conditions on the boundary of the domain then the solution is sufficiently smooth away from $t = 0$ but has a singularity of the form $t^{-\beta}$ with some β positive. This kind of smoothing property has been proved also for the standard Galerkin parabolic finite element equations and used to derive optimal error estimates for problems with rough initial data, see, e.g., Luskin and Rannacher [11], Thomée [16] and Rannacher [13]. Furthermore, superconvergence results for the gradient for the standard Galerkin finite element methods with initial data of low regularity were obtained by Thomée, Xu and Zhang in [17] and by Chen in [4]. The main goal of this paper is to establish superconvergence estimates for the mixed finite element methods when applied to parabolic problems with rough initial data.

2 Problem Formulation

Let $\Omega \in R^2$ be a bounded domain with boundary $\partial\Omega$. We consider the following parabolic problem:

$$\begin{cases} p_t - \nabla \cdot (a \nabla p) = f & \text{in } \Omega \times (0, \infty), \\ p = 0, \text{ on } \partial\Omega \times (0, \infty), \quad p|_{t=0} \text{ is given,} \end{cases} \quad (1)$$

where $a = a(x, t)$ is a sufficiently smooth function that is bounded below by a positive constant on $\Omega \times (0, \infty)$. Our purpose is to solve problem (1) using mixed finite element methods. To describe the mixed variational form for (1), as usual, we introduce two Hilbert spaces. Let

$$W = L^2(\Omega), \quad \mathbf{V} = \left\{ \boldsymbol{\varphi} \in L^2(\Omega)^2, \nabla \cdot \boldsymbol{\varphi} \in L^2(\Omega) \right\},$$

and the space \mathbf{V} be equipped with the norm $\|\boldsymbol{\varphi}\|_{\mathbf{V}} = (\|\boldsymbol{\varphi}\|^2 + \|\nabla \cdot \boldsymbol{\varphi}\|^2)^{1/2}$. The inner product and the norm in $L^2(\Omega)$ are denoted by (\cdot, \cdot) and $\|\cdot\|$, respectively. For the sake of simplicity, (\cdot, \cdot) and $\|\cdot\|$ are also used as the inner product and norm, respectively, in the product space $L^2(\Omega)^2$.

Let $\mathbf{u} = a \nabla p$; then the pair $(p, \mathbf{u}) \in W \times \mathbf{V}$ satisfies the following mixed variational equation:

$$\begin{cases} (p_t, \psi) - (\nabla \cdot \mathbf{u}, \psi) = (f, \psi), \quad \forall \psi \in W, t \in (0, \infty), \\ (\alpha \mathbf{u}, \boldsymbol{\varphi}) + (\nabla \cdot \boldsymbol{\varphi}, p) = 0, \quad \forall \boldsymbol{\varphi} \in \mathbf{V}, t \in (0, \infty), \end{cases} \quad (2)$$

where $p_t = \partial p / \partial t$, $\alpha = a^{-1}$ and $p(0)$ is given. We note that the boundary condition $p = 0$ on $\partial\Omega$ is implicitly contained in (2).

Given the finite-dimensional spaces $W_h \subset W$ and $\mathbf{V}_h \subset \mathbf{V}$, $0 < h < 1$, the so-called mixed finite element approximation $(p_h, \mathbf{u}_h) \in W_h \times \mathbf{V}_h$ to the pair $(p, \mathbf{u}) \in W \times \mathbf{V}$ is the solution of the following problem:

$$\begin{cases} (p_{h,t}, \psi_h) - (\nabla \cdot \mathbf{u}_h, \psi_h) = (f, \psi_h), \quad \forall \psi_h \in W_h, t \in (0, \infty), \\ (\alpha \mathbf{u}_h, \boldsymbol{\varphi}_h) + (\nabla \cdot \boldsymbol{\varphi}_h, p_h) = 0, \quad \forall \boldsymbol{\varphi}_h \in \mathbf{V}_h, t \in (0, \infty), \end{cases} \quad (3)$$

where, $p_h(0) \in W_h$ is given. We note that $\mathbf{u}_h(0)$ is determined by $p_h(0)$ through the second equation of (3).

To ensure the existence and convergence of the solution of the above formulation, we assume that $\nabla \cdot \mathbf{V}_h \subset W_h$ and there exists a linear operator

$\mathbf{\Pi}_h: \mathbf{V} \rightarrow \mathbf{V}_h$ such that $\nabla \cdot \mathbf{\Pi}_h = Q_h \nabla \cdot \cdot$. Here, the operator $Q_h: W \rightarrow W_h$ is the L^2 -projection. The classical inf-sup condition is then satisfied. Further, we assume that there exists an integer $r \geq 0$ such that the following approximation properties are satisfied:

$$\|\boldsymbol{\varphi} - \mathbf{\Pi}_h \boldsymbol{\varphi}\| \leq ch^i \|\boldsymbol{\varphi}\|_{\mathbf{H}^i(\Omega)}, \quad \forall \boldsymbol{\varphi} \in \mathbf{H}^i(\Omega), \quad 1 \leq i \leq r+1, \quad (4)$$

$$\|\psi - Q_h \psi\| \leq ch^i \|\psi\|_{H^i(\Omega)}, \quad \forall \psi \in H^i(\Omega), \quad 0 \leq i \leq r+1. \quad (5)$$

Here, and throughout the paper, the letter c is used as a generic constant, which is independent of h , p , \mathbf{u} , etc. $H^i(\Omega)$ denotes the standard Sobolev space $W^{i,2}(\Omega)$ with $H^0(\Omega) = L^2(\Omega)$. Examples of spaces of piece-wise polynomials that satisfy the conditions stated above are the triangular and rectangular Raviat-Thomas elements from [14] (for other examples see Brezzi and Fortin [3]).

3 L^2 -error estimates

This section is designed to provide the preliminaries for deriving the super-convergence results obtained in the next section. The main results are contained in Theorem 1, where L^2 - estimates for $(p - p_h)^{(k)}(t)$ and $(\mathbf{u} - \mathbf{u}_h)^{(k)}(t)$ are derived for homogeneous equations with initial data from $H^i(\Omega)$, $0 \leq i \leq k+1$, including the case of data just in $L^2(\Omega)$.

Lemma 1 *Let $(p, \mathbf{u}) \in W \times \mathbf{V}$ and $(p_h, \mathbf{u}_h) \in W_h \times \mathbf{V}_h$ be the solutions of problems (2) and (3), respectively. If $p_h(0) = Q_h p(0)$, then*

$$\|(Q_h p - p_h)(t)\|^2 + \int_0^t \|\mathbf{u} - \mathbf{u}_h\|^2 ds \leq c \int_0^t \|\mathbf{u} - \mathbf{\Pi}_h \mathbf{u}\|^2 ds. \quad (6)$$

$$\|(p - p_h)(t)\|_{H^{-j}(\Omega)} \leq ch^{i+j} \|p(0)\|_{H^i(\Omega)}, \quad 0 \leq i, j \leq r+1. \quad (7)$$

$$\int_0^t \|p - p_h\|^2 ds \leq ch^2 \|p(0)\|^2. \quad (8)$$

$$\|(p - p_h)(t)\| \leq cht^{-1/2} \|p(0)\|. \quad (9)$$

Based on the above lemmas we prove the main result of this section stated in the following theorems. Below, for a function $\theta(t)$, we denote its k -th derivative with respect to t as $\theta^{(k)}(t)$.

Theorem 1 *Let $(p, \mathbf{u}) \in W \times \mathbf{V}$ and $(p_h, \mathbf{u}_h) \in W_h \times \mathbf{V}_h$ be the solutions of problems (2) and (3), respectively. If $f = 0$ and $q_h(0) = Q_h q(0)$, then,*

$$\|(p - p_h)^{(k)}(t)\| \leq ch^m t^{-(m-i)/2-k} \|p(0)\|_{H^i(\Omega)}, \quad (10)$$

$$\|(\mathbf{u} - \mathbf{u}_h)^{(k)}(t)\| \leq ch^m t^{-(m+1-i)/2-k} \|p(0)\|_{H^i(\Omega)}, \quad (11)$$

where $0 \leq i \leq m \leq r + 1$, $k \geq 0$ and $p(0) \in H_0^i(\Omega)$.

Proof : This is the most technical result. We prove it by induction and using the results of the lemmas and theorems stated above.

4 Superconvergence

The estimate obtained in the previous sections can be interpreted in the following way: the maximum rate of convergence for $p - p_h$ is $O(h^{r+1}t^{-(r+1-i)/2})$ while for $\mathbf{u} - \mathbf{u}_h$ is $O(h^{r+1}t^{r+2-i}/2)$ for initial data in $H^i(\Omega)$. Obviously, for any fixed $t > 0$, the convergence is asymptotically $O(h^{r+1})$, due to the smoothing properties of the parabolic operator. For smooth initial data, i.e., $p(0)$ from $H^{r+1}(\Omega)$, the error for p is asymptotically $O(h^{r+1})$ for any t . It should be pointed out that the estimates (10) and (11) cover the whole range of smoothness of the initial data, including the worst case of data only in $L^2(\Omega)$.

The estimates (10) and (11) are of optimal type and cannot be improved in terms of the norms involved even if the solutions are smoother. However, in the case of a smooth solution one can select special points or postprocess the finite element solution in order to obtain higher order convergence. Such estimates, called superconvergence estimates, have been obtained for the standard Galerkin method for parabolic problems by Thomée, Xu and Zhang [17].

For Raviart-Thomas finite elements of order $r+1$, which fit into the framework of this paper, Ewing and Lazarov in [8] derive $O(h^{r+2})$ error estimates for both $p - p_h$ and $\mathbf{u} - \mathbf{u}_h$ at the Gauss points in the case of sufficiently smooth solution for all $t \geq 0$. Our goal in this section is to obtain superconvergent type estimates for more realistic situations, namely, for data in $H^i(\Omega)$, for $0 \leq i \leq r + 2$, including the case of data in $L^2(\Omega)$.

In order to bound the temporal derivatives of $Q_h p - p_h$ we need estimates for $Q_h p - p_h$ and $\mathbf{\Pi}_h \mathbf{u} - \mathbf{u}_h$ in the L^2 -norm. We begin with an estimate

for $Q_h p - p_h$ and then we derive estimates for the derivatives of $\mathbf{\Pi}_h \mathbf{u} - \mathbf{u}_h$ and $Q_h p - p_h$ with respect to time. Next, the duality argument for the error estimates discussed in Theorem 2, requires estimates in negative Sobolev norm. These estimates are obtained in the following lemma.

Lemma 2 *Let $(p, \mathbf{u}) \in W \times \mathbf{V}$ and $(p_h, \mathbf{u}_h) \in W_h \times \mathbf{V}_h$ be the solutions of problems (2) and (3), respectively. If $f = 0$ and $p_h(0) = Q_h p(0)$, then*

$$\|(Q_h p - p_h)(t)\|_{H^{-j}(\Omega)} \leq ch^{r+2+j} \|p(0)\|_{H^{r+2}(\Omega)}, \quad 0 \leq j \leq r-1, \quad (12)$$

$$\|(Q_h p - p_h)(t)\|_{H^{-r}(\Omega)} \leq ch^{2r+2} |\log h| \|p(0)\|_{H^{r+2}(\Omega)}. \quad (13)$$

Now, we are ready to obtain the main results of this section, namely, estimates of $Q_h p - p_h$ for problems with nonsmooth initial data.

Theorem 2 *Let $(p, \mathbf{u}) \in W \times \mathbf{V}$ and $(p_h, \mathbf{u}_h) \in W_h \times \mathbf{V}_h$ be the solutions of problems (2) and (3), respectively. If $f = 0$ and $p_h(0) = Q_h p(0)$, and $k \geq 0$, then*

$$\|(Q_h p - p_h)^{(k)}(t)\| \leq ch^{r+2} t^{-(r+2-i)/2-k} \|p(0)\|_{H^i(\Omega)}, \quad (14)$$

$$\|(\mathbf{\Pi}_h \mathbf{u} - \mathbf{u}_h)^{(k)}(t)\| \leq ch^{r+2} t^{-(r+3-i)/2-k} \|p(0)\|_{H^i(\Omega)}, \quad (15)$$

where $1 \leq i \leq r+2$.

Now we present one possible way of using the results of Theorem 2 for superconvergent recovery of the solution p and \mathbf{u} from their finite element approximations p_h and \mathbf{u}_h in the case of Raviart-Thomas rectangular elements in rectangular domains Ω . Let T_h be partition of the domain Ω into rectangles with sides parallel to the coordinate axes.

Let $P_r(e)$ be the restriction of the polynomials of total degree r to the set e and $P_{r,l}(e)$ the restriction of $P_r(R^1) \times P_l(R^1)$ to element e . We set

$$\mathbf{V}(e) = P_{r+1,r}(e) \times P_{r,r+1}(e), \quad W(e) = P_{r,r}(e),$$

$$\mathbf{V}_h = \{\mathbf{v} \in \mathbf{V} : \mathbf{v}|_e \in \mathbf{V}(e), e \in T_h\}, \quad W_h = \{w \in W : w|_e \in W(e), e \in T_h\}.$$

The spaces \mathbf{V}_h and W_h defined above satisfy the approximation properties (4) and (5) and the projection operators $\mathbf{\Pi}_h$ and Q_h are defined element by element.

Next, we introduce some discrete seminorms in W and \mathbf{V} . Let \hat{g}_i , $i = 1, \dots, r+1$ be the Gauss points in the interval $[-1, 1]$, i.e., $L_{r+1}(\hat{g}_i) = 0$, where L_{r+1} is the Legendre polynomial of degree $r+1$ orthogonal on the interval $[-1, 1]$. Each finite element $e \in T_h$ can be transferred by an affine mapping F to the reference element $\hat{e} = [-1, 1] \times [-1, 1]$. Then, the inverse mapping F^{-1} introduces in e the set of points $G(e) = \{F^{-1}(\hat{g}_i, \hat{g}_j), i, j = 1, \dots, r+1\}$. The seminorms in W and \mathbf{V} are defined by

$$\|w\| = \left(\sum_{e \in T_h} \sum_{x \in G(e)} h^2 w^2(x) \right)^{1/2}, \quad \|\mathbf{v}\| = \|v_1\| + \|v_2\|, \quad \mathbf{v} = (v_1, v_2).$$

It is easy to see that there is a constant $c > 0$ such that

$$\|w_h\| \leq c \|w_h\|, \quad \|\mathbf{v}_h\| \leq c \|\mathbf{v}_h\|, \quad \forall w_h \in W_h, \mathbf{v}_h \in \mathbf{V}_h.$$

Now, we bound the errors for $p - p_h$ and $\mathbf{u} - \mathbf{u}_h$ in the discrete seminorms defined above, which are the desired superconvergence estimates.

Theorem 3 *Let $(p, \mathbf{u}) \in W \times \mathbf{V}$ and $(p_h, \mathbf{u}_h) \in W_h \times \mathbf{V}_h$ be the solutions of problems (2) and (3), respectively. If $f = 0$ and $p_h(0) = Q_h p(0)$, then*

$$\|(p - p_h)(t)\| \leq ch^{r+2} t^{-(r+2-i)/2} \|p(0)\|_{H^i(\Omega)}, \quad (16)$$

$$\|(\mathbf{u} - \mathbf{u}_h)(t)\| \leq ch^{r+2} t^{-(r+3-i)/2} \|p(0)\|_{H^i(\Omega)}, \quad (17)$$

where $1 \leq i \leq r+2$.

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