FINITE VOLUME METHODS FOR CONVECTION-DIFFUSION PROBLEMS*

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Abstract. Derivation, stability and error analysis in both discrete H^1 and L^2 norms for cell-centered finite volume approximations of convection-diffusion problems are presented. Various upwind strategies are investigated. The theoretical results are illustrated by numerical examples.

Key words. cell-centered grid, non-selfadjoint elliptic problems, convection-diffusion problem, upwind and modified upwind approximations, error estimates

AMS(MOS) subject classifications. 65N05, 65N15

1. Introduction.

In this paper we consider cell-centered finite difference approximations for second order convection-diffusion equations of divergence type. Our goal is to construct finite difference methods of second order of approximation that satisfy the discrete maximum principle. The error estimates are in the discrete Sobolev spaces associated with the considered boundary value problem.

Approximation of the convection term in convection-diffusion problems by central finite differences leads to schemes of second order, which are stable only for sufficiently small mesh size h. The upwinding has been used to avoid the conditional stability, but these approximations are of first order and add substantial numerical diffusion to the physical problem. Various modifications of the upwind schemes have been proposed aiming at a second order of accuracy and unconditional stability, cf., e.g., Samarskii [20] (see also Axelsson and Gustafson [3]). We investigate a number of modified upwind finite difference strategies which provide both a second order of accuracy and that are unconditionally (i.e., not only for small h) stable.

There is a variety of techniques to derive and study finite difference discretizations for diffusion and convection-diffusion problems (see, e.g., Samarskii [20], Axelsson and Gustavson [3], Spalding [22], Il'in [14], etc.). In [20] an error estimate of order $O(h^2)$ in the discrete maximum norm for smooth solutions (four continuous derivatives required) is derived. Another modified upwind finite difference strategy leading to a second order scheme was considered in Axelsson and Gustafson [3]. Runchal [19] and also Spalding [22] have proposed and tested numerically upwind finite difference schemes that can be used in both convection dominated and diffusive limits. For one dimensional problems Il'in [14] has proposed finite difference schemes for convection–dominated second order equations and proved an $O(h^2)$ error estimate in the maximum norm.

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A systematic treatment of finite difference schemes on triangular meshes was presented in Heinrich [12]. For self-adjoint problems the schemes in [12] are similar to those obtained by the finite element method. Cell-centered finite difference schemes on triangular meshes (including the case of locally refined meshes) were considered by Vassilevski, Petrova and Lazarov [24]. The error estimates derived in [24] are in a discrete H^1-norm and for uniform triangulations include superconvergent rates, namely, $O(h^2)$. Cell-centered discretizations on tensor-product nonuniform meshes were considered by Weiser and Wheeler [25] and superconvergence error estimates were derived. H^1-error estimates of order $O(h^{1+\alpha})$, $\frac{1}{2} < \alpha \leq 1$ for the Poisson equation were proved by Sũli [23]. Morton and Sũli [17] considered point-centered finite difference schemes for one and two-dimensional hyperbolic equations.

A method closely related to the finite element approximations is the finite volume element method proposed and analyzed by Cai [6], Cai, Mandel and McCormick [7], and McCormick [16]; see also an early formulation by Baliga and Patankar [4] that includes the convection-diffusion case. The relationship of the similar box method and the finite element method in the symmetric positive definite case has been investigated by Bank and Rose [5] and by Hackbusch [11]. In Hackbusch [11] second order error estimates in an H^1-norm on uniform meshes has been proved.

This paper is devoted to filling in the lack of results for nonsymmetric equations and cell-centered finite differences. We construct a number of upwind finite difference schemes and prove error estimates in a discrete H^1-norm of order $O(h^{m-1})$, $\frac{3}{2} < m \leq 3$ for solution $u \in H^m(\Omega)$. These results can be viewed as a natural extension of the results from Ewing, Lazarov, and Vassilevski [9], to non-selfadjoint equations. In addition, we provide error estimates in an L^2-norm elaborating the discrete "Aubin-Nitsche trick" of duality argument proposed by Samarskii, Lazarov, and Makarov [21] and used in the case of finite difference schemes for general self-adjoint elliptic equations in Lazarov, Makarov and Weinelt [15]. For the original duality technique in the finite element method, refer to Aubin [2], Nitsche [18], and Ciarlet [8].

The remainder of the paper is organized as follows. In §1.1 the boundary value problem is stated; the notation used is introduced in §1.2. The discretization schemes are presented in §2. The stability (a priori estimates) and error estimates in an H^1-norm are derived in §3.1. The error estimates in an L^2-norm are proved in §3.2. Finally, in §4, the numerical results are presented.

1.1. Boundary value problem. We use the standard notation for Sobolev spaces [1]:

$$W_p^m(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega),\ |\alpha| \leq m\}, \quad m \geq 0,\ 1 \leq p \leq \infty$$

and $W_2^m(\Omega) = H^m(\Omega)$. The norm in $H^m(\Omega)$ is denoted $\|\cdot\|_{m,\Omega}$ and defined by

$$\|u\|_{m,\Omega} \equiv \left(\sum_{i=0}^{m} |u|_{i,\Omega}^2\right)^{1/2}, \qquad |u|_{i,\Omega} \equiv \left(\sum_{|\alpha|=i} \|D^\alpha u\|_{0,\Omega}^2\right)^{1/2},$$

$$\|u\|_{m,\infty,\Omega} \equiv \max_{|\alpha|\leq m} \sup_{x\in\Omega} |D^\alpha u|,$$

where $\|\cdot\|_{0,\Omega}$ is the standard L^2-norm in Ω. We also use Sobolev spaces with real index $m > 0$ [1].

We consider the following convection-diffusion boundary value problem: find a function $u(x)$ which satisfies the following differential equation and boundary condition:

$$\begin{cases} \operatorname{div}(-a(x)\nabla u(x) + \underline{b}(x)u(x)) = f(x), & \text{in } \Omega \\ u(x) = 0, & \text{on } \Gamma \end{cases} \tag{1}$$

where $\Omega \subset R^2$ is a bounded domain, $\Gamma = \partial\Omega$, $f(x)$, and the velocity vector $\underline{b}(x) = (b_1(x), b_2(x))$ are given functions in Ω. We introduce the bilinear form

$$\begin{aligned} a(u,v) = &\int_\Omega a(x)\sum_{i=1}^{2} \partial_i u(x)\partial_i v(x)\,dx \\ &+ \int_\Omega (\underline{b}(x)\cdot\nabla u(x))v(x)\,dx + \int_\Omega u(x)v(x)(\nabla\cdot\underline{b}(x))\,dx \end{aligned}$$

and the linear form

$$f(v) = \int_\Omega f(x)v(x)\,dx\,.$$

Here and hereafter ∂_i denotes the partial derivative with respect to x_i.

The problem (1) can also be formulated in the following weak form:

Find $u \in H_0^1(\Omega)$ such that $a(u,v) = f(v)$ for all $v \in H_0^1(\Omega)$. From

$$\begin{aligned} \int_\Omega (\underline{b}(x)\cdot\nabla u(x))u(x)\,dx &= -\int_\Omega \nabla\cdot(\underline{b}(x)u(x))u(x)\,dx \\ &= -\int_\Omega (\nabla\cdot\underline{b}(x))u^2(x)\,dx - \int_\Omega (\underline{b}(x)\cdot\nabla u(x))u(x)\,dx \end{aligned}$$

we obtain

$$\int_\Omega (\underline{b}(x)\cdot\nabla u(x))u(x)\,dx = -\frac{1}{2}\int_\Omega (\nabla\cdot\underline{b}(x))u^2(x)\,dx$$

and hence

$$a(u,u) = \int_\Omega a(x)\sum_{i=1}^{2}(\partial_i u(x))^2\,dx + \frac{1}{2}\int_\Omega (\nabla\cdot\underline{b}(x))u^2(x)\,dx\,. \tag{2}$$

Let the coefficients $a(x)$, $\underline{b}(x)$ satisfy the conditions:

(i) $a(x) \geq \alpha > 0$, $a(x) \in W_\infty^1(\Omega)$,

(ii) $(\nabla\cdot\underline{b}(x)) \geq \beta_0 > 0$, $|b_i(x)| \leq \beta_1$, $b_i \in W_\infty^1(\Omega)$.

Then from (2) it follows that there exists a constant $C > 0$ such that $a(u,u) \geq C\|u\|_{1,\Omega}^2$; i.e., $a(u,v)$ is H_0^1-coercive and by the Lax-Milgram lemma argument the problem (1) has a unique solution in $H_0^1(\Omega)$.

For the stability analysis (Propositions 2.1, 2.3, 2.5, 2.7) we will need higher smoothness, i.e., $b_i(x) \in W_\infty^{1+\alpha}(\Omega)$, $\alpha > 0$. Condition (ii) can be weakened to $\beta_0 = 0$: then the bilinear form $a(u,v)$ is coercive in H^1 and consequently the finite difference approximations will have the same property for sufficiently small h. However, $\beta_0 > 0$ is needed to prove the discrete maximum principle.

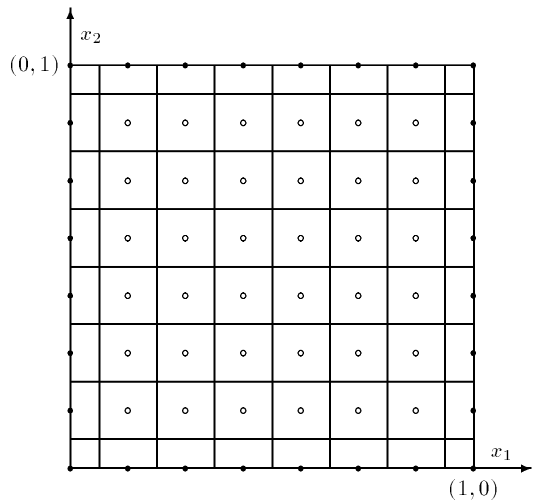

FIG. 1. *Cell-centered Mesh*

1.2. Grids and grid functions. We suppose that Ω is a rectangle with sides parallel to the axes x_1 and x_2. We consider the case of cell-centered grids, which owing to their good conservation properties, are very popular in reservoir simulation, weather prediction, heat transfer, etc. We cover the plane R^2 by square cells with sides of length h. The grid points are the centers of the cells (see Fig. 1). We suppose that the Dirichlet boundary Γ passes through the grid points, as shown in Fig. 1.

The grid points are denoted by $x = (x_1, x_2) = (x_{1,i}, x_{2,j}) = (ih, jh)$, where $i, j = 0, 1, 2, ..., N$ are integer indices. We introduce the following notation for various grids in $\bar{\Omega}$:

$$\bar{\omega} = \left\{(x_{1,i}, x_{2,j}) \in \bar{\Omega} : i, j = 0, 1, 2, ..., N\right\};$$

$$\omega = \bar{\omega} \cap \Omega, \qquad \gamma = \bar{\omega} \backslash \omega;$$

$$\omega_i^{\pm} = \omega \cup \gamma_i^{\pm}, \quad \text{where } \gamma_i^{\pm} = \{x \in \gamma : \cos(x_i, \underline{n}) = \pm 1\}, \quad i = 1, 2.$$

Here $\underline{n}$ is the unit outer normal to the boundary Γ.

Functions defined for $x \in \omega$ are called grid functions. We consistently use the dual notation for the value of the function y at the grid point $x = (x_{1,i}, x_{2,j})$; $y(x) = y(x_{1,i}, x_{2,j}) = y_{i,j}$ and in the points $(x_{1,i}, x_{2,j} \pm h/2) = (x_{1,i}, x_{2,j\pm 1/2})$ and $(x_{1,i} \pm h/2, x_{2,j}) = (x_{1,i\pm 1/2}, x_{2,j})$, $y_{i,j\pm 1/2} = y(x_{1,i}, x_{2,j\pm 1/2})$, $y_{i\pm 1/2,j} = y(x_{1,i\pm 1/2}, x_{2,j})$.

For a given function $y(x)$, $x \in \bar{\omega}$ we use the following discrete inner products and norms:

$$(y, v) = \sum_{x \in \omega} h^2 y(x) v(x), \qquad \|y\|_{0,\omega} = (y, y)^{1/2};$$

$$(y, v]_s = \sum_{x \in \omega_s^+} h^2 y(x) v(x), \qquad \|y]|_s = (y, y]_s^{1/2}, \quad s = 1, 2.$$

We introduce the following finite differences for grid functions $y(x)$:

(i) forward difference $\Delta_1 y_{i,j} = y_{i+1,j} - y_{i,j}$ and divided forward difference $y_{x_1,i,j} = \Delta_1 y_{i,j}/h$;

(ii) backward difference $\bar{\Delta}_1 y_{i,j} = y_{i,j} - y_{i-1,j}$ and divided backward difference $y_{\bar{x}_1,i,j} = \bar{\Delta}_1 y_{i,j}/h$;

(iii) divided central difference of second order

$$y_{\bar{x}_1 x_1} = \frac{\Delta_1 y_{i,j} - \bar{\Delta}_1 y_{i,j}}{h^2}.$$

Similarly, differences are defined in x_2 and in combination of x_1 and x_2 coordinate directions.

We also introduce the discrete analogues of H^1 and H^2-norms:

$$|y|^2_{1,\omega} = \|y_{\bar{x}_1}]|^2_1 + \|y_{\bar{x}_2}]|^2_2,$$

$$\|y\|^2_{1,\omega} = |y|^2_{1,\omega} + \|y\|^2_{0,\omega},$$

and

$$|y|^2_{2,\omega} = |y_{\bar{x}_1 x_1}|^2 + 2|y_{\bar{x}_1 \bar{x}_2}|^2 + |y_{\bar{x}_2 x_2}|^2,$$

$$\|y\|^2_{2,\omega} = |y|^2_{2,\omega} + \|y\|^2_{1,\omega}.$$

We will also need the negative norm:

$$\|y\|_{-1,\omega} = \sup_{v \neq 0} \frac{|(y,v)|}{\|v\|_{1,\omega}}.$$

Any grid function $y(x)$ can be considered as an element of a vector space of dimension equal to n, the number of the grid points in ω. In this case, we denote $y(x)$ by $\mathbf{y} \in R^n$ and consider it as an n-dimensional column vector. Then $\mathbf{y}^T$ will be the row vector transpose of $\mathbf{y}$.

2. Discretization schemes. The finite difference approximation is derived from the balance equation. We integrate (1) over each cell e

$$\int_e \operatorname{div}[(-a(x)\nabla u(x) + \underline{b}(x)u(x)]\,dx = \int_e f(x)\,dx$$

and then using Green's formula and dividing by h^2 we get

$$\frac{1}{h^2}\int_{\partial e}[-a\nabla u.\underline{n} + u\underline{b}\cdot\underline{n}]\,d\gamma = \frac{1}{h^2}\int_e f(x)\,dx \tag{3}$$

where $\underline{n}$ is the unit outward vector normal to the boundary of e. Splitting $\partial e = s_1^+ \cup s_2^+ \cup s_1 \cup s_2$ (see Fig. 2), the left-hand side of this identity can be written in the following form:

$$\begin{aligned}\frac{1}{h^2}\left[\int_{\partial e} W\,d\gamma + \int_{\partial e} V\,d\gamma\right] = \frac{1}{h^2}\Bigg[&\int_{s_1^+} W\,d\gamma + \int_{s_1^+} V\,d\gamma + \int_{s_1} W\,d\gamma + \int_{s_1} V\,d\gamma \\ &+ \int_{s_2^+} W\,d\gamma + \int_{s_2^+} V\,d\gamma + \int_{s_2} W\,d\gamma + \int_{s_2} V\,d\gamma\Bigg]\end{aligned} \tag{4}$$

where we have denoted

$$W = -a(\gamma)\nabla u(\gamma)\cdot\underline{n} \quad \text{and} \quad V = \underline{b}(\gamma)\cdot\underline{n}u(\gamma).$$

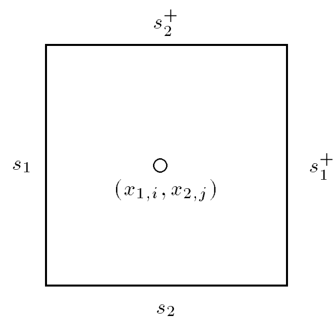

FIG. 2. *Cell* $e(x)$

In order to construct the finite difference scheme we approximate the balance equation (4). We split the approximation of the balance equation (4) in two parts,

$$A^{(2)}y + A^{(1)}y, \tag{5}$$

where $A^{(2)}$ is the part arising from the approximation of the second derivatives, and $A^{(1)}$ comes from the approximation of the first derivatives; y is an approximation to the exact solution u. We have the expressions

$$\begin{aligned} A^{(2)}y &= w_{1,i,j}^+ - w_{1,i,j} + w_{2,i,j}^+ - w_{2,i,j}\,, \quad x \in \omega\,, \\ A^{(1)}y &= v_{1,i,j}^+ - v_{1,i,j} + v_{2,i,j}^+ - v_{2,i,j}\,, \quad x \in \omega\,. \end{aligned} \tag{6}$$

In these formulae w_l^+, w_l, v_l^+, v_l, $l = 1, 2$ are some approximations of the corresponding integrals $\int_{s_1^+} W$, $\int_{s_1} W$, $\int_{s_2^+} W$, $\int_{s_2} W$, and $\int_{s_1^+} V$, $\int_{s_1} V$, $\int_{s_2^+} V$, $\int_{s_2} V$, respectively. Now, in order to complete the finite difference scheme we have to express the approximate fluxes w_l^+, w_l, v_l^+, v_l by the approximate values $y(x)$ of the solution $u(x)$ at the grid points. We consider the following approximations:

1. central difference scheme **CDS**
2. upwind difference scheme **UDS**
3. modified upwind difference scheme **MUDS**
4. Il'in's difference scheme **IDS**

2.1. Central difference scheme (CDS). We call this scheme "central" because of the analogy of $A^{(1)}$ and a central difference approximation of the first derivatives. We first rewrite the fluxes $-a(x)\nabla u(x) = (W_1(x), W_2(x))$ in the form

$$\frac{\partial u}{\partial x_l} = -\frac{W_l(x)}{a(x)}, \qquad x \in \Omega, \quad l = 1,\, 2.$$

Next, we integrate the equation for $l = 1$ along the interval with endpoints $(x_{1,i-1}, x_{2,j})$ and $(x_{1,i}, x_{2,j})$. We get

$$u_{i,j} - u_{i-1,j} = -\int_{x_{1,i-1}}^{x_{1,i}} \frac{W_1(s, x_{2,j})}{a(s, x_{2,j})}\, ds \approx -W_{1,i-1/2,j} \int_{x_{1,i-1}}^{x_{1,i}} \frac{ds}{a(s, x_{2,j})}.$$

We can now write the following approximate relations

$$\frac{1}{h^2}\int_{s_1} W_1(x)\,ds \approx \frac{hW_{1,i-1/2,j}}{h^2} \approx -\frac{1}{h}\left(\frac{1}{h}\int_{x_{1,i-1}}^{x_{1,i}} \frac{ds}{a(s,x_{2,j})}\right)^{-1} \frac{[u_{i,j}-u_{i-1,j}]}{h},$$

$$\frac{1}{h^2}\int_{s_2} W_2(x)\,ds \approx \frac{hW_{2,i,j-1/2}}{h^2} \approx -\frac{1}{h}\left(\frac{1}{h}\int_{x_{2,j-1}}^{x_{2,j}} \frac{ds}{a(x_{1,i},s)}\right)^{-1} \frac{[u_{i,j}-u_{i,j-1}]}{h}.$$

These approximate relations allow us to define:

$$w_l^+(x) \equiv w_{l,i,j}^+ = -\frac{k_{l,i,j}^+}{h} y_{x_l,i,j}, \qquad l=1,2, \tag{7}$$
$$w_l(x) \equiv w_{l,i,j} = -\frac{k_{l,i,j}}{h} y_{\bar{x}_l,i,j}, \qquad l=1,2,$$

where

$$k_{1,i,j} = \left(\frac{1}{h}\int_{x_{1,i-1}}^{x_{1,i}} \frac{ds}{a(s,x_{2,j})}\right)^{-1}, \quad k_{1,i,j}^+ = k_{1,i+1,j}, \tag{8}$$
$$k_{2,i,j} = \left(\frac{1}{h}\int_{x_{2,j-1}}^{x_{2,j}} \frac{ds}{a(x_{1,i},s)}\right)^{-1}, \quad k_{2,i,j}^+ = k_{2,i,j+1}.$$

The integrals $\int_{s_l^+} V_l^+(x)\,ds$, $\int_{s_l} V_l(x)\,ds$ can be approximated as follows ($l=1$):

$$\frac{1}{h^2}\int_{s_1^+} V_1^+(x)\,ds \approx \frac{b_{1,i+1/2,j}}{h}\left[\frac{u_{i,j}+u_{i+1,j}}{2}\right],$$

$$\frac{1}{h^2}\int_{s_1} V_1(x)\,ds \approx \frac{b_{1,i-1/2,j}}{h}\left[\frac{u_{i-1,j}+u_{i,j}}{2}\right].$$

And thus we can define the approximations:

$$v_{1,i,j}^+ = B_{1,i,j}^+(y_{i+1,j}+y_{i,j}), \quad B_{1,i,j}^+ = \frac{b_{1,i+1/2,j}}{2h},$$
$$v_{1,i,j} = B_{1,i,j}(y_{i,j}+y_{i-1,j}), \quad B_{1,i,j} = \frac{b_{1,i-1/2,j}}{2h}, \tag{9}$$
$$v_{2,i,j}^+ = B_{2,i,j}^+(y_{i,j+1}+y_{i,j}), \quad B_{2,i,j}^+ = \frac{b_{2,i,j+1/2}}{2h},$$
$$v_{2,i,j} = B_{2,i,j}(y_{i,j}+y_{i,j-1}), \quad B_{2,i,j} = \frac{b_{2,i,j-1/2}}{2h}.$$

Substituting (7) and (9) in (5) we get **CDS**. This scheme is stable if the local Peclet number satisfies the inequality [13], [20]:

$$\frac{|b_l(.,.)|h}{2k_l(.,.)} \le 1. \tag{10}$$

Obviously this is true only for sufficiently small h. We will not further consider the **CDS** because of its conditional stability.

2.2. Upwind difference scheme (UDS). One of the ways to find a stable finite difference approximation for the convection-diffusion boundary value problem is to use an upwind approximation for the first derivatives. In this case, $A^{(2)}$ is defined as in **CDS** and the terms v_1 and v_1^+ in $A^{(1)}$ are approximated in the following way:

$$
\begin{aligned}
v^+_{1,i,j} &= (B^+_{1,i,j} - |B^+_{1,i,j}|)y_{i+1,j} + (B^+_{1,i,j} + |B^+_{1,i,j}|)y_{i,j}, \\
v_{1,i,j} &= (B_{1,i,j} - |B_{1,i,j}|)y_{i,j} + (B_{1,i,j} + |B_{1,i,j}|)y_{i-1,j}.
\end{aligned}
\tag{11}
$$

In order to investigate the properties of the **UDS** we need the following auxiliary result.

PROPOSITION 2.1. *Let* $\underline{b}(x) \in (W^{1+\alpha}_\infty(\Omega))^2$*,* $\alpha > 0$ *and* $\nabla \cdot \underline{b}(x) \geq \beta_0$ *for some* $\beta_0 > 0$*. Then there exists* h_0 *such that for* $h \in (0, h_0)$ *the following inequality holds:*

$$
\left[\left(B^+_{1,i,j} - B_{1,i,j}\right) + \left(B^+_{2,i,j} - B_{2,i,j}\right) \right] \geq c_0,
$$

where $c_0 = \beta_0 - O(h^\alpha)$*,* $0 < \alpha \leq 2$*.*

Proof. Consider the linear functional:

$$
l(b_1) := \frac{b_{1,i+1/2,j} - b_{1,i-1/2,j}}{h} - \frac{\partial b_{1,i,j}}{\partial x_1}.
$$

This functional is bounded for $b_1 \in W^{1+\alpha}_\infty(\Omega)$, $0 \leq \alpha \leq 2$ and vanishes for all polynomials of second degree. Therefore, by the Bramble-Hilbert lemma argument we get

$$
|l(b_1)| \leq Ch^\alpha |b_1|_{1+\alpha,\infty,e}.
$$

A similar inequality holds for b_2. Using the triangle inequality and the assumption $\nabla \cdot \underline{b} \geq \beta_0$, the desired inequality is obtained. □

Remark 2.1. The above proposition means that, if the divergence of the vector $\underline{b}$ is greater than $\beta_0 > 0$, then the discrete analog of $\nabla \cdot \underline{b}$, defined by

$$
\frac{b_{1,i+1/2,j} - b_{1,i-1/2,j}}{h} + \frac{b_{2,i,j+1/2} - b_{2,i,j-1/2}}{h},
$$

is also positive for sufficiently small h.

First we will prove that the considered scheme is monotone.

PROPOSITION 2.2. **UDS** *satisfies the discrete maximum principle and the corresponding matrix* A *is an* **M***–matrix.*

Proof. Let $a_{i+k,j+l}$ be the coefficient in front of $y_{i+k,j+l}$, $k, l = -1, 0, 1$ in the finite difference scheme. Then it is enough to check the conditions [12]:

1. $a_{i,j} > 0$;
2. $a_{i-1,j}$, $a_{i+1,j}$, $a_{i,j-1}$, and $a_{i,j+1}$ are negative;
3. $a_{i,j} - \sum_{k,l=\pm 1} a_{i+k,j+l} > 0$, i.e., A is strictly diagonally dominant.

We have

1.

$$
a_{i,j} = \sum_{l=1}^{2} \left[\left(\frac{k^+_{l,i,j}}{h} + \frac{k_{l,i,j}}{h} \right) + (B^+_{l,i,j} - B_{l,i,j}) + |B^+_{l,i,j}| + |B_{l,i,j}| \right] > 0
$$

2.

$$|B_{l,i,j}| + B_{l,i,j} \geq 0 \Rightarrow -\left(\frac{k_{l,i,j}}{h} + |B_{l,i,j}| + B_{l,i,j}\right) < 0,$$

$$B^+_{l,i,j} - |B_{l,i,j}| \leq 0 \Rightarrow -\frac{k^+_{l,i,j}}{h} + B^+_{l,i,j} - |B_{l,i,j}| < 0$$

3.

$$a_{i,j} - \sum_{k,l=\pm 1} a_{i+k,j+l} = 2\sum_{l=1}^{2}(B^+_{l,i,j} - B_{l,i,j}) \geq 2c_0 > 0.$$

□

Now we concentrate on the positive definiteness of the operator A_h and the matrix A. In Section 1 we showed that the bilinear form, corresponding to the continuous problem (1) is H^1_0-elliptic. In the following proposition we establish that the discrete analog of the bilinear form inherits this property.

Proposition 2.3. *Let $\underline{b}(x) \in (W^{1+\alpha}_\infty(\Omega))^2$, $\alpha > 0$ and $\nabla \cdot \underline{b}(x) \geq \beta_0$. Then the matrix A of* **UDS** *is a positive real matrix and there exists a constant C such that the following inequality is true:*

$$(A_h y, y) \geq C\|y\|^2_{1,\omega}, \quad \text{for all } y \in D^0 = \{y,\, y_{|\gamma} = 0\}.$$

The constant C depends only on the ratio $a(x)/|\underline{b}(x)|$.

Proof. Let $z(x)$ and $y(x)$ be grid functions from D^0. Then

$$(12)\qquad \begin{aligned}(A_h y, z) = &-\sum_{x\in\omega}\sum_{l=1}^{2} h^2\left[\frac{k^+_{l,i,j}}{h} y_{x_l,i,j} - \frac{k_{l,i,j}}{h} y_{\bar{x}_l,i,j}\right] z_{i,j}\\ &+\sum_{x\in\omega}\sum_{l=1}^{2} h^2\left[v^+_{l,i,j} - v_{l,i,j}\right] z_{i,j} \equiv I + \sum_{l=1}^{2} J_l.\end{aligned}$$

We transform the sums in formula (12) for $l = 1, 2$ using summation by parts thus obtaining

$$I = \sum_{l=1}^{2}\sum_{x\in\omega} k_{l,i,j} y_{\bar{x}_l,i,j} z_{\bar{x}_l,i,j}.$$

Using (11) we rewrite J_1 in the following way

$$(13)\qquad \begin{aligned}J_1 = &\sum_{x\in\omega} h^2\left[(B^+_{1,i,j} - |B^+_{1,i,j}|)y_{i+1,j} + (B^+_{1,i,j} + |B^+_{1,i,j}|)y_{i,j}\right.\\ &\left.-(B_{1,i,j} - |B_{1,i,j}|)y_{i,j} - (B_{1,i,j} + |B_{1,i,j}|)y_{i-1,j}\right] z_{i,j}\\ = &\sum_{x\in\omega} h^2\left[B^+_{1,i,j} y_{i+1,j} - B_{1,i,j} y_{i-1,j}\right] z_{i,j} + \sum_{x\in\omega} h^2\left[B^+_{1,i,j} - B_{1,i,j}\right] y_{i,j} z_{i,j}\\ &-\sum_{x\in\omega} h^2\left[|B^+_{1,i,j}|\Delta_1 y_{i,j} - |B_{1,i,j}|\bar{\Delta}_1 y_{i,j}\right] z_{i,j}.\end{aligned}$$

We now transform the first term in the last identity in (13)

$$\begin{aligned}&\sum_{x\in\omega} h^2\left[B^+_{1,i,j} y_{i+1,j} - B_{1,i,j} y_{i-1,j}\right] z_{i,j}\\ &\quad = \sum_{x\in\omega} h^2\left[B^+_{1,i,j} y_{i+1,j} - B_{1,i,j} y_{i,j}\right] z_{i,j} + \sum_{x\in\omega} B_{1,i,j}(y_{i,j} - y_{i-1,j}) z_{i,j}.\end{aligned}$$

Using summation by parts for the first term above we obtain

$$\sum_{x\in\omega} h^2 \left[B^+_{1,i,j} y_{i+1,j} - B_{1,i,j} y_{i-1,j}\right] z_{i,j} = \sum_{x\in\omega} h^2 B_{1,i,j} (z_{i,j}\bar{\Delta}_1 y_{i,j} - y_{i,j}\bar{\Delta}_1 z_{i,j}).$$

Finally we get

$$\begin{aligned}(A_h y, z) &= \sum_{l=1}^{2}\sum_{x\in\omega} h^2 \left(k_{l,i,j} + |B_{l,i,j} h|\right) y_{\bar{x}_l,i,j} z_{\bar{x}_l,i,j} \\ &\quad + \sum_{l=1}^{2}\sum_{x\in\omega} h^2 B_{l,i,j} (z_{i,j}\bar{\Delta}_1 y_{i,j} - y_{i,j}\bar{\Delta}_1 z_{i,j}) \\ &\quad + \sum_{l=1}^{2}\sum_{x\in\omega} h^2 \left(B^+_{l,i,j} - B_{l,i,j}\right) y_{i,j} z_{i,j}.\end{aligned}$$

Letting $z = y$ in the above formula the desired result follows using Proposition 2.1. □

2.3. Modified upwind difference scheme (MUDS). As we will later show the **UDS** is only $O(h)$ accurate. In order to obtain a diagonally dominant matrix and achieve $O(h^2)$ order of accuracy we modify the upwind scheme in the following way [3] (see also [20]):

$$\begin{aligned}\int_{s_1} & b_1 u \, d\gamma \\ &= \left(B_{1,i,j} h^2 - |B_{1,i,j} h^2|\right) u_{i,j} + \left(B_{1,i,j} h^2 + |B_{1.i,j} h^2|\right) u_{i-1,j} + O(h) \\ &= I_1 + O(h), \\ \int_{s_1} & b_1 u \, d\gamma \\ &= B_{1,i,j} h^2 (u_{i,j} + u_{i-1,j}) + O(h^2) \\ &= I_2 + O(h^2), \\ \int_{s_1} & \left(-a\frac{\partial u}{\partial x_1} + b_1 u\right) d\gamma \\ &= -k_{1,i,j}\bar{\Delta}_1 u_{i,j} + I_2 + O(h^2) \\ &= -(k_{1,i,j} - |B_{1,i,j} h^2|)\bar{\Delta}_1 u_{i,j} + I_1 + O(h^2) \\ &= -\frac{k_{1,i,j}}{1 + |B_{1,i,j} h^2|/k_{1,i,j}}\bar{\Delta}_1 u_{i,j} \\ &\quad - \left(k_{1,i,j} - |B_{1,i,j} h^2| - \frac{k^2_{1i,j}}{k_{1,i,j} + |B_{1,i,j} h^2|}\right)\bar{\Delta}_1 u_{i,j} + I_1 + O(h^2) \\ &= -\frac{k_{1,i,j}}{1 + |B_{1,i,j} h^2|/k_{1,i,j}}\bar{\Delta}_1 u_{i,j} + \frac{B^2_{1,i,j} h^4}{k_{1,i,j} + |B_{1,i,j} h^2|}\bar{\Delta}_1 u_{i,j} + I_1 + O(h^2) \\ &= -\frac{k_{1,i,j}}{1 + |B_{1,i,j} h^2|/k_{1,i,j}}\bar{\Delta}_1 u_{i,j} + I_1 + O(h^2).\end{aligned}$$

In the last step we have taken into account that $B_1 h^2 = O(h)$. These heuristic formulae show that if we want to get a second order finite difference scheme we should choose w_1^+, w_1, v_1^+, and v_1 in such a way that they satisfy the following conditions:

$$w_{1,i,j}^+ + v_{1,i,j}^+ = -\frac{\tilde{k}_{1,i,j}^+}{h} y_{x_l,i,j} + (B_{1,i,j}^+ - |B_{1,i,j}^+|) y_{i+1,j} + (B_{1,i,j}^+ + |B_{1,i,j}^+|) y_{i,j},$$
$$w_{1,i,j} + v_{1,i,j} = -\frac{\tilde{k}_{1,i,j}}{h} y_{\bar{x}_1,i,j} + (B_{1,i,j} - |B_{1,i,j}|) y_{i,j} + (B_{1,i,j} + |B_{1,i,j}|) y_{i-1,j}.$$

We remark here that we split the scheme into two parts only for convenience of the error analysis. Then we define **MUDS** as follows: $A^{(1)}$ is the same as in **CDS** and the expressions w_l and w_l^+ in $A^{(2)}$ are defined by

$$\begin{aligned} w_{l,i,j}^+ &= -\frac{1}{h}\left(\tilde{k}_{l,i,j}^+ + |B_{l,i,j}^+ h^2|\right) y_{x_l,i,j}, \quad l = 1,2, \\ w_{l,i,j} &= -\frac{1}{h}\left(k_{l,i,j} + |B_{l,i,j} h^2|\right) y_{\bar{x}_l i,j}, \quad l = 1,2. \end{aligned} \tag{14}$$

where

$$\begin{aligned} \tilde{k}_{1,i,j} &= \frac{k_{1,i,j}}{1 + |B_{1,i,j} h^2| / k_{1,i,j}}, \qquad \tilde{k}_{1,i,j}^+ = \tilde{k}_{1,i+1,j}, \\ \tilde{k}_{2,i,j} &= \frac{k_{2,i,j}}{1 + |B_{2,i,j} h^2| / k_{2,i,j}}, \qquad \tilde{k}_{2,i,j}^+ = \tilde{k}_{2,i,j+1}. \end{aligned} \tag{15}$$

Using a similar argument as in Propositions 2.2 and 2.3 we can prove the following.

PROPOSITION 2.4. **MUDS** *satisfies the discrete maximum principle and the corresponding matrix A is an* $\mathbf{M}$*–matrix.*

PROPOSITION 2.5. *Let $\underline{b}(x) \in (W_\infty^{1+\alpha}(\Omega))^2$, $\alpha > 0$ and $\nabla \cdot \underline{b}(x) \geq \beta_0$. Then the matrix A of the* **MUDS** *is a positive real matrix and there exists a constant C such that the following inequality is true:*

$$(A_h y, y) \geq C \|y\|_{1,\omega}^2, \quad \text{for all } y \in D^0 = \{y,\, y_{|\gamma} = 0\}.$$

The constant C depends only on the ratio $a(x)/|\underline{b}(x)|$.

2.4. Il'in's difference scheme (IDS). Another approximation we derive in a similar way as in [14] is

$$\frac{1}{h^2}\int_{s_1^+} (-a(\gamma)\frac{\partial u(\gamma)}{\partial x} + b_1(\gamma) u(\gamma))\, d\gamma \approx -\frac{\gamma_{1,i,j}^+}{h} y_{x_l,i,j} + B_{1,i,j}^+ y_{i+1,j} + B_{1,i,j}^+ y_{i,j}$$

or

$$w_{l,i,j}^+ = -\frac{\gamma_{l,i,j}^+}{h} y_{x_l,i,j}, \qquad w_{l,i,j} = -\frac{\gamma_{l,i,j}}{h} y_{\bar{x}_l,i,j}, \quad l = 1,2, \tag{16}$$

and v_l^+ and v_l are defined as in **CDS**. We choose the coefficient γ such that the above approximate relation is exact for $u = e^{b_1 x/a}$ when $a(x)$ and $b_1(x)$ are constants. We get

$$\gamma_{l,i,j}^+ = B_{l,i,j}^+ h^2 \coth\left(\frac{B_{l,i,j}^+ h^2}{k_{l,i,j}^+}\right), \qquad \gamma_{l,i,j} = B_{l,i,j} h^2 \coth\left(\frac{B_{l,i,j} h^2}{k_{l,i,j}}\right). \tag{17}$$

It is easy to see that γ_l^+ and $\gamma_l > 0$ are positive regardless of the sign of b_l. From $|\coth(x)| > 1$ we have $\gamma_l^+ > |B_l^+|h^2$ and $\gamma_l > |B_l|h^2$. Using the same technique as in previous propositions we have:

PROPOSITION 2.6. **IDS** *satisfies the discrete maximum principle and the corresponding matrix A is an* $\mathbf{M}$*–matrix.*

PROPOSITION 2.7. *Let $\underline{b}(x) \in (W_\infty^{1+\alpha}(\Omega))^2$, $\alpha > 0$ and $\nabla \cdot \underline{b}(x) \geq \beta_0$. Then the matrix A of the* **IDS** *is a positive real matrix and there exists a constant C such that the following inequality is true:*

$$(A_h y, y) \geq C\|y\|_{1,\omega}^2, \quad \text{for all } y \in D^0 = \{y, y_{|\gamma} = 0\}.$$

The constant C depends only on the ratio $a(x)/|\underline{b}(x)|$.

Remark 2.2. If $\beta_0 = 0$ the **UDS**, **MUDS** and **IDS** in general does not satisfy the discrete maximum principle, but for sufficiently small h the constructed finite difference operators are coercive in discrete H^1-norm. Therefore, all error estimates which we prove in the next sections hold for $\beta_0 = 0$.

Summarizing these approximations we formulate the following discrete problem for (1): find a grid function $y(x)$, which satisfies the finite difference equations:

$$\sum_{l=1}^{2}(w_l^+(x) - w_l(x)) + \sum_{l=1}^{2}(v_l^+(x) - v_l(x)) = \phi, \quad \text{in } \omega, \qquad y(x) = 0, \quad \text{on } \gamma, \tag{18}$$

where w_l and v_l are defined by (7), (14), (16), (9) and (11), respectively and $\phi = \frac{1}{h^2}\int_e f(x)\,dx$. These schemes can be written as systems of linear algebraic equations

$$A\mathbf{y} = \phi. \tag{19}$$

3. Stability and error analysis.

The stability of problem (19) is a simple consequence of the positive definiteness of matrix A. Namely, we prove the following lemma.

LEMMA 3.1. *For all considered difference schemes the following a priori estimate is valid:*

$$\|y\|_{1,\omega} \leq C\|\phi\|_{-1,\omega}$$

where y is the discrete solution and ϕ is the right-hand side of (19). (The constant C does not depend on y or ϕ.)

Proof. The proof follows from the inequalities based on the coercivity of the operator A and on the definition of the norm $\|\cdot\|_{-1,\omega}$:

$$\|y\|_{1,\omega}^2 \leq C(A_h y, y) = C(\phi, y) \leq C\|\phi\|_{-1,\omega}\ \|y\|_{1,\omega}.$$

□

Remark 3.1. Since $\|\phi\|_{-1,\omega} \leq \|\phi\|_{0,\omega}$ and $\|y\|_{0,\omega} \leq \|y\|_{1,\omega}$ we also can obtain the following estimate:

$$\|y\|_{0,\omega} \leq C\|\phi\|_{0,\omega}.$$

3.1. Error estimates in discrete H^1-norm. The error analysis presented here is done in the general framework of the methods developed in [21] and [9]. We consider only the case when $a(x) \equiv 1$. Let

$$z(x) = y(x) - u(x), \qquad x \in \omega$$

be the error of the finite difference method. Substituting $y = z + u$ in (18) we obtain

$$Az = \phi - Au \equiv \psi. \tag{20}$$

Then using (4)–(19) we transform ψ in the following form

$$\sum_{l=1}^{2} \frac{1}{h} \left\{ \left[\frac{1}{h} \int_{s_l^+} -\frac{\partial u}{\partial x_l}\, d\gamma - h w_l^+ \right] - \left[\frac{1}{h} \int_{s_l} -\frac{\partial u}{\partial x_l}\, d\gamma - h w_l \right] \right\}$$
$$+ \sum_{l=1}^{2} \frac{1}{h} \left\{ \left[\frac{1}{h} \int_{s_l^+} b_l u \, d\gamma - h v_l^+ \right] - \left[\frac{1}{h} \int_{s_l} b_l u \, d\gamma - h v_l \right] \right\} \equiv \psi_1 + \psi_2 = \psi,$$

where the local truncation error ψ has been split up into two terms:

$$\begin{aligned} \psi_1 &\equiv \frac{1}{h} \sum_{l=1}^{2} \left[\eta_l^+(x) - \eta_l(x) \right], \qquad & \psi_2 &\equiv \frac{1}{h} \sum_{l=1}^{2} \left[\mu_l^+(x) - \mu_l(x) \right], \\ \eta_l &= \frac{1}{h} \int_{s_l} -\frac{\partial u}{\partial x_l}\, d\gamma - h w_l, \qquad & \mu_l &= \frac{1}{h} \int_{s_l} b_l u \, d\gamma - h v_l. \end{aligned} \tag{21}$$

Here ψ_1 is the error of approximation of the first derivatives, and ψ_2 is the error of approximation of the second derivatives.

Note that the components of the local truncation error η_l and μ_l are defined on the shifted grids ω_l^+, $l = 1, 2$. Using summation by parts and the Schwarz inequality, we get

$$(\psi_2, z) = \sum_{l=1}^{2} \sum_{x \in \omega} h^2 \left[\frac{\eta_l^+(x) - \eta_l(x)}{h} \right] z(x) = - \sum_{l=1}^{2} \sum_{x \in \omega_l^+} h^2 \eta_l(x) z_{\bar{x}_l}$$
$$\le \left(\sum_{l=1}^{2} \sum_{x \in \omega_l^+} h^2 \eta_l^2(x) \right)^{1/2} \left(\sum_{l=1}^{2} \sum_{x \in \omega_l^+} h^2 z_{\bar{x}_l}^2 \right)^{1/2} \le \left(\|\eta_1]|_1 + \|\eta_2]|_2 \right) \|z\|_{1,\omega}.$$

Likewise

$$(\psi_1, z) \le \left(\|\mu_1]|_1 + \|\mu_2]|_2 \right) \|z\|_{1,\omega}.$$

Summarizing these results and using Propositions 2.3, 2.5, and 2.7 we obtain the following main result.

LEMMA 3.2. *The error $z(x) = y(x) - u(x)$, $x \in \omega$ of all considered finite difference schemes satisfies the a priori estimate*

$$\|z\|_{1,\omega} \le C \sum_{l=1}^{2} \left(\|\eta_l]|_l + \|\mu_l]|_l \right), \tag{22}$$

where the components η_l, μ_l, $l = 1, 2$ of the local truncation error are defined by (21) with approximate fluxes w_l^+, w_l, v_l^+, v_l, $l = 1, 2$ determined by (7), (11), (14) and (16) for the **UDS**, **MUDS** *and* **IDS**, *correspondingly. (The constant C does not depend on h or z.)*

In order to use the estimate (22) of Lemma 3.2 we have to bound the corresponding norms of the local truncation error components η_l, μ_l, $l = 1, 2$ defined by (21). These estimates are provided in the lemma given below.

LEMMA 3.3. *Let the solution of the problem (1) be H^m-regular, $\frac{3}{2} < m$, and the components of the local truncation error η_l, μ_l, $l = 1, 2$ be defined by (21) with approximate fluxes w_l^+, w_l, v_l^+, v_l, $l = 1, 2$ determined by (7), (11), (14), and (16). Then the following estimates are valid ($l = 1, 2$):*

$$|\eta_l| \le Ch^{m-2}|u|_{m,\bar{e}}, \qquad \frac{3}{2} < m \le 3, \tag{23}$$

$$|\mu_l| \le \begin{cases} Ch^{m-1}\|b_l\|_{1,\infty,\Omega}\ |u|_{m,\bar{e}} & \text{for } \mathbf{MUDS} \text{ and } \mathbf{IDS}, \\ C\left[|b_l|_{0,\infty,\Omega}\ |u|_{1,\bar{e}} + h^{m-1}\|b_l\|_{1,\infty,\Omega}\ |u|_{m,\bar{e}}\right] & \text{for } \mathbf{UDS}, \end{cases} \tag{24}$$

where $1 < m \le 2$; $\bar{e} = e_{i-1,j} \bigcup e_{i,j}$ for $l = 1$ and $\bar{e} = e_{i,j-1} \bigcup e_{i,j}$ for $l = 2$.

Proof. Consider first the component $\eta_1(x) = \eta_1(x_{1,i}, x_{2,j})$ for the **UDS**. Then

$$\eta_1(x) = \frac{1}{h}\int_{s_1} -\frac{\partial u}{\partial x_1}\,d\gamma - hw_1(x) = \frac{1}{h}\left[\int_{s_1} -\frac{\partial u}{\partial x_1}(x_{1,i-1/2},\gamma)\,d\gamma + (u_{i,j} - u_{i-1,j})\right].$$

For a fixed $x \in \omega_1^+$, η_1 is a linear functional of u. Using the embedding of Sobolev spaces $H^m(\Omega) \subset L_\infty(\Omega)$, $1 < m$ (see for example [1]), we conclude that this functional is bounded in $H^m(\bar{e})$, for $\frac{3}{2} < m$; i.e. $|\eta_1(x)| \le \frac{C}{h}\|u\|_{m,\bar{e}}$ for every $u \in H^m(\bar{e})$, $\frac{3}{2} < m$. It is easy to check that η_1 vanishes if u is a polynomial of second degree. Therefore, by the Bramble-Hilbert lemma argument we get

$$|\eta_1(x)| \le Ch^{m-2}|u|_{m,\bar{e}}, \qquad \frac{3}{2} < m \le 3. \tag{25}$$

Now we consider η_1 for the **MUDS**. By construction

$$k_1(x) = \left(\frac{1}{1 + |B_{1,i,j}|} + |B_{1,i,j}|\right) = 1 + C_1(x)h^2, \qquad C_1(x) \sim b_1^2(x).$$

Then

$$\eta_1(x) = \frac{1}{h}\int_{s_1} -\frac{\partial u}{\partial x_1}\,d\gamma - w_1(x) = \frac{1}{h}\left[\int_{s_1} -\frac{\partial u}{\partial x_1}\,d\gamma + (1 + C_1h^2)(u_{i,j} - u_{i-1,j})\right].$$

We consider $u_{i,j} - u_{i-1,j}$ as a linear functional of u for a fixed $x \in \omega^+$. This functional is bounded in $H^m(\bar{e})$, $1 < m \le 3$ and vanishes for all polynomials of zero degree. Therefore, by the corollary of the Bramble-Hilbert lemma [12] we get

$$|u_{i,j} - u_{i-1,j}| \le C(|u|_{1,\bar{e}} + h^{m-1}|u|_{m,\bar{e}}), \qquad 1 < m \le 3. \tag{26}$$

Hence the estimate (25) is valid in this case as well. Finally for the **IDS** the result follows from the fact that

$$B_{1,i,j}\coth(B_{1,i,j}) = 1 + \tilde{C}_1(x)h^2, \qquad \tilde{C}_1(x) \sim b_1^2(x)$$

and for the same reasons as in the case for the **MUDS**. In a similar way we can estimate $\eta_2(x)$.

For the component $\mu_1(x)$ let us begin again with the **UDS**. We have,

$$\mu_1(x) = \frac{1}{h}\int_{s_1} b_1 u\, ds - hv_1 = \frac{|b_{1,i-1/2,j}|}{2}\bar{\Delta}_1 u_{i,j} - l(b_1,u) \tag{27}$$

where $l(b_1,u)$ is defined by

$$l(b_1,u) = b_{1,i-1/2,j}\left[\frac{u_{i,j}+u_{i-1,j}}{2}\right] - \frac{1}{h}\int_s b_1\left(x_{1,i}-\frac{h}{2},\gamma\right) u\left(x_{1,i}-\frac{h}{2},\gamma\right). \tag{28}$$

Now we can estimate the first term in (27) by

$$\frac{|b_{1,i-1/2,j}|}{2}|u_{i,j}-u_{i-1,j}| \le C|b_1|_{1,\infty,\Omega}\left(|u|_{1,\bar{e}} + h^{m-1}\,|u|_{m,\bar{e}}\right), \qquad 1<m\le 3.$$

The functional $l(b_1,u)$ is estimated in the following lemma, which concludes the proof for the second component of the truncation error μ_1. We note that for **MUDS** and **IDS** we have only the first term l in the formula (27). □

LEMMA 3.4. *If the solution of problem (1) is H^m-regular, $1<m$, then for the bilinear functional $l(b_1,u)$ defined by (28) the following estimate is valid:*

$$|l(b_1,u)| \le Ch^{m-1}\|b_1\|_{1,\infty,\Omega}\,\|u\|_{m,\bar{e}}, \qquad 1<m\le 2.$$

Proof. After the change of variables $x_i + s_i h = \gamma_i$ we get the domain $E = \{(s_1,s_2) : -1<s_1<0,\ |s_2|<\frac{1}{2}\}$ and the functions $\tilde{u}(s_1,s_2) = u(x_1+s_1h, x_2+s_2h)$, $\tilde{b_1}(s_1,s_2) = b_1(x_1+s_1h,x_2+s_2h)$,

$$\begin{aligned} l(b_1,u) &= l(\tilde{b}_1,\tilde{u}) \\ &= \tilde{b}_1\left(-\frac{1}{2},0\right)\frac{\tilde{u}(0,0)+\tilde{u}(-1,0)}{2} - \int_{-1/2}^{1/2}\tilde{b}_1\left(-\frac{1}{2},s_2\right)\tilde{u}\left(-\frac{1}{2},s_2\right)ds_2. \end{aligned}$$

We rewrite l in the following way

$$\begin{aligned} l(\tilde{b}_1,\tilde{u}) &= \tilde{b}_1\left(-\frac{1}{2},0\right)\left[\frac{\tilde{u}(0,0)+\tilde{u}(-1,0)}{2} - \int_{-1/2}^{1/2}\tilde{u}\left(-\frac{1}{2},s_2\right)ds_2\right] \\ &\quad + \left[\int_{-1/2}^{1/2}\left[\tilde{b}_1\left(-\frac{1}{2},s_2\right) - \tilde{b}_1\left(-\frac{1}{2},0\right)\right]\left[\tilde{u}\left(-\frac{1}{2},s_2\right)\right]ds_2\right] \\ &= \tilde{b}_1\left(-\frac{1}{2},0\right)\left[\frac{\tilde{u}(0,0)+\tilde{u}(-1,0)}{2} - \int_{-1/2}^{1/2}\tilde{u}\left(-\frac{1}{2},s_2\right)ds_2\right] \\ &\quad + \int_{-1/2}^{1/2}\left[\tilde{b}_1\left(-\frac{1}{2},s_2\right) - \tilde{b}_1\left(-\frac{1}{2},0\right)\right]\left[\tilde{u}\left(-\frac{1}{2},s_2\right) - \tilde{u}\left(-\frac{1}{2},0\right)\right]ds_2 \\ &\quad + \int_{-1/2}^{1/2}\left[\tilde{b}_1\left(-\frac{1}{2},0\right) - \tilde{b}_1\left(-\frac{1}{2},s_2\right)\right]\tilde{u}\left(-\frac{1}{2},0\right)ds_2 \\ &= \tilde{b}_1\left(-\frac{1}{2},0\right)p(\tilde{u}) + c(\tilde{b}_1,\tilde{u}) + \tilde{u}\left(-\frac{1}{2},0\right)q(\tilde{b}_1), \end{aligned}$$

where the linear functionals $p(\tilde{u})$, $q(\tilde{b}_1)$ and the bilinear functional $c(\tilde{b}_1, \tilde{u})$ are defined by

$$p(\tilde{u}) = \frac{\tilde{u}(0,0) + \tilde{u}(-1,0)}{2} - \int_{-1/2}^{1/2} \tilde{u}\left(-\frac{1}{2}, s_2\right) ds_2,$$

$$c(\tilde{b}_1, \tilde{u}) = \int_{-1/2}^{1/2} \left[\tilde{b}_1\left(-\frac{1}{2}, s_2\right) - \tilde{b}_1\left(-\frac{1}{2}, 0\right)\right] \left[\tilde{u}\left(-\frac{1}{2}, s_2\right) - \tilde{u}\left(-\frac{1}{2}, 0\right)\right] ds_2,$$

and

$$q(\tilde{b_1}) := \tilde{b}_1\left(-\frac{1}{2}, 0\right) - \int_{-1/2}^{1/2} \tilde{b}_1\left(-\frac{1}{2}, s_2\right) ds_2.$$

Hence

$$|l(\tilde{b}_1, \tilde{u})| \le |\tilde{b}_{0,\infty,E}|p(\tilde{u})| + |c(\tilde{b}_1, \tilde{u})| + |\tilde{u}_{0,\infty,E}|q(\tilde{b}_1)|.$$

First we consider the linear functional $p(\tilde{u})$. It is bounded for $u \in H^m(E)$, $1 < m$ and vanishes for all polynomials of first degree. Hence $|p(u)| \le Ch^{m-1}|u|_{m,\bar{e}}$, $1 < m \le 2$. Obviously $c(\tilde{b}_1, \tilde{u})$ is a bilinear functional bounded for $(\tilde{b}_1, \tilde{u}) \in W^1_\infty(E) \times H^1(E)$ and vanishes for r, s polynomials of zero degree; i.e., $c(r, \tilde{u}) = 0$ for $\tilde{u} \in H^1(E)$ and $c(\tilde{b}_1, s) = 0$ for $\tilde{b}_1 \in W^1_\infty(E)$. Then by the bilinear variant of the Bramble-Hilbert lemma we have $|c(b_1, u)| \le Ch|b_1|_{1,\infty,\bar{e}}\, |u|_{1,\bar{e}}$. And finally the linear functional $q(\tilde{b}_1)$ fulfills $q(\tilde{b}_1) = 0$ for all polynomials of first degree and therefore the estimate $|q(b_1)| \le Ch|b_1|_{1,\infty,\bar{e}}$ holds. Combining the above estimates we have

$$|l_1(b_1, u)| \le Ch^{m-1}\left[|u|_{m,\bar{e}}\, |b_1|_{0,\infty,\bar{e}} + h^{2-m}|b_1|_{1,\infty,\bar{e}}(|u|_{1,\bar{e}} + |u|_{0,\infty,\bar{e}})\right].$$

Hence by the embedding $H^m(\Omega) \subset L^\infty(\Omega)$, $m > 1$ we get the desired assertion. □

Now we are ready to prove the main result of this subsection.

THEOREM 3.5. *If the solution $u(x)$ of the problem (1) is H^m-regular, with $\frac{3}{2} < m \le 3$ then:*

(i) the **MUDS** *and the* **IDS** *defined by (14), (9), (16), and (9) have $O(h^{m-1})$ rate of convergence in the H^1-discrete norm, and*

$$\|y - u\|_{1,\omega} \le Ch^{m-1}\left(1 + h^\delta(\|b_1\|_{1,\infty,\Omega} + \|b_2\|_{1,\infty,\Omega})\right)\|u\|_{m,\Omega}.$$

(ii) the **UDS** *defined by (7) and (11) has at most first order of convergence in the H^1-discrete norm, and*

$$\|y - u\|_{1,\omega} \le Ch\left(|b_1|_{0,\infty,\Omega} + |b_2|_{0,\infty,\Omega}\right)|u|_{1,\Omega} + Ch^{m-1}\left(1 + h^\delta(\|b_1\|_{1,\infty,\Omega} + \|b_2\|_{1,\infty,\Omega})\right)\|u\|_{m,\Omega}.$$

Here

$$\delta = \begin{cases} 1, & \frac{3}{2} < m \le 2, \\ 3 - m, & 2 \le m \le 3. \end{cases}$$

Proof. In Lemma 3.3 we have proved the estimates for the components η_l, μ_l, $l = 1, 2$ of the local truncation error. Hence

$$\|\eta_l]| = \left(\sum_{x\in\omega_l^+} h^2\eta_l^2(x)\right)^{1/2} \le Ch^{m-1}\left(\sum_{x\in\omega_l^+} |u|^2_{m,\bar{e}}\right)^{1/2} \le Ch^{m-1}|u|_{m,\Omega}.$$

In the same way we get for $\|\mu_l]|_l$

$$\|\mu_1]|_1 \le Ch^m \|b_1\|_{1,\infty,\Omega}\ \|u\|_{m,\Omega}$$

when **MUDS** or **IDS** are used, and

$$\|\mu_1]|_1 \le C\,(h^m \|b_1\|_{1,\infty,\Omega}\ \|u\|_{m,\Omega} + h|b_1|_{0,\infty,\Omega}\ |u|_{1,\Omega})$$

otherwise. This completes the proof. □

3.2. Error estimates in discrete L^2-norm. Here we elaborate the discrete Aubin-Nitsche "trick" for finite difference operators obtained in §2 (see also [21]). Since the issue of constructing and studying monotone approximations to convection-diffusion operators is our main goal we disregard the differences that may occur from the approximation of the right hand side. Thus we consider the following homogeneous problem:

$$\begin{cases} \operatorname{div}(-a(x)\nabla u(x) + \underline{b}(x)u(x)) = 0, & \text{in } \Omega, \\ u(x) = g(x), & \text{on } \Gamma, \end{cases} \tag{29}$$

where $g(x) \in L^2(\Gamma)$. In order to simplify our presentation we consider only the case $a(x) \equiv 1$. First, we introduce the following averaging operators [21]:

$$\begin{aligned} S_i u &= \frac{1}{h}\int_{x_{i-h/2}}^{x_{i+h/2}} u(x_1, \ldots, \xi_i, \ldots, x_n)\, d\xi_i, \\ S_i^+ u &= \frac{1}{h}\int_{x_i}^{x_i+h} u(x_1, \ldots, \xi_i, \ldots, x_n)\, d\xi_i, \\ S_i^- u &= \frac{1}{h}\int_{x_i-h}^{x_i} u(x_1, \ldots, \xi_i, \ldots, x_n)\, d\xi_i, \\ T_i &= S_i^2 = S_i^+ S_i^-, \quad T = T_1 T_2. \end{aligned}$$

Then applying T to the differential equation (29) at any grid point $x \in \omega$ and using the properties,

$$T_i\left(\frac{\partial^2 u}{\partial x_i^2}\right)(x) = u_{\bar{x}_i, x_i}, \quad S_i^+\left(\frac{\partial u}{\partial x_i}\right)(x) = u_{x_i},$$

we get

$$-(T_2 u)_{\bar{x}_1, x_1} - (T_1 u)_{\bar{x}_2, x_2} + T_2 S_1^-(b_1 u)_{x_1} + T_1 S_2^-(b_2 u)_{x_2} = 0. \tag{30}$$

We express the operator A_h in the form

$$h w_{1,x_1} + h w_{2,x_2} + A^{(1)} y = 0, \quad x \in \omega, \tag{31}$$

$$y(x) = T_{3-l}\, g(x), \quad x \in \gamma_l^{\pm}, \quad l = 1, 2. \tag{32}$$

Let $z(x) = y(x) - u(x)$, $x \in \bar{\omega}$ be the error of the finite difference method. We define

$$\bar{u} = \begin{cases} u(x), & x \in \omega \\ T_{3-l}\, g(x), & x \in \gamma_l^{\pm}, \quad l = 1, 2. \end{cases}$$

Then $z = (y - \bar{u}) + (\bar{u} - u) = \bar{z} + \bar{u} - u$. Note that $\bar{z} = 0$ on γ. Substituting $y = \bar{z} + \bar{u}$ in (31) we obtain

$$A_h \bar{z} = A_h y - A_h \bar{u}. \tag{33}$$

The right-hand side of (33) is the local truncation error. In order to obtain an a priori estimate we represent the local truncation error in a divergence or almost divergence form (depending upon the choice of the difference scheme). Next, we rewrite (33) as

$$\begin{aligned} A_h \bar{z} &= \sum_{l=1}^{2} \left[h w_l + (T_{3-l} u)_{\bar{x}_l} \right]_{x_l} + \sum_{l=1}^{2} \left[h v_l - T_{3-l} S_l^- (b_l u) \right]_{x_l} \\ &= \sum_{l=1}^{2} (T_{3-l} u - u)_{\bar{x}_l x_l} + \sum_{l=1}^{2} \left[h v_l - T_{3-l} S_l^- (b_l u) \right]_{x_l} \\ &\quad + \sum_{l=1}^{2} \left[-(k_l - 1) u_{\bar{x}_l} \right]_{x_l}. \end{aligned}$$

Finally, we find the expression for the local truncation error

$$A_h \bar{z} = \eta_{1,\bar{x}_1 x_1} + \eta_{2,\bar{x}_2 x_2} + \mu_{1,x_1} + \mu_{2,x_2} + \xi_{1,x_1} + \xi_{2,x_2} \tag{34}$$

where

$$\eta_l = T_{3-l} u - \bar{u}, \quad x \in \omega_l^{\pm}, \tag{35}$$

$$\mu_l = h \bar{v}_l - T_{3-l} S_l^- (b_l u), \qquad \xi_l = -(k_l - 1) \bar{u}_{\bar{x}_l}, \quad x \in \omega_l^+, \tag{36}$$

where $\bar{v}$ is gotten by replacing u with $\bar{u}$ in formulas (9) and (11).

Let us introduce the solution of the following auxiliary discrete problem

$$\begin{aligned} A_h^T w &= \bar{z}, \quad \text{in } \omega, \\ w &= 0, \quad \text{on } \gamma. \end{aligned} \tag{37}$$

Note that similarly to the Aubin-Nitsche "trick", w is a solution of a discrete second order problem with right-hand side the error $\bar{z}(x)$ of the method. Obviously,

$$(A_h \bar{z}, w) = (A_h^T w, \bar{z}) = (\bar{z}, \bar{z}) = \|\bar{z}\|_{0,\omega}^2. \tag{38}$$

On the other hand from (34) we get

$$\begin{aligned} (A_h \bar{z}, w) &= \sum_{l=1}^{2} \left[(\eta_{l,\bar{x}_l x_l}, w) + (\mu_{l,x_l}, w) + (\xi_{l,x_l}, w) \right] \\ &= \sum_{l=1}^{2} (\eta_l, w_{\bar{x}_l x_l}) - \sum_{l=1}^{2} \{ (\mu_l, w_{\bar{x}_l}]_l + (\xi_l, w_{\bar{x}_l}]_l \} \\ &\le \sum_{l=1}^{2} (\|\eta_l\|_{0,\omega} + \|\mu_l]|_l + \|\xi_l]|_l)(\|w_{\bar{x}_l x_l}\|_{0,\omega} + \|w_{\bar{x}_l}]|_l). \end{aligned} \tag{39}$$

To complete the proof of the a priori estimate we need the following lemma.

LEMMA 3.6. *Let $\underline{b} \in \left(W_\infty^1\right)^2$. Then for the error $\bar{z}(x) = y(x) - \bar{u}(x)$, $x \in \omega$ of all considered schemes and the solution w of the problem (37) the inequalities are valid:*

$$\|w\|_{2,\omega} \le C_1 \|A_h^{(2)} w\|_{0,\omega} \le C_2 \|\bar{z}\|_{0,\omega} \tag{40}$$

for sufficiently small h.

Proof. Using the definition of $A_h^{(2)}$ and the triangle inequality we get

$$\begin{aligned}
\|A_h^{(2)} w\|_{0,\omega} &= \|[k_1 w_{\bar{x}_1}]_{x_1} + [k_2 w_{\bar{x}_2}]_{x_2}\|_{0,\omega} \\
&= \left\| \left[\left(1 + C_1(x)h^2\right) w_{\bar{x}_1}\right]_{x_1} + \left[\left(1 + C_2(x)\right) w_{\bar{x}_2}\right]_{x_2} \right\|_{0,\omega} \\
&\ge \|w_{\bar{x}_1 x_1} + w_{\bar{x}_2 x_2}\|_{0,\omega} \\
&\quad - h^2 \|C_{1,x_1} w_{x_1} + C_1 w_{\bar{x}_1 x_1} + C_{2,x_2} w_{x_2} + C_2 w_{\bar{x}_2 x_2}\|_{0,\omega} \\
&\ge \|w_{\bar{x}_1 x_1} + w_{\bar{x}_2 x_2}\|_{0,\omega} - D_2 h^2 \|w\|_{2,\omega}.
\end{aligned}$$

Here $k_l = 1$, $C_l = 0$, $l = 1, 2$ for the **UDS** and

$$k_l = 1 + C_l(x)h^2, \quad C_l(x) \sim b_l^2(x), \quad l = 1, 2,$$

otherwise. We also use the fact that C_1, C_2 and C_{1,x_1}, C_{2,x_2} are bounded.

Finally using the equivalence of $\|w_{\bar{x}_1 x_1} + w_{\bar{x}_2 x_2}\|_{0,\omega}$ and $\|w\|_{2,\omega}$ in the space D^0 we obtain

$$\left\|A_h^{(2)}\right\|_{0,\omega} \ge \left(D_1 - D_2 h^2\right) \|w\|_{2,\omega},$$

where D_1 and D_2 are positive constants. Hence for sufficiently small h the lower bound in (40) is proved.

An upper estimate for $\left\|A_h^{(2)}\right\|_{0,\omega}$ is derived by using the standard a priori estimate in $W_2^1(\omega)$, $\|w\|_{1,\omega} \le C\|\bar{z}\|_{0,\omega}$. Then

$$\begin{aligned}
\left\|A_h^{(2)} w\right\|_{0,\omega} &= \left\|A_h^{(2)^T} w\right\|_{0,\omega} = \left\|\left(A_h - A_h^{(1)}\right)^T w\right\|_{0,\omega} \\
&\le \|A_h^T w\|_{0,\omega} + \left\|A_h^{(1)^T} w\right\|_{0,\omega} \le \|\bar{z}\|_{0,\omega} + C\|w\|_{1,\omega} \le C\|\bar{z}\|_{0,\omega}.
\end{aligned}$$

□

Remark 3.2. Lemma 3.6 is actually a discrete regularity result in $W_2^2(\omega)$ (cf., Hackbusch [10]):

$$\|w\|_{2,\omega} \le C\|\bar{z}\|_{0,\omega}.$$

Then (38) and (39) yield

$$\|\bar{z}\|_{0,\omega}^2 = (A_h \bar{z}, w) \le C \sum_{l=1}^{2} \left(\|\eta_l\|_{0,\omega} + \|\mu_l]|_l + \|\xi_l]|_l\right) \|\bar{z}\|_{0,\omega}.$$

Thus, we have proved the following a priori estimate.

LEMMA 3.7. *The error $\bar{z}(x) = y(x) - \bar{u}(x)$, $x \in \omega$ of all considered finite difference schemes satisfies the a priori estimate,*

$$\|\bar{z}\|_{0,\omega} \le C \sum_{l=1}^{2} \left(\|\eta_l\|_{0,\omega} + \|\mu_l]|_l + \|\xi_l]|_l\right),$$

where the components η_l, μ_l and ξ_l, $l = 1, 2$ of the local truncation error are defined by (35) and (36). The constant C does not depend on h or $\bar{z}$.

Now we are ready to prove the following basic lemma.

LEMMA 3.8. *If the solution u of the problem (1) with constant coefficient $a(x)$ is $H^m(\Omega)$-regular, $1 < m \le 2$, then the components of the local truncation error η_l and μ_l, $l = 1, 2$, defined by (35) and (36), respectively, satisfy the following estimates:*

(i)

$$\|\eta_l\|_{0,\omega} \le Ch^m \|u\|_{m,\Omega},$$

(ii)

$$\|\mu_l]|_l \le \begin{cases} Ch^m \|b_l\|_{1,\infty,\Omega} \|u\|_{m,\Omega}, & \text{for } \textbf{MUDS} \text{ and } \textbf{IDS}, \\ C\left(h|b_l|_{0,\infty,\Omega}|u|_{1,\Omega} + h^m \|b_l\|_{1,\infty,\Omega} \|u\|_{m,\Omega}\right) & \text{for } \textbf{UDS}, \end{cases}$$

(iii)

$$\|\xi_l]|_l \le \begin{cases} Ch^2 \|u\|_{m,\Omega}, & \text{for } \textbf{MUDS} \text{ and } \textbf{IDS} \\ 0, & \text{for } \textbf{UDS}, \end{cases}$$

Proof. Consider $e_{i,j} = \{(x_1, x_2) : x_{1,i-1} \le x_1 \le x_{1,i+1}, x_{2,j-1} \le x_2 \le x_{2,j+1}\}$. We begin with **UDS**. To obtain (i) we rewrite (35) in the form

$$\eta_1 = u(x_{1,i}, x_{2,j}) - \int_{-1}^{1} (1 - |s_2|) u(x_{1,i}, x_{2,j} + s_2 h)\, ds_2.$$

It suffices to prove the estimate for $x \in \omega$ because by construction $\eta_l = 0$ on $\omega_l^{\pm}$. We have that η_1 is a linear functional of $u(x)$, bounded for $u \in H^m(\Omega)$, $1 < m \le 2$. This functional vanishes for all polynomial of first degree. Therefore, by the Bramble-Hilbert lemma argument we get

$$|\eta_1(x)| \le Ch^{m-1} |u|_{m,e}, \quad 1 < m \le 2,$$

(41)
$$\|\eta_1\|_{0,\omega} = \left(\sum_{x \in \omega} \eta_1^2(x) h^2 \right)^{1/2} \le Ch^m |u|_{m,e}.$$

We note that in this case $\xi_1(x) \equiv 0$. Now, let us take the component $\eta_1(x)$ for the **MUDS** and the **IDS**. In both schemes the coefficients $\tilde{k}_1(x)$ and $\gamma_1(x)$ are perturbations of the coefficient $k_1(x) \equiv 1$ of the **UDS** with a term of order $O(h^2)$. More precisely,

$$\tilde{k}_1(x) = \frac{1}{1 + |b_1(x)h/2|} + \frac{|b_1(x)|h}{2} = 1 + C_1 h^2, \qquad (\textbf{MUDS})$$

and

$$\gamma_1(x) = \frac{b_1(x)h}{2} \coth\left(\frac{b_1(x)h}{2} \right) = 1 + \tilde{C}_1 h^2, \qquad (\textbf{IDS}).$$

Since

$$\xi_1(x) = -(k_1(x) - 1)\bar{u}_{\bar{x}_1} = -C_1 h^2 \frac{[\bar{u}_{i,j} - \bar{u}_{i,j-1}]}{h},$$

we have

$$|\xi(x)| \le Ch(|u|_{1,e} + h^{m-1}|u|_{m,e}), \quad 1 < m \le 2$$

for the interior points and hence

$$\|\xi_1]|_1 \le Ch^2\|u\|_{m,\Omega}, \qquad 1 < m \le 2.$$

For the boundary points we have $\xi_1(x) = -Ch[u_{i,j} - u_{i-1,j}] + Ch[u_{i,j} - \bar{u}_{i,j}]$ and the second term is estimated with the approach used in the proof of Theorem 3.9.

For the second component $\mu_1(x)$ we proceed in the same way as in Lemma 3.3. First, we need the equality (see [21]):

$$\begin{aligned} T_2 S_1^-(b_1 u)(x_{1,i}, x_{2,j}) \\ = \int_{-1}^{1} (1-|s_2|) \left[\int_{-1}^{0} b_1(x_{1,i}+s_1 h, x_{2,j}+s_2 h) u(x_{1,i}+s_1 h, x_{2,j}+s_2 h)\, ds_1 \right] ds_2. \end{aligned}$$

Now, let us consider the component for the **MUDS** and **IDS**:

$$\mu_1(x) = \frac{b_{1,i-1/2,j}}{2}[\bar{u}_{i,j} + \bar{u}_{i-1,j}] - T_2 S_1^-(b_1 u)(x_{1,i}, x_{2,j}).$$

We can represent μ_1 in the following way

$$\mu_1(x) = b_{1,i-1/2,j}\, p(u) - c(b_1, u) + u_{i,j}\, q(b_1)$$

where

$$\begin{aligned} p(u) &= \frac{[\bar{u}_{i,j} + \bar{u}_{i-1,j}]}{2} - \int_{-1}^{1} (1-|s_2|) \left[\int_{-1}^{0} u(x_{1,i}+s_1 h, x_{2,j}+s_2 h)\, ds_1 \right] ds_2, \\ c(b_1, u) &= \int_{-1}^{1} (1-|s_2|) \left[\int_{-1}^{0} u(x_{1,i}+s_1 h, x_{2,j}+s_2 h) - \bar{u}_{i,j} \right] \\ &\quad \cdot \left[b_1(x_{1,i}+s_1 h, x_{2,j}+s_2 h) - b_{1,i-1/2,j} \right] ds_1\, ds_2 \end{aligned}$$

and

$$\begin{aligned} q(b_1) &= b_{1,i-1/2,j} \\ &\quad - \int_{-1}^{1} (1-|s_2|) \left[\int_{-1}^{0} b_1(x_{1,i}+s_1 h, x_{2,j}+s_2 h) - b_{1,i-1/2,j}\, ds_1 \right] ds_2. \end{aligned}$$

We have the estimates:

$$\begin{aligned} |p(u)| &\le Ch^{m-1}|u|_{m,e}, \qquad 1 < m \le 2, \\ |c(b_1, u)| &\le Ch|b_1|_{1,\infty,e}|u|_{1,e}, \\ |q(u)| &\le Ch|b_1|_{1,\infty,e}, \end{aligned}$$

Hence

$$|\mu_1(x)| \le Ch^{m-1}\|b_1\|(|u|_{m,e} + h^{2-m}(|u|_{1,e} + |u|_{0,\infty,e})).$$

For **UDS** we have to add the error of the term $-|b_{1,i,j}|u_{\bar{x}_1}$ which is $h\left(|b_1|_{0,\infty,e}(|u|_{1,e} + h^{m-1}|u|_{m,e})\right)]$. Combining the above results we obtain the assertions of the lemma. ☐

Now we can prove the main result in this subsection.

THEOREM 3.9. *If the solution of problem (1) is H^m-regular, $1 < m \le 2$ then:*

(i) *the* **MUDS** *and* **IDS** *defined by (14), (9), (16) and (9) have $O(h^m)$ rate of convergence in the L^2-discrete norm, i.e.,*

$$\|y-u\|_{0,\omega} \le Ch^m \left[(1+\|b_1\|_{1,\infty,\Omega} + \|b_2\|_{1,\infty,\Omega}) \|u\|_{m,\Omega} + \|g\|_{m-\frac{1}{2},\Gamma} \right].$$

(ii) *the* **UDS** *defined by (7) and (11) has at most first order of convergence in the L^2-discrete norm, i.e.,*

$$\|y-u\|_{0,\omega} \le Ch\left(|b_1|_{0,\infty,\Omega} + |b_2|_{0,\infty,\Omega}\right)|u|_{1,\Omega} \\ +Ch^m \left[(1+\|b_1\|_{1,\infty,\Omega} + \|b_2\|_{1,\infty,\Omega}) \|u\|_{m,\Omega} + \|g\|_{m-\frac{1}{2},\Gamma} \right].$$

Proof. We have $\|y-u\|_{0,\omega} \le \|y-\bar{u}\|_{0,\omega} + \|u-\bar{u}\|_{0,\omega}$. From Lemma 3.6 and Lemma 3.7 we get immediately the estimate for $\|y-\bar{u}\|_{0,\omega}$. To find the upper bound of the second term,

$$\|u-\bar{u}\|^2_{0,\omega} = \sum_{l=1}^{2} \sum_{\gamma_l^\pm} h^2 \left(T_{3-l}g(x) - g(x)\right)^2,$$

we observe that we can consider $T_{3-l}g - g$ as a linear functional of g which is bounded in $H^{m-1/2}(\Gamma)$ and vanishes for all polynomials of first degree. Then $|T_{3-l}g - g| \le Ch^{m-1}\|g\|_{m-1/2,e_\gamma}$ where $e_\gamma = (x_l - h, x_l + h)$, which shows that $\|u-\bar{u}\|_{0,\omega} \le Ch^m\|g\|_{m-1/2,\Gamma}$. ☐

Remark 3.3. The technique used in §3.1 and §3.2 directly gives the same estimates for the **CDS** as for **MUDS** and **IDS**, when this scheme is stable, i.e., when (10) holds.

4. Numerical results. In this section on the basis of model test examples we study the error behavior of our three schemes (**UDS**, **MUDS**, and **IDS**) in both H^1 and L^2 discrete norms.

We consider

$$\begin{cases} \operatorname{div}(-\varepsilon\nabla u(x,y) + \underline{b}(x,y)u(x,y)) = f(x,y), & \text{in } \Omega \\ u(x,y) = 0, & \text{on } \Gamma, \end{cases} \tag{42}$$

and for velocity vector $\underline{b}$ we choose

$$b_1 = -(1-x\cos\alpha)\cos\alpha, \qquad b_2 = -(1-y\sin\alpha)\sin\alpha,$$

where the angle is $\alpha = 15^0$.

Problem 1. $f(x,y)$ is chosen such that the solution is

$$u(x,y) = x(1-x)y(1-y)e^{d(x+2y)}, \quad \text{for } d=0 \text{ or } d=1.$$

In Tables 1–6 we display the error for smooth solutions without boundary layer behavior. In the first and the second rows we show the $L^2(\omega)$ and $H^1(\omega)$-norms of the error $z = y - u$ and the "numerical" rate of convergence is β, i.e., h^β. Our computational experiments clearly show that **MUDS** and **IDS** exhibit a second order of convergence both in L^2 and H^1-norms for problems with moderate convection (i.e., not too small $\varepsilon > 0$); the factor β is in the range of 1.822–1.995, correspondingly. For these problems **UDS** is only a first order accurate: β is between 0.947–1.260. For highly dominating convection all schemes show about a first order of accuracy. The results for $\varepsilon = 10^{-2}$, 10^{-5} show that all considered schemes are stable.

TABLE 1
UDS, $\alpha = 15^0$, $d = 0$

$\epsilon \backslash N$		16	32	64	128	256
1	L^2	$0.389.10^{-1}$	$0.198.10^{-3}$	$0.100.10^{-3}$	$0.503.10^{-4}$	$0.252.10^{-4}$
	β	0.947	0.974	0.986	0.991	0.997
	H^1	$0.154.10^{-2}$	$0.859.10^{-3}$	$0.454.10^{-3}$	$0.233.10^{-3}$	$0.118.10^{-3}$
	β	0.699	0.842	0.920	0.962	0.982
10^{-2}	L^2	$0.149.10^{-1}$	$0.811.10^{-2}$	$0.425.10^{-2}$	$0.218.10^{-2}$	$0.110.10^{-2}$
	β	0.780	0.878	0.932	0.963	0.987
	H^1	$0.633.10^{-1}$	$0.462.10^{-1}$	$0.288.10^{-1}$	$0.163.10^{-1}$	$0.868.10^{-2}$
	β	0.298	0.454	0.682	0.821	0.909
10^{-5}	L^2	$0.233.10^{-1}$	$0.135.10^{-1}$	$0.737.10^{-2}$	$0.388.10^{-2}$	$0.200.10^{-2}$
	β	0.667	0.787	0.873	0.926	0.956
	H^1	$0.110.10^{0}$	$0.779.10^{-1}$	$0.505.10^{-1}$	$0.305.10^{-1}$	$0.180.10^{-1}$
	β	0.338	0.498	0.625	0.727	0.761

TABLE 2
MUDS, $\alpha = 15^0$, $d = 0$

$\epsilon \backslash N$		16	32	64	128	256
1	L^2	$0.213.10^{-4}$	$0.567.10^{-5}$	$0.146.10^{-5}$	$0.372.10^{-6}$	$0.940.10^{-7}$
	β	1.822	1.909	1.957	1.973	1.985
	H^1	$0.818.10^{-4}$	$0.239.10^{-4}$	$0.649.10^{-5}$	$0.169.10^{-5}$	$0.431.10^{-6}$
	β	1.559	1.775	1.881	1.941	1.971
10^{-2}	L^2	$0.102.10^{-1}$	$0.416.10^{-2}$	$0.148.10^{-2}$	$0.468.10^{-3}$	$0.134.10^{-3}$
	β	1.100	1.294	1.491	1.661	1.804
	H^1	$0.436.10^{-1}$	$0.240.10^{-1}$	$0.101.10^{-1}$	$0.347.10^{-2}$	$0.104.10^{-2}$
	β	0.609	0.861	1.249	1.541	1.738
10^{-5}	L^2	$0.233.10^{-1}$	$0.135.10^{-1}$	$0.736.10^{-2}$	$0.387.10^{-2}$	$0.198.10^{-2}$
	β	0.667	0.787	0.875	0.927	0.967
	H^1	$0.110.10^{0}$	$0.784.10^{-1}$	$0.511.10^{-1}$	$0.309.10^{-1}$	$0.174.10^{-1}$
	β	0.338	0.489	0.618	0.728	0.820

TABLE 3
IDS, $\alpha = 15^0$, $d = 0$

$\epsilon \backslash N$		16	32	64	128	256
1	L^2	$0.169.10^{-4}$	$0.451.10^{-5}$	$0.116.10^{-5}$	$0.295.10^{-6}$	$0.740.10^{-7}$
	β	1.840	1.906	1.959	1.975	1.995
	H^1	$0.650.10^{-4}$	$0.189.10^{-4}$	$0.511.10^{-5}$	$0.133.10^{-5}$	$0.338.10^{-6}$
	β	1.578	1.782	1.887	1.942	1.976
10^{-2}	L^2	$0.860.10^{-2}$	$0.288.10^{-2}$	$0.816.10^{-3}$	$0.213.10^{-3}$	$0.540.10^{-4}$
	β	1.253	1.578	1.819	1.937	1.980
	H^1	$0.366.10^{-1}$	$0.166.10^{-1}$	$0.557.10^{-2}$	$0.158.10^{-2}$	$0.420.10^{-3}$
	β	0.786	1.141	1.575	1.818	1.911
10^{-5}	L^2	$0.233.10^{-1}$	$0.133.10^{-1}$	$0.736.10^{-2}$	$0.387.10^{-2}$	$0.198.10^{-2}$
	β	0.667	0.787	0.875	0.927	0.967
	H^1	$0.110.10^{0}$	$0.770.10^{-1}$	$0.511.10^{-1}$	$0.309.10^{-1}$	$0.175.10^{-1}$
	β	0.338	0.515	0.592	0.728	0.820

Table 4
UDS, $\alpha = 15^0$, $d = 1$

$\epsilon \backslash N$		16	32	64	128	256
1	L^2	$0.232.10^{-2}$	$0.102.10^{-2}$	$0.470.10^{-3}$	$0.223.10^{-3}$	$0.113.10^{-3}$
	β	1.260	1.186	1.118	1.040	0.981
	H^1	$0.930.10^{-2}$	$0.451.10^{-2}$	$0.218.10^{-2}$	$0.106.10^{-2}$	$0.545.10^{-3}$
	β	0.984	1.044	1.049	1.040	0.960
10^{-2}	L^2	$0.486.10^{-1}$	$0.267.10^{-1}$	$0.141.10^{-1}$	$0.725.10^{-2}$	$0.368.10^{-2}$
	β	0.769	0.864	0.921	0.960	0.978
	H^1	$0.228.10^{0}$	$0.170.10^{0}$	$0.110.10^{0}$	$0.637.10^{-1}$	$0.345.10^{-1}$
	β	0.291	0.423	0.628	0.788	0.885
10^{-5}	L^2	$0.719.10^{-1}$	$0.408.10^{-1}$	$0.219.10^{-1}$	$0.114.10^{-1}$	$0.585.10^{-2}$
	β	0.690	0.817	0.898	0.942	0.963
	H^1	$0.329.10^{0}$	$0.215.10^{0}$	$0.138.10^{0}$	$0.847.10^{-1}$	$0.496.10^{-1}$
	β	0.536	0.614	0.640	0.704	0.772

Table 5
MUDS, $\alpha = 15^0$, $d = 1$

$\epsilon \backslash N$		16	32	64	128	256
1	L^2	$0.768.10^{-3}$	$0.204.10^{-4}$	$0.526.10^{-4}$	$0.134.10^{-4}$	$0.337.10^{-5}$
	β	1.830	1.911	1.956	1.973	1.991
	H^1	$0.302.10^{-2}$	$0.900.10^{-3}$	$0.246.10^{-3}$	$0.646.10^{-4}$	$0.165.10^{-4}$
	β	1.531	1.747	1.871	1.929	1.969
10^{-2}	L^2	$0.291.10^{-1}$	$0.117.10^{-1}$	$0.415.10^{-2}$	$0.130.10^{-2}$	$0.374.10^{-3}$
	β	1.101	1.315	1.495	1.675	1.797
	H^1	$0.132.10^{0}$	$0.721.10^{-1}$	$0.306.10^{-1}$	$0.106.10^{-1}$	$0.319.10^{-2}$
	β	0.677	0.872	1.236	1.529	1.732
10^{-5}	L^2	$0.719.10^{-1}$	$0.407.10^{-1}$	$0.218.10^{-1}$	$0.114.10^{-1}$	$0.576.10^{-2}$
	β	0.690	0.821	0.901	0.935	0.985
	H^1	$0.329.10^{0}$	$0.216.10^{0}$	$0.138.10^{0}$	$0.840.10^{-1}$	$0.432.10^{-1}$
	β	0.536	0.607	0.646	0.716	0.822

Table 6
IDS, $\alpha = 15^0$, $d = 1$

$\epsilon \backslash N$		16	32	64	128	256
1	L^2	$0.752.10^{-3}$	$0.200.10^{-3}$	$0.515.10^{-4}$	$0.131.10^{-4}$	$0.330.10^{-5}$
	β	1.828	1.911	1.957	1.975	1.989
	H^1	$0.296.10^{-2}$	$0.883.10^{-3}$	$0.242.10^{-3}$	$0.634.10^{-4}$	$0.162.10^{-4}$
	β	1.532	1.754	1.867	1.932	1.968
10^{-2}	L^2	$0.227.10^{-1}$	$0.755.10^{-2}$	$0.212.10^{-2}$	$0.553.10^{-3}$	$0.138.10^{-3}$
	β	1.277	1.588	1.832	1.939	2.002
	H^1	$0.100.10^{0}$	$0.454.10^{-1}$	$0.153.10^{-1}$	$0.443.10^{-2}$	$0.116.10^{-2}$
	β	0.880	1.139	1.569	1.788	1.933
10^{-5}	L^2	$0.718.10^{-1}$	$0.407.10^{-1}$	$0.218.10^{-1}$	$0.114.10^{-1}$	$0.576.10^{-2}$
	β	0.690	0.819	0.901	0.935	0.985
	H^1	$0.329.10^{0}$	$0.216.10^{0}$	$0.138.10^{0}$	$0.840.10^{-1}$	$0.475.10^{-1}$
	β	0.536	0.607	0.646	0.716	0.822

TABLE 7
UDS, $\alpha = 15^0$, *boundary layer*

$\epsilon \backslash N$		16	32	64	128	256
10^{-3}	L^2	$0.427.10^{-2}$	$0.252.10^{-2}$	$0.147.10^{-2}$	$0.894.10^{-3}$	$0.594.10^{-3}$
	β	0.622	0.761	0.778	0.717	0.590
	H^1	$0.414.10^{-1}$	$0.276.10^{-1}$	$0.172.10^{-1}$	$0.109.10^{-1}$	$0.744.10^{-2}$
	β	0.332	0.585	0.682	0.658	0.551
10^{-4}	L^2	$0.393.10^{-2}$	$0.225.10^{-2}$	$0.122.10^{-2}$	$0.645.10^{-3}$	$0.342.10^{-3}$
	β	0.641	0.805	0.883	0.920	0.920
	H^1	$0.365.10^{-1}$	$0.233.10^{-1}$	$0.134.10^{-1}$	$0.733.10^{-2}$	$0.396.10^{-2}$
	β	0.380	0.648	0.798	0.870	0.888
10^{-5}	L^2	$0.319.10^{-2}$	$0.223.10^{-2}$	$0.119.10^{-2}$	$0.621.10^{-3}$	$0.318.10^{-3}$
	β	0.642	0.810	0.906	0.938	0.965
	H^1	$0.361.10^{-1}$	$0.229.10^{-1}$	$0.130.10^{-1}$	$0.700.10^{-2}$	$0.364.10^{-2}$
	β	0.387	0.657	0.817	0.893	0.943

TABLE 8
MUDS, $\alpha = 15^0$, *boundary layer*

$\epsilon \backslash N$		16	32	64	128	256
10^{-3}	L^2	$0.392.10^{-2}$	$0.224.10^{-2}$	$0.122.10^{-2}$	$0.682.10^{-3}$	$0.340.10^{-3}$
	β	0.640	0.807	0.877	0.839	0.652
	H^1	$0.364.10^{-1}$	$0.233.10^{-1}$	$0.135.10^{-1}$	$0.790.10^{-2}$	$0.525.10^{-2}$
	β	0.381	0.644	0.787	0.773	0.415
10^{-4}	L^2	$0.390.10^{-2}$	$0.223.10^{-2}$	$0.119.10^{-2}$	$0.618.10^{-3}$	$0.315.10^{-3}$
	β	0.643	0.806	0.906	0.945	0.972
	H^1	$0.361.10^{-1}$	$0.228.10^{-1}$	$0.130.10^{-1}$	$0.696.10^{-2}$	$0.361.10^{-2}$
	β	0.384	0.663	0.811	0.901	0.947
10^{-5}	L^2	$0.390.10^{-2}$	$0.223.10^{-2}$	$0.1191.10^{-2}$	$0.618.10^{-3}$	$0.315.10^{-3}$
	β	0.643	0.806	0.906	0.945	0.972
	H^1	$0.360.10^{-1}$	$0.229.10^{-1}$	$0.130.10^{-1}$	$0.696.10^{-2}$	$0.361.10^{-2}$
	β	0.388	0.653	0.806	0.901	0.947

Problem 2.

$$f(x, y) = \nabla \cdot (\underline{b} u_0), \quad u_0(x, y) = x^2 y(1 - y).$$

Here u_0 is the solution of equation (42) when $\varepsilon = 0$. In Tables 7–9 we show $\|y - u_0\|_{0,\bar{\omega}}$, where $\bar{\omega}$ is a grid in $\bar{\Omega} = [0, 7/8] \times [0, 1]$, i.e., away from the boundary layer. This gives us reasonable information since for small ε the function u_0 is close to the exact solution of problem 2, except within the boundary layer. In fact we have an estimate $\|u - u_0\|_{0,\bar{\Omega}} \leq C\varepsilon$, and when ε is significantly less than h we may use u_0 instead of the unknown exact solution u in $\bar{\Omega}$. In case h and ε are of the same order, this is inappropriate as is shown by Tables 7–9, $h = 1/256$ and $\varepsilon = 10^{-3}$. Our experiments show very weak dependence of the numerical solution with respect to $\varepsilon \to 0$ in $\bar{\Omega}$. This means that if we use a more sophisticated method near the boundary layer, e.g., local refinement, defect-correction, in combination with the proposed schemes outside the layer we can get better results.

TABLE 9
IDS, $\alpha = 15^0$, *boundary layer*

$\epsilon\backslash N$		16	32	64	128	256
10^{-3}	L^2	$0.390.10^{-2}$	$0.222.10^{-2}$	$0.116.10^{-2}$	$0.594.10^{-3}$	$0.376.10^{-3}$
	β	0.643	0.813	0.936	0.966	0.660
	H^1	$0.361.10^{-1}$	$0.227.10^{-1}$	$0.124.10^{-1}$	$0.665.10^{-2}$	$0.454.10^{-2}$
	β	0.384	0.669	0.872	0.899	0.551
10^{-4}	L^2	$0.390.10^{-2}$	$0.223.10^{-2}$	$0.119.10^{-2}$	$0.619.10^{-3}$	$0.314.10^{-3}$
	β	0.643	0.806	0.906	0.943	0.979
	H^1	$0.360.10^{-1}$	$0.229.10^{-1}$	$0.130.10^{-1}$	$0.697.10^{-2}$	$0.360.10^{-2}$
	β	0.384	0.653	0.817	0.899	0.953
10^{-5}	L^2	$0.390.10^{-2}$	$0.223.10^{-2}$	$0.119.10^{-2}$	$0.619.10^{-3}$	$0.315.10^{-3}$
	β	0.643	0.806	0.906	0.943	0.975
	H^1	$0.360.10^{-1}$	$0.229.10^{-1}$	$0.130.10^{-1}$	$0.697.10^{-2}$	$0.361.10^{-2}$
	β	0.384	0.653	0.817	0.899	0.949

5. Concluding remarks.

We studied finite difference approximations of convection-diffusion problems on square meshes with step size h. The extension on rectangular meshes with step sizes h_1 and h_2 in the directions x_1 and x_2 is almost immediate and all obtained results will be true with $h^2 = h_1^2 + h_2^2$ as long as the ratio h_1/h_2 is bounded from above and by constants when $h \to 0$.

Extension of some of the results can be made to rectangular nonuniform grids. However, this requires a different technique (see e.g., Weiser and Wheeler [25]) that is beyond the scope of this paper.

For more general domains, similar results can be accomplished using the technique described in Samarskii, Lazarov and Makarov [21, Chapter III, p. 123] with introduction of new notation and considering a few different cases.

Although our theoretical results are for the diffusion coefficient ε comparable to the convection coefficient $\underline{b}$, the numerical experiments show that the constructed schemes are very robust with respect to ε. For very small ε (down to $\varepsilon = 10^{-5}$) the schemes produce reasonable results and the convergence rates are of first order.

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