AN INVERSE PROBLEM FROM PETROLEUM AND HYDROLOGY APPLICATIONS

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ISC–94–01–MATH

1. Problem Formulation.

1.1. The model and the inverse problem. The steady-state flow of a single fluid through the core sample is modeled as follows. The interior of the sample is represented by an open set $\Omega \subset \mathbf{R}^n$, $n = 2$ or 3, with the boundary of Ω denoted as $\partial\Omega$. When $n = 3$, Ω is the cylinder with base radius 1 and central axis having endpoints $(0,0)$ and $(1,0)$. When $n = 2$, Ω is the square formed by the intersection of this cylinder with the xy plane.

Fluid is forced into the core sample along the face $x = 0$, which is denoted Γ_{in}, with the pressure there held constant at p_{in}. The fluid is extracted through the face $x = 1$, which is denoted Γ_{out}, with the pressure there held constant at p_{out}, with $p_{out} < p_{in}$. No flow is permitted on the remainder of the boundary, which is denoted Γ_N.

The flow through the sample is described by the following equation for the pressure potential p^*:

$$\nabla \cdot (a^* \nabla p^*) = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$(a^* \nabla p^*) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N, \tag{1.2}$$

$$p^* = p_{in} \quad \text{on } \Gamma_{in}, \tag{1.3}$$

$$p^* = p_{out} \quad \text{on } \Gamma_{out}. \tag{1.4}$$

The function a^* represents the absolute permeability of the medium. It is assumed here that the medium is isotropic, and that it has no "voids" of infinite permeability and no "obstacles" of zero permeability. These assumptions are built into the mathematical model by the requirement that a^* is a scalar function and

$$a^* \in \mathcal{A} \stackrel{\text{def}}{=} \{a \in L^\infty(\Omega) \,|\, m_a \le a^*(\mathbf{x}) \le M_a \text{ a.e. in } \Omega\} \tag{1.5}$$

for two positive constants m_a and M_a.

Highly accurate measurements of the fluid velocity, $\psi^* \stackrel{\text{def}}{=} -a^* \nabla p^*$ are available by way of NMR imaging techniques. The goal is to estimate $a^*(x)$ based on this velocity data and the model (1.1)–(1.4).

1.2. Functional-analytic considerations. For $\mathbf{u}$ and $\mathbf{v}$ in $[L^2(\Omega)]^n$, the inner product will be denoted $\langle\, \mathbf{u}, \mathbf{v} \,\rangle$. That is,

$$\langle\, bfu, \mathbf{v} \,\rangle \stackrel{\text{def}}{=} \int_\Omega \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \, d\Omega.$$

If $a^* \in \mathcal{A}$ and is related to p^* through (1.1)–(1.4), then the observed flux ψ^* lies in the divergence-free subspace of $[L^2(\Omega)]^n$. Recall that any $[L^2(\Omega)]^n$ vector field may be expressed as the sum gradient vector field and a divergence-free field. In particular, in connection with the essential boundary conditions (1.3) and (1.4), define

$$H_e^1(\Omega) \stackrel{\text{def}}{=} \{\phi \in H^1(\Omega) \mid \phi|_{\Gamma_{in}} = \phi|_{\Gamma_{out}} = 0\}.$$

Then,

$$G(\Omega) \stackrel{\text{def}}{=} \{\sigma \in [L^2(\Omega)]^2 \mid \sigma = \nabla\phi \text{ for some } \phi \in H_e^1(\Omega)\},$$

which is the "space of gradient fields". The divergence-free subspace of $[L^2(\Omega)]^n$ is the orthogonal complement $G^\perp$ of G.

For each given $a \in \mathcal{A}$ there is a unique $p \in H^1(\Omega)$ which is related to a through the state equations (1.1)–(1.4). It is convenient to express this p as the sum of a term from G and a term from $G^\perp$. To this end, define p_0 by

$$p_0(x, y, z) = p_{in} + x(p_{out} - p_{in}),$$

so that p_0 satisfies the boundary conditions (1.3) and (1.4). Then for any $a \in \mathcal{A}$ the corresponding p can be written as $p = \phi + p_0$ for some $\phi \in H_e^1(\Omega)$ which satsifies the homogeneous boundary conditions corresponding to (1.2)–(1.4) and

$$\nabla \cdot (a\nabla\phi) = -\nabla \cdot (a\nabla p_0). \tag{1.6}$$

It will also be useful to introduce the idea of the "permeability to velocity" map $\mathcal{A} \mapsto \mathcal{R}$. The "action" of $\mathcal{F}$ is given by $\mathcal{F}(a) = \psi$ where $\psi = -a\nabla p$ and p satisfies (1.1)–(1.4) with a replacing a^*. In terms of this map, the inverse problem may be stated:

> Given $\psi^* \in G^\perp$, (attempt to) find a permeability $a^* \in L^\infty(\Omega)$ such that

$$\psi^* = \mathcal{F}(a^*). \tag{1.7}$$

2. Nonuniqueness of Solutions. The main content of this section is a theorem which, for a given solution of the inverse problem, gives conditions for duplicate solutions. In the next section, these conditions will be shown to be nonvoid, and will also shed light on the proposed solution method.

THEOREM 1. *For a fixed ψ^* in the range of $\mathcal{F}$, suppose (1.7) holds for a given $a^* \in \mathcal{A}$. Let $\phi^* \in H^1_e(\Omega)$ be such that $p^* = \phi^* + p_0$, where p^* is related to a^* through (1.1)–(1.4). Suppose there is a nonconstant function δq which satsifies*

$$\delta q(\mathbf{x}) > 0 \quad \forall \mathbf{x} \in \bar{\Omega}, \tag{2.1}$$

$$(\nabla \delta q(\mathbf{x})) \times (\nabla p^*(\mathbf{x})) = \mathbf{0} \quad \forall \mathbf{x} \in \Omega. \tag{2.2}$$

Then the solution of the inverse problem (1.7) is nonunique.

Proof: Define $q^*(\mathbf{x}) = 1/(a^*(\mathbf{x}))$. Then clearly, $(q^*\psi^* - \nabla p_0) = \phi^* \in G$. Also, for some constant c to be defined later, let

$$\begin{aligned} \sigma &\stackrel{\text{def}}{=} c(\delta q)q^*\psi^* - \nabla p_0 \\ & \quad c(\delta q)\nabla p^* - \nabla p_0. \end{aligned} \tag{2.3}$$

The goal is to show that c can be chosen so that σ is also in G. Then this will then be shown to imply nonuniqueness.

First note that

$$\begin{aligned} \nabla \times \sigma(\mathbf{x}) &= \Big(c\nabla\delta q(\mathbf{x})\Big) \times \Big(q * (\mathbf{x})\psi^*(\mathbf{x})\Big) \;+\; c\delta q(\mathbf{x})\,\nabla \times \Big(q^*(\mathbf{x})\psi^*(\mathbf{x})\Big) \\ &= 0. \end{aligned}$$

This is because the first term is zero by (2.2), since $q^*\psi(\mathbf{x})$ is parallel to $\nabla p^*(\mathbf{x})$, and the second term is zero because the curl of a gradient, namely $q^*\psi^* = \nabla p^*$, is zero.

By classical arguements involving Stokes' Theorem, the fact that $\nabla \times \sigma \equiv 0$ means that σ is a conservative vector field and there is a $w \in H^1(\Omega)$ such that $\nabla w = \sigma$. For any $\mathbf{x} \in \bar{\Omega}$, any piecewise smooth oriented path $\Pi \subset \bar{\Omega}$ from $\mathbf{0}$ to $\mathbf{x}$ and any piecewise smooth parameterization $\mathbf{c}(t), t \in [0, t_f]$ of Π, $w(\mathbf{x})$ is given by

$$\begin{aligned} w(\mathbf{x}) &= \int_\Pi \sigma \cdot d\Pi \\ &= \int_0^{t_f} \sigma(\mathbf{c}(t) \cdot \mathbf{c}'(t)dt. \end{aligned} \tag{2.4}$$

It remains to be shown that $w \in H^1_e(\Omega)$ so that $\sigma \in G$. That is, w must be shown to satsify the homogeneous Dirichlet boundary conditions on Γ_{in} and Γ_{out}.

For any point $\mathbf{x} \in \Gamma_{in}$, let the path Π in (2.4) be chosen to lie completely in Γ_{in}. By (2.3) and the boundary conditions on p^* and p_0, it is clear that the tangential component of $\sigma(\mathbf{x})$ at any $\mathbf{x} \subset \Gamma_{in}$ is zero. Thus $w(\mathbf{x}) = 0$.

For any point $\mathbf{x} \in \Gamma_{out}$, let the path in (2.4) be chosen as

$$\Pi = \Pi_0 \cup \Pi_{out},$$

where Π_0 is the intersection of the x axis with $\bar{\Omega}$ and $\Pi_{out} \subset \Gamma_{out}$ connects the endpoint of Π_0 at $x = 1$ to $\mathbf{x}$. Then since all tangential components of $\boldsymbol{\sigma}$ along Γ_{out} are zero, $w(\mathbf{x})$ as given by (2.4) reduces to

$$w(\mathbf{x}) = \int_{\Pi_0} \Big(c\delta q \nabla p^* - \nabla p_0 \Big) \cdot d\Pi.$$

If c is chosen as

$$c \stackrel{\text{def}}{=} \frac{\int_{\Pi_0} (\nabla p_0) \cdot d\Pi}{\int_{\Pi_0} (\nabla p_0) \cdot d\Pi},$$

then $w(\mathbf{x}) = 0$ for $\mathbf{x} \in \Gamma_{out}$. Thus $w \in H^1_e(\Omega)$, so that $\boldsymbol{\sigma} \in G$.

The existence of w will now be shown to imply nonuniqueness. Since $\nabla w = c(\delta q) q^* \psi^* - \nabla p_0$ and since $q* = 1/a^*$ and $\nabla \psi^* = 0$, it follows that

$$\frac{a}{c\delta q} \nabla (w + p_0) = \psi,$$

and

$$\nabla \cdot \Big(\frac{a}{c\delta q} \nabla (w + p_0) \Big) = 0,$$

so that the pair $[\ (a/(c\delta q)),\ (w + p_0)\ ]$ is a solution of the inverse problem. This proves the nonuniqueness. ∎

3. Solution Methods. Given a ψ^* in the range of $\mathcal{F}$, we seek a^* and p^* which satisfy (1.1)–(1.4) with $a^* \nabla p^* = \psi^*$. As in the proof of Theorem 1, define q^* by $q^*(\mathbf{x}) \stackrel{\text{def}}{=} 1/a^*(\mathbf{x})$, so that the problem becomes that of finding q^* such that

$$(q^* \psi^* - p_0) = \nabla \phi^*$$

for some $\phi^* \in H^1_e(\Omega)$.

This suggests that the problem be posed as that of minimizing the following quadratic functional over an approriate pair of function spaces:

$$J(q, \phi) \stackrel{\text{def}}{=} \frac{1}{2} \int_\Omega |\nabla \phi + \nabla p_0 - q \psi^*|^2 \, d\Omega.$$

Taking the first variation of J with respect to q and ϕ leads to the first order necessary condtions

$$\langle \nabla\phi^* + \nabla p_0 - q^*\psi^*,\, \nabla v\rangle = 0 \quad \forall v \in H_e^1, \tag{3.1}$$

$$\langle \nabla\phi^* + \nabla p_0 - q^*\psi^*,\, u\psi\rangle = 0 \quad \forall u \in L^2 \tag{3.2}$$

Since (3.2) must hold for all $u \in L^2(\Omega)$, it implies that

$$q^*(\mathbf{x}) = \frac{\big(\nabla\phi^*(\mathbf{x}) + \nabla p_0(\mathbf{x})\big)\cdot\psi^*(\mathbf{x})}{|\psi^*(\mathbf{x})|^2}. \tag{3.3}$$

Replacing q^* by this expression in (3.1) yields

$$\langle \nabla\phi^* + \nabla p_0 - P_{\psi}(\nabla\phi^* + \nabla p_0),\, \nabla v\rangle = 0 \quad \forall v \in H_e^1. \tag{3.4}$$

Here, P_{ψ} is the operator for which, at each $\mathbf{x} \in \Omega$, $[P_{\psi}\sigma](\mathbf{x})$ is the orthogonal projection (in $\mathbf{R}^n$) of $\sigma(\mathbf{x})$ upon $\psi^*(\mathbf{x})$. It is an orthogonal projection in $[L^2(\Omega)]^n$.

Since (3.4) must hold for all $v \in H_e^1(\Omega)$, it is equivalent (via integration by parts) to

$$\nabla\cdot[(I - P_{\psi})\nabla\phi] = \nabla\cdot(P_{\psi}\nabla p_0), \tag{3.5}$$

and the boundary condtions are the homogeneous counterparts of (1.2)–(1.4).

This equation has the *form* of a standard elliptic PDE, where the left-hand side is of the form

$$\nabla\cdot(\,A(\mathbf{x})\nabla\phi(\mathbf{x})\,).$$

For example, in the case $n = 2$ and with $\psi^* = (psi^{(1)}, \psi^{(2)})^T$ and with the explicit dependence on $\mathbf{x}$ suppressed from the notation, the matrix A at $\mathbf{x}$ is given by

$$A = \begin{pmatrix} 1 - (\psi^{(1)})^2/|\psi|^2 & \psi^{(1)}\psi^{(2)}/|\psi|^2 \\ \psi^{(1)}\psi^{(2)}/|\psi|^2 & 1 - (\psi^{(2)})^2/|\psi|^2 \end{pmatrix}. \tag{3.6}$$

However, it is *not* strongly elliptic; at any fixed $\mathbf{x} \in \Omega$, $\xi^T A(\mathbf{x})\xi = 0$ for any ξ which is parallel to $\psi^*(\mathbf{x})$.

3.1. Relationship to Theorem 1. The fact that $A(\mathbf{x})$ is singular implies that(3.5) and the quadratic minimization problem are ill-posed. Now the question might arise as to whether it is "more" ill-posed than the original inverse problem, in the sense some duplicated solutions of (3.5) do not correspond to any duplicated solutions of (1.7). However, this is not the case. That is, the duplicate solutions of (3.5) correspond directly to δq which were discussed in Theorem 1. This is shown in the following theorem.

THEOREM 2. *Let ϕ^* satisfy (3.5) and let q^* be related to ϕ^* through (3.3). Then any $\delta\phi \in H_e^1(\Omega)$ such that $\phi^* + \delta\phi$ also satisfies (3.5) gives rise to a δq which satisfies the hypotheses of Theorem 1.*

Proof: If $(\phi^* + \delta^*)$ satisfies (3.5) and $\delta\phi \in H_e^1(\Omega)$, then $\nabla(\delta\phi)(\mathbf{x})$ is parallel to $\psi^*(\mathbf{x})$ at each $x \in \Omega$. Then $\delta\phi$ is related to ψ^* by

$$\nabla(\delta\phi)(\mathbf{x}) = \beta(\mathbf{x})\psi^*(\mathbf{x}) \tag{3.7}$$

some scalar function β. Then $\phi^* + \delta\phi$ satisfies (3.5) but gives rise through (3.3) to

$$q(\mathbf{x}) = q^*(\mathbf{x}) + \beta(\mathbf{x}). \tag{3.8}$$

Also,

$$\begin{aligned}\nabla \times \Big(\nabla(\delta\phi) \Big) = \mathbf{0} \;&=\; \nabla \times \Big(\beta\psi^* \Big) \\ &=\; \nabla \times \Big[\Big(\frac{\beta}{q^*}\Big) q^*\psi^* \Big] \\ &=\; \nabla\Big(\frac{\beta}{q^*}\Big) \times (q^*\psi^*)\end{aligned} \tag{3.9}$$

Define δq so that

$$\frac{\beta(\mathbf{x})}{q^*(\mathbf{x})} = \delta q(\mathbf{x}) - 1. \tag{3.10}$$

Then the q in (3.8) is also given by

$$q(\mathbf{x}) = (\delta q(\mathbf{x}))q^*(\mathbf{x}),$$

and (3.9) becomes

$$(\nabla\delta q(\mathbf{x})) \times (\nabla p^*(\mathbf{x})) = \mathbf{0},$$

which is (2.2). Also, by (3.10), δq satisfies (2.1). Thus this δq satisfies all of the hypothheses of Theorem 1. ∎

3.2. Regularization. By Theorem 2, the gradient of any "ill-posed component" $\delta\phi$ of the (3.5) is parallel to ψ^*. This suggests that the objective functional J should be regularized by a term which supresses these components. In particular, for $\alpha > 0$, define

$$J_\alpha(q,\phi) \stackrel{\text{def}}{=} \frac{1}{2}\int_\Omega |\nabla\phi + \nabla p_0 - q\psi^*|^2 \, d\Omega \; + \; \frac{\alpha}{2}\int_\Omega (\psi^* \cdot \nabla\phi)^2 \, d\Omega. \tag{3.11}$$

By reasoning similar to that leading to (3.5), the first-order necessary conditions for minima of J_α lead to

$$\nabla \cdot \left[\left(I + (\alpha - 1) P_\psi \right) \nabla\phi \right] = \nabla \cdot (P_\psi \nabla p_0). \tag{3.12}$$

In contrast to (3.5), this is a strongly elliptic self-adjoint PDE. The numerical investigation of the inverse problem reported below is based on a discretization of the equation.