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# THE APPROXIMATION OF THE PRESSURE BY A MIXED METHOD IN THE SIMULATION OF MISCIBLE DISPLACEMENT (*)

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Résumé. — *Le déplacement miscible d'un fluide incompressible par un autre dans un milieu poreux est gouverné par un système de deux équations, l'une elliptique pour la pression, et l'autre parabolique pour la concentration de l'un des fluides. La pression apparaît dans la concentration seulement par son champ de vitesses, et il est recommandé de choisir une méthode numérique qui approche directement la vitesse. La pression est approchée par une méthode d'éléments finis mixte et la concentration par une méthode de Galerkin usuelle. On obtient des estimations d'erreur optimales lorsque les écoulements extérieurs imposés sont distribués régulièrement. On propose une modification de la méthode mixte lorsque l'écoulement est localisé à des sources et des puits, et on établit la convergence à des taux réduits dans le cas particulier où la viscosité du mélange est indépendante de la concentration.*

Abstract. — *The miscible displacement of one incompressible fluid by another in a porous medium is governed by a system of two equations, one of elliptic form for the pressure and the other of parabolic form for the concentration of one of the fluids. The pressure appears in the concentration only through its velocity field, and it is appropriate to choose a numerical method that approximates the velocity directly. The pressure is approximated by a mixed finite element method and the concentration by a standard Galerkin method. Optimal order estimates are derived when the imposed external flows are smoothly distributed. A modification of the mixed method is proposed when the flow is located at sources and sinks (i.e., wells), and convergence is established at reduced rates in the special case when the viscosity of the mixture is independent of the concentration.*

## 1. INTRODUCTION

We shall consider the miscible displacement of one incompressible fluid by another in a reservoir $\Omega \subset \mathbb{R}^2$ of unit thickness and local elevation $z(x)$, $x \in \Omega$. The Darcy velocity of the fluid mixture is given by

$$u = -\frac{k(x)}{\mu(c)}(\nabla p - \gamma_0(c)\,\nabla z), \tag{1.1}$$

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where $p$ is the pressure, $k$ the permeability of the medium, $\mu$ the concentration-dependent viscosity, and $\gamma_0$ the density of the fluid. Incompressibility implies that

$$\nabla . u = q, \tag{1.2}$$

where $q = q(x, t)$ is the imposed external flow, positive for injection and negative for production. For convenience of notation, we shall write the pressure equation (1.2) in the slightly more general form

$$\nabla . u = - \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[ a_i(x, c)\left( \frac{\partial p}{\partial x_i} - \gamma_i(x, c) \right)\right] = q \tag{1.3}$$

for $x \in \Omega$ and $t \in J = [0, T]$. We shall impose the boundary condition

$$u . \nu = 0, \qquad x \in \partial\Omega, \quad t \in J, \tag{1.4}$$

where $\nu$ is the exterior normal to $\partial\Omega$. Compatibility requires that

$$(q, 1) = \int_\Omega q(x, t)\, dx = 0, \qquad t \in J. \tag{1.5}$$

The equation for the concentration can be put in the form [8, 9]

$$\phi(x) \frac{\partial c}{\partial t} + u . \nabla c - \nabla . (D \nabla c) = (\tilde{c} - c)\, q, \tag{1.6}$$

where $\phi$ is the porosity of the medium and $D = D(\phi, u)$ is a $2 \times 2$ matrix,

$$D = \phi(x)\left[ d_m I + | u | (d_l E(u) + d_t E^{\perp}(u)) \right]. \tag{1.7}$$

In (1.7), the matrix $E$ is the projection along the direction of flow given by

$$E(u) = (u_i u_j / | u |^2), \tag{1.8}$$

$E^{\perp} = I - E$, $d_m$ is the molecular diffusion coefficient, and $d_l$ and $d_t$ are, respectively, the longitudinal and transverse dispersion coefficients. The tensor dispersion is more important physically than the molecular diffusion; also, $d_l$ is usually considerably larger than $d_t$. The term $\tilde{c}$ must be specified where $q > 0$; it is the concentration of the injected fluid. In addition, it will be assumed that $\tilde{c} = c$ where $q < 0$. The no-flow boundary condition (1.4) can be carried over in the form

$$\sum_{i,j} D_{ij}(\phi, u) \frac{\partial c}{\partial x_j} \nu_i = 0, \qquad x \in \partial\Omega, \quad t \in J. \tag{1.9}$$

Finally, it is necessary to specify the initial concentration,

$$c(x, 0) = c_0(x), \qquad x \in \Omega. \tag{1.10}$$

The initial pressure, modulo an additive constant, can be computed from (1.3) and (1.4). For physical relevance, $0 \leqslant c_0(x) \leqslant 1$. Again, we generalize slightly by taking a modified right-hand side, $g = g(x, t, c)$ :

$$\phi \frac{\partial c}{\partial t} + u . \nabla c - \nabla . (D \nabla c) = g, \qquad x \in \Omega, \quad t \in J. \tag{1.11}$$

The object of this paper is to design and analyze a finite element method for approximating the solution of the system (1.3), (1.11), subject to the initial and boundary conditions, that is particularly suitable to it. Note that the pressure does not appear explicitly in the equation (1.11) for the concentration; however, velocity does. Consequently, the motivation exists for choosing a numerical method for the pressure equation (1.3) that gives a direct approximation of the velocity, rather than one that requires differentiation or differencing of the approximation of the pressure and multiplication by $k(x)/\mu(c)$. In reasonable physical examples the velocity at a point varies much slower in time than the gradient of the pressure and the ratio $k/\mu$, and the direct evaluation of the velocity can be expected to give improved accuracy for the same computational effort. A mixed finite element procedure [10, 13] will be adopted here to approximate the pressure and the velocity simultaneously.

The concentration will be approximated using an essentially standard parabolic finite element method. Other methods could be employed for the concentration in combination with the mixed method for the pressure. In particular, a finite difference-method of characteristics scheme has been considered [5] by one of the authors; the analysis of that combination is based in part on the results of this paper. It would also be possible to use a finite element-method of characteristics procedure, as was discussed in the thesis of T. F. Russell [11], or an interior penalty Galerkin method [1, 3, 4, 14]; however, since the main point of this paper is to show the feasibility of the use of the mixed method for the pressure, we shall confine our treatment of the concentration to the single case.

Discretization of the time variable will not be discussed in this paper; one procedure, similar to that employed in [4], will be developed by the authors elsewhere. The analysis to be presented below follows the general outline of the argument given by two [6] of the authors for a standard Galerkin method for (1.3), (1.11); it also makes strong use of the results and arguments of Brezzi [2] and Raviart and Thomas [10].

The organization of the paper is as follows. A weak form of the problem involving a saddle-point replacement of the pressure equation will be presented, and the mixed finite element-Galerkin, continuous-time approximation method will be described. The existence and uniqueness of the approximate solution will be demonstrated. Two projections that will be valuable in the convergence analysis will be introduced and analyzed. Then, the convergence analysis will be given for the case of smoothly distributed external flow. Finally, a modification of the procedure will be indicated to treat the case of imposed flow at sources and sinks, and convergence will be shown under the restriction that the viscosity of the fluid mixture is independent of the concentration.

## 2. A WEAK FORM OF THE PROBLEM

Let $H(\mathrm{div}\,;\Omega)$ be the set of vector functions $v \in L^2(\Omega)^2$ such that $\nabla . v \in L^2(\Omega)$, and let

$$V = H(\mathrm{div}; \Omega) \cap \{ v . \nu = 0 \text{ on } \partial\Omega \} . \tag{2.1}$$

The solution $p$ of (1.3) is determined only to an additive constant, and we shall avoid this trivial difficulty by considering

$$W = L^2(\Omega)/\{ \varphi \equiv \text{constant on } \Omega \} . \tag{2.2}$$

For $\alpha, \beta \in V$, $\varphi \in W$, and $\theta \in L^\infty(\Omega)$ define the bilinear forms

$$\begin{aligned} &(a) \quad A(\theta; \alpha, \beta) = \left(\frac{1}{a(\theta)}\alpha, \beta\right) = \sum_{i=1}^{2} \left(\frac{1}{a_i(\theta)}\alpha_i, \beta_i\right), \\ &(b) \quad B(\alpha, \varphi) = -(\nabla . \alpha, \varphi) . \end{aligned} \tag{2.3}$$

Then, the pressure equation is equivalent to solving the family of saddle-point problems for a map $\{ u, p \} : J \to V \times W$ given by

$$\begin{aligned} &(a) \quad A(c; u, v) + B(u, p) = (\gamma(c), v) , \qquad v \in V , \\ &(b) \quad B(u, \varphi) = -(q, \varphi) , \qquad \varphi \in W . \end{aligned} \tag{2.4}$$

The first of these two equations expresses the relation $\partial p/\partial x_i = -a_i^{-1} u_i + \gamma_i$ and the second that the divergence of the velocity is the external flow rate. The no-flow boundary condition (1.4) that was incorporated into $V$ was used in the integration by parts to get (2.4*a*).

The concentration equation can be put in the weak form of finding a differentiable map $c : J \to H^1(\Omega)$ such that

$$\left(\phi \frac{\partial c}{\partial t}, z\right) + (u.\nabla c, z) + (D(u) \nabla c, \nabla z) = (g(c), z) \tag{2.5}$$

for $z \in H^1(\Omega)$ and $0 < t \leqslant T$ and such that $c(0, t) = c_0(x)$. In fact, in order that (2.5) make sense, it is necessary that $u.\nabla c \in L^2(\Omega)$; standard elliptic regularity theory implies the boundedness of $u$ under sufficient smoothness of the imposed flow rate $q$ and of the gravity term $\gamma(c)$. These conditions are rarely met in the practical simulation of a miscible flood in a petroleum reservoir, since wells normally must be treated as point sources and sinks. In the convergence analysis given below for the numerical procedure, we shall assume that the external flow is smoothly distributed, instead of being concentrated at points, and that the coefficients and domain are sufficiently regular as to allow a smooth solution of the differential problem. Later, we shall indicate a modification of the numerical method to make use of the asymptotic behavior of the velocity in the neighborhood of a well. No analysis has been constructed to cover the non-smooth data case except under the equally unphysical assumption that the viscosity $\mu(c)$ is independent of the concentration. Two methods have been studied [7, 12] under this latter assumption, and we shall indicate an extension of the argument of our method to this case.

A number of other assumptions will be needed in the analysis. In particular, the functions $a_i(x, c)$ should be bounded above and below by positive constants and the matrix $D$ should be uniformly positive-definite :

$$\sum_{i,j=1}^{2} D_{ij}(\phi, u)\, \xi_i\, \xi_j \geqslant D_* \,|\, \xi \,|^2 , \qquad \xi \in \mathbb{R}^2 , \tag{2.6}$$

with $D_*$ being independent of $x$ and $u$. The various bounds that are used for the coefficients and their derivatives need hold only in a neighborhood of the solution of the differential problem, for the quasi-regularity that will be imposed on the triangulations (or quadrilateralizations) associated with the finite element spaces and the optimal order convergence estimates that we shall derive under such an assumption will imply at least sub-optimal uniform convergence and, thus, only values near the solution ever are encountered in the calculation for sufficiently small parameter size.

## 3. THE APPROXIMATION PROCEDURE

Let $h = (h_c, h_p)$, with $h_c$ and $h_p$ being positive and, in general, different. Let $\tilde{V}_h \times \tilde{W}_h$ be one of the Raviart-Thomas spaces [10, 13] associated with a quasi-regular triangulation or quadrilateralization of $\Omega$ such that the elements have diameters bounded by $h_p$. Let the index of this space be the integer $k$, so that approximation to order $0(h_p^{k+1})$ is possible for both the vector and scalar components, as will be expressed below in (3.2). Let

$$\begin{aligned} &(a) \quad V_h = \{ v \in \tilde{V}_h : v.\nu = 0 \text{ on } \partial\Omega \}, \\ &(b) \quad W_h = \tilde{W}_h / \{ \varphi \equiv \text{constant on } \Omega \}. \end{aligned} \tag{3.1}$$

The approximation of $V \times W$ by $V_h \times W_h$ is described by the following relations. If $v \in V$ and $w \in W$, then

$$\begin{aligned} &(a) \quad \inf_{v_h \in V_h} \| v - v_h \|_{L^2(\Omega)^2} \leqslant M \| v \|_{H^{k+1}(\Omega)^2} h_p^{k+1}, \\ &(b) \quad \inf_{v_h \in V_h} \| v - v_h \|_V \leqslant M \{ \| v \|_{H^{k+1}(\Omega)^2} + \| \nabla . v \|_{H^{k+1}(\Omega)} \} h_p^{k+1}, \\ &(c) \quad \inf_{w_h \in W_h} \| w - w_h \|_W \leqslant M \| w \|_{H^{k+1}(\Omega)} h_p^{k+1}, \end{aligned} \tag{3.2}$$

whenever the norms on the right-hand side are finite.

Let $M_h \subset H^1(\Omega)$ be a standard finite element space for Galerkin methods. Assume that $M_h$ is associated with a quasi-regular polygonalizaton of $\Omega$ and that

$$\inf_{z_h \in M_h} \| z - z_h \|_{H^1(\Omega)} \leqslant M \| z \|_{H^{l+1}(\Omega)} h_c^l \tag{3.3}$$

for $z \in H^{l+1}(\Omega)$. This implies approximation in $L^2(\Omega)$ of order $0(h_c^{l+1})$.

Our continuous-time approximation procedure for the problem given by (1.3), (1.4), (1.9), (1.11) is defined by finding the map

$$\{ C, U, P \} : J \to M_h \times V_h \times W_h$$

such that

$$\begin{aligned} &(a) \quad C(0) - c_0 \text{ small} : L^2(\Omega)\text{- or } H^1(\Omega)\text{-projection of } c_0 \text{ into } M_h \text{ or} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{some interpolation of } c_0 \text{ into } M_h, \\ &(b) \quad \left( \phi \frac{\partial C}{\partial t}, z \right) + (U . \nabla C, z) + (D(U) \nabla C, \nabla z) = (g(C), z), \; z \in M_h, \; t \in J, \\ &(c.\text{i}) \quad A(C; U, v) + B(v, P) = (\gamma(C), v), \quad v \in V_h, \quad t \in J, \\ &(c.\text{ii}) \quad B(U, \varphi) = -(q, \varphi), \quad \varphi \in W_h, \quad t \in J. \end{aligned} \tag{3.4}$$

## 4. EXISTENCE AND UNIQUENESS OF THE APPROXIMATE SOLUTION

The demonstration of the existence and uniqueness of the solutions of (3.4) relies on the known theory for approximating the solution of a saddle point problem, with Brezzi's treatment [2] combined with the general ideas of Raviart and Thomas [10] being most convenient for us. We shall give the argument for a slightly restricted case by taking $\Omega$ to be a square and by thinking of the spaces $V_h$, $W_h$, and $M_h$ as being related to tensor product grids, for which it is very easy to apply the boundary condition $v.\nu = 0$ on $V_h$. Thus, let $\delta = \{ x_0, x_1, \dots, x_n \}$, $x_i > x_{i-1}$, and

$$\mathcal{M}_j(m, \delta) = \{ \psi \in C^j([x_0, x_n]) : \psi |_{[x_{i-1}, x_i]} \in P_m \},$$

$P_m$ denoting the polynomials of degree not greater than $m$. Then, with $\delta_p$ and $\delta_c$ being quasi-regular on a side of $\Omega$, set

$$(a) \quad V_h = [\{ \mathcal{M}_0(k+1, \delta_p) \otimes \mathcal{M}_{-1}(k, \delta_p)\}] \times \\ \times [\{ \mathcal{M}_{-1}(k, \delta_p) \otimes \mathcal{M}_0(k+1, \delta_p)\}] \cap \{ v.\nu = 0 \text{ on } \partial\Omega \}, \tag{4.1}$$

$$(b) \quad W_h = \mathcal{M}_{-1}(k, \delta_p) \otimes \mathcal{M}_{-1}(k, \delta_p)/\{ \varphi \equiv \text{constant on } \Omega \},$$

$$(c) \quad M_h = \mathcal{M}_0(l, \delta_c) \otimes \mathcal{M}_0(l, \delta_c).$$

Now, let

$$Z = \{ v \in V : B(v, \varphi) = 0, \varphi \in W \},$$

$$Z_h = \{ v \in V_h : B(v, \varphi) = 0, \varphi \in W_h \}.$$

Note that the boundary condition $v.\nu = 0$ for $v \in V_h$ implies that $\operatorname{div} V_h \subset W_h$. Hence, $v \in Z_h$ implies that $\operatorname{div} v = 0$, and, since

$$Z = H(\operatorname{div}; \Omega) \cap \{ \operatorname{div} v = 0 \text{ in } \Omega \text{ and } v.\nu = 0 \text{ on } \partial\Omega \},$$

it follows that $Z_h \subset Z$. Thus,

$$\| v \|_V = \| v \|_{L^2(\Omega)^2}, \quad v \in Z_h.$$

So, if $v \in Z_h$,

$$a(C; v, v) = \sum_{i=1}^{2} \left( \frac{1}{a_i(C)} v_i, v_i \right) \geqslant \eta \| v \|_V^2, \tag{4.2}$$

where $\eta = \left( \max_{i,x,C} a_i \right)^{-1}$. This inequality establishes the hypothesis H2 of Brezzi's paper [2] with $\gamma_h = \gamma_h' = \eta$.

Next, we wish to show that

$$\sup_{v \in V_h \setminus \{0\}} \frac{B(v, \varphi)}{\| v \|_V} \geqslant \beta \| \varphi \|_W, \qquad \varphi \in W_h. \tag{4.3}$$

This result is contained in Theorem 4 of Raviart-Thomas [10] and appears explicitly as the first remark in the proof of their Theorem 5 in their treatment of the Dirichlet problem for the Laplace equation ; it is easy to see that their argument requires but trivial modification to handle the variable coefficient and the Neumann boundary condition, as the imposition of the boundary condition on $V_h$ and the reduction modulo constants for $W_h$ are in balance. Hence, the hypothesis H1 of Brezzi holds, and his Proposition 2.1 provides existence and uniqueness of a solution $\{ U, P \}$ of (3.4*c*) for any function $C \in L^\infty(\Omega)$. Moreover, the assumption that $a_i(C)$ is bounded independently of $C$ implies that the inverse operator is bounded independently of $\delta_p$ and $C$ :

$$\| U \|_V + \| P \|_W \leqslant M \{ \| \gamma(C) \|_{L^2(\Omega)^2} + \| q \|_W \}. \tag{4.4}$$

Since $\gamma(C)$ represents the density of the fluid mixture and the slope of the reservoir and since we have assumed $q$ to be well-behaved, we shall assume the right-hand side of (4.4) to be bounded.

The quasi-regularity of the partition $\delta_p$ implies that

$$\| U \|_{L^\infty(\Omega)^2} \leqslant M h_p^{-1}, \tag{4.5}$$

a distinctly sub-optimal result that will be used only in the demonstration of the existence of a solution of (3.4). If the test function in (3.4*b*) is taken to be $C$ and (4.5) is used to bound $U$, then it follows that

$$\frac{1}{2} \frac{d}{dt}(\phi C, C) + (D \nabla C, \nabla C) \leqslant M(h_p) \| C \|^2_{L^2(\Omega)}, \tag{4.6}$$

where $M(h_p) = 0(h_p^{-1})$. The positive-definiteness of $(D \nabla C, \nabla C)$ and the nonsingularity of $\phi$, which we assume, give an a priori estimate in

$$L^\infty(J ; L^2(\Omega)) \cap L^2(J ; H^1(\Omega))$$

for $C$ which can be used in a standard way to demonstrate existence and uniqueness for a solution of (3.4). We can turn to analyzing the convergence of the method.

## 5. TWO TECHNICALLY USEFUL PROJECTIONS

It is frequently valuable to decompose the analysis of the convergence of finite element methods by passing through a projection of the solution of the differential problem into the finite element space. Consider first the map $\{\tilde{U}, \tilde{P}\} : J \to V_h \times W_h$ given by

$$\begin{aligned} &(a) \quad A(c; \tilde{U}, v) + B(v, \tilde{P}) = (\gamma(c), v), \quad v \in V_h, \\ &(b) \qquad\qquad B(\tilde{U}, \varphi) = -(q, \varphi), \quad \varphi \in W_h. \end{aligned} \tag{5.1}$$

The constants $\eta$ and $\beta$ of the last section are independent of the argument $c$ occurring in the $A$-form; thus, the map exists and, by Theorem 2.1 of Brezzi [2],

$$\| u - \tilde{U} \|_V + \| p - \tilde{P} \|_W \leqslant M \{ \inf_{v \in V_h} \| u - V \|_V + \inf_{\varphi \in W_h} \| p - \varphi \|_W \}, \tag{5.2}$$

where the constant $M$ does not depend on $c$. Since we have assumed that $p$ (and, consequently, $u$) is smooth, then for $t \in J$

$$\begin{aligned} &(a) \quad \inf_{v \in V_h} \| u - v \|_V \leqslant M \| p \|_{H^{k+3}(\Omega)} h_p^{k+1}, \\ &(b) \quad \inf_{\varphi \in W_h} \| p - \varphi \|_W \leqslant M \| p \|_{H^{k+1}(\Omega)} h_p^{k+1}, \end{aligned} \tag{5.3}$$

and it follows that

$$\| u - \tilde{U} \|_V + \| p - \tilde{P} \|_W \leqslant M \| p \|_{L^\infty(J; H^{k+3}(\Omega))} h_p^{k+1}, \tag{5.4}$$

with $M$ depending only on uniform bounds for $a_i(c)$, but not on $c$ itself.

Next, let $\tilde{C} : J \to M_h$ be the projection of $c$ given by

$$(D(u) \nabla(\tilde{C} - c), \nabla z) + (u.\nabla(\tilde{C} - c), z) + (\lambda(\tilde{C} - c), z) = 0, \quad z \in M_h. \tag{5.5}$$

The function $\lambda$ will be chosen to assure coercivity of the form. Since

$$(u.\nabla f, f) = \frac{1}{2}(u, \nabla(f^2)) = -\frac{1}{2}(\nabla.u, f^2) + \frac{1}{2}\langle u.\nu, f^2 \rangle = -\frac{1}{2}(qf, f), \tag{5.6}$$

it suffices to take

$$\lambda = 1 + \frac{1}{2} q^+. \tag{5.7}$$

Then, it follows that

$$(D(u)\,\nabla\zeta, \nabla\zeta) + (u.\nabla\zeta, \zeta) + (\lambda\zeta, \zeta) \geqslant (\phi(d_m + d_t\,|\,u\,|)\,\nabla\zeta, \nabla\zeta) + (\zeta, \zeta)\,, \quad (5.8)$$

since, at any point $x \in \Omega$, $\nabla\zeta$ can be decomposed into orthogonal components $\alpha$ and $\beta$, respectively parallel to $u$ and orthogonal to $u$, for which under the assumption that $d_l \geqslant d_t$

$$\langle\, d_l\, E(u)\,\nabla\zeta + d_t\, E(u)^{\perp}\,\nabla\zeta, \nabla\zeta\,\rangle_{R^2} = d_l\,|\,\alpha\,|^2 + d_t\,|\,\beta\,|^2 \geqslant d_t\,|\,\nabla\zeta\,|^2\,. \quad (5.9)$$

So long as both $q$ and its time derivative are smooth functions of position, standard arguments show that, for $t \in J$,

$$\|\, c - \tilde{C}\,\|_{L^2(\Omega)} + h_c\,\|\, c - \tilde{C}\,\|_{H^1(\Omega)} \leqslant M\,\|\, c\,\|_{H^{l+1}(\Omega)}\, h_c^{l+1}\,; \quad (5.10)$$

here, the constant $M$ depends, in particular, on the $L^\infty$-norm of $u$ and the ellipticity constant derivable from $d_m\,\phi(x)$. Differentiation in time of (5.5) leads to the additional estimate

$$\left\|\frac{\partial(\tilde{C} - c)}{\partial t}\right\|_{L^2(\Omega)} \leqslant M\left\{\|\, c\,\|_{H^{l+1}(\Omega)} + \left\|\frac{\partial c}{\partial t}\right\|_{H^{l+1}(\Omega)}\right\} h_c^{l+1}\,, \quad (5.11)$$

where the constant now depends on the $L^\infty$-norm of $\partial u/\partial t$ as well.

Since we have already estimated $U - \tilde{U}, p - \tilde{P}$, and $c - \tilde{C}$, the convergence argument is reduced to bounding $U - \tilde{U}$, $P - \tilde{P}$, and $C - \tilde{C}$.

## 6. AN ESTIMATE OF $U - \tilde{U}$

We derive first an estimate of $U - \tilde{U}$ and $P - \tilde{P}$ of a nature similar to that given in Lemma 3.1 of Ewing and Wheeler [6]. Manipulation of (3.4*c*) and (5.1) leads to the equations

$$\begin{aligned} &(a)\quad A(C; U - \tilde{U}, v) + B(v, P - \tilde{P}) = \\ &\qquad\qquad = A(c; \tilde{U}, v) - A(C; \tilde{U}, v) + (\gamma(C) - \gamma(c), v)\,, \quad v \in V_h\,, \\ &(b)\quad B(U - \tilde{U}, \varphi) = 0\,, \quad \varphi \in W_h\,. \end{aligned} \quad (6.1)$$

We have already seen by Brezzi's Proposition 2.1 that the solution operator for (6.1) is bounded; hence

$$\|\, U - \tilde{U}\,\|_V + \|\, P - \tilde{P}\,\|_W \leqslant M\,\{\, 1 + \|\,\tilde{U}\,\|_{L^\infty(\Omega)}\,\}\,\|\, c - C\,\|_{L^2(\Omega)}\,, \quad (6.2)$$

with again only bounds on $a_i(c)$ being involved in the constant $M$. The quasi-regularity of the grid and the bound (5.4) imply that $\tilde{U}$ is bounded in $L^\infty(J; L^\infty(\Omega))$, so that the right-hand side of (6.2) is bounded by

$$M \| p \|_{L^\infty(J;H^3(\Omega))} \| c - C \|_{L^2(\Omega)}, \quad t \in J, \tag{6.3}$$

where the optimal index $k$ has been replaced by zero in the estimate (5.4).

## 7. AN ESTIMATE OF $C - \tilde{C}$

Let $\xi = C - \tilde{C}$ and $\eta = c - \tilde{C}$. Then, (2.5), (3.4*b*), and (5.5) can be used to see that

$$\left(\phi \frac{\partial \xi}{\partial t}, z\right) + (U . \nabla \xi, z) + (D(U) \nabla \xi, \nabla z) = \left(\phi \frac{\partial \eta}{\partial t}, z\right) - (\lambda \eta, z) - \\ - ((U - u). \nabla \tilde{C}, z) - ((D(U) - D(u)) \nabla \tilde{C}, \nabla z) + (g(C) - g(c), z), \quad z \in M_h. \tag{7.1}$$

We shall begin by estimating the right-hand side, term-by-term. The most difficult term is the one involving $D(U) - D(u)$, which can be written in the form

$$D(U) - D(u) = \phi \{ d_l(| U | E(U) - | u | E(u)) + \\ + d_t(| U | E^\perp(U) - | u | E^\perp(u)) \}. \tag{7.2}$$

Momentarily assume that both $| U |$ and $| u |$ are nonzero; the conclusion given below in (7.4) clearly holds if one or both vanish. The $(i, j)$-entry of $| U | E(U) - | u | E(u)$ satisfies the inequality

$$\left| \frac{U_i U_j}{| U |} - \frac{u_i u_j}{| u |} \right| = \left| \frac{1}{| u |} \{ U_j(U_i - u_i) \} - (u_j - U_j) u_i + \right. \\ \left. + (| u | - | U |) \frac{U_i U_j}{| u | | U |} \right| \leqslant \left(1 + \frac{2 | U |}{| u |}\right) | u - U |, \tag{7.3}$$

pointwise. Obviously, the roles of $| U |$ and $| u |$ can be reversed in the inequality, so that we can choose, pointwise, the minimum of the factors $| U |/| u |$ and $| u |/| U |$. Hence,

$$\left| \frac{U_i U_j}{| U |} - \frac{u_i u_j}{| u |} \right| \leqslant 3 | u - U |. \tag{7.4}$$

The same bound holds for the entries of $| U | E^{\perp}(U) - | u | E^{\perp}(u)$, so that, with $M$ depending only on the dispersion coefficients,

$$| ((D(U) - D(u)) \nabla\tilde{C}, \nabla\zeta) | \leqslant M \| \nabla\tilde{C} \|_{L^\infty(\Omega)} \| u - U \|_{L^2(\Omega)^2} \| \nabla\xi \|_{L^2(\Omega)} . \tag{7.5}$$

It is trivial that

$$| ((U - u).\nabla\tilde{C}, \xi) | \leqslant \| \nabla\tilde{C} \|_{L^\infty(\Omega)} \| u - U \|_{L^2(\Omega)^2} \| \xi \|_{L^2(\Omega)} \tag{7.6}$$

and

$$| (g(C) - g(c), \xi) | \leqslant M \| c - C \|_{L^2(\Omega)} \| \xi \|_{L^2(\Omega)} , \tag{7.7}$$

provided that $g$ is Lipschitz continuous. The $L^\infty$-norm of $\tilde{C}$ is bounded as a consequence of (5.10) and the assumed regularity of $c$.

Now, we need to bound the left-hand side of (7.1) from below for the choice of the test function being $\xi$. First note that

$$(U.\nabla\xi, \xi) = -\frac{1}{2}(\nabla.U, \xi^2) = -\frac{1}{2}(q\xi, \xi) - \frac{1}{2}B(u - U, \xi^2 - \varphi) \tag{7.8}$$

for any $\varphi \in W_h$, using (2.4*b*) and (3.4*c*). The space $W_h$ possesses optimal approximation properties in $L^1(\Omega)$ as well as in $L^2(\Omega)$ for functions that are orthogonal to constants, as $\nabla.(u - U)$ is. Hence,

$$\begin{aligned}
\inf_{\varphi \in W_h} | (\nabla.(u - U), \xi^2 - \varphi) | &\leqslant Mh_p \| \nabla.(u - U) \|_{L^\infty(\Omega)} \| \nabla(\xi^2) \|_{L^1(\Omega)} \\
&\leqslant Mh_p \{ \| \nabla.(u - \tilde{U}) \|_{L^\infty(\Omega)} + \| \nabla.(\tilde{U} - U) \|_{L^\infty(\Omega)} \} \| \xi \|_{L^2(\Omega)} \| \nabla\xi \|_{L^2(\Omega)} \\
&\leqslant M \{ \| p \|_{H^{k+3}(\Omega)} h_p^k + \| c - C \|_{L^2(\Omega)} \} \| \xi \|_{L^2(\Omega)} \| \nabla\xi \|_{L^2(\Omega)} \\
&\leqslant M \{ 1 + \| c - C \|^2_{L^2(\Omega)} \} \| \xi \|^2_{L^2(\Omega)} + \varepsilon \| \nabla\xi \|^2_{L^2(\Omega)} \\
&\leqslant M \{ 1 + \| \xi \|^2_{L^2(\Omega)} \} \| \xi \|^2_{L^2(\Omega)} + \varepsilon \| \nabla\xi \|^2_{L^2(\Omega)} ,
\end{aligned} \tag{7.9}$$

where quasi-regularity, (5.4), best approximation of $u$ in $V_h$ in $L^\infty(\Omega)$, and (5.10) have been used to accomplish the steps of (7.9). Thus, the left-hand side of (7.1) is bounded below by

$$\frac{1}{2}\frac{d}{dt}(\phi\xi, \xi) + (\{ \phi(d_m + d_t | U |) - \varepsilon \} \nabla\xi, \nabla\xi) - \\ - M \{ 1 + \| \xi \|^2_{L^2(\Omega)} \} \| \xi \|^2_{L^2(\Omega)} . \tag{7.10}$$

The application of (7.5)-(7.7) and (7.10) to (7.1) lead to the evolution inequality

$$\frac{d}{dt}(\phi\xi, \xi) + (\phi(d_m + d_t \mid U \mid) \nabla\xi, \nabla\xi) \leqslant M\Big[(1 + \| \xi \|^2_{L^2(\Omega)}) \| \xi \|^2_{L^2(\Omega)} +$$

$$+ \| u - U \|^2_{L^2(\Omega)^2} + \| \eta \|^2_{L^2(\Omega)} + \left\| \frac{\partial \eta}{\partial t} \right\|^2_{L^2(\Omega)} + \| c - C \|^2_{L^2(\Omega)}\Big]. \quad (7.11)$$

Then, (5.4), (5.10), (5.11), (6.2), and (6.3) imply that

$$\| \eta \|^2_{L^2(\Omega)} + \left\| \frac{\partial \eta}{\partial t} \right\|^2_{L^2(\Omega)} + \| u - U \|^2_{L^2(\Omega)^2} \leqslant M\Big[\Big\{ \| c \|^2_{H^{l+1}(\Omega)}$$

$$+ \left\| \frac{\partial c}{\partial t} \right\|^2_{H^{l+1}(\Omega)} \Big\} h_c^{2l+2} + \| p \|^2_{H^{k+3}(\Omega)} h_p^{2k+2} + \| \xi \|^2_{L^2(\Omega)}\Big]. \quad (7.12)$$

So,

$$\frac{d}{dt}(\phi\xi, \xi) + (\phi(d_m + d_t \mid U \mid) \nabla\xi, \nabla\xi) \leqslant$$

$$\leqslant M[\{ 1 + \| \xi \|^2_{L^2(\Omega)} \} \| \xi \|^2_{L^2(\Omega)} + M_1(c) h_c^{2l+2} + M_2(p) h_p^{2k+2}], \quad (7.13)$$

where $M$ depends on certain lower norms of the solution of the differential problem but not on the solution of the approximation problem and $M_1(c)$ and $M_2(p)$ have the forms required by (7.12).

Let us make the induction hypothesis that

$$\| \xi \|_{L^\infty(J; L^2(\Omega))} \leqslant 1; \quad (7.14)$$

certainly, for any reasonable choice of the initial condition (7.14) holds for $t = 0$. Thus, (7.14) will hold for $t \leqslant T_h$ for some $T_h > 0$; we shall show for $h = (h_c, h_p)$ sufficiently small that $T_h = T$ and that convergence will take place asymptotically at an optimal rate.

Assume that

$$\| \xi(0) \|_{L^2(\Omega)} \leqslant M \| c_0 \|_{H^{l+1}(\Omega)} h_c^{l+1}. \quad (7.15)$$

It follows from (7.13), (7.14), and the Gronwall lemma that

$$\| \xi \|^2_{L^\infty(J; L^2(\Omega))} \leqslant e^{2MT} \{ M_1(c) h_c^{2l+2} + M_2(p) h_p^{2k+2} + \| \xi(0) \|^2_{L^2(\Omega)} \}; \quad (7.16)$$

thus,

$$\| \xi \|_{L^\infty(J; L^2(\Omega))} \leqslant M_3\Big[\Big\{ \| c_0 \|_{H^{l+1}(\Omega)} + \| c \|_{L^2(J; H^{l+1}(\Omega))} +$$

$$+ \left\| \frac{\partial c}{\partial t} \right\|_{L^2(J; H^{l+1}(\Omega))} \Big\} h_c^{l+1} + \| p \|_{L^2(J; H^{k+3}(\Omega))} h_p^{k+1}\Big], \quad (7.17)$$

where $M_3$ depends on $T$, bounds on $a_i(c)$ and the other coefficients, and certain lower norms of the solution of the differential problem. Note that (7.17) implies that the induction hypothesis (7.14) holds for small $h$, so that the entire argument is validated.

It is of particular importance that the estimate (7.17) holds without a constraint being imposed to relate $h_c$ and $h_p$ and that it holds for the minimal index cases $k = 0$ for the Raviart-Thomas space and $l = 1$ for the standard finite element space used for the concentration, as well as for higher order spaces, again independently of any relation between $k$ and $l$.

We can summarize our results by combining (7.17) with the inequalities of sections 5 and 6. It follows that

$$\begin{aligned}
(a)\quad & \| c - C \|_{L^\infty(J;L^2(\Omega))} \leqslant M\Bigg[\Bigg\{ \| c \|_{L^\infty(J;H^{l+1}(\Omega))} + \\
& \qquad + \left\| \frac{\partial c}{\partial t} \right\|_{L^2(J;H^{l+1}(\Omega))} \Bigg\} h_c^{l+1} + \| p \|_{L^2(J;H^{k+3}(\Omega))} h_p^{k+1} \Bigg], \\
(b)\quad & \| u - U \|_{L^\infty(J;V)} + \| p - P \|_{L^\infty(J;W)} \leqslant M\Bigg[\Bigg\{ \| c \|_{L^\infty(J;H^{l+1}(\Omega))} + \\
& \qquad + \left\| \frac{\partial c}{\partial t} \right\|_{L^2(J;H^{l+1}(\Omega))} \Bigg\} h_c^{l+1} + \| p \|_{L^\infty(J;H^{k+3}(\Omega))} h_p^{k+1} \Bigg].
\end{aligned} \tag{7.18}$$

This completes the treatment of the continuous-time problem for smooth data. We shall formulate a modification of the procedure that recognizes the existence of point sources and sinks (i.e., wells); however, the analysis given above does not extend to this case. Elsewhere, we present and analyze a time-stepping method for the procedure studied here.

## 8. MODIFICATION OF THE MIXED METHOD IN THE PRESENCE OF WELLS

In any realistic reservoir simulation, the external flow is concentrated at wells :

$$q(x, t) = \sum_{j=1}^{N} q_j(t)\, \delta_{x_j}\,, \tag{8.1}$$

where $\delta_{x_j}$ is the Dirac mass at the point $x_j$ and

$$\sum_{j=1}^{N} q_j(t) = 0\,, \tag{8.2}$$

as required by incompressibility. The Darcy velocity $u$ has a singularity at each well, and $u$ can be represented in the form

$$u = u_s + u_r , \tag{8.3}$$

where $u_s$, the singular part, is given by

$$u_s = \sum_{j=1}^{N} q_j(t) \nabla N_j , \qquad N_j = \frac{1}{2\pi} \log | x - x_j | , \tag{8.4}$$

and $u_r$, the regular part, satisfies the relations

$$\begin{aligned} &(a) \quad \nabla . u_r = 0 , && x \in \Omega , \\ &(b) \quad u_r . \nu = - u_s . \nu , && x \in \partial\Omega , \end{aligned} \tag{8.5}$$

for $t \in J$. The pressure equation (3.4$c$) can be replaced by

$$\begin{aligned} &(a) \quad A(C; U_r, v) + B(v, P) = - A(C; u_s, v), \quad v \in V_h , \\ &(b) \quad B(U_r, \varphi) = 0 , \qquad \varphi \in W_h , \\ &(c) \quad \langle (U_r + u_s).\nu, v.\nu \rangle = 0 , \qquad v \in \tilde{V}_h , \end{aligned} \tag{8.6}$$

where now $U_r(t) \in \tilde{V}_h$. The boundary equations (8.6$b$) require, in particular, that the net flow across the boundary edge of each boundary element be zero, along with the moments up to order $k$ when $k > 0$.

Set

$$U = U_r + u_s \tag{8.7}$$

and use (3.4$a$) and (3.4$b$) as the equations for the concentration to complete the definition of the method.

The convergence of this modification has not been demonstrated when the viscosity depends on the concentration, as it does in any realistic physical example; however, convergence can be established when $\mu(c) = \mu$, a constant, by following through the proof given by two [7] of the authors for the constant viscosity problem when the concentration is approximated in the same manner as in this method and the pressure by a Galerkin method incorporating logarithmic singular terms.

First, it follows from the Brezzi [2] argument that, if $\{ \tilde{U}_r, \tilde{P} \}$ corresponds to the projection $\{ \tilde{U}, \tilde{P} \}$ of Section 5,

$$\begin{aligned} \| u_r - \tilde{U}_r \|_{L^2(\Omega)^2} + \| p - \tilde{P} \|_{L^2(\Omega)} &\leqslant \\ \leqslant M \{ \inf_{v \in V_h} \| u_r - v \|_V + \| (u_r - v).\nu \|_{L^2(\partial\Omega)} \} &+ \\ + \{ \inf_{\varphi \in W_h} \| p - \varphi \|_{L^2(\Omega)} \} . \end{aligned} \tag{8.8}$$

It is easy to see that, since the boundary data is smooth, the approximation of $u_r$ is of optimal order; however, the pressure is not smooth and its approximation is the limiting factor. For any $k \geqslant 0$,

$$\inf_{\varphi \in W_h} \| p - \varphi \|_{L^2(\Omega)} \leqslant M h_p \log h_p^{-1} \tag{8.9}$$

and

$$\| u - \tilde{U} \|_{L^\infty(J; L^2(\Omega)^2)} \leqslant M h_p \log h_p^{-1} . \tag{8.10}$$

Now, since $\mu(c)$ is independent of $c$, the approximate solution for $\{ u_r, p \}$ is exactly $\{ \tilde{U}_r, \tilde{P} \}$. Hence,

$$\| u - U \|_{L^2(\Omega)^2} = \| u_r - \tilde{U}_r \|_{L^2(\Omega)^2} \leqslant M h_p \log h_p^{-1} . \tag{8.11}$$

The argument in [7] leading to their Theorems 3.1 and 3.2 did not distinguish between $h_p$ and $h_c$, and for convenience here we shall not either. Thus, if $h$ denotes the larger of $h_p$ and $h_c$ and if $h_p$ is comparable, above and below, with $h_c^\alpha$ for some positive $\alpha$, the proofs of Theorems 3.1 and 3.2 can be repeated without modification to show that, if $D = d_m \phi(x) I$,

$$\| c - C \|_{L^\infty(J; L^2(\Omega))} \leqslant \| c - C \|_{L^2(J; H^1(\Omega))} + \\ + \left[ \sum_{j=1}^{N} \int_0^T | q_j(t) | (c - C)(x_j, t)^2 \, dt \right]^{1/2} \leqslant M h^{1-\varepsilon} \tag{8.12}$$

and that, if $D$ includes the tensorial dispersion, the bound becomes $M h^{1/2-\varepsilon}$. The convergence rates have orders independent of the indices $k \geqslant 0$ and $l \geqslant 1$ of the method.

The modification of the mixed method discussed above was suggested to the authors by Douglas N. Arnold.

## REFERENCES

1. D. N. ARNOLD, *An interior penalty finite element method with discontinuous elements,* Thesis, University of Chicago, June 1979.
2. F. BREZZI, *On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers,* R.A.I.R.O., Anal. Numér. 2 (1974), pp. 129-151.
3. J. DOUGLAS, Jr. and T. DUPONT, Interior penalty procedure for elliptic and parabolic Galerkin methods, *Computing Methods in Applied Science,* Lecture Notes in Physics 58, Springer-Verlag, 1976.

4. J. DOUGLAS, Jr., M. F. WHEELER, B. L. DARLOW and R. P. KENDALL, *Self-adaptive finite element simulation of miscible displacement in porous media,* to appear in SIAM J. Sci. Stat. Computing.

5. J. DOUGLAS, Jr., *Simulation of miscible displacement in porous media by a modified method of characteristics procedure,* to appear in the proceedings of the 1981 Dundee Conference on Numerical Analysis.

6. R. E. EWING and M. F. WHEELER, *Galerkin methods for miscible displacement problems in porous media*, SIAM J. Numer. Anal. 17 (1980), pp. 351-365.

7. R. E. EWING and M. F. WHEELER, *Galerkin methods for miscible displacement problems with point sources and sinks, unit mobility ratio case*, to appear.

8. D. W. PEACEMAN, *Improved treatment of dispersion in numerical calculation of multidimensional miscible displacement,* Soc. Pet. Eng. J. (1966), pp. 213-216.

9. D. W. PEACEMAN, *Fundamentals of Numerical Reservoir Simulation,* Elsevier Publishing Co., 1977.

10. P. A. RAVIART and J. M. THOMAS, A mixed finite element method for 2nd order elliptic problems, *Mathematical Aspects of the Finite Element Method,* Lecture Notes in Mathematics 606, Springer-Verlag, 1977.

11. T. F. RUSSELL, *An incompletely iterated characteristic finite element method for a miscible displacement problem,* Thesis, University of Chicago, June 1980.

12. P. H. SAMMON, *Numerical approximations for a miscible displacement process in porous media,* to appear.

13. J. M. THOMAS, *Sur l'analyse numérique des méthodes d'éléments finis hybrides et mixtes,* Thèse, Université Pierre et Marie Curie, 1977.

14. M. F. WHEELER and B. L. DARLOW, Interior penalty Galerkin methods for miscible displacement problems in porous media, *Computational Methods in Nonlinear Mechanics* (J. T. Oden, ed.), North-Holland Publishing Co., 1980.