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EFFICIENT TIME-STEPPING METHODS FOR MISCIBLE DISPLACEMENT PROBLEMS IN POROUS MEDIA*

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Abstract. Efficient procedures for time-stepping Galerkin methods for approximating the solution of a coupled system for $c = c(x, t)$ and $p = p(x, t)$, with nonlinear Neumann boundary conditions, of the form

$$-\nabla \cdot [a(x, c)\{\nabla p - \gamma(x, c)\nabla g\}] \equiv \nabla \cdot u = f_1(x, t), \qquad x \in \Omega, \quad t \in (0, T],$$

$$\nabla \cdot [b(x, c, \nabla p)\nabla c] - u \cdot \nabla c = \phi(x)\frac{\partial c}{\partial t} - f_2(x, t, c), \qquad x \in \Omega, \quad t \in (0, T],$$

$$u \cdot \nu = q_1(x, t), \qquad x \in \partial\Omega, \quad t \in (0, T],$$

$$b\frac{\partial c}{\partial \nu} = q(x, t, c), \qquad x \in \partial\Omega, \quad t \in (0, T],$$

$$c(x, 0) = c_0(x), \qquad x \in \Omega,$$

where $\Omega \subset \mathbb{R}^d$, $2 \leqq d \leqq 3$, are presented and analyzed. This system is a possible model system for describing the miscible displacement of one incompressible fluid by another in a porous medium when flow conditions are prescribed on the boundary. The procedures involve the use of a preconditioned iterative method for approximately solving the algebraic problem at each time step. The iteration need be performed only long enough to stabilize the scheme. Motivated by the fact that the pressure is smoother in time than the concentration, larger time steps are used for the pressure than for the concentration. Under certain smoothness assumptions on the solution, optimal order convergence rates and almost optimal order work estimates are obtained.

1. Introduction. We consider a numerically efficient modification of a backward difference-Galerkin procedure to solve a coupled system of partial differential equations which has been employed as a model for the miscible displacement of one incompressible fluid by another in a porous medium [14]. An elliptic equation simulates the pressure in the fluid mixture, and a quasilinear parabolic equation models the relative concentration of one of the fluids. One application of this model is to oil reservoirs, where an external fluid may be injected in order to push oil out of the reservoir and into production.

This work extends the results of Ewing and Wheeler [14] in several respects. We generalize the differential problem of [14], in which homogeneous boundary conditions were assumed, by including a nonlinear boundary condition in the concentration equation. This will be seen below to be natural for this problem. We modify the time-stepping procedure by using a larger time step for the pressure than for the concentration, motivated by the physical fact that the pressure is smoother in time than the concentration, and we show that work is saved by doing this. Finally, we replace the direct matrix solution of [14], which requires the factorization of two matrices at each time step and is expensive in a problem with more than one space dimension, by a preconditioned iterative procedure. The iterative method approximately solves the algebraic problem at each time step, and it need be performed only

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long enough to stabilize the scheme. We analyze a preconditioned conjugate gradient procedure employing a fixed preconditioning matrix for each equation, although our analysis will apply equally well to more general iterative methods. Only two matrix factorizations are necessary in the entire procedure.

The analysis will require several techniques not employed in [14]. We demonstrate that the accuracy of the underlying backward difference-Galerkin method of [14] is maintained, while the differential problem is generalized and the work requirements are greatly reduced. Of particular note is the result that, in many cases, a fixed number of preconditioned iterations per time step, independent of all mesh parameters, is sufficient to stabilize the procedure. In other cases, the number of iterations need grow no faster than the logarithm of the time step. We therefore have optimal or nearly optimal order work estimates for our method, a sizable improvement over previous results. This work is an extension of some of the results of Ewing [11], [12]; reference will be made to [11], [12] for some details of the proofs.

The employment of preconditioned conjugate gradient iteration in quasilinear parabolic problems is not new; treatments appear in [6], [10]. However, the application of this procedure to an elliptic equation coupled with a time-dependent equation appears to be unprecedented. We emphasize that, unlike standard iterative procedures for elliptic equations, our method will require only a fixed number of iterations per time step in most cases. Parabolic problems with nonlinear boundary conditions have also been treated before [3], [19], but there is little analysis for such problems as part of a coupled system.

The model for miscible displacement is given by

$$
(1.1)\quad
\begin{array}{lll}
\text{(a)} & -\nabla\cdot[a(\nabla p-\gamma\nabla d)]\equiv\nabla\cdot u=f_1(x,t), & x\in\Omega,\quad t\in(0,T],\\[4pt]
\text{(b)} & u\cdot n=g_1(x,t), & x\in\partial\Omega,\quad t\in(0,T],\\[4pt]
\text{(c)} & \phi\dfrac{\partial c}{\partial t}-\nabla\cdot(b\nabla c-uc)=\hat{c}(x,t,c)f_1(x,t), & x\in\Omega,\quad t\in(0,T],\\[4pt]
\text{(d)} & b\dfrac{\partial c}{\partial n}-(u\cdot n)c=g_2(x,t), & x\in\partial\Omega,\quad t\in(0,T],\\[4pt]
\text{(e)} & c(x,0)=c_0(x), & x\in\Omega.
\end{array}
$$

The solution functions are the pressure $p(x,t)$ and the concentration $c(x,t)$. The pressure is determined only up to an additive constant, so we normalize it to have mean value zero on Ω. We assume that Ω is a bounded domain in $\mathbb{R}^d$, $d=2$, with boundary $\partial\Omega$, and we let $J=(0,T]$. We consider the case $d=3$ at the end of § 4 and show that the results are essentially unchanged. We take the coefficients to be of the form $a=a(x,c)$, $\gamma=\gamma(x,c)$, $d=d(x)$, $\phi=\phi(x)$, and $b=b(x,c,\nabla p)$, $b=b(x,c)$ or $b=b(x)$. The convergence results in [14] depended on the nature of b, and we demonstrate that the same holds here; we also show how the work estimates depend on b. The physical significance of these functions is discussed in [14], [20], [23]. The function $u=u(x,c,\nabla p)$ is known as the Darcy velocity of the flow, and the boundary conditions (1.1b) and (1.1d) describe the flow rates across the boundary.

Experimental results have shown that it is preferable to approximate a nondivergence form of the equation (1.1c). Differentiation of the Darcy velocity term in (1.1c) and use of (1.1b) leads to the alternative form

$$
(1.2)\quad
\begin{aligned}
&\text{(a), (b) as in (1.1),}\\
&\text{(c)}\quad \phi\frac{\partial c}{\partial t}-\nabla\cdot(b\nabla c)+u\cdot\nabla c=(\hat{c}-c)f_1\equiv f(x,t,c), && x\in\Omega,\quad t\in J,\\
&\text{(d)}\quad b\frac{\partial c}{\partial n}=g_1c+g_2\equiv g(x,t,c), && x\in\partial\Omega,\quad t\in J,\\
&\text{(e)}\quad \text{as in (1.1).}
\end{aligned}
$$

We note that the natural Neumann boundary condition is necessarily nonlinear unless g_1 is assumed to be homogeneous, as was done in [14]. Thus we are naturally led to the nonlinear boundary condition by practical considerations.

It will be clear that, if f_1 and f satisfy the smoothness assumptions to be placed on g_1 and g respectively in § 2, then the analysis of the f_1 and f terms will follow from that of the g_1 and g terms. We therefore assume that $f_1\equiv f\equiv 0$ in what follows for simplicity of exposition. In the oil reservoir problem, it must be pointed out that f_1 is normally taken to be a singular distribution of point sources and sinks, modeling the effect of small injection and production wells. The analysis will fail if f_1 is a singular distribution, so we are analyzing a model with smoothly distributed sources and sinks.

Possible extensions of this work include the use of interior penalties in approximating the concentration, combining with the method of characteristics to treat the physically dominant first-order term of the concentration equation, and the use of a mixed method to approximate the pressure. Wheeler and Darlow [25] have shown that the results of [14] are compatible with penalties on the jumps of normal derivatives of continuous piecewise polynomials across inter-element boundaries. Experiments indicate that penalties improve the accuracy of standard Galerkin approximations. The use of a mixed method, simultaneously approximating both the pressure and the Darcy velocity, is suggested by the fact that the concentration equation (1.2c) depends on the pressure only through the Darcy velocity u. Details of the analysis of using mixed methods together with interior penalties on both the function values and normal derivatives will appear in [7]. In [22] Russell has proved that for a spatially periodic version of the miscible displacement problem, the methods of this paper may be combined with the method of characteristics, preserving the asymptotic order of the errors while expecting to reduce their actual size.

A brief outline of this paper is as follows. In § 2, we define our finite element spaces, list our constraints on the domain, coefficients, and solutions, introduce elliptic projections of the solutions which will aid the convergence analysis, and define our modified backward difference-Galerkin method. In § 3, we consider the algebraic problem, describing the matrices, the preconditioned conjugate gradient iterative procedures, and the stability conditions on which the convergence results depend. In § 4, we obtain global L^2 and H^1 error estimates for the procedures described in § 2 and § 3, and for any other iterative procedure achieving the stability conditions of § 3. Section 5 contains a discussion of the computational work estimates obtained from the results in § 4.

2. Preliminaries and description of Galerkin methods. Let $W_p^k=W_p^k(\Omega)=\{\psi|(\partial^\alpha\psi/\partial x^\alpha)\in L^p(\Omega)\text{ for }|\alpha|\leqq k\}$ be the Sobolev space on Ω with the usual norm. If $p=2$, we write $H^k=H^k(\Omega)=W_2^k(\Omega)$ with norm $\|\psi\|_k=\|\psi\|_{H^k}=\|\psi\|_{W_2^k(\Omega)}$. We write $\|\psi\|$

for $\|\psi\|_0$. On the boundary $\partial\Omega$, we let $H^s(\partial\Omega)$ denote the Sobolev space with norm $|\psi|_s$. We define the inner products $(\varphi, \psi) = \int_\Omega \varphi\psi\, dx$ and $\langle \varphi, \psi\rangle = \int_{\partial\Omega} \varphi\psi\, d\sigma$.

We also wish to consider spaces of the form

$$W_p^l((a, b); X) = \left\{\psi: (a, b) \to X \,\middle|\, \left\|\frac{\partial^\alpha \psi}{\partial t^\alpha}(t)\right\|_X \in L^p((a, b))\right\},$$

with norm

$$\|\psi\|_{W_p^l(a,b;X)} = \left[\sum_{\alpha=0}^{l} \left\|\left\|\frac{\partial^\alpha \psi}{\partial t^\alpha}(t)\right\|_X\right\|_{L^p(a,b)}^p\right]^{1/p},$$

where X is a Banach space. In the applications, X will be a Sobolev space $W_q^k(\Omega)$, and we will write $\|\psi\|_{W_p^l(a,b;W_q^k)} = \|\psi\|_{W_p^l(a,b;W_q^k(\Omega))}$. If $(a, b) = (0, T)$, we suppress the time interval and write $\|\psi\|_{W_p^l(W_q^k)} = \|\psi\|_{W_p^l(0,T;W_q^k(\Omega))}$.

We approximate the concentration c and the pressure p, respectively, by families $\mathcal{M}_h$ and $\mathcal{N}_h$ of finite-dimensional subspaces of $H^1(\Omega)$. We assume that there exist integers $r \geqq 2$ and $s \geqq 2$ and a constant K_0 independent of h such that these subspaces satisfy the following approximation properties and inverse hypotheses:

(A)
$$\inf_{\chi\in\mathcal{M}_h} (\|\psi-\chi\| + h\|\psi-\chi\|_1 + h^{d/2}(\|\psi-\chi\|_{L^\infty} + h\|\psi-\chi\|_{W_\infty^1}))$$
$$\leqq K_0 h^k \|\psi\|_k, \quad \text{all } \psi \in H^k(\Omega), \quad 2 \leqq k \leqq r,$$
$$\inf_{\varphi\in\mathcal{N}_h} (\|\psi-\varphi\| + h\|\psi-\varphi\|_1 + h^{d/2}(\|\psi-\varphi\|_{L^\infty} + h\|\psi-\varphi\|_{W_\infty^1}))$$
$$\leqq K_0 h^k \|\psi\|_k, \quad \text{all } \psi \in H^k(\Omega), \quad 2 \leqq k \leqq s,$$

where $h^{d/2} = h^{2/2} = h$;

(I)
$$\|\chi\|_{W_\infty^j} \leqq K_0 h^{-d/2}\|\chi\|_j = K_0 h^{-1}\|\chi\|_j, \quad \text{all } \chi \in \mathcal{M}_h, \quad j = 0, 1,$$
$$\|\chi\|_1 \leqq K_0 h^{-1}\|\chi\|, \quad \text{all } \chi \in \mathcal{M}_h,$$
$$\|\nabla\varphi\|_{L^\infty} \leqq K_0 h^{-d/2}\|\nabla\varphi\| = K_0 h^{-1}\|\nabla\varphi\|, \quad \text{all } \varphi \in \mathcal{N}_h.$$

We note that it is entirely permissible to associate distinct spatial mesh parameters h_c and h_p with the concentration and pressure, respectively. Since this would not significantly affect the analysis, we use a single h for economy of notation.

We recall that Ω is said to be H^k*-regular* if

$$-\Delta v + v = \zeta, \qquad x \in \Omega,$$
$$\frac{\partial v}{\partial n} = \eta, \qquad x \in \partial\Omega$$

implies that $\|v\|_k \leqq K(\Omega)[\|\zeta\|_{k-2} + |\eta|_{k-3/2}]$. We assume that Ω satisfies the smoothness constraints

(S) Ω is H^2-regular (in certain cases we will suppose that Ω is H^3-regular); $\partial\Omega$ is Lipschitz.

Our assumptions about the coefficients require some comments. In the physical problem under consideration, the concentration c lies between 0 and 1, inclusive. For c within this range, the coefficients depend smoothly on c. We will therefore extend the coefficients to values of c outside $[0, 1]$ by truncating c to $[0, 1]$ before evaluating them. Accordingly, we may restrict κ to lie in $[0, 1]$ in the assumptions below. The physical coefficients also depend smoothly on ∇p provided that ∇p remains bounded. Assuming sufficient regularity of the solutions, our arguments will show that the approximate pressure gradient remains bounded. Thus we can require (π_1, π_2) to be bounded in the assumptions below. The assumptions are then consistent with the nature of the physical problem. We will suppose that

(C) There exist uniform constants such that, for $0 \leqq \kappa \leqq 1$, (π_1, π_2) bounded in $\mathbb{R}^2$, $x \in \Omega$, $t \in J$, and $i = 1, 2$,

$$0 < a_* \leqq a(x, \kappa) \leqq a^* \leqq K_1,$$
$$0 < \phi_* \leqq \phi(x) \leqq \phi^* \leqq K_1,$$
$$0 < b_* \leqq b(x, \kappa, \pi_1, \pi_2) \leqq b^* \leqq K_1,$$
$$|\gamma(x, \kappa)| \leqq K_1,$$
$$|\nabla d(x)| \leqq K_1,$$
$$|u_i(x, \kappa, \pi_1, \pi_2)| \leqq K_1(1 + |(\pi_1, \pi_2)|),$$
$$|g_1(x, t)| \leqq K_1,$$
$$|g(x, t, \kappa)| \leqq K_1,$$

and, for arguments evaluated at κ, π_1, π_2 where appropriate,

$$\left|\frac{\partial a}{\partial x_i}\right|, \quad \left|\frac{\partial a}{\partial c}\right|, \quad \left|\frac{\partial^2 a}{\partial x_i\, \partial c}\right|, \quad \left|\frac{\partial^2 a}{\partial c^2}\right|, \quad \left|\frac{\partial \gamma}{\partial c}\right|, \quad \left|\frac{\partial^4 g}{\partial x_i^2\, \partial c\, \partial t}\right|, \quad \left|\frac{\partial^4 g}{\partial x_i^2\, \partial c^2}\right|$$

and all partial derivatives of b, u_i, and g of order up to 3 are uniformly bounded by K_1.

We organize our regularity hypotheses on the solution (p, c) of the differential problem according to the results in which they are used. We assume

$$(\mathrm{R}_1) \quad c \in L^\infty(H^r) \cap H^1(H^{r-1}) \cap W^1_\infty(H^2) \cap W^1_\infty(W^1_\infty) \cap H^2(H^1) \cap W^2_\infty(L^\infty),$$
$$p \in L^2(H^s) \cap W^1_\infty(H^3) \cap W^1_\infty(W^2_\infty) \cap W^2_\infty(W^1_\infty);$$
$$(\mathrm{R}_6) \quad c \in (R_1) \cap L^\infty(H^{2+\varepsilon}) \cap H^1(H^r),$$
$$p \in (R_1) \cap H^1(H^s);$$
$$(\mathrm{R}_3) \quad c \in (R_6) \cap H^1(W^2_\infty) \cap W^1_\infty(H^{2+\varepsilon}),$$
$$p \in (R_6).$$

Let K_2 be a bound for the norms of the functions in all of the spaces in (R_1), (R_6), and (R_3).

Our analysis will use the technique of Wheeler [24] in examining two auxiliary elliptic problems. For each $t \in J$, we define $\tilde{p} \in \mathcal{N}_h$ to be the elliptic projection of p given by

$$(2.1) \quad \begin{aligned} &(a(c(t))\nabla \tilde{p}, \nabla \varphi) = (a(c(t))\nabla p, \nabla \varphi) = (a(c)\gamma(c)\nabla d, \nabla \varphi) - \langle g_1(t), \varphi \rangle, \quad \text{all } \varphi \in \mathcal{N}_h, \\ &(p - \tilde{p}, 1) = 0. \end{aligned}$$

As in [8], [24], the restrictions (A), (S), (C), and (R_1) imply the following result. Let $\theta = p - \tilde{p}$.

LEMMA 2.1. *There exists $K_3 = K_3(\Omega, a_*, K_0, K_1, K_2)$ such that, for $1 \leqq k \leqq s$ and $t \in J$,*

$$\|\theta\| + h\|\theta\|_1 \leqq K_3 h^k \|p\|_k,$$

$$\left\|\frac{\partial\theta}{\partial t}\right\| + h\left\|\frac{\partial\theta}{\partial t}\right\|_1 \leqq K_3 h^k\left(\|p\|_k + \left\|\frac{\partial p}{\partial t}\right\|_k\right), \tag{2.2}$$

$$\left\|\frac{\partial^2\theta}{\partial t^2}\right\|_1 \leqq K_3.$$

For the concentration, we define a nonlinear projection $\tilde{c}$ for each $t \in J$ satisfying, for $\chi \in \mathcal{M}_h$,

$$(b(c, \nabla p)\nabla(\tilde{c} - c), \nabla\chi) + (u(c, \nabla p) \cdot \nabla(\tilde{c} - c), \chi) + \lambda(\tilde{c} - c, \chi) - \langle g(t, \tilde{c}) - g(t, c), \chi\rangle = 0, \tag{2.3}$$

where λ is a positive constant to be fixed sufficiently large that existence and uniqueness of $\tilde{c}$ are assured. Define

$$G(x, t) = \int_0^1 \frac{\partial g}{\partial c}(x, t, \alpha c(x, t) + (1 - \alpha)\tilde{c}(x, t))\, d\alpha, \tag{2.4}$$

and

$$B(\varphi, \chi) = (b(c, \nabla p)\nabla\varphi, \nabla\chi) + (u(c, \nabla p) \cdot \nabla\varphi, \chi) + \lambda(\varphi, \chi) - \langle G\varphi, \chi\rangle. \tag{2.5}$$

Then for $\xi = c - \tilde{c}$, we also restrict λ to be sufficiently large that there exists a constant $K_* > 0$ satisfying

$$B(\xi, \xi) \geqq K_*\|\xi\|_1^2$$

(see [3] for such a choice of λ). Then, using the techniques of [8], [19], [24] with the assumptions (A), (S), (C), and (R_1), we have the following result which will allow us to estimate the error of our approximate concentration solution by estimating the difference between that solution and the elliptic projection $\tilde{c}$.

LEMMA 2.2. *If $\xi = c - \tilde{c}$, with $\tilde{c}$ defined in (2.3), there exists $K_4 = K_4(\Omega, b_*, \lambda, K_0, K_1, K_2, K_*)$ such that, for $1 \leqq k \leqq r$ and $p = 2$ or $p = \infty$,*

$$\|\xi\|_{L^p(L^2)} + h\|\xi\|_{L^p(H^1)} \leqq K_4 h^k \|c\|_{L^p(H^k)},$$

$$\left\|\frac{\partial\xi}{\partial t}\right\|_{L^p(L^2)} + h\left\|\frac{\partial\xi}{\partial t}\right\|_{L^p(H^1)} \leqq K_4 h^k\left(\|c\|_{L^p(H^k)} + \left\|\frac{\partial c}{\partial t}\right\|_{L^p(H^k)}\right), \tag{2.6}$$

$$\left\|\frac{\partial^2\xi}{\partial t^2}\right\|_{L^2(H^1)} \leqq K_4.$$

We can now argue as in [3], [6], [24] to obtain bounds for the projections $\tilde{p}$ and $\tilde{c}$. Using (A), (S), (C), and Lemmas 2.1 and 2.2, we have the following result.

LEMMA 2.3. *There exists $K_5 = K_5(K_0, K_2, K_3, K_4)$ such that if (R_1) holds, then*

$$\|\tilde{c}\|_{L^\infty(W^1_\infty)} + \left\|\frac{\partial\tilde{c}}{\partial t}\right\|_{L^2(H^1)} + \left\|\frac{\partial\tilde{c}}{\partial t}\right\|_{L^\infty(L^\infty)} + \left\|\frac{\partial^2\tilde{c}}{\partial t^2}\right\|_{L^2(H^1)} + \|\tilde{p}\|_{L^\infty(W^1_\infty)} + \left\|\frac{\partial\tilde{p}}{\partial t}\right\|_{L^\infty(W^1_\infty)} + \left\|\frac{\partial^2\tilde{p}}{\partial t^2}\right\|_{L^2(H^1)} \leqq K_5. \tag{2.7}$$

If (R_6) *holds, then also*

$$\left\|\frac{\partial \tilde{c}}{\partial t}\right\|_{L^\infty(W^1_\infty)} \leqq K_5. \tag{2.8}$$

The nonlinear boundary condition and some integrations by parts will require us to estimate certain boundary integrals. To do this most efficiently we will need some negative-norm estimates in Sobolev spaces on $\partial\Omega$. We will also use negative-norm estimates on Ω to achieve minimal smoothness requirements for the solution of the differential problem. For $k \geqq 0$, $s \geqq 0$, define these negative norms by

$$\|\psi\|_{-k} = \sup\{(\psi, \varphi) \,|\, \|\varphi\|_k = 1\}, \qquad |\psi|_{-s} = \sup\{\langle \psi, \varphi\rangle \,|\, |\varphi|_s = 1\}.$$

We collect these estimates in the following result.

LEMMA 2.4. *Assume that the regularity* (R_1) *holds. Then there exists* $K_6 = K_6(\Omega, a_*, b_*, K_0, K_1, K_2, K_3, K_4)$ *such that the following statements hold*:

For each $t \in J$ *and* $1 \leqq k \leqq s$,

$$|\theta|_{-1/2} \leqq K_6 h^k \|p\|_k, \tag{2.9}$$

$$\left|\frac{\partial \theta}{\partial t}\right|_{-1/2} \leqq K_6 h^k\left(\|p\|_k + \left\|\frac{\partial p}{\partial t}\right\|_k\right). \tag{2.10}$$

If Ω *is* H^3*-regular then, for each* $t \in J$ *and* $1 \leqq k \leqq r$, $r \geqq 3$,

$$|\xi|_{-3/2} \leqq K_6 h^{k+1} \|c\|_k, \tag{2.11}$$

$$\|\xi\|_{-1} \leqq K_6 h^{k+1} \|c\|_k, \tag{2.12}$$

$$\left\|\frac{\partial \xi}{\partial t}\right\|_{-1} \leqq \begin{cases} K_6 h^{k+1}\left(\|c\|_k + \left\|\dfrac{\partial c}{\partial t}\right\|_k\right), & r \geqq 3, \\ K_6 h^{k+1}\left(\|c\|_k^2 + \|c\|_k \left\|\dfrac{\partial c}{\partial t}\right\|_k\right), & r = 2. \end{cases} \tag{2.13}$$

Proof. We first consider θ. Let $\psi \in H^{1/2}(\partial\Omega)$, and let f satisfy

$$\begin{aligned} -\nabla \cdot (a(c)\nabla f) + f &= 0, \qquad && x \in \Omega, \\ a(c)\frac{\partial f}{\partial n} &= \psi, && x \in \partial\Omega. \end{aligned} \tag{2.14}$$

Under our smoothness assumptions, we then have $\|f\|_2 \leqq K|\psi|_{1/2}$ as in [18]. Then

$$\begin{aligned} \langle \theta, \psi\rangle = \left\langle \theta, a(c)\frac{\partial f}{\partial n}\right\rangle &= (\nabla\theta, a(c)\nabla f) + (\theta, \nabla\cdot(a(c)\nabla f)) \\ &= (\nabla \theta, a(c)\nabla(f-\varphi)) + (\theta, f), \quad \text{all } \varphi \in \mathcal{N}_h, \end{aligned} \tag{2.15}$$

by (2.1) and (2.14). Thus, by (A), (2.2), and the H^2-regularity of Ω, for $1 \leqq k \leqq s$

$$\begin{aligned} |\langle \theta, \psi\rangle| &\leqq K\left(\|\theta\|_1 \inf_{\varphi \in \mathcal{N}_h} \|f-\varphi\|_1 + \|\theta\|\|f\|\right) \\ &\leqq K(h^{k-1}\|p\|_k h\|f\|_2 + h^k\|p\|_k\|f\|) \\ &\leqq K h^k \|p\|_k |\psi|_{1/2}, \end{aligned} \tag{2.16}$$

from which (2.9) follows.

We now differentiate the equation below, obtained from (2.1),

$$(a(c)\nabla\theta, \nabla\varphi) = 0, \qquad \varphi \in \mathcal{N}_h, \tag{2.17}$$

with respect to t to obtain

$$\left(a(c)\nabla\frac{\partial\theta}{\partial t}, \nabla\varphi\right) = -\left(\frac{\partial a}{\partial c}\frac{\partial c}{\partial t}\nabla\theta, \nabla\varphi\right), \qquad \varphi \in \mathcal{N}_h. \tag{2.18}$$

Then

$$\begin{aligned}\left\langle\frac{\partial\theta}{\partial t}, \psi\right\rangle &= \left(\nabla\frac{\partial\theta}{\partial t}, a(c)\nabla f\right) + \left(\frac{\partial\theta}{\partial t}, \nabla\cdot(a(c)\nabla f)\right)\\ &= \left(\nabla\frac{\partial\theta}{\partial t}, a(c)\nabla(f-\varphi)\right) + \left(\frac{\partial\theta}{\partial t}, f\right) - \left(\frac{\partial a}{\partial c}\frac{\partial c}{\partial t}\nabla\theta, \nabla\varphi\right), \qquad \varphi \in \mathcal{N}_h,\end{aligned} \tag{2.19}$$

by (2.18). This then yields, for $1 \leqq k \leqq s$,

$$\begin{aligned}\left|\left\langle\frac{\partial\theta}{\partial t}, \psi\right\rangle\right| &\leqq K\left(\left\|\frac{\partial\theta}{\partial t}\right\|_1 \|f-\varphi\|_1 + \left\|\frac{\partial\theta}{\partial t}\right\|\|f\|\right)\\ &\qquad + \left|\left(\frac{\partial a}{\partial c}\frac{\partial c}{\partial t}\nabla\theta, \nabla(f-\varphi)\right)\right| + \left|\left(\frac{\partial a}{\partial c}\frac{\partial c}{\partial t}\nabla\theta, \nabla f\right)\right|\\ &\leqq K\left[h^{k-1}\left(\|p\|_k + \left\|\frac{\partial p}{\partial t}\right\|_k\right)h\|f\|_2 + h^k\left(\|p\|_k + \left\|\frac{\partial p}{\partial t}\right\|_k\right)\|f\|\right.\\ &\qquad \left. + h^{k-1}\|p\|_k h\|f\|_2 + |(a(c)\nabla\theta, \nabla f)|\right]\\ &\leqq K\left[h^k\left(\|p\|_k + \left\|\frac{\partial p}{\partial t}\right\|_k\right)\|f\|_2 + |(a(c)\nabla\theta, \nabla(f-\varphi))|\right]; \qquad \varphi \in \mathcal{N}_h.\end{aligned}$$

We then have

$$\left|\left\langle\frac{\partial\theta}{\partial t}, \psi\right\rangle\right| \leqq Kh^k\left(\|p\|_k + \left\|\frac{\partial p}{\partial t}\right\|_k\right)|\psi|_{1/2},$$

from which (2.10) follows.

Next, we consider ξ. For the remainder of the proof, assume that Ω is H^3-regular. From (2.3) and (2.5), we have

$$B(\xi, \chi) = 0, \qquad \chi \in \mathcal{M}_h. \tag{2.20}$$

Differentiating (2.20) with respect to t, we find that

$$\begin{aligned}B\left(\frac{\partial\xi}{\partial t}, \chi\right) &= -\left(\left(\frac{\partial b}{\partial c}\frac{\partial c}{\partial t} + \frac{\partial b}{\partial\nabla p}\frac{\partial\nabla p}{\partial t}\right)\nabla\xi, \nabla\chi\right) - \left(\left(\frac{\partial u}{\partial c}\frac{\partial c}{\partial t} + \frac{\partial u}{\partial\nabla p}\frac{\partial\nabla p}{\partial t}\right)\cdot\nabla\xi, \chi\right)\\ &\qquad + \left\langle\int_0^1\left[\frac{\partial^2 g}{\partial t\,\partial c} + \frac{\partial^2 g}{\partial c^2}\left\{\alpha\frac{\partial c}{\partial t} + (1-\alpha)\frac{\partial\tilde{c}}{\partial t}\right\}\right]d\alpha\,\xi, \chi\right\rangle\\ &\equiv N(\chi), \qquad \chi \in \mathcal{M}_h.\end{aligned} \tag{2.21}$$

Let $\psi \in H^{1/2}(\partial\Omega)$ and let f be the solution of

$$\begin{aligned}-\nabla\cdot[b(c, \nabla p)\nabla f + u(c, \nabla p)f] + \lambda f &= 0, \qquad x \in \Omega,\\ b(c, \nabla p)\frac{\partial f}{\partial n} + [u(c, \nabla p)\cdot n]f - Gf &= \psi, \qquad x \in \partial\Omega,\end{aligned} \tag{2.22}$$

where G is defined in (2.4). By our smoothness assumptions, we then have $\|f\|_3 \leqq K|\psi|_{3/2}$ [18]. We also see that, by (2.3), (2.20) and (2.22),

$$
\begin{aligned}
\langle \xi, \psi\rangle &= \left\langle \xi, b(c, \nabla p)\frac{\partial f}{\partial n} + [u(c, \nabla p)\cdot n]f - Gf\right\rangle \\
&= (\nabla \xi, b(c, \nabla p)\nabla f + u(c, \nabla p)f) + (\xi, \nabla\cdot(b(c, \nabla p)\nabla f + u(c, \nabla p)f)) - \langle Gf, \xi\rangle \\
&= (b(c, \nabla p)\nabla \xi, \nabla f) + (u(c, \nabla p)\cdot \nabla \xi, f) + (\xi, \lambda f) - \langle g(t, c) - g(t, \tilde{c}), f\rangle \\
&= B(\xi, f) = B(\xi, f-\chi), \qquad \chi \in \mathcal{M}_h.
\end{aligned} \tag{2.23}
$$

Thus, for $1 \leqq k \leqq r$,

$$
\begin{aligned}
|\langle \xi, \psi\rangle| &\leqq K\|\xi\|_1 \inf_{\chi\in\mathcal{M}_h} \|f-\chi\|_1 \leqq Kh^{k-1}\|c\|_k h^2\|f\|_3 \\
&\leqq Kh^{k+1}\|c\|_k|\psi|_{3/2},
\end{aligned} \tag{2.24}
$$

from which (2.11) follows.

Let $\psi \in H^1(\Omega)$ and let f be the solution of

$$
\begin{aligned}
-\nabla\cdot[b(c, \nabla p)\nabla f + u(c, \nabla p)f] + \lambda f &= \psi, \qquad x \in \Omega, \\
b(c, \nabla p)\frac{\partial f}{\partial n} + [u(c, \nabla p)\cdot n]f - Gf &= 0, \qquad x \in \partial\Omega.
\end{aligned} \tag{2.25}
$$

By H^3-regularity, $\|f\|_3 \leqq K\|\psi\|_1$. We see at once that, emulating (2.23),

$$
(\xi, \psi) = B(\xi, f) = B(\xi, f-\chi), \qquad \chi \in \mathcal{M}_h. \tag{2.26}
$$

Thus

$$
|(\xi, \psi)| \leqq K\|\xi\|_1 \inf_{\chi\in\mathcal{M}_h} \|f-\chi\|_1 \leqq Kh^{k-1}\|c\|_k h^2\|f\|_3 \leqq Kh^{k+1}\|c\|_k\|\psi\|_1, \tag{2.27}
$$

provided that $1 \leqq k \leqq r$ and $r \geqq 3$. Then (2.12) follows. We also have that

$$
\begin{aligned}
\left(\frac{\partial \xi}{\partial t}, \chi\right) &= B\left(\frac{\partial \xi}{\partial t}, f\right) = B\left(\frac{\partial \xi}{\partial t}, f-\chi\right) + N(\chi) \\
&= B\left(\frac{\partial \xi}{\partial t}, f-\chi\right) - N(f-\chi) + N(f).
\end{aligned} \tag{2.28}
$$

Then, for $1 \leqq k \leqq r$,

$$
\begin{aligned}
|N(f-\chi)| &\leqq K\left(K_1, \left\|\frac{\partial c}{\partial t}\right\|_{L^\infty(L^\infty)}, \left\|\frac{\partial p}{\partial t}\right\|_{L^\infty(W^1_\infty)}\right)\|\xi\|_1 \inf_{\chi\in\mathcal{M}_h} \|f-\chi\|_1 \\
&\leqq Kh^{k+1}\|c\|_k\|f\|_3 \leqq Kh^{k+1}\|c\|_k\|\psi\|_1.
\end{aligned} \tag{2.29}
$$

We then see that

$$
N(f) = (\xi, \nabla\cdot(\alpha_1\nabla f)) + (\xi, (\nabla\cdot\alpha_2)f) - \left\langle \xi, \alpha_1\frac{\partial f}{\partial n}\right\rangle - \langle \xi, [\alpha_2\cdot n]f\rangle + \langle \alpha_3\xi, f\rangle, \tag{2.30}
$$

where

$$
\begin{aligned}
\alpha_1 &= \frac{\partial b}{\partial c}\frac{\partial c}{\partial t} + \frac{\partial b}{\partial \nabla p}\frac{\partial \nabla p}{\partial t},\\
\alpha_2 &= \frac{\partial u}{\partial c}\frac{\partial c}{\partial t} + \frac{\partial u}{\partial \nabla p}\frac{\partial \nabla p}{\partial t},\\
\alpha_3 &= \int_0^1 \left[\frac{\partial^2 g}{\partial t\,\partial c} + \frac{\partial^2 g}{\partial c^2}\frac{\partial c}{\partial t} + \frac{\partial^2 g}{\partial c^2}(\beta-1)\frac{\partial \xi}{\partial t}\right] d\beta\\
&\equiv \alpha_4 + \alpha_5 + \alpha_6.
\end{aligned} \tag{2.31}
$$

We note that

$$
\begin{aligned}
&\|\alpha_1\|_{L^\infty(H^2)} \leqq K\left(K_1, \left\|\frac{\partial c}{\partial t}\right\|_{L^\infty(H^2)}, \left\|\frac{\partial p}{\partial t}\right\|_{L^\infty(H^3)}\right),\\
&\|\alpha_2\|_{L^\infty(H^2)} \leqq K\left(K_1, \left\|\frac{\partial c}{\partial t}\right\|_{L^\infty(H^2)}, \left\|\frac{\partial p}{\partial t}\right\|_{L^\infty(H^3)}\right),\\
&\|\alpha_4\|_{L^\infty(H^2)} \leqq K(K_1),\\
&\|\alpha_5\|_{L^\infty(H^2)} \leqq K\left(K_1, \left\|\frac{\partial c}{\partial t}\right\|_{L^\infty(H^2)}\right).
\end{aligned} \tag{2.32}
$$

Also we have, by Lemma 2.2 and the trace theorem,

$$
\begin{aligned}
\langle \alpha_6 \xi, f\rangle &\leqq \left\|\frac{\partial^2 g}{\partial c^2} f\right\|_{L^\infty} |\xi_t|\,|\xi|\\
&\leqq K\left\|\frac{\partial^2 g}{\partial c^2} f\right\|_{L^\infty} \|\xi_t\|^{1/2}\|\xi_t\|_1^{1/2}\|\xi\|^{1/2}\|\xi\|_1^{1/2}\\
&\leqq K\|f\|_{L^\infty} h^{2r-1}\left\{\|c\|_k^2 + \|c\|_k\left\|\frac{\partial c}{\partial t}\right\|_k\right\}\\
&\leqq \begin{cases} K\|f\|_{L^\infty} h^{r+1}\left\{\|c\|_k^2 + \|c\|_k\left\|\frac{\partial c}{\partial t}\right\|_k\right\}, & r=2,\\ \varepsilon h^{k+1}, & r \geqq 3.\end{cases}
\end{aligned} \tag{2.33}
$$

Then we have, using (2.11), (2.12) and the bounds on α_i,

$$
\begin{aligned}
|N(f)| &\leqq \|\xi\|_{-1}\|\nabla\cdot(\alpha_1\nabla f)\|_1 + \|\xi\|_{-1}\|(\nabla\cdot\alpha_2)f\|_1\\
&\quad + |\xi|_{-3/2}\left\{\left|\alpha_1\frac{\partial f}{\partial n}\right|_{3/2} + |[\alpha_2\cdot n]f|_{3/2} + |[\alpha_4+\alpha_5]f|_{3/2}\right\} + \varepsilon h^{k+1}\\
&\leqq K h^{k+1}\|c\|_k\|f\|_3 \leqq K h^{k+1}\|c\|_k\|\psi\|_1
\end{aligned} \tag{2.34}
$$

if $r \geqq 3$, with the norm on c modified as above if $r=2$. Then, combining Lemma 2.2, (2.28), (2.29) and (2.34), we obtain

$$
\begin{aligned}
\left|\left(\frac{\partial\xi}{\partial t},\psi\right)\right| &\leqq \left\|\frac{\partial\xi}{\partial t}\right\|_1 \inf_{\chi\in\mathcal{M}_h}\|f-\chi\|_1+Kh^{k+1}\|c\|_k\|\psi\|_1\\
&\leqq Kh^{k-1}\left\{\|c\|_k+\left\|\frac{\partial c}{\partial t}\right\|_k\right\}h^2\|f\|_3+Kh^{k+1}\|c\|_k\|\psi\|_1 \qquad (2.35)\\
&\leqq Kh^{k+1}\left\{\|c\|_k+\left\|\frac{\partial c}{\partial t}\right\|_k\right\}\|\psi\|_1
\end{aligned}
$$

and (2.13) follows. Again, the norms on c are modified if $r=2$.

We now turn to the definition of our discrete-time Galerkin method. In [14], it was shown that in the case of homogeneous boundary conditions, a continuous-time Galerkin method will yield, where C is the numerical concentration approximation,

$$
(2.36)\qquad
\begin{aligned}
&\|C-c\|_{L^\infty(L^2)}=O(h^r+h^{s-1}) &&\text{if } b=b(x,c,\nabla p),\quad s\geqq 3,\\
&\|C-c\|_{L^\infty(L^2)}=O(h^r+h^{s}) &&\text{if } b=b(x,c),\quad r\geqq 3 \text{ or } s\geqq 3.
\end{aligned}
$$

Standard backward difference time-stepping procedures were also analyzed, and it was shown that they introduced the expected $O(\Delta t)$ time discretization error. We now modify the time-stepping procedures of [14] as indicated in the introduction to this paper.

Let $\Delta t_c>0$, $\Delta t_p>0$, $\Delta t_c^0>0$, $\Delta t_p^0>0$. Here Δt_c and Δt_p are the time steps for the concentration and pressure, respectively. We will see that the first pressure step must be smaller than the later ones, and we denote it by Δt_p^0. In Theorem 4.6, we will require two smaller initial concentration steps, denoted by Δt_c^0. We let the integer j denote the ratio $\Delta t_p/\Delta t_c$.

We use superscripts to denote concentration steps and subscripts for pressure steps. Thus, $t^n=n\Delta t_c$ and $t_m=\Delta t_p^0+(m-1)\Delta t_p$ prior to Theorem 4.6. We let $\psi^n=\psi(t^n)$, $\psi_m=\psi(t_m)$, and we denote difference quotients and differences by

$$
(2.37)\qquad
\begin{aligned}
&d_t\psi^n=\frac{\psi^{n+1}-\psi^n}{\Delta t_c}, && d_t\psi_m=\frac{\psi_{m+1}-\psi_m}{\Delta t_p}\quad\text{for } m>0,\\
& && d_t\psi_0=\frac{\psi_1-\psi_0}{\Delta t_p^0},\\
&\delta\psi^n=\psi^{n+1}-\psi^n, && \delta\psi_m=\psi_{m+1}-\psi_m,\\
&\delta^2\psi^n=\psi^{n+1}-2\psi^n+\psi^{n-1}, && \delta^2\psi_m=\psi_{m+1}-2\psi_m+\psi_{m-1}\quad\text{for } m\geqq 2,\\
& && \delta^2\psi_1=\psi_2-\left(1+\frac{\Delta t_p}{\Delta t_p^0}\right)\psi_1+\frac{\Delta t_p}{\Delta t_p^0}\psi_0.
\end{aligned}
$$

We make the obvious concentration modifications for Theorem 4.6.

The standard backward difference Galerkin scheme in [14] employed approximations $P\colon\{0=t^0,t^1,\cdots,t^N=T\}\to\mathcal{N}_h$ and $C\colon\{0=t^0,t^1,\cdots,t^N\}\to\mathcal{M}_h$, given by

$$
(2.38)\qquad
\begin{aligned}
&C^0(x)=\tilde{c}(x,0),\\
&(\phi d_tC^n,\chi)+(b(C^n,\nabla P^n)\nabla C^{n+1},\nabla\chi)=-(u(C^n,\nabla P^n)\cdot\nabla C^{n+i},\chi), &&\chi\in\mathcal{M}_h,\\
&(a(C^{n+1})\nabla P^{n+1},\nabla\varphi)=(a(C^{n+1})\gamma(C^{n+1})\nabla d,\nabla\varphi), &&\varphi\in\mathcal{N}_h,\\
&(P^{n+1},1)=0,
\end{aligned}
$$

where $i=0$ or 1. We see that P^0 can be determined from C^0, and then C^{n+1} and P^{n+1} can be found once C^n and P^n are known. By lagging the coefficients in the concentration equation, the scheme uncouples the system and reduces the algebraic problem to the solution of two separate linear systems. We note that if $i=0$, then the coefficient matrix arising from (2.38) is symmetric. In the physical problem, however, the transport term dominates the diffusion term, and it may be numerically advantageous to carry the transport term at the advanced time level by taking $i=1$, even though the matrix is no longer symmetric. In our methods, since we wish to consider a preconditioned conjugate gradient iterative scheme for the algebraic problem, we will consider only the case $i=0$.

We now describe our modifications of (2.38). With the time step ratio j chosen, we will linearly extrapolate the pressure in the evaluation of the coefficients in the concentration equation. Let F be a function of time, consider concentration time level t^n, and let m be the greatest integer such that $t_m < t^n$. We approximate F^n by extrapolating linearly from F_m and F_{m-1} if $m \geqq 1$. Define $\nu \in \{1, 2, \cdots, j\}$ by $t^n = t_m + \nu \, \Delta t_c$ if $m \geqq 1$. Then set

$$EF^n = \begin{cases} F_0, & \text{if } m=0, \\ \left(1+\frac{\nu}{j}\frac{\Delta t_p}{\Delta t_p^0}\right)F_1 - \frac{\nu}{j}\frac{\Delta t_p}{\Delta t_p^0}F_0, & \text{if } m=1, \\ \left(1+\frac{\nu}{j}\right)F_m - \frac{\nu}{j}F_{m-1}, & \text{if } m \geqq 2. \end{cases} \tag{2.39}$$

This will give an approximation of F^n with error $O((\Delta t_p)^2 \|d^2F/dt^2\|_{L^2(L^2)})$.

This procedure requires the computation of a new pressure extrapolation at each concentration time level. Other less accurate methods, such as an extrapolation to the midpoint of the current pressure time interval suggested by Todd Dupont, will demand somewhat less work. We will analyze only the linear extrapolation, noting that first-order methods need smaller pressure time steps in all our results except Theorem 4.6, where they fail completely.

An alternative procedure in evaluating the Darcy velocity $u(x, c, \nabla p)$ in the concentration equation is to extrapolate the velocity itself instead of the pressure argument. This is motivated by the fact that the velocity is smooth in time, while the individual factors depending on the concentration and pressure may be quite rough. This alternative should take greater advantage of the use of different time steps for the two equations. We will describe this alternative and obtain the same convergence results as for the extrapolation of the pressure.

Next, we recall that $0 \leqq c \leqq 1$ and that the coefficients satisfied the bounds in (C) for concentrations in this range. If C^n is the numerical approximation to the concentration, we define the truncation $C^{*n} = \min\{1, \max\{C^n, 0\}\}$ and replace C^n by C^{*n} in evaluating the coefficients. This type of truncation has been discussed earlier in [4, 11], and we will analyze the resulting error. Combining these modifications of (2.38), we obtain the scheme

$$\begin{aligned} &C^0(x) = \tilde{c}(x, 0), \\ &(\phi d_t C^n, \chi) + (b(C^{*n}, E\nabla P^{n+1})\nabla C^{n+1}, \nabla \chi) \\ &\quad = -(u(C^{*n}, E\nabla P^{n+1}) \cdot \nabla C^n, \chi) + \langle g(t^{n+1}, C^{*n}), \chi \rangle, \qquad \chi \in \mathcal{M}_h, \\ &(a(C_m^*)\nabla P_m, \nabla \varphi) = (a(C_m^*)\gamma(C_m^*)\nabla d, \nabla \varphi) - \langle g_1(t_m), \varphi \rangle, \qquad \varphi \in \mathcal{N}_h, \\ &(P_m, 1) = 0, \end{aligned} \tag{2.40}$$

which is to be implemented in the order $C^0, P_0, C^1, C^2, \cdots, C^j, P_1, C^{j+1}, C^{j+2}, \cdots, C^{2j}, P_2$, etc. In Theorem 4.3 and subsequent results, the argument C^{*n} of g will be changed back to C^n.

We conclude this section with a technical lemma which will expedite the estimation of errors arising from pressure conjugate gradient residuals.

LEMMA 2.5. *Let F_m and G_m, $2 \leqq m \leqq k-1$, be nonnegative numbers satisfying the relations*

$$\begin{aligned} &F_2 \leqq \bar{K}RG_2, \\ &F_3 \leqq \bar{K}(RG_3 + 2R^2G_2), \\ &F_m \leqq \bar{K}RG_m + 2RF_{m-1} + RF_{m-2}, \qquad 4 \leqq m \leqq k-1, \end{aligned} \tag{2.41}$$

where $\bar{K}$ and $0 \leqq R < 1/(1+\sqrt{5})$ are constants. Then there exists a constant $K = K(\bar{K}, R)$, independent of R bounded away from $1/(1+\sqrt{5})$, such that

$$\begin{aligned} &\sum_{m=2}^{k-1} F_m \leqq KR \sum_{m=2}^{k-1} G_m, \\ &F_m \leqq K \max_{2 \leqq n \leqq k-1} G_n, \qquad 2 \leqq m \leqq k-1. \end{aligned} \tag{2.42}$$

Proof. Since G_{m-1} does not appear in (2.41) until F_{m-1}, we see that

$$\begin{aligned} &F_{m-1} \leqq \bar{K}RG_{m-1} + T_{m-2}R, \\ &F_m \leqq \bar{K}RG_m + \bar{K}(2R^2)G_{m-1} + T_{m-2}R^2 \end{aligned} \tag{2.43}$$

for $m \geqq 4$, where T_{m-2} represents terms involving G_i for $2 \leqq i \leqq m-2$. For $\alpha \geqq 1$ and $\beta \geqq 0$, we let $c_{\alpha\beta}$ denote the coefficient of $\bar{K}R^\alpha G_{m-\beta}$ in the estimate for F_m obtained recursively. From (2.43), we have

$$\begin{aligned} &c_{10} = 1, && c_{1\beta} = 0, \quad \beta \geqq 1, \\ &c_{20} = 0, && c_{21} = 2, \\ &c_{\alpha 0} = 0, \quad \alpha \geqq 3, && c_{\alpha 1} = 0, \quad \alpha \geqq 3, \end{aligned} \tag{2.44}$$

and from the recursion estimate in (2.41) we see that

$$c_{\alpha+1,\beta} = 2c_{\alpha,\beta-1} + c_{\alpha,\beta-2}; \tag{2.45}$$

(2.44) and (2.45) show that $c_{\alpha\beta}$ is well defined.

We claim that $c_{\alpha\beta}$ is given by the formula

$$c_{\alpha\beta} = \begin{cases} \binom{\alpha-1}{\beta-(\alpha-1)} 2^{(2\alpha-2)-\beta}, & \alpha - 1 \leqq \beta \leqq 2\alpha - 2, \\ 0, & \text{otherwise}. \end{cases} \tag{2.46}$$

This will be demonstrated by induction on α. The case $\alpha = 1$ is verified in (2.44). For $\alpha \geqq 2$, we may assume $\beta \geqq 2$ since $\beta = 0$ and $\beta = 1$ are contained in (2.44). We check the relevant cases, using (2.46) and induction:

$$\begin{aligned} &\beta < \alpha - 1: && c_{\alpha\beta} = 2c_{\alpha-1,\beta-1} + c_{\alpha-1,\beta-2} = 0 + 0 = 0; \\ &\beta = \alpha - 1: && c_{\alpha\beta} = 2c_{\alpha-1,\alpha-2} + c_{\alpha-1,\alpha-3} = 2\binom{\alpha-2}{0} 2^{(2\alpha-4)-(\alpha-2)} + 0 = 2^{\alpha-1}; \\ &\alpha - 1 < \beta < 2\alpha - 2: && c_{\alpha\beta} = 2\binom{\alpha-2}{\beta-\alpha+1} 2^{2\alpha-\beta-3} + \binom{\alpha-2}{\beta-\alpha} 2^{2\alpha-\beta-2} = \binom{\alpha-1}{\beta-\alpha+1} 2^{(2\alpha-2)-\beta}; \\ &\beta = 2\alpha - 2: && c_{\alpha\beta} = 2c_{\alpha-1,2\alpha-3} + c_{\alpha-1,2\alpha-4} = 0 + \binom{\alpha-2}{\alpha-2} 2^{(2\alpha-4)-(2\alpha-4)} = 1; \\ &\beta > 2\alpha - 2: && c_{\alpha\beta} = 2c_{\alpha-1,\beta-1} + c_{\alpha-1,\beta-2} = 0 + 0 = 0. \end{aligned}$$

Thus the claim is proved.

Finally, we note that the coefficient of G_i in $\sum_{m=2}^{k-1} F_m$ is bounded by

$$\begin{aligned}(2.47)\qquad \sum_{m=i}^{\infty} \bar{K} \sum_{\alpha=1}^{\infty} R^\alpha c_{\alpha,m-i} &\leqq \bar{K} \sum_{\alpha=1}^{\infty}\left(R^\alpha \sum_{\beta=0}^{\infty} c_{\alpha\beta}\right)\\ &= \bar{K}\sum_{\alpha=1}^{\infty} R^\alpha(2+1)^{\alpha-1} = \bar{K}R\sum_{\alpha=0}^{\infty}(3R)^\alpha = KR,\end{aligned}$$

where $K=\bar{K}/(1-3R)$. Thus,

$$(2.48)\qquad \sum_{m=2}^{k-1} F_m \leqq \frac{\bar{K}R}{1-3R}\sum_{m=2}^{k-1} G_m,$$

and the first statement is proved.

To show the second assertion, we note first that, by the definition of $c_{\alpha\beta}$, we have

$$(2.49)\qquad F_m \leqq \bar{K}\left(\sum_{\alpha=1}^{\infty} c_{\alpha\beta}R^\alpha\right)\left(\max_n G_n\right),$$

so we require a bound independent of β for the sum in (2.49). To obtain this estimate, we use the fact that

$$(2.50)\qquad \sum_{\alpha=1}^{\infty}\binom{\alpha-1}{\beta-(\alpha-1)} = \sum_{\alpha=[(\beta+3)/2]}^{\beta+1}\binom{\alpha-1}{\beta-(\alpha-1)} = f_\beta \leqq K_0\left(\frac{1+\sqrt{5}}{2}\right)^\beta,$$

where f_β denotes the βth Fibonacci number. The first equality holds because all omitted terms are zero, and the second can be proved by an easy induction argument, since each term of the sum of f_β is itself the sum of two terms, one each from the sums for $f_{\beta-1}$ and $f_{\beta-2}$. For convenience, we now assume that β is even; the case where β is odd can be handled similarly. Using (2.46) and (2.50), we see that

$$\begin{aligned}\sum_{\alpha=1}^{\infty} c_{\alpha\beta}R^\alpha &= \sum_{\alpha=(\beta/2)+1}^{\beta+1}\binom{\alpha-1}{\beta-(\alpha-1)}2^{(2\alpha-2)-\beta}R^\alpha\\ (2.51)\qquad &\leqq K_0R^{(\beta/2)+1}\left(\frac{1+\sqrt{5}}{2}\right)^\beta \sum_{\alpha=(\beta/2)+1}^{\beta+1} 2^{(2\alpha-2)-\beta}R^{(\alpha-1)-\beta/2}\\ &\leqq K_0R^{(\beta/2)+1}\left(\frac{1+\sqrt{5}}{2}\right)^\beta \sum_{\gamma=0}^{\beta/2}(4R)^\gamma.\end{aligned}$$

This last expression is increasing in R, so it suffices to show that it is bounded independently of β for $1/(1+\sqrt{5})-\varepsilon_0 \leqq R \leqq 1/(1+\sqrt{5})-\varepsilon_1$. For such values of R, the sum grows as $(4R)^{(\beta/2)+1}$, and the entire expression is bounded by

$$\begin{aligned}(2.52)\qquad K_0R^{(\beta/2)+1}\left(\frac{1+\sqrt{5}}{2}\right)^\beta K_1(4R)^{(\beta/2)+1} &= K_0K_1\left(\frac{1+\sqrt{5}}{2}\right)^\beta (2R)^{\beta+2}\\ &= 4K_0K_1R^2((1+\sqrt{5})R)^\beta.\end{aligned}$$

This proves the second statement.

The estimates of this lemma are not sharp, since it should be possible to replace $1/(1+\sqrt{5})$ by $\frac{1}{3}$. The argument would require a deeper analysis of the binomial coefficients in the proof of the second statement of (2.42). In applications of this lemma via the theorems of § 4, the constants in the error estimates will be enormous if values of R near the limit are used. A practical margin of safety is not significantly affected by the possible improvement.

3. Approximate solution of the linear equations by iteration. In this section, we present the linear equations arising from (2.40). Since the coefficient matrices change at each time step and have bandwidths which increase as the meshes become finer, we wish to avoid direct factorization of them. Accordingly, we will consider a preconditioned conjugate gradient iterative method for the solution of the linear equations. The analysis will extend results of [6], [10], [11].

The convergence results of the next section will depend only upon the norm reduction inequalities defined here. The conjugate gradient algorithm is only one example of an iterative method meeting these criteria.

We now define some matrices and vectors. Let $\{\chi_i\}_{i=1}^{M_c}$ and $\{\varphi_i\}_{i=1}^{M_p}$ be bases for $\mathcal{M}_h$ and $\mathcal{N}_h$, respectively. We denote the exact solution of (2.40) by $(\bar{C}^n, \bar{P}_m)$, given in terms of the bases by

$$\bar{C}^n = \sum_{i=1}^{M_c} \bar{\kappa}_i^n \chi_i, \tag{3.1}$$
$$\bar{P}_m = \sum_{i=1}^{M_p} \bar{\pi}_{mi} \varphi_i.$$

The matrices and vectors in the linear problems are denoted by

$$\Phi = (\phi_{ij}) = ((\phi \chi_j, \chi_i)),$$

$$B^n(\kappa, \pi) = (b_{ij}^n(\kappa, \pi)) = \left(\left(b\left(\left(\sum_{l=1}^{M_c} \kappa_l^n \chi_l\right)^*, E\sum_{l=1}^{M_p} \pi_l^{n+1} \nabla \varphi_l\right) \nabla \chi_j, \nabla \chi_i\right)\right),$$

$$U^n(\kappa, \pi) = (u_i^n(\kappa, \pi)) = -\left(\left(u\left(\left(\sum_{l=1}^{M_c} \kappa_l^n \chi_l\right)^*, E\sum_{l=1}^{M_p} \pi_l^{n+1} \nabla \varphi_l\right) \cdot \sum_{l=1}^{M_c} \kappa_l^n \chi_l, \chi_i\right)\right)$$
$$+\left(\left\langle g\left(t^{n+1}, \left(\sum_{l=1}^{M_c} \kappa_l^n \chi_l\right)^*\right), \chi_i\right\rangle\right), \qquad i, j = 1, \cdots, M_c; \tag{3.2}$$

$$A_m(\kappa) = ((a_m)_{ij}(\kappa)) = \left(\left(a\left(\left(\sum_{l=1}^{M_c} \kappa_{ml} \chi_l\right)^*\right) \nabla \varphi_j, \nabla \varphi_i\right)\right),$$

$$\Gamma_m(\kappa) = ((\gamma_m)_i(\kappa)) = \left(\left(a\left(\left(\sum_{l=1}^{M_c} \kappa_{ml} \chi_l\right)^*\right) \gamma\left(\left(\sum_{l=1}^{M_c} \kappa_{ml} \chi_l\right)^*\right) \nabla d, \nabla \varphi_i\right)\right) - (\langle g_1(t_m), \varphi_i \rangle),$$
$$i, j = 1, \cdots, M_p.$$

Then we can write (2.40) in the form

$$L^n(\kappa, \pi)(\bar{\kappa}^{n+1} - \kappa^n) \equiv (\Phi + \Delta t_c B^n(\kappa, \pi))(\bar{\kappa}^{n+1} - \kappa^n)$$
$$= (\Delta t_c) U^n(\kappa, \pi) - (\Delta t_c) B^n(\kappa, \pi) \kappa^n, \tag{3.3}$$
$$A_m(\kappa) \bar{\pi}_m = \Gamma_m(\kappa).$$

We will not solve (3.3) exactly; instead, we will use a predetermined number of conjugate gradient [1], [2], [5], [6], [9], [10] iterations to advance the solution one time step. The iterative procedure will be stable provided that a sufficient norm reduction is achieved. The magnitude of this norm reduction requirement will be analyzed in the next section. In order to speed the iterative process, we will precondition by matrices which are known to be reasonably close to L^n and A_m. Specifically, define

$$B_0 = ((b_0 \nabla \chi_j, \nabla \chi_i)), \qquad i, j = 1, \cdots, M_c, \tag{3.4}$$
$$A_0 = ((a_0 \nabla \varphi_j, \nabla \varphi_i)), \qquad i, j = 1, \cdots, M_p,$$

where the functions $b_0(x)$, $a_0(x)$ can be chosen arbitrarily. We then use as preconditioners the matrices

$$L_0 \equiv \Phi + (\Delta t_c) B_0 \tag{3.5}$$

for the concentration and A_0 for the pressure. These preconditioners are independent of time, so only two matrix factorizations need be done in the entire procedure. Assuming that starting procedures for C^0 and P_0, which we discuss later, have been performed, good choices for b_0 and a_0 might be $b_0(x) = b(x, C^0(x), \nabla P_0(x))$ and $a_0 = a(x, C^0(x))$. We note that in practice, it may be more efficient to update and refactor the preconditioner from time to time.

Denote by

$$C^n = \sum_{i=1}^{M_c} \kappa_i^n \chi_i \quad \text{and} \quad P_m = \sum_{i=1}^{M_p} \pi_{mi} \varphi_i \tag{3.6}$$

the approximations to $\bar{C}^n$ and $\bar{P}_m$, respectively, obtained by iteratively solving (3.3). Assuming that C^n and P_m are known, we describe preconditioned conjugate gradient iteration procedures to approximate $\bar{C}^{n+1}$ and $\bar{P}_{m+1}$ from (3.3). Our initial guesses will be C^0 for $n = 0$, P_0 for $m = 0$, and, in most cases, linear extrapolations for $n \geqq 1$ and $m \geqq 1$. Specifically, our concentration iteration for $\kappa^{n+1} - \kappa^n$ will proceed as follows:

$$\begin{aligned}
n = 0\colon \quad & x_0 = 0, \\
n \geqq 1\colon \quad & x_0 = \kappa^n - \kappa^{n-1}, \\
& q_0 = L^n(\kappa, \pi) x_0 - (\Delta t_c) U^n(\kappa, \pi) + (\Delta t_c) B^n(\kappa, \pi) \kappa^n, \\
& s_0 = L_0^{-1} q_0, \\
& x_{k+1} = x_k + \alpha_k s_k, \quad \text{where } \alpha_k = \frac{-(L_0^{-1} q_k, q_k)_e}{(s_k, L^n(\kappa, \pi) s_k)_e}, \quad k \geqq 0, \\
& q_{k+1} = q_k + \alpha_k L^n(\kappa, \pi) s_k, \\
& s_{k+1} = L_0^{-1} q_{k+1} + \beta_k s_k, \quad \text{where } \beta_k = \frac{(L_0^{-1} q_{k+1}, q_{k+1})_e}{(L_0^{-1} q_k, q_k)_e}.
\end{aligned} \tag{3.7}$$

Here $(\cdot,\cdot)_e$ denotes the Euclidean inner product, and the x_k, q_k, and s_k are the iterates, residuals, and search directions, respectively. Finally, after some predetermined number of iterations N_c, set

$$\kappa^{n+1} = \kappa^n + x_{N_c}. \tag{3.8}$$

Our analysis will show that for the results prior to Theorem 4.3, the linear extrapolation for the concentration is actually unnecessary and may be replaced by $x_0 = 0$. The pressure iteration for π_{m+1} will obey the algorithm

$$\begin{aligned}
m = 0\colon \quad & x_0 = \pi_0, \\
m \geqq 1\colon \quad & x_0 = 2\pi_m - \pi_{m-1}, \\
& q_0 = A_{m+1}(\kappa) x_0 - \Gamma_{m+1}(\kappa), \\
& s_0 = A_0^{-1} q_0, \\
& x_{k+1}, q_{k+1}, s_{k+1} \text{ determined as in (3.7) with } A_0, A_{m+1} \\
& \qquad \text{replacing } L_0, L^n, \text{ respectively}, \\
& \pi_{m+1} = x_{N_p}, \text{ where } N_p \text{ is predetermined.}
\end{aligned} \tag{3.9}$$

We will see that the linear extrapolation is unnecessary for Theorem 4.6, where $x_0 = \pi_m$ may be used.

It is well known [1], [2], [6], [10] that the preconditioned conjugate gradient methods yield constants ρ_c and ρ_p at each time step such that

$$
(3.10)\qquad
\begin{aligned}
&\|L^0(\kappa,\pi)^{1/2}(\bar\kappa^1-\kappa^1)\|_e \leqq \rho_c\|L^0(\kappa,\pi)^{1/2}(\bar\kappa^1-\kappa^0)\|_e, && n=0,\\
&\|L^n(\kappa,\pi)^{1/2}(\bar\kappa^{n+1}-\kappa^{n+1})\|_e \leqq \rho_c\|L^n(\kappa,\pi)^{1/2}(\bar\kappa^{n+1}-2\kappa^n+\kappa^{n-1})\|_e, && n\geqq 1,\\
&\|A_1(\kappa)^{1/2}(\bar\pi_1-\pi_1)\|_e \leqq \rho_p\|A_1(\kappa)^{1/2}(\bar\pi_1-\pi_0)\|_e, && m=0,\\
&\|A_{m+1}(\kappa)^{1/2}(\bar\pi_{m+1}-\pi_{m+1})\|_e \leqq \rho_p\|A_{m+1}(\kappa)^{1/2}(\bar\pi_{m+1}-2\pi_m+\pi_{m-1})\|_e, && m\geqq 1,
\end{aligned}
$$

where the subscript e indicates the Euclidean norm of the vector. Given the functions b_0 and a_0, we denote the comparability constants between the preconditioners L_0 and A_0 and the matrices L^n and A_m by δ_L, D_L, δ_A, and D_A, where these satisfy the inequalities

$$
(3.11)\qquad
\begin{aligned}
&0<\delta_L\leqq \frac{x^TL^n(\kappa,\pi)x}{x^TL_0x}\leqq D_L, && x\in\mathbb{R}^{M_c}-\{0\},\\
&0<\delta_A\leqq \frac{y^TA_m(\kappa)y}{y^TA_0y}\leqq D_A, && y\in\mathbb{R}^{M_p}-\{0\}.
\end{aligned}
$$

We note that these constants are independent of h and t, depending only on the bounds for the coefficients in (C). Letting

$$
(3.12)\qquad
\begin{aligned}
Q_c&=\frac{1-(\delta_L/D_L)^{1/2}}{1+(\delta_L/D_L)^{1/2}},\\
Q_p&=\frac{1-(\delta_A/D_A)^{1/2}}{1+(\delta_A/D_A)^{1/2}},
\end{aligned}
$$

we know from [1], [2], [5], [6], [10] that

$$
(3.13)\qquad \rho_c\leqq 2Q_c^{N_c},\qquad \rho_p\leqq 2Q_p^{N_p}.
$$

Since $Q_c<1$ and $Q_p<1$, it follows that norm reductions of ρ_c and ρ_p can be achieved in $O(\log(1/\rho_c))$ and $O(\log(1/\rho_p))$ iterations. In particular, a fixed norm reduction is reached in a fixed number of iterations.

Our analysis will be aided by the definition of some weighted norms. We set

$$
(3.14)\qquad
\begin{aligned}
&\|\psi\|_\phi^2=(\phi\psi,\psi),\\
&\|\psi\|_{b^n}^2=(b(C^{*n},E\nabla P^{n+1})\nabla\psi,\nabla\psi),\\
&\|\psi\|_{a_m}^2=(a(C_m^*)\nabla\psi,\nabla\psi),\\
&|||\psi|||_n^2=\|\psi\|_\phi^2+\Delta t_c\|\psi\|_{b^n}^2=(\phi\psi,\psi)+\Delta t_c(b(C^{*n},E\nabla P^{n+1})\nabla\psi,\nabla\psi).
\end{aligned}
$$

We note that $\|\cdot\|_\phi$ is equivalent to $\|\cdot\|$, and $\|\cdot\|_{b^n}$ and $\|\cdot\|_{a_m}$ are equivalent to $\|\nabla\cdot\|$. We see that by (3.10),

$$
\begin{aligned}
|||\bar C^1-C^1|||_0^2&=\|\bar C^1-C^1\|_\phi^2+\Delta t_c\|\bar C^1-C^1\|_{b^n}^2\\
&=(\bar\kappa^1-\kappa^1)^TL^0(\kappa,\pi)(\bar\kappa^1-\kappa^1)\\
&=\|L^0(\kappa,\pi)^{1/2}(\bar\kappa^1-\kappa^1)\|_e^2\\
&\leqq\rho_c^2\|L^0(\kappa,\pi)^{1/2}(\bar\kappa^1-\kappa^0)\|_e^2\\
&=\rho_c^2|||\bar C^1-C^0|||_0^2,
\end{aligned}
$$

so that

$$\|\!|\bar{C}^1 - C^1\|\!|_0 \leqq \frac{\rho_c}{1-\rho_c}\|\!|C^1 - C^0\|\!|_0 \equiv \rho_c'\|\!|\delta C^0\|\!|_0. \tag{3.15}$$

In a similar fashion, we can derive

$$\|\!|\bar{C}^{n+1} - C^{n+1}\|\!|_n \leqq \rho_c'\|\!|\delta^2 C^n\|\!|_n, \qquad n \geqq 1, \quad \text{where } \delta^2 C^n = C^{n+1} - 2C^n + C^{n-1},$$

$$\|\bar{P}_1 - P_1\|_{a_1} \leqq \rho_p'\|\delta P_0\|_{a_1}, \quad \text{where } \rho_p' = \frac{\rho_p}{1-\rho_p}, \tag{3.16}$$

$$\|\bar{P}_m - P_m\|_{a_m} \leqq \rho_p'\|\delta^2 P_{m-1}\|_{a_m}, \qquad m \geqq 2.$$

The convergence results of the next section depend only on the norm reductions (3.15)–(3.16) and not on the particular iterative method used to achieve those reductions.

We note here that we must alter the scheme (2.40) slightly since we are not solving the equations exactly. We have

$$\begin{aligned} &\text{(a)} \quad \left(\phi\frac{\bar{C}^{n+1} - C^n}{\Delta t_c}, \chi\right) + (b(C^{*n}, E\nabla P^{n+1})\nabla\bar{C}^{n+1}, \nabla\chi) \\ &\qquad = -(u(C^{*n}, E\nabla P^{n+1}) \cdot \nabla C^n, \chi) + \langle g(t^{n+1}, C^{*n}), \chi\rangle, \qquad \chi \in \mathcal{M}_h, \\ &\text{(b)} \quad (a(C_m^*)\nabla\bar{P}_m, \nabla\varphi) = (a(C_m^*)\gamma(C_m^*)\nabla d, \nabla\varphi) - \langle g_1(t_m), \varphi\rangle, \qquad \varphi \in \mathcal{N}_h. \end{aligned} \tag{3.17}$$

In Theorem 4.3 and afterward, the argument C^{*n} of g will be changed to C^n. We also note for future reference that, since $0 \leqq c \leqq 1$, we have

$$\begin{aligned} \|c^n - C^{*n}\| &\leqq \|c^n - C^n\| \leqq \|\xi^n\| + \|\tilde{c}^n - C^n\|, \\ \|\tilde{c}^n - C^{*n}\| &\leqq \|\xi^n\| + \|c^n - C^{*n}\| \leqq 2\|\xi^n\| + \|\tilde{c}^n - C^n\|. \end{aligned} \tag{3.18}$$

Finally, we consider starting procedures to obtain C^0 and P_0. Our analysis will require C^0 to approximate $\tilde{c}^0$ well enough so that

$$\|C^0 - \tilde{c}^0\|_1 \leqq Kh^r. \tag{3.19}$$

This can be obtained by factoring the elliptic projection matrix and solving directly, or by iterating the conjugate gradient procedure sufficiently many times. If iteration is used, a good preconditioner would be the matrix L_0, which by (2.4) is comparable to the elliptic projection matrix. For most of our results, the necessary estimate on P_0 is

$$\|P_0 - \bar{P}_0\|_1 \leqq Kh^r, \tag{3.20}$$

which can trivially be procured by factoring the matrix A_0, to be used as a preconditioner for future pressure time steps. If this option is not chosen, again a sufficiently lengthy iteration will work. Similar comments apply to the estimate needed for Theorem 4.6, which is

$$\|P_0 - \bar{P}_0\|_1 \leqq Kh^r\,\Delta t_p. \tag{3.21}$$

We remark that a detailed argument appears in [12] which weakens the estimate (3.19) to

$$\|\!|C^0 - \tilde{c}^0\|\!|_0 \leqq Kh^r. \tag{3.22}$$

In many cases, $c(x, 0)$ is identically zero, and this sharpening is unimportant; it may be of considerable interest if a simulation is stopped and restarted.

4. A priori error estimates. In this section we obtain a priori error bounds for the procedures described in §§ 2 and 3. We focus our attention on the concentration error $c-C$, since it is the quantity of physical interest. As noted in (2.36), it was found in [14] that the convergence results were affected by the dependence of b on ∇p, even in the continuous-time case with homogeneous boundary conditions. We obtain here the same convergence rates, with time truncation errors, for our iterative procedures.

In the case $b=b(x, c, \nabla p)$, Theorem 4.1 shows that if at least piecewise quadratic polynomials approximate the pressure ($s \geqq 3$), then $O(\log(1/\Delta t_c))$ concentration iterations and a fixed number of pressure iterations per time step yield an L^2 error estimate of the form $O(h^r+h^{s-1}+\Delta t_c)$. Theorem 4.6, under slightly stronger regularity assumptions, fixes the number of concentration iterations while requiring $O(\log(1/\Delta t_p))$ per step for the pressure. This reduces the asymptotic work estimate by a factor of $\Delta t_p/\Delta t_c$, since the pressure is computed less often than the concentration, and improves the H^1 bound from $O(h^{r-1}+h^{s-2}+h^{-1}\Delta t_c)$ to $O(h^{r-1}+h^{s-1}+\Delta t_c)$. Whether the improved work estimate applies for practical values of the mesh parameters is not clear.

In the case $b=b(x, c)$, Corollary 4.2 finds the optimal L^2 estimate $O(h^r+h^s+\Delta t_c)$ with the iterations of Theorem 4.1, provided that either $r \geqq 3$ or $s \geqq 3$. If $r=s=2$, the nearly optimal bound $O(h^2|\log h|+\Delta t_c)$ is demonstrated. Under slightly more regularity, Theorem 4.3 reduces the work to a fixed number of iterations per time step for both equations and improves the H^1 estimate. Corollary 4.4 proves an intermediate work estimate under intermediate regularity assumptions which are balanced for the solutions and their time derivatives.

Corollary 4.5 points out that the concentration iteration may be suppressed if $b=b(x)$. Corollary 4.7 remarks upon the possible benefits of updating the preconditioners. Corollary 4.8 shows that the preceding convergence results are unaffected if the Darcy velocity u, instead of its pressure argument, is linearly extrapolated in the concentration equation (3.17a). We close this section by considering the minor modifications needed to extend the results to three space dimensions, and by applying the analysis to a single quasilinear parabolic equation with nonlinear boundary condition in Theorem 4.9. The principal results of this paper are Theorems 4.1, 4.3 and 4.6.

We now proceed to derive our error estimates. Throughout, we denote generic concentration and pressure time levels by the superscript n and the subscript m, respectively. If we need a correspondence between these symbols, $m=m(n)$ will be the latest pressure time level satisfying $t_m \leqq t^n$. A particular concentration level, often the top index in summations on n, will be denoted by t^l. We define $k=k(l)$ such that t_{k-1} is the last pressure time level satisfying $t_{k-1}<t^l$. If m occurs in summations on n, m is understood to represent $m(n)$.

The symbol K will denote a generic constant, not necessarily the same at different occurrences. The explicit dependence of K on norms of c, p, $\tilde{c}$, and $\tilde{p}$ will often be indicated. For economy, we suppress dependences on these norms in $W^1_\infty(W^1_\infty)$ in intermediate estimates, unless desired for emphasis. The symbol ε will represent a generic small positive constant.

THEOREM 4.1. *Suppose that* $b=b(x, c, \nabla p)$, (R_1) *holds, and* $s \geqq 3$. *If* $r \geqq 3$, *assume also that* Ω *is* H^3*-regular. Suppose that the space and time discretizations satisfy the relation*

$$\Delta t_c = o(h), \tag{4.1}$$

and that the pressure and concentration time steps are related by

$$\Delta t_p^0 = O((\Delta t_c)^{2/3}), \qquad \Delta t_p = O(D(\Delta t_c)^{1/2}), \tag{4.2}$$

where $D = (\|\tilde{c}\|_{H^2(L^2)}/\|\tilde{p}\|_{H^2(H^1)})^{1/2}$. *If we achieve norm reductions of the form*

$$\begin{aligned} \rho_c &= O(\Delta t_c), && \text{all } n,\\ \rho_p' &= O((\Delta t_p^0)^{1/8}), && m=1,\\ \rho_p' &< \left(\frac{1}{8+4\sqrt{5}}\right)^{1/2}\left(\frac{a_*}{a^*}\right)^{1/2}, && m\geqq 2, \end{aligned} \tag{4.3}$$

then for h sufficiently small,

$$\sup_n (\|C^n - c^n\| + h\|C^n - c^n\|_1) \leqq K_7(h^r + h^{s-1} + \Delta t_c), \tag{4.4}$$

where $K_7 = K_7(\Omega, a_*, b_*, \phi_*, \lambda, K_*, K_0, K_1, K_2, K_3, K_4, K_5, K_6)$.

Proof. We recall that $\xi = c - \tilde{c}$, and we set $\zeta^n = C^n - \tilde{c}^n$. Then $C^n - c^n = \zeta^n - \xi^n$, so by (I) and (2.6), it suffices to show that

$$\sup_n \|\zeta^n\| \leqq K(h^r + h^{s-1} + \Delta t_c). \tag{4.5}$$

We look first at the pressure equation. Set $\eta_m = P_m - \tilde{p}_m$. Subtract (2.1) from (3.17b) to obtain

$$\begin{aligned} (a(C_m^*)\nabla\eta_m, \nabla\varphi) &= ([a(c_m) - a(C_m^*)]\nabla\tilde{p}_m, \nabla\varphi)\\ &\quad + ([a(C_m^*)\gamma(C_m^*) - a(c_m)\gamma(c_m)]\nabla d, \nabla\varphi)\\ &\quad + (a(C_m^*)\nabla(P_m - \bar{P}_m), \nabla\varphi), \qquad \varphi\in\mathcal{N}_h. \end{aligned} \tag{4.6}$$

Choosing the test function $\varphi = \eta_m$ in (4.6), we have

$$\begin{aligned} \|\eta_m\|_{a_m}^2 &\leqq K(\|\nabla\tilde{p}_m\|_{L^\infty} + \|\nabla d\|_{L^\infty})\|c_m - C_m^*\|\|\nabla\eta_m\| + \|P_m - \bar{P}_m\|_{a_m}\|\eta_m\|_{a_m}\\ &\leqq K(\|\xi_m\|^2 + \|\zeta_m\|^2) + \tfrac{1}{2}\|P_m - \bar{P}_m\|_{a_m}^2 + (\tfrac{1}{2}+\varepsilon)\|\eta_m\|_{a_m}^2, \end{aligned} \tag{4.7}$$

so that

$$\begin{aligned} \|\eta_m\|_{a_m}^2 &\leqq K(\|\tilde{p}\|_{L^\infty(W_\infty^1)}, \|c\|_{L^\infty(H^r)})h^{2r} + K(\|\tilde{p}\|_{L^\infty(W_\infty^1)})\|\zeta_m\|^2\\ &\quad + (1+\varepsilon)\|P_m - \bar{P}_m\|_{a_m}^2. \end{aligned} \tag{4.8}$$

For the concentration, we subtract (2.3) from (3.17a) to find

$$\begin{aligned} &(\phi d_t\zeta^n, \chi) + (b(C^{*n}, E\nabla P^{n+1})\nabla\zeta^{n+1}, \nabla\chi)\\ &\quad = \left(\phi\left(\frac{\partial c^{n+1}}{\partial t} - d_t\tilde{c}^n\right), \chi\right) - \lambda(\xi^{n+1}, \chi)\\ &\qquad + ([b(c^{n+1}, \nabla p^{n+1}) - b(C^{*n}, E\nabla P^{n+1})]\nabla\tilde{c}^{n+1}, \nabla\chi)\\ &\qquad + (u(c^{n+1}, \nabla p^{n+1})\cdot\nabla\tilde{c}^{n+1} - u(C^{*n}, E\nabla P^{n+1})\cdot\nabla C^n, \chi)\\ &\qquad + \langle g(t^{n+1}, C^{*n}) - g(t^{n+1}, \tilde{c}^{n+1}), \chi\rangle + \left(\phi\frac{C^{n+1} - \bar{C}^{n+1}}{\Delta t_c}, \chi\right)\\ &\qquad + (b(C^{*n}, E\nabla P^{n+1})\nabla(C^{n+1} - \bar{C}^{n+1}), \nabla\chi), \qquad \chi\in\mathcal{M}_h. \end{aligned} \tag{4.9}$$

In (4.9) we choose $\chi = \zeta^{n+1}$ as test function in order to obtain an L^2 estimate for ζ.

We multiply by Δt_c and sum from $n=0$ to $n=l-1$ to obtain

$$
\begin{aligned}
&\sum_{n=0}^{l-1} (\phi(\zeta^{n+1}-\zeta^n), \zeta^{n+1}) + \sum_{n=0}^{l-1} \|\zeta^{n+1}\|_{b^n}^2 \Delta t_c \\
&\quad= \sum_{n=0}^{l-1} \left(\phi\left(\frac{\partial c^{n+1}}{\partial t} - d_t\tilde{c}^n\right), \zeta^{n+1}\right) \Delta t_c - \sum_{n=0}^{l-1} \lambda(\xi^{n+1}, \zeta^{n+1}) \Delta t_c \\
&\qquad+ \sum_{n=0}^{l-1} ([b(c^{n+1}, \nabla p^{n+1}) - b(C^{*n}, E\nabla P^{n+1})]\nabla\tilde{c}^{n+1}, \nabla\zeta^{n+1}) \Delta t_c \\
&\qquad+ \sum_{n=0}^{l-1} (u(c^{n+1}, \nabla p^{n+1}) \cdot \nabla\tilde{c}^{n+1} - u(C^{*n}, E\nabla P^{n+1}) \cdot \nabla C^n, \zeta^{n+1}) \Delta t_c \\
&\qquad+ \sum_{n=0}^{l-1} \langle g(t^{n+1}, C^{*n}) - g(t^{n+1}, \tilde{c}^{n+1}), \zeta^{n+1}\rangle \Delta t_c \\
&\qquad+ \sum_{n=0}^{l-1} (\phi(C^{n+1} - \bar{C}^{n+1}), \zeta^{n+1}) \\
&\qquad+ \sum_{n=0}^{l-1} (b(C^{*n}, E\nabla P^{n+1})\nabla(C^{n+1} - \bar{C}^{n+1}), \nabla\zeta^{n+1}) \Delta t_c \\
&\quad\equiv S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7.
\end{aligned}
\tag{4.10}
$$

Using the inequality $(a-b)a \geqq \frac{1}{2}(a^2-b^2)$, we see that the left-hand side of (4.10) dominates

$$
\begin{aligned}
&\sum_{n=0}^{l-1} \frac{1}{2}[(\phi\zeta^{n+1}, \zeta^{n+1}) - (\phi\zeta^n, \zeta^n)] + \sum_{n=0}^{l-1} \|\zeta^{n+1}\|_{b^n}^2 \Delta t_c \\
&\quad= \frac{1}{2}(\|\zeta^l\|_\phi^2 - \|\zeta^0\|_\phi^2) + \sum_{n=0}^{l-1} \|\zeta^{n+1}\|_{b^n}^2 \Delta t_c.
\end{aligned}
\tag{4.11}
$$

In our estimates, we can therefore handle L^2-norms of ζ with the discrete Gronwall lemma, and we can hide small multiples of H^1-norms of ζ on the left-hand side.

We now estimate the right-hand side of (4.10). First, we note that

$$
\begin{aligned}
S_1 + S_2 &= \sum_{n=0}^{l-1} \left(\phi\frac{\partial\xi^{n+1}}{\partial t}, \zeta^{n+1}\right) \Delta t_c + \sum_{n=0}^{l-1} \left(\phi\left(\frac{\partial\tilde{c}^{n+1}}{\partial t} - d_t\tilde{c}^n\right), \zeta^{n+1}\right) \Delta t_c \\
&\qquad- \sum_{n=0}^{l-1} \lambda(\xi^{n+1}, \zeta^{n+1}) \Delta t_c,
\end{aligned}
$$

$$
\begin{aligned}
|S_1 + S_2| &\leqq K \sum_{n=0}^{l-1} \left\|\frac{\partial\xi^{n+1}}{\partial t}\right\|_{-1} \|\zeta^{n+1}\|_1 \Delta t_c + K \sum_{n=0}^{l-1} \left\|\frac{\partial\tilde{c}^{n+1}}{\partial t} - d_t\tilde{c}^n\right\| \|\zeta^{n+1}\| \Delta t_c \\
&\qquad+ K \sum_{n=0}^{l-1} \|\xi^{n+1}\| \|\zeta^{n+1}\| \Delta t_c \\
&\leqq K(\|c\|_{H^1(H^{r-1})}, \|c\|_{L^2(H^r)}) h^{2r} + K(\|\tilde{c}\|_{H^2(L^2)})(\Delta t_c)^2 \\
&\qquad+ K \sum_{n=0}^{l-1} \|\zeta^{n+1}\|^2 \Delta t_c + \varepsilon \sum_{n=0}^{l-1} \|\zeta^{n+1}\|_1^2 \Delta t_c.
\end{aligned}
\tag{4.12}
$$

Here, if $r \geqq 3$, we have used H^3-regularity in Lemma 2.4 to estimate $\|\partial\xi^{n+1}/\partial t\|_{-1}$. If $r=2$, we do not have an estimate on $\|\partial\xi^{n+1}/\partial t\|_{-1}$, so we must substitute $\|\partial\xi^{n+1}/\partial t\|$ and require a bound on $\|c\|_{H^1(H^2)}$.

Next, we see that

$$
\begin{aligned}
S_3 = &\sum_{n=0}^{l-1} ([b(c^{n+1}, \nabla p^{n+1}) - b(C^{*n}, \nabla p^{n+1})]\nabla \tilde{c}^{n+1}, \nabla\zeta^{n+1})\,\Delta t_c \\
&+\sum_{n=0}^{l-1} ([b(C^{*n}, \nabla p^{n+1}) - b(C^{*n}, \nabla \tilde{p}^{n+1})]\nabla \tilde{c}^{n+1}, \nabla\zeta^{n+1})\,\Delta t_c \\
&+\sum_{n=0}^{l-1} ([b(C^{*n}, \nabla \tilde{p}^{n+1}) - b(C^{*n}, E\nabla \tilde{p}^{n+1})]\nabla \tilde{c}^{n+1}, \nabla\zeta^{n+1})\,\Delta t_c \\
&+\sum_{n=0}^{l-1} ([b(C^{*n}, E\nabla \tilde{p}^{n+1}) - b(C^{*n}, E\nabla P^{n+1})]\nabla \tilde{c}^{n+1}, \nabla\zeta^{n+1})\,\Delta t_c \\
\equiv\ & T_1+T_2+T_3+T_4.
\end{aligned} \tag{4.13}
$$

Now we have

$$
\begin{aligned}
|T_1| &\leqq K \sum_{n=0}^{l-1} (\|c^{n+1}-c^n\| + \|\xi^n\| + \|\zeta^n\|)\|\nabla\zeta^{n+1}\|\,\Delta t_c \\
&\leqq K(\|c\|_{L^2(H^r)})h^{2r} + K(\Delta t_c)^2 + K\sum_{n=0}^{l-1}\|\zeta^n\|^2\,\Delta t_c + \varepsilon\sum_{n=0}^{l-1}\|\zeta^{n+1}\|_1^2\,\Delta t_c, \\
|T_2| &\leqq K\sum_{n=0}^{l-1}\|\nabla\theta^{n+1}\|\|\nabla\zeta^{n+1}\|\,\Delta t_c \\
&\leqq K(\|p\|_{L^2(H^s)})h^{2s-2} + \varepsilon\sum_{n=0}^{l-1}\|\zeta^{n+1}\|_1^2\,\Delta t_c, \\
|T_4| &\leqq K\sum_{n=0}^{l-1}(\|\nabla\eta_m\| + \|\nabla\eta_{m-1}\|)\|\nabla\zeta^{n+1}\|\,\Delta t_c.
\end{aligned} \tag{4.14}
$$

In estimating T_4, we note that η_0 will appear for $\Delta t_p^0/\Delta t_c$ values of n, while η_m, $m \geqq 1$, will appear for $\Delta t_p/\Delta t_c$ values of n. Using (4.8), we have

$$
\begin{aligned}
|T_4| &\leqq K\left[\frac{\Delta t_p^0}{\Delta t_c}\|\nabla\eta_0\|^2\,\Delta t_c + \sum_{m=1}^{k-1}\|\nabla\eta_m\|^2\left(\frac{\Delta t_p}{\Delta t_c}\right)\Delta t_c\right] + \varepsilon\sum_{n=0}^{l-1}\|\zeta^{n+1}\|_1^2\,\Delta t_c \\
&\leqq K(\|c\|_{L^\infty(H^r)})h^{2r} + K\|\zeta_0\|^2\,\Delta t_p^0 + K\sum_{m=1}^{k-1}\|\zeta_m\|^2\,\Delta t_p \\
&\quad + K\|P_0-\bar{P}_0\|_{a_0}^2\,\Delta t_p^0 + K\sum_{m=1}^{k-1}\|P_m-\bar{P}_m\|_{a_m}^2\,\Delta t_p + \varepsilon\sum_{n=0}^{l-1}\|\zeta^{n+1}\|_1^2\,\Delta t_c.
\end{aligned} \tag{4.15}
$$

To bound T_3, we recall that

$$
\begin{aligned}
&\|\nabla\tilde{p}^{n+1} - E\nabla\tilde{p}^{n+1}\|^2 \leqq K(\Delta t_p^0)^2\left\|\frac{\partial\tilde{p}}{\partial t}\right\|_{L^\infty(0,t_1;H^1)}^2, && m=0, \\
&\|\nabla\tilde{p}^{n+1} - E\nabla\tilde{p}^{n+1}\|^2 \leqq K(\Delta t_p)^3\left\|\frac{\partial^2\tilde{p}}{\partial t^2}\right\|_{L^2(t_{m-1},t_{m+1};H^1)}^2, && m\geqq 1.
\end{aligned} \tag{4.16}
$$

Then

$$|T_3| \leqq K\Big[\sum_{m=0} \|\nabla\tilde{p}^{n+1} - E\nabla\tilde{p}^{n+1}\|\|\nabla\zeta^{n+1}\|\,\Delta t_c$$

$$+\sum_{m\geqq 1} \|\nabla\tilde{p}^{n+1} - E\nabla\tilde{p}^{n+1}\|\|\nabla\zeta^{n+1}\|\,\Delta t_c\Big]$$

$$\leqq K\Big(\frac{\Delta t_p^0}{\Delta t_c}\Big)(\Delta t_p^0)^2(\Delta t_c) + K\Big(\frac{\Delta t_p}{\Delta t_c}\Big)(\Delta t_p)^3 \sum_{m=1}^{k-1} \Big\|\frac{\partial^2 \tilde{p}}{\partial t^2}\Big\|^2_{L^2(t_{m-1},t_{m+1};H^1)}\Delta t_c \tag{4.17}$$

$$+\varepsilon \sum_{n=0}^{l-1} \|\zeta^{n+1}\|_1^2\,\Delta t_c$$

$$\leqq K(\|\tilde{p}\|_{W^1_\infty(H^1)})(\Delta t_p^0)^3 + K(\|\tilde{p}\|_{H^2(H^1)})(\Delta t_p)^4 + \varepsilon \sum_{n=0}^{l-1} \|\zeta^{n+1}\|_1^2\,\Delta t_c.$$

Collecting the estimates for T_1 through T_4, we have

$$|S_3| \leqq K(\|c\|_{L^\infty(H^r)})h^{2r} + K(\|p\|_{L^2(H^s)})h^{2s-2} + K(\Delta t_c)^2$$

$$+K(\|\tilde{p}\|_{W^1_\infty(H^1)})(\Delta t_p^0)^3 + K(\|\tilde{p}\|_{H^2(H^1)})(\Delta t_p)^4$$

$$+K\|P_0 - \bar{P}_0\|^2_{a_0}\,\Delta t_p^0 + K\sum_{m-1}^{k=1} \|P_m - \bar{P}_m\|^2_{a_m}\,\Delta t_p \tag{4.18}$$

$$+K\|\zeta_0\|^2\,\Delta t_p^0 + K\sum_{m=1}^{k-1} \|\zeta_m\|^2\,\Delta t_p$$

$$+K\sum_{n=0}^{l-1} \|\zeta^n\|^2\,\Delta t_c + \varepsilon\sum_{n=0}^{l-1} \|\zeta^{n+1}\|_1^2\,\Delta t_c.$$

Next, we split S_4 as follows:

$$S_4 = \sum_{n=0}^{l-1} ([u(c^{n+1}, \nabla p^{n+1}) - u(C^{*n}, E\nabla P^{n+1})]\cdot\nabla\tilde{c}^{n+1}, \zeta^{n+1})\,\Delta t_c$$

$$+\sum_{n=0}^{l-1} (u(C^{*n}, E\nabla P^{n+1})\cdot\nabla(\tilde{c}^{n+1} - C^n), \zeta^{n+1})\,\Delta t_c \tag{4.19}$$

$$\equiv T_5 + T_6.$$

We observe at once that T_5 has the same form as S_3 with $\nabla\zeta^{n+1}$ replaced by ζ^{n+1}, so the same bounds will hold. To handle T_6 we use an induction argument. We assume that $\|u(C^{*n}, E\nabla P^{n+1})\|_{L^\infty}$ is uniformly bounded for $n = 0, 1, \cdots, l-1$. By assumptions on u, we know that

$$|u(C^{*n}, E\nabla P^{n+1})| \leqq K(1 + |E\nabla P^{n+1}|) \leqq K(1 + |\nabla P_m| + |\nabla P_{m-1}|). \tag{4.20}$$

It therefore suffices to bound $\|\nabla P_m\|_{L^\infty}$ for $m = 0, 1, \cdots, k-1$. We assume that

$$\|\nabla P_m\|_{L^\infty} \leqq 2K_5, \qquad 0 \leqq m \leqq k-1. \tag{4.21}$$

To start the induction, we see easily that from (2.7), (I), (4.8), (3.19), and (3.20),

$$\|\nabla P_0\|_{L^\infty} \leqq \|\nabla\tilde{p}_0\|_{L^\infty} + \|\nabla\eta_0\|_{L^\infty} \leqq K_5 + K_0 h^{-1}\|\nabla\eta_0\|$$

$$\leqq K_5 + K(K_0, \|c\|_{L^\infty(H^r)})h^{-1}(\|\zeta_0\| + h^r + \|P_0 - \bar{P}_0\|_1) \tag{4.22}$$

$$\leqq K_5 + Kh^{-1}h^r$$

$$\leqq 2K_5,$$

for h sufficiently small. If $t^l = t_k$, we will show at the end of the argument that $\|\nabla P_k\|_{L^\infty} \leqq 2K_5$. This will verify the comment allowing (π_1, π_2) to be bounded in the assumptions (C) of § 2. With (4.21) in hand, we can write

$$
\begin{aligned}
|T_6| &\leqq K \sum_{n=0}^{l-1} (\|\nabla(\tilde{c}^{n+1} - \tilde{c}^n)\| + \|\nabla \zeta^n\|)\|\zeta^{n+1}\| \,\Delta t_c \\
&\leqq K(\Delta t_c)^2 + K \sum_{n=0}^{l-1} \|\zeta^{n+1}\|^2 \,\Delta t_c + \varepsilon \sum_{n=0}^{l-1} \|\zeta^n\|_1^2 \,\Delta t_c,
\end{aligned} \tag{4.23}
$$

$$|S_4| \leqq |S_3| + |T_6|.$$

Next, we find that the boundary term can be estimated by

$$
\begin{aligned}
|S_5| &\leqq K \sum_{n=0}^{l-1} |\langle \zeta^n - (\tilde{c}^{n+1} - \tilde{c}^n), \zeta^{n+1} \rangle| \Delta t_c \\
&\leqq K \sum_{n=0}^{l-1} \|\zeta^n\| \|\zeta^n\|_1 \,\Delta t_c + K \sum_{n=0}^{l-1} \|\tilde{c}^{n+1} - \tilde{c}^n\|_1 \|\zeta^{n+1}\|_1 \,\Delta t_c \\
&\quad + \sum_{n=0}^{l-1} \|\zeta^{n+1}\| \|\zeta^{n+1}\|_1 \,\Delta t_c \\
&\leqq K(\Delta t_c)^2 + K \sum_{n=0}^{l} \|\zeta^n\|^2 \,\Delta t_c + \varepsilon \sum_{n=0}^{l} \|\zeta^n\|_1^2 \,\Delta t_c,
\end{aligned} \tag{4.24}
$$

where we have used Lemma 2.4 and the trace theorem.

Finally, we have by (3.15) and (3.16) that

$$
\begin{aligned}
|S_6 + S_7| &\leqq \sum_{n=0}^{l-1} |||C^{n+1} - \bar{C}^{n+1}|||_n |||\zeta^{n+1}|||_n \\
&\leqq \rho_c' |||\delta C^0|||_0 |||\zeta^1|||_0 + \rho_c' \sum_{n=1}^{l-1} |||\delta^2 C^n|||_n |||\zeta^{n+1}|||_n \\
&\equiv T_7 + T_8,
\end{aligned} \tag{4.25}
$$

and

$$
\begin{aligned}
|T_7| &\leqq \rho_c' (|||\delta \tilde{c}^0|||_0 + |||\delta \zeta^0|||_0)^2 + \rho_c' |||\zeta^1|||_0^2 \\
&\leqq K \,\Delta t_c \left[\left\| \frac{\partial \tilde{c}}{\partial t} \right\|_{L^2(0,t^1;H^1)}^2 \Delta t_c + |||\zeta^0|||_0^2 + |||\zeta^1|||_0^2 \right] \\
&\leqq K(\|\tilde{c}\|_{H^1(H^1)})(\Delta t_c)^2 + K(\|\zeta^0\|^2 + \|\zeta^1\|^2) \,\Delta t_c + K(\|\zeta^0\|_1^2 + \|\zeta^1\|_1^2)(\Delta t_c)^2, \\
|T_8| &\leqq K\rho_c' \sum_{n=1}^{l-1} (|||\delta \tilde{c}^n|||_n^2 + |||\delta \tilde{c}^{n-1}|||_n^2 + |||\zeta^{n+1}|||_n^2 + |||\zeta^n|||_n^2 + |||\zeta^{n-1}|||_n^2) \\
&\leqq K \,\Delta t_c \sum_{n=1}^{l-1} \left(\left\| \frac{\partial \tilde{c}}{\partial t} \right\|_{L^2(t^{n-1},t^{n+1};H^1)}^2 \Delta t_c + \|\zeta^{n+1}\|^2 + \|\zeta^n\|^2 + \|\zeta^{n-1}\|^2 \right. \\
&\qquad \left. + \|\zeta^{n+1}\|_1^2 \,\Delta t_c + \|\zeta^n\|_1^2 \,\Delta t_c + \|\zeta^{n-1}\|_1^2 \,\Delta t_c \right) \\
&\leqq K(\|\tilde{c}\|_{H^1(H^1)})(\Delta t_c)^2 + K \sum_{n=0}^{l} \|\zeta^n\|^2 \,\Delta t_c + K \,\Delta t_c \sum_{n=0}^{l} \|\zeta^n\|_1^2 \,\Delta t_c \\
&\leqq K(\Delta t_c)^2 + K \sum_{n=0}^{l} \|\zeta^n\|^2 \,\Delta t_c + \varepsilon \sum_{n=0}^{l} \|\zeta^n\|_1^2 \,\Delta t_c,
\end{aligned} \tag{4.26}
$$

for $\rho_c' = O(\Delta t_c)$ and Δt_c sufficiently small. Thus,

$$|S_6+S_7| \leqq K(\Delta t_c)^2 + K\sum_{n=0}^{l} \|\zeta^n\|^2\,\Delta t_c + \varepsilon \sum_{n=0}^{l} \|\zeta^n\|_1^2\,\Delta t_c. \tag{4.27}$$

A glance at (4.26) reveals that the argument goes through with $|||\delta C^n|||_n$ in place of $|||\delta^2 C^n|||_n$, so that the linearly extrapolated initial concentration guess is unnecessary for this theorem, as noted in § 3.

Collecting the estimates (4.10), (4.11), (4.12), (4.18), (4.23), (4.24) and (4.27), we have

$$\begin{aligned}
&\frac{1}{2}\|\zeta^l\|_\phi^2 + \sum_{n=0}^{l-1} \|\zeta^{n+1}\|_{b^n}^2\,\Delta t_c \\
&\quad\leqq K(\|c\|_{L^\infty(H^r)}, \|c\|_{H^1(H^{r-1})}, \|\tilde{c}\|_{L^\infty(W_\infty^1)}, \|\tilde{p}\|_{L^\infty(W_\infty^1)})h^{2r} + K(\|\tilde{c}\|_{L^\infty(W_\infty^1)}, \|p\|_{L^2(H^s)})h^{2s-2} \\
&\qquad + K(\|c\|_{H^1(H^1)}, \|\tilde{c}\|_{L^\infty(W_\infty^1)}, \|\tilde{c}\|_{H^1(H^1)}, \|\tilde{c}\|_{H^2(L^2)})(\Delta t_c)^2 \\
&\qquad + K(\|\tilde{c}\|_{L^\infty(W_\infty^1)}, \|\tilde{p}\|_{W_\infty^1(H^1)})(\Delta t_p^0)^3 + K(\|\tilde{c}\|_{L^\infty(W_\infty^1)}, \|\tilde{p}\|_{H^2(H^1)})(\Delta t_p)^4 \\
&\qquad + K(\|\tilde{c}\|_{L^\infty(W_\infty^1)})\|P_0 - \bar{P}_0\|_{a_0}^2\,\Delta t_p^0 + K(\|\tilde{c}\|_{L^\infty(W_\infty^1)}) \sum_{m=1}^{k-1} \|P_m - \bar{P}_m\|_{a_m}^2\,\Delta t_p \\
&\qquad + K(\|\tilde{p}\|_{L^\infty(W_\infty^1)}, \|\tilde{c}\|_{L^\infty(W_\infty^1)}) \sum_{m=1}^{k-1} \|\zeta_m\|^2\,\Delta t_p \\
&\qquad + K(\|\tilde{c}\|_{L^\infty(W_\infty^1)}) \sum_{n=0}^{l} \|\zeta^n\|^2\,\Delta t_c + \varepsilon \sum_{n=0}^{l} \|\zeta^n\|_1^2\,\Delta t_c + K\|\zeta^0\|^2 \\
&\quad\equiv K(h^{2r} + h^{2s-2} + (\Delta t_c)^2 + (\Delta t_p^0)^3 + (\Delta t_p)^4) + A_1 + A_2 + A_3 + A_4 + A_5 + A_6.
\end{aligned} \tag{4.28}$$

A_1 and A_6 are estimated by the starting bounds (3.19) and (3.20). A_3 and A_4 will disappear when the discrete Gronwall lemma is applied, at the cost of allowing the other constants to depend on $\|\tilde{p}\|_{L^\infty(W_\infty^1)}$ and $\|\tilde{c}\|_{L^\infty(W_\infty^1)}$. A_5 hides on the left-hand side of (4.28). We now proceed to handle A_2.

Considering $m = 1$ first, we see that

$$\begin{aligned}
\|P_1 - \bar{P}_1\|_{a_1} &\leqq \rho_p' \|\delta P_0\|_{a_1} \leqq \rho_p'(\|\delta \tilde{p}_0\|_{a_1} + \|\eta_0\|_{a_1} + \|\eta_1\|_{a_1}) \\
&\leqq \left(\frac{a^*}{a_*}\right)^{1/2} \rho_p'(\|\eta_0\|_{a_0} + \|\eta_1\|_{a_1}) + K\rho_p'\,\Delta t_p^0 \\
&\leqq K(\Delta t_p^0)^{1/8}[K(\|c\|_{L^\infty(H^r)})h^r + K(\|\zeta_0\| + \|\zeta_1\|) \\
&\qquad + \|P_0 - \bar{P}_0\|_{a_0} + \|P_1 - \bar{P}_1\|_{a_1}] + K(\Delta t_p^0)^{9/8},
\end{aligned} \tag{4.29}$$

and, using (3.19), (3.20), and (4.8), letting $\rho_p' = O((\Delta t_p^0)^{1/8})$, and taking Δt_p^0 sufficiently small, we obtain

$$\|P_1 - \bar{P}_1\|_{a_1}^2 \leqq K(\|c\|_{L^\infty(H^r)})h^{2r} + K\|\zeta_1\|^2 + K(\Delta t_p^0)^{9/4}. \tag{4.30}$$

Since $(\Delta t_p^0)^{9/4}\,\Delta t_p = O((\Delta t_c)^{9/4\cdot 2/3+1/2}) = O((\Delta t_c)^2)$, we see that the first summand $\|P_1 - \bar{P}_1\|_{a_1}^2\,\Delta t_p$ of A_2 is bounded in the proper fashion. For future reference, we let

$$F_m = \|P_m - \bar{P}_m\|_{a_m}^2, \tag{4.31}$$

and we note that for suitable constants $\tilde{K}_0$ and $\tilde{K}_1$ we have shown that

$$F_0 \leqq \tilde{K}_0 h^{2r},\qquad(4.32)$$
$$F_1 \leqq \tilde{K}_1(h^{2r} + \|\zeta_1\|^2 + (\Delta t_p)^3).$$

Now we consider $m \geqq 2$, and we note, using $(a+b+c+d)^2 \leqq 4(a^2+b^2+c^2+d^2)$, that

$$\begin{aligned}
F_m &\leqq (\rho_p')^2 \|\delta^2 P_{m-1}\|_{a_m}^2 \\
&\leqq (\rho_p')^2(1+\varepsilon)\|\delta^2 \eta_{m-1}\|_{a_m}^2 + K\|\delta^2 \tilde{p}_{m-1}\|_{a_m}^2 \\
&\leqq (\rho_p')^2(1+\varepsilon)(\|\eta_m\|_{a_m} + \|\eta_{m-1}\|_{a_m} + \|\eta_{m-1}\|_{a_m} + \|\eta_{m-2}\|_{a_m})^2 \\
&\quad + K(\Delta t_p)^3 \left\|\frac{\partial^2 \tilde{p}}{\partial t^2}\right\|_{L^2(t_{m-2},t_m;H^1)}^2 \\
&\leqq 4\left(\frac{a^*}{a_*}\right)(\rho_p')^2(1+\varepsilon)(\|\eta_m\|_{a_m}^2 + 2\|\eta_{m-1}\|_{a_{m-1}}^2 + \|\eta_{m-2}\|_{a_{m-2}}^2) \qquad(4.33)\\
&\quad + K(\Delta t_p)^3 \left\|\frac{\partial^2 \tilde{p}}{\partial t^2}\right\|_{L^2(t_{m-2},t_m;H^1)}^2 \\
&\leqq \left(\frac{1}{2+\sqrt{5}} - \varepsilon\right)(F_m + 2F_{m-1} + F_{m-2}) + K(\|c\|_{L^\infty(H^r)})h^{2r} \\
&\quad + K(\|\zeta_m\|^2 + \|\zeta_{m-1}\|^2 + \|\zeta_{m-2}\|^2) + K(\Delta t_p)^3 \left\|\frac{\partial^2 \tilde{p}}{\partial t^2}\right\|_{L^2(t_{m-2},t_m;H^1)}^2.
\end{aligned}$$

Letting

$$\begin{aligned}
G_m &= K(\|\tilde{p}\|_{L^\infty(W_\infty^1)}, \|c\|_{L^\infty(H^r)})h^{2r} + K(\|\tilde{p}\|_{L^\infty(W_\infty^1)})(\|\zeta_m\|^2 + \|\zeta_{m-1}\|^2 + \|\zeta_{m-2}\|^2) \qquad(4.34)\\
&\quad + K(\Delta t_p)^3 \left\|\frac{\partial^2 \tilde{p}}{\partial t^2}\right\|_{L^2(t_{m-2},t_m;H^1)}^2,
\end{aligned}$$

we have shown that

$$F_m \leqq R(G_m + 2F_{m-1} + F_{m-2}),\qquad(4.35)$$

where

$$R = \frac{\dfrac{1}{2+\sqrt{5}} - \varepsilon}{1 - \left(\dfrac{1}{2+\sqrt{5}} - \varepsilon\right)} < \frac{1}{1+\sqrt{5}}.$$

Using (4.32) and (4.35), we can see that, since $F_1 \leqq \tilde{K}_1 G_2$ and $F_0 \leqq \tilde{K}_0 G_2$,

$$\begin{aligned}
F_2 &\leqq R(G_2 + 2F_1 + F_0) \leqq R(1 + 2\tilde{K}_1 + \tilde{K}_0)G_2, \\
F_3 &\leqq R(G_3 + 2F_2 + F_1) \qquad(4.36)\\
&\leqq RG_3 + 2R^2(1 + 2\tilde{K}_1 + \tilde{K}_0)G_2 + R\tilde{K}_1 G_2 \\
&\leqq RG_3 + 2R^2\left(1 + 2\tilde{K}_1 + \tilde{K}_0 + \frac{\tilde{K}_1}{2R}\right)G_2.
\end{aligned}$$

Letting $\bar{K}=1+2\tilde{K}_1+\tilde{K}_0+\tilde{K}_1/2R$, we see that all hypotheses of Lemma 2.5 are satisfied. The lemma tells us that

$$\sum_{m=2}^{k-1}\|P_m-\bar{P}_m\|_{a_m}^2\,\Delta t_p\leqq K\sum_{m=2}^{k-1}G_m\,\Delta t_p\leqq K(h^{2r}+(\Delta t_p)^4+A_3. \tag{4.37}$$

Thus A_2 is bounded by terms already on the right-hand side of (4.28).

We now apply the discrete Gronwall lemma to (4.28), obtaining

$$\begin{aligned}\|\zeta^l\|^2&+\sum_{n=0}^{l-1}\|\zeta^{n+1}\|_{b^n}^2\,\Delta t_c\\ &\leqq K(\|\tilde{p}\|_{L^\infty(W_\infty^1)},\|\tilde{c}\|_{L^\infty(W_\infty^1)})[K(\|c\|_{L^\infty(H^r)},\|c\|_{H^1(H^{r-1})})h^{2r}+K(\|p\|_{L^2(H^s)})h^{2s-2}\\ &\qquad+K(\|c\|_{H^1(L^2)},\|\tilde{c}\|_{H^1(H^1)},\|\tilde{c}\|_{H^2(L^2)})(\Delta t_c)^2\\ &\qquad+K(\|\tilde{p}\|_{W_\infty^1(H^1)})(\Delta t_p^0)^3+K(\|\tilde{p}\|_{H^2(H^1)})(\Delta t_p)^4].\end{aligned} \tag{4.38}$$

With the time step choices

$$\begin{aligned}\Delta t_p^0&\approx\left(\frac{\|\tilde{c}\|_{W_\infty^2(L^2)}}{\|\tilde{p}\|_{W_\infty^1(H^1)}}\right)^{2/3}(\Delta t_c)^{2/3},\\ \Delta t_p&\approx\left(\frac{\|\tilde{c}\|_{H^2(L^2)}}{\|\tilde{p}\|_{H^2(H^1)}}\right)^{1/2}(\Delta t_c)^{1/2},\end{aligned} \tag{4.39}$$

we obtain the desired result (4.5), assuming that the chosen constants are the dominant ones.

It remains to check the induction hypothesis (4.21) if $t^l=t_k$. We have

$$\begin{aligned}\|\nabla P_k\|_{L^\infty}^2&\leqq(\|\nabla\tilde{p}_k\|_{L^\infty}+\|\nabla\eta_k\|_{L^\infty})^2\leqq 2(\|\nabla\tilde{p}_k\|_{L^\infty}^2+\|\nabla\eta_k\|_{L^\infty}^2)\\ &\leqq 2K_5^2+2K_0h^{-2}\|\nabla\eta_k\|^2\\ &\leqq 2K_5^2+Kh^{-2}(\|\zeta_k\|^2+h^{2r}+\|P_k-\bar{P}_k\|_{a_k}^2).\end{aligned} \tag{4.40}$$

We now know that $\|\zeta_k\|^2\leqq K(h^{2r}+h^{2s-2}+(\Delta t_c)^2)$. Furthermore, we can apply Lemma 2.5, with a term of the form

$$K(\Delta t_p)^4\left\|\frac{\partial^2\tilde{p}}{\partial t^2}\right\|_{L^\infty(t_{m-2},t_m;H^1)}^2 \tag{4.41}$$

replacing the last term of G_m. The second statement of Lemma 2.5, with m running from 2 to k, yields

$$\|P_k-\bar{P}_k\|_{a_k}^2\leqq K(h^{2r}+h^{2s-2}+(\Delta t_c)^2+(\Delta t_p)^4), \tag{4.42}$$

so that

$$\|\nabla P_k\|_{L^\infty}^2\leqq 2K_5^2+Kh^{-2}(h^{2r}+h^{2s-2}+(\Delta t_c)^2)\leqq 4K_5^2 \tag{4.43}$$

for h sufficiently small, provided that $s\geqq 3$ and $\Delta t_c=o(h)$.

We remark that the optimal relationship between h and Δt_c is $\Delta t_c=O(h^r+h^{s-1})$, and that for all values of $r\geqq 2$ and $s\geqq 3$ this lies well within the restrictions of the theorem.

Examination of the proof of Theorem 4.1 shows that the term h^{s-1} arose only in the estimation of S_3 and S_4. It is obvious that, if $b=b(x,c)$ instead of $b=b(x,c,\nabla p)$, then h^{s-1} will not appear in S_3. We remark in the next result that in the estimation of S_4, we can use the form of u to integrate by parts and improve the error estimate from h^{s-1} to h^s.

COROLLARY 4.2. *Suppose that $b=b(x,c)$, (R_1) is strengthened by adding $c\in L^2(W^2_\infty)\cap L^\infty(H^{2+\varepsilon})$ and $p\in L^\infty(H^s)$, and allow $r=s=2$. Assume that the other hypotheses of Theorem 4.1 hold. Then, for h sufficiently small,*

$$\sup_n(\|C^n-c^n\|+h\|C^n-c^n\|_1)\leqq\begin{cases}K_8(h^r+h^s+\Delta t_c), & r\geqq 3 \text{ or } s\geqq 3,\\ K_8(h^2|\log h|+\Delta t_c), & r=s=2.\end{cases}\tag{4.44}$$

Proof. It suffices to revise the estimate for S_4. We write

$$\begin{aligned}S_4=&\sum_{n=0}^{l-1}([u(c^{n+1},\nabla p^{n+1})-u(c^{n+1},\nabla\tilde p^{n+1})]\cdot\nabla\tilde c^{n+1},\zeta^{n+1})\,\Delta t_c\\&+\sum_{n=0}^{l-1}([u(c^{n+1},\nabla\tilde p^{n+1})-u(c^{n+1},E\nabla\tilde p^{n+1})]\cdot\nabla\tilde c^{n+1},\zeta^{n+1})\,\Delta t_c\\&+\sum_{n=0}^{l-1}([u(c^{n+1},E\nabla\tilde p^{n+1})-u(c^{n+1},E\nabla P^{n+1})]\cdot\nabla\tilde c^{n+1},\zeta^{n+1})\,\Delta t_c\\&+\sum_{n=0}^{l-1}([u(c^{n+1},E\nabla P^{n+1})-u(C^{*n},E\nabla P^{n+1})]\cdot\nabla\tilde c^{n+1},\zeta^{n+1})\,\Delta t_c\\\equiv&\ U_1+U_2+U_3+U_4.\end{aligned}\tag{4.45}$$

U_2, U_3, and U_4 are estimated analogously to T_3, T_4, and T_1, respectively, in (4.13); these bounds do not involve h^{2s-2}. For U_1, we use the form of $u(c,\nabla p)=-a(c)[\nabla p-\gamma(c)\nabla d]$ to write

$$\begin{aligned}U_1&=\sum_{n=0}^{l-1}(a(c^{n+1})\nabla(\tilde p-p)^{n+1}\cdot\nabla\tilde c^{n+1},\zeta^{n+1})\,\Delta t_c\\&=\sum_{n=0}^{l-1}(a(c^{n+1})\nabla\theta^{n+1}\cdot\nabla\xi^{n+1},\zeta^{n+1})\,\Delta t_c\\&\quad-\sum_{n=0}^{l-1}(a(c^{n+1})\nabla\theta^{n+1}\cdot\nabla c^{n+1},\zeta^{n+1})\,\Delta t_c\\&\equiv U_5-U_6.\end{aligned}\tag{4.46}$$

Integration by parts in U_6 will improve the convergence rate. We could not integrate U_1 by parts directly because we could not place two spatial derivatives on $\tilde c$. We have

$$\begin{aligned}|U_5|&\leqq K\sum_{n=0}^{l-1}\|\nabla\theta^{n+1}\|\|\nabla\xi^{n+1}\|\|\zeta^{n+1}\|_{L^\infty}\,\Delta t_c\\&\leqq K\sum_{n=0}^{l-1}h^{s-1}\|p^{n+1}\|_s h^{r-1}\|c^{n+1}\|_r h^{-1}\|\zeta^{n+1}\|\,\Delta t_c\\&\leqq K(\|p\|_{L^2(H^s)},\|c\|_{L^\infty(H^r)})h^{2r+2s-6}+K\sum_{n=0}^{l-1}\|\zeta^{n+1}\|^2\,\Delta t_c\\&\leqq K(\|p\|_{L^2(H^s)},\|c\|_{L^\infty(H^r)})(h^{2r}+h^{2s})+K\sum_{n=0}^{l-1}\|\zeta^{n+1}\|^2\,\Delta t_c,\qquad r\geqq 3\text{ or }s\geqq 3,\end{aligned}\tag{4.47}$$

$$\begin{aligned}|U_5|&\leqq K\sum_{n=0}^{l-1}\|\nabla\theta^{n+1}\|_{L^\infty}\|\nabla\xi^{n+1}\|\|\zeta^{n+1}\|\,\Delta t_c\\&\leqq K\sum_{n=0}^{l-1}h|\log h|\|p^{n+1}\|_{W^2_\infty}h\|c^{n+1}\|_2\|\zeta^{n+1}\|\,\Delta t_c\\&\leqq K(\|p\|_{L^2(W^2_\infty)},\|c\|_{L^\infty(H^2)})h^4|\log h|^2+K\sum_{n=0}^{l-1}\|\zeta^{n+1}\|^2\,\Delta t_c,\qquad r=s=2,\end{aligned}$$

where the L^∞ estimate in the latter case depends on results of [15], [21], and

$$
\begin{aligned}
U_6 &= \sum_{n=0}^{l-1} \left\langle \theta^{n+1}, a(c^{n+1})\zeta^{n+1} \frac{\partial c^{n+1}}{\partial n} \right\rangle \Delta t_c \\
&\quad - \sum_{n=0}^{l-1} (\theta^{n+1}, \nabla \cdot (a(c^{n+1})\zeta^{n+1} \nabla c^{n+1})) \Delta t_c \qquad (4.48)\\
&\equiv U_7 - U_8.
\end{aligned}
$$

Using Lemma 2.2 of [3] and $c \in L^\infty(H^{2+\varepsilon})$, we obtain

$$
\begin{aligned}
|U_7| &\leqq K \sum_{n=0}^{l-1} |\theta^{n+1}|_{-1/2} |\zeta^{n+1}|_{1/2} \|c^{n+1}\|_{2+\varepsilon} \Delta t_c \\
&\leqq K \sum_{n=0}^{l-1} h^s \|p\|_s \|\zeta^{n+1}\|_1 \Delta t_c \\
&\leqq K(\|c\|_{L^\infty(H^{2+\varepsilon})}, \|p\|_{L^2(H^s)}) h^{2s} + \varepsilon \sum_{n=0}^{l-1} \|\zeta^{n+1}\|_1^2 \Delta t_c, \qquad (4.49)\\
|U_8| &\leqq \sum_{n=0}^{l-1} \|\theta^{n+1}\| \|a(c^{n+1})\|_{W^1_\infty} \|\zeta^{n+1}\|_1 \|c^{n+1}\|_{W^2_\infty} \Delta t_c \\
&\leqq K(\|p\|_{L^\infty(H^s)}, \|c\|_{L^2(W^2_\infty)}) h^{2s} + \varepsilon \sum_{n=0}^{l-1} \|\zeta^{n+1}\|_1^2 \Delta t_c.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|U_1| &\leqq K(\|p\|_{L^2(H^s)}, \|c\|_{L^\infty(H^r)}) h^{2r} \\
&\quad + K(\|p\|_{L^\infty(H^s)}, \|c\|_{L^\infty(H^r)}, \|c\|_{L^2(W^2_\infty)}, \|c\|_{L^\infty(H^{2+\varepsilon})}) h^{2s} \qquad (4.50)\\
&\quad + K \sum_{n=0}^{l-1} \|\zeta^{n+1}\|^2 \Delta t_c + \varepsilon \sum_{n=0}^{l-1} \|\zeta^{n+1}\|_1^2 \Delta t_c.
\end{aligned}
$$

The rest of the proof of Theorem 4.1 goes through as before, except that $Kh^{-2}(h^4|\log h|^2)$ appears in (4.43) if $r = s = 2$.

The use of the test function $\chi = \zeta^{n+1}$ does not produce the best possible results with our time-stepping methods. In (4.26), we were unable to take advantage of the differences of the form $|||\delta\zeta^n|||_n$; we would hope to obtain a factor of Δt_c, but we had to appeal to the norm reduction for this. In (4.29) and (4.33), we had to introduce factors of the form a^*/a_* in order to perturb the indices on the weighted norms. If we could estimate the change of ζ (and hence C) with time, we could avoid this problem also. By using the test function $\chi = \zeta^{n+1} - \zeta^n = \Delta t_c\, d_t\zeta^n$ instead, we can obtain an a priori estimate on $d_t\zeta^n$ in the discrete $L^2(L^2)$-norm and find better results. In addition, we derive an estimate for ζ in the discrete $L^\infty(H^1)$ norm, which gives us a better H^1 error estimate by removing the need to appeal to the inverse assumption (I). We need a bit more regularity to do this, as the next result will show. We first consider $b = b(x, c)$.

THEOREM 4.3. *Suppose that $b = b(x, c)$, (R_3) holds, and allow $r = s = 2$. Suppose that the discretizations satisfy the relation*

$$
\Delta t_c = o(h), \qquad (4.51)
$$

and that the pressure and concentration time steps are related by

$$\Delta t_p^0 = O((\Delta t_c)^{2/3}), \qquad \Delta t_p = O(D(\Delta t_c)^{1/2}), \tag{4.52}$$

where $D = (\|\tilde{c}\|_{H^2(H^1)}/\|\tilde{p}\|_{H^2(H^1)})^{1/2}$. *If we achieve norm reductions of the form*

$$\begin{aligned} \rho_c &= O((\Delta t_c)^{1/2}), && n=1,\\ \rho_c &\leqq \delta_c < \tfrac{1}{5}, && n \geqq 2,\\ \rho_p &= O((\Delta t_p^0)^{1/8}), && m=1,\\ \rho_p &\leqq \delta_p < \frac{1}{3+\sqrt{5}}, && m \geqq 2, \end{aligned} \tag{4.53}$$

with δ_c *and* δ_p *independent of* n *and* m, *then for* h *sufficiently small*

$$\begin{aligned} \sup_n \|C^n - c^n\| &\leqq \begin{cases} K_9(h^r + h^s + \Delta t_c), & r \geqq 3 \text{ or } s \geqq 3,\\ K_9(h^2|\log h| + \Delta t_c), & r = s = 2, \end{cases}\\ \sup_n \|C^n - c^n\|_1 &\leqq \begin{cases} K_9(h^{r-1} + h^s + \Delta t_c), & r \geqq 3 \text{ or } s \geqq 3,\\ K_9(h|\log h| + \Delta t_c), & r = s = 2. \end{cases} \end{aligned} \tag{4.54}$$

Proof. It suffices to show that

$$\sup_n \|\zeta^n\|_1 \leqq \begin{cases} K(h^r + h^s + \Delta t_c), & r \geqq 3 \text{ or } s \geqq 3,\\ K(h^2|\log h| + \Delta t_c), & r = s = 2. \end{cases} \tag{4.55}$$

We obtain (4.8) exactly as in Theorem 4.1. We also have (4.9) with b independent of its second argument. Taking $\chi = \zeta^{n+1} - \zeta^n = (\Delta t_c)\, d_t\zeta^n$ and summing from $n=0$ to $n = l-1$, we obtain

$$\begin{aligned} &\sum_{n=0}^{l-1} (\phi d_t\zeta^n, d_t\zeta^n)\,\Delta t_c + \sum_{n=0}^{l-1} (b(C^{*n})\nabla\zeta^{n+1}, \nabla d_t\zeta^n)\,\Delta t_c\\ &\quad = \sum_{n=0}^{l-1}\left(\phi\left(\frac{\partial c^{n+1}}{\partial t} - d_t\tilde{c}^n\right), d_t\zeta^n\right)\Delta t_c - \sum_{n=0}^{l-1} \lambda(\xi^{n+1}, d_t\zeta^n)\,\Delta t_c\\ &\qquad + \sum_{n=0}^{l-1} ([b(c^{n+1}) - b(C^{*n})]\nabla\tilde{c}^{n+1}, \nabla d_t\zeta^n)\,\Delta t_c\\ &\qquad + \sum_{n=0}^{l-1} (u(c^{n+1}, \nabla p^{n+1})\cdot\nabla\tilde{c}^{n+1} - u(C^{*n}, E\nabla P^{n+1})\cdot\nabla C^n, d_t\zeta^n)\,\Delta t_c\\ &\qquad + \sum_{n=0}^{l-1} \langle g(t^{n+1}, C^n) - g(t^{n+1}, \tilde{c}^{n+1}), d_t\zeta^n\rangle\,\Delta t_c\\ &\qquad + \sum_{n=0}^{l-1} (\phi(C^{n+1} - \bar{C}^{n+1}), d_t\zeta^n)\\ &\qquad + \sum_{n=0}^{l-1} (b(C^{*n})\nabla(C^{n+1} - \bar{C}^{n+1}), \nabla d_t\zeta^n)\,\Delta t_c\\ &\quad \equiv S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7. \end{aligned} \tag{4.56}$$

We see that the left-hand side of (4.56) is equal to

$$
\begin{aligned}
&\sum_{n=0}^{l-1}(\phi d_t\zeta^n, d_t\zeta^n)\,\Delta t_c+\frac{1}{2}\sum_{n=0}^{l-1}(b(C^{*n})\nabla[(\zeta^{n+1}-\zeta^n)+(\zeta^{n+1}+\zeta^n)], \nabla d_t\zeta^n)\,\Delta t_c\\
&\quad=\frac{1}{2}\sum_{n=0}^{l-1}[2(\phi d_t\zeta^n, d_t\zeta^n)+\Delta t_c(b(C^{*n})\nabla d_t\zeta^n, \nabla d_t\zeta^n)]\,\Delta t_c\\
&\qquad+\frac{1}{2}\sum_{n=0}^{l-1}(b(C^{*n})\nabla(\zeta^{n+1}+\zeta^n), \nabla(\zeta^{n+1}-\zeta^n))\\
&\quad\geqq\frac{1}{2}\sum_{n=0}^{l-1}|||d_t\zeta^n|||_n^2\,\Delta t_c+\frac{1}{2}\sum_{n=0}^{l-1}(\|\zeta^{n+1}\|_{b^n}^2-\|\zeta^n\|_{b^n}^2).
\end{aligned}
\tag{4.57}
$$

Thus we can hide small multiples of L^2-norms of $d_t\zeta$ on the left-hand side, and we will be able to apply Gronwall's lemma to H^1-norms of ζ after perturbing the weighted-norm indices.

We proceed to estimate S_1 through S_7. First, we have

$$
\begin{aligned}
S_1+S_2&=\sum_{n=0}^{l-1}\left(\phi\frac{\partial\xi^{n+1}}{\partial t}, d_t\zeta^n\right)\Delta t_c+\sum_{n=0}^{l-1}\left(\phi\left(\frac{\partial\tilde{c}^{n+1}}{\partial t}-d_t\tilde{c}^n\right), d_t\zeta^n\right)\Delta t_c\\
&\quad-\sum_{n=0}^{l-1}\lambda(\xi^{n+1}, d_t\zeta^n)\,\Delta t_c,\\
|S_1+S_2|&\leqq K(\|c\|_{H^1(H^r)})h^{2r}+K(\|\tilde{c}\|_{H^2(L^2)})(\Delta t_c)^2+\varepsilon\sum_{n=0}^{l-1}\|d_t\zeta^n\|^2\,\Delta t_c.
\end{aligned}
\tag{4.58}
$$

We note that since we cannot estimate $\|d_t\zeta^n\|_1$, we were unable to use the H^{-1}-norm on $\partial\xi/\partial t$, and hence were forced to assume $\partial c/\partial t\in L^2(H^r)$.

In handling S_3, we again must avoid having to bound $\|\nabla d_t\zeta^n\|$. We therefore sum by parts in time, reducing $\nabla d_t\zeta^n$ to $\nabla\zeta^n$, which can be treated. We see that

$$
\begin{aligned}
S_3&=([b(c^l)-b(C^{*l-1})]\nabla\tilde{c}^l, \nabla\zeta^l)-([b(c^1)-b(C^{*0})]\nabla\tilde{c}^1, \nabla\zeta^0)\\
&\quad-\sum_{n=1}^{l-1}([b(c^{n+1})-b(C^{*n})]\nabla\tilde{c}^{n+1}-[b(c^n)-b(C^{*n-1})]\nabla\tilde{c}^n, \nabla\zeta^n)\\
&\equiv T_1-T_2-T_3.
\end{aligned}
\tag{4.59}
$$

Now

$$
\begin{aligned}
|T_1|&\leqq K\|c^l-C^{*l-1}\|\|\nabla\tilde{c}^l\|_{L^\infty}\|\nabla\zeta^l\|\\
&\leqq K(\|c^l-c^{l-1}\|+\|\xi^{l-1}\|+\|\zeta^{l-1}\|)\|\nabla\zeta^l\|\\
&\leqq K(\|c\|_{L^\infty(H^r)})h^{2r}+K(\Delta t_c)^2+K\|\zeta^{l-1}\|^2+\varepsilon\|\zeta^l\|_1^2,
\end{aligned}
\tag{4.60}
$$

and a similar estimate holds for T_2. To bound T_3, we write

$$(4.61)\quad \begin{aligned} T_3 &= \sum_{n=1}^{l-1} ([b(c^{n+1})-b(C^{*n})]\nabla(\tilde{c}^{n+1}-\tilde{c}^n), \nabla\zeta^n) \\ &\quad + \sum_{n=1}^{l-1} (\{[b(c^{n+1})-b(C^{*n})]-[b(c^n)-b(C^{*n-1})]\}\nabla\tilde{c}^n, \nabla\zeta^n) \\ &\equiv T_4+T_5, \\ T_4 &= \sum_{n=1}^{l-1} ([b(c^{n+1})-b(C^{*n})]\nabla d_t\tilde{c}^n, \nabla\zeta^n)\,\Delta t_c, \\ |T_4| &\leqq K\sum_{n=1}^{l-1} (\|c^{n+1}-c^n\|+\|\xi^n\|+\|\zeta^n\|)\|\nabla d_t\tilde{c}^n\|_{L^\infty}\|\nabla\zeta^n\|\,\Delta t_c \\ &\leqq K(\|c\|_{L^2(H^r)})h^{2r}+K(\Delta t_c)^2+K\sum_{n=1}^{l-1}(\|\zeta^n\|^2+\|\zeta^n\|_1^2)\,\Delta t_c. \end{aligned}$$

To bound T_5, we define

$$(4.62)\quad \begin{aligned} b_{1,n} &= \int_0^1 \frac{\partial b}{\partial c}(\alpha c^{n+1}+(1-\alpha)c^n)\,d\alpha, \\ b_{2,n} &= \int_0^1 \frac{\partial b}{\partial c}(\alpha C^{*n}+(1-\alpha)C^{*n-1})\,d\alpha. \end{aligned}$$

Then

$$(4.63)\quad \begin{aligned} b(c^{n+1})-b(c^n) &= (c^{n+1}-c^n)b_{1,n}, \\ b(C^{*n})-b(C^{*n-1}) &= (C^{*n}-C^{*n-1})b_{2,n}. \end{aligned}$$

Thus

$$(4.64)\quad \begin{aligned} T_5 &= \sum_{n=1}^{l-1} ([b_{1,n}d_tc^n-b_{2,n}d_tC^{*n-1}]\nabla\tilde{c}^n, \nabla\zeta^n)\,\Delta t_c \\ &= \sum_{n=1}^{l-1} (d_tc^n[b_{1,n}-b_{2,n}]\nabla\tilde{c}^n, \nabla\zeta^n)\,\Delta t_c \\ &\quad + \sum_{n=1}^{l-1} (b_{2,n}\{[d_tc^n-d_tc^{n-1}]+d_t\xi^{n-1}+d_t\zeta^{n-1}\}\nabla\tilde{c}^n, \nabla\zeta^n)\,\Delta t_c \\ &\equiv T_6+T_7. \end{aligned}$$

We note that

$$(4.65)\quad \begin{aligned} b_{1,n}-b_{2,n} &= \int_0^1 \left[\frac{\partial b}{\partial c}(\alpha c^{n+1}+(1-\alpha)c^n)-\frac{\partial b}{\partial c}(\alpha c^n+(1-\alpha)c^{n-1})\right]d\alpha \\ &\quad + \int_0^1 \left[\frac{\partial b}{\partial c}(\alpha c^n+(1-\alpha)c^{n-1})-\frac{\partial b}{\partial c}(\alpha C^{*n}+(1-\alpha)C^{*n-1})\right]d\alpha \\ &= \int_0^1 \Delta t_c d_t(\alpha c^n+(1-\alpha)c^{n-1}) \\ &\quad \cdot \int_0^1 \frac{\partial^2 b}{\partial c^2}(\beta[\alpha c^{n+1}+(1-\alpha)c^n]+(1-\beta)[\alpha c^n+(1-\alpha)c^{n-1}])\,d\beta\,d\alpha \end{aligned}$$

$$+\int_0^1 [\alpha(c-C^*)^n+(1-\alpha)(c-C^*)^{n-1}]$$
$$\cdot\int_0^1 \frac{\partial^2 b}{\partial c^2}(\beta[\alpha c^n+(1-\alpha)c^{n-1}]+(1-\beta)[\alpha C^{*n}+(1-\alpha)C^{*n-1}])\,d\beta\,d\alpha,$$

so that

$$|b_{1,n}-b_{2,n}|\le K\,\Delta t_c(|d_tc^n|+|d_tc^{n-1}|)+K(|\xi^n|+|\xi^{n-1}|+|\zeta^n|+|\zeta^{n-1}|). \tag{4.66}$$

Thus

$$\begin{aligned}|T_6|&\le\sum_{n=1}^{l-1}\|d_tc^n\|_{L^\infty}\|b_{1,n}-b_{2,n}\|\|\nabla\tilde c^n\|_{L^\infty}\|\nabla\zeta^n\|\,\Delta t_c\\&\le K(\|c\|_{L^2(H^r)})h^{2r}+K(\Delta t_c)^2+K\sum_{n=0}^{l-1}\|\zeta^n\|^2\,\Delta t_c+K\sum_{n=1}^{l-1}\|\zeta^n\|_1^2\,\Delta t_c,\\|T_7|&\le K\sum_{n=1}^{l-1}(\|d_tc^n-d_tc^{n-1}\|+\|d_t\xi^{n-1}\|+\|d_t\zeta^{n-1}\|)\|\nabla\tilde c^n\|_{L^\infty}\|\nabla\zeta^n\|\,\Delta t_c\\&\le K(\|c\|_{H^1(H^r)})h^{2r}+K(\|c\|_{H^2(L^2)})(\Delta t_c)^2+K\sum_{n=1}^{l-1}\|\zeta^n\|_1^2\,\Delta t_c+\varepsilon\sum_{n=0}^{l-2}\|d_t\zeta^n\|^2\,\Delta t_c.\end{aligned} \tag{4.67}$$

Combining the estimates (4.60) through (4.67), we have shown

$$\begin{aligned}|S_3|\le{}&K(\|c\|_{H^1(H^r)})h^{2r}+K(\|c\|_{H^2(L^2)})(\Delta t_c)^2+K\sum_{n=0}^{l-1}\|\zeta^n\|^2\,\Delta t_c\\&+K\sum_{n=1}^{l-1}\|\zeta^n\|_1^2\,\Delta t_c+\varepsilon\sum_{n=0}^{l-2}\|d_t\zeta^n\|^2\,\Delta t_c\\&+K\|\zeta^0\|_1^2+K\|\zeta^{l-1}\|^2+\varepsilon\|\zeta^l\|_1^2.\end{aligned} \tag{4.68}$$

Next, we see that

$$\begin{aligned}S_4={}&\sum_{n=0}^{l-1}([u(c^{n+1},\nabla p^{n+1})-u(C^{*n},E\nabla P^{n+1})]\cdot\nabla\tilde c^{n+1},d_t\zeta^n)\,\Delta t_c\\&+\sum_{n=0}^{l-1}(u(C^{*n},E\nabla P^{n+1})\cdot\nabla(\tilde c^{n+1}-C^n),d_t\zeta^n)\,\Delta t_c\\\equiv{}&T_8+T_9.\end{aligned} \tag{4.69}$$

As in Theorem 4.1, we make the induction hypothesis (4.21) to treat T_9. This allows us to write

$$\begin{aligned}|T_9|&\le\sum_{n=0}^{l-1}\|u(C^{*n},E\nabla P^{n+1})\|_{L^\infty}(\|\nabla(\tilde c^{n+1}-\tilde c^n)\|+\|\nabla\zeta^n\|)\|d_t\zeta^n\|\,\Delta t_c\\&\le K(\Delta t_c)^2+K\sum_{n=0}^{l-1}\|\zeta^n\|_1^2\,\Delta t_c+\varepsilon\sum_{n=0}^{l-1}\|d_t\zeta^n\|^2\,\Delta t_c,\end{aligned}$$

$$\begin{aligned}T_8={}&\sum_{n=0}^{l-1}([u(c^{n+1},\nabla p^{n+1})-u(c^{n+1},\nabla\tilde p^{n+1})]\cdot\nabla\tilde c^{n+1},d_t\zeta^n)\,\Delta t_c\\&+\sum_{n=0}^{l-1}([u(c^{n+1},\nabla\tilde p^{n+1})-u(c^{n+1},E\nabla\tilde p^{n+1})]\cdot\nabla\tilde c^{n+1},d_t\zeta^n)\,\Delta t_c\end{aligned} \tag{4.70}$$

$$+\sum_{n=0}^{l-1}([u(c^{n+1}, E\nabla\tilde{p}^{n+1})-u(c^{n+1}, E\nabla P^{n+1})]\cdot\nabla\tilde{c}^{n+1}, d_t\zeta^n)\,\Delta t_c$$

$$+\sum_{n=0}^{l-1}([u(c^{n+1}, E\nabla P^{n+1})-u(C^{*n}, E\nabla P^{n+1})]\cdot\nabla\tilde{c}^{n+1}, d_t\zeta^n)\,\Delta t_c$$

$$\equiv T_{10}+T_{11}+T_{12}+T_{13}.$$

To estimate T_{11}, recall that

$$\|\nabla\tilde{p}^{n+1}-E\nabla\tilde{p}^{n+1}\|^2\leqq K(\Delta t_p^0)^2\left\|\frac{\partial\tilde{p}}{\partial t}\right\|^2_{L^\infty(0,t_1;H^1)}, \qquad m=0,$$

$$\|\nabla\tilde{p}^{n+1}-E\nabla\tilde{p}^{n+1}\|^2\leqq K(\Delta t_p)^3\left\|\frac{\partial^2\tilde{p}}{\partial t^2}\right\|^2_{L^2(t_{m-1},t_{m+1};H^1)}, \qquad m\geqq 1. \tag{4.71}$$

Thus

$$|T_{11}|\leqq K\sum_{n=0}^{l-1}\|\nabla\tilde{p}^{n+1}-E\nabla\tilde{p}^{n+1}\|\|\nabla\tilde{c}^{n+1}\|_{L^\infty}\|d_t\zeta^n\|\Delta t_c$$

$$\leqq K\Big[\sum_{m=0}\|\nabla\tilde{p}^{n+1}-E\nabla\tilde{p}^{n+1}\|^2\,\Delta t_c+\sum_{m\geqq 1}\|\nabla\tilde{p}^{n+1}-E\nabla\tilde{p}^{n+1}\|^2\,\Delta t_c\Big]$$

$$+\varepsilon\sum_{n=0}^{l-1}\|d_t\zeta^n\|^2\,\Delta t_c$$

$$\leqq K\left(\frac{\Delta t_p^0}{\Delta t_c}\right)(\Delta t_p^0)^2\,\Delta t_c+K\left(\frac{\Delta t_p}{\Delta t_c}\right)(\Delta t_p)^3\sum_{m=1}^{k-1}\left\|\frac{\partial^2\tilde{p}}{\partial t^2}\right\|^2_{L^2(t_{m-1},t_{m+1};H^1)}\Delta t_c \tag{4.72}$$

$$+\varepsilon\sum_{n=0}^{l-1}\|d_t\zeta^n\|^2\,\Delta t_c$$

$$\leqq K(\|\tilde{p}\|_{W^1_\infty(H^1)})(\Delta t_p^0)^3+K(\|\tilde{p}\|_{H^2(H^1)})(\Delta t_p)^4+\varepsilon\sum_{n=0}^{l-1}\|d_t\zeta^n\|^2\,\Delta t_c.$$

Next, we have

$$|T_{12}|\leqq K\sum_{n=0}^{l-1}\|E\nabla\eta^{n+1}\|\|d_t\zeta^n\|\,\Delta t_c$$

$$\leqq K\|\eta_0\|^2_{a_0}\left(\frac{\Delta t_p^0}{\Delta t_c}\right)\Delta t_c+K\sum_{m=1}^{k-1}\|\eta_m\|^2_{a_m}\left(\frac{\Delta t_p}{\Delta t_c}\right)\Delta t_c+\varepsilon\sum_{n=0}^{l-1}\|d_t\zeta^n\|^2\,\Delta t_c$$

$$\leqq K(\|c\|_{L^\infty(H^r)})h^{2r}+K\|P_0-\bar{P}_0\|^2_{a_0}\,\Delta t_p^0$$

$$+K\sum_{m=1}^{k-1}\|P_m-\bar{P}_m\|^2_{a_m}\,\Delta t_p+K\|\zeta_0\|^2\,\Delta t_p^0 \tag{4.73}$$

$$+K\sum_{m=1}^{k-1}\|\zeta_m\|^2\,\Delta t_p+\varepsilon\sum_{n=0}^{l-1}\|d_t\zeta^n\|^2\,\Delta t_c,$$

$$|T_{13}|\leqq K\sum_{n=0}^{l-1}(\|c^{n+1}-c^n\|+\|\xi^n\|+\|\zeta^n\|)\|d_t\zeta^n\|\,\Delta t_c$$

$$\leqq K(\|c\|_{L^2(H^r)})h^{2r}+K(\Delta t_c)^2+K\sum_{n=0}^{l-1}\|\zeta^n\|^2\,\Delta t_c+\varepsilon\sum_{n=0}^{l-1}\|d_t\zeta^n\|^2\,\Delta t_c.$$

We now analyze T_{10} in a fashion similar to U_1 in Corollary 4.2. We see that

$$
\begin{aligned}
T_{10} &= \sum_{n=0}^{l-1} (a(c^{n+1})\nabla(\tilde{p}-p)^{n+1} \cdot \nabla(\tilde{c}-c)^{n+1}, d_t\zeta^n)\,\Delta t_c \\
&\quad + \sum_{n=0}^{l-1} (a(c^{n+1})\nabla(\tilde{p}-p)^{n+1} \cdot \nabla c^{n+1}, d_t\zeta^n)\,\Delta t_c \qquad (4.74)\\
&\equiv T_{14}+T_{15}.
\end{aligned}
$$

Now in the manner of (4.47), we have

$$
(4.75)\qquad |T_{14}| \leqq \begin{cases} K(\|p\|_{L^2(H^s)}, \|c\|_{L^\infty(H^r)})(h^{2r}+h^{2s}) + \varepsilon \displaystyle\sum_{n=0}^{l-1} \|d_t\zeta^n\|^2\,\Delta t_c, & r \geqq 3 \text{ or } s \geqq 3, \\ K(\|p\|_{L^2(W^2_\infty)}, \|c\|_{L^\infty(H^2)})h^4|\log h|^2 + \varepsilon \displaystyle\sum_{n=0}^{l-1} \|d_t\zeta^n\|^2\,\Delta t_c, & r=s=2. \end{cases}
$$

We wish to integrate by parts in T_{15}. However, this would demand an estimate for $\|d_t\zeta^n\|_1$, which we do not have. We avoid this difficulty by first summing by parts in time to obtain

$$
\begin{aligned}
T_{15} &= -(a(c^l)\nabla\theta^l \cdot \nabla c^l, \zeta^l) + (a(c^1)\nabla\theta^1 \cdot \nabla c^1, \zeta^0) \\
&\quad + \sum_{n=1}^{l-1} (a(c^{n+1})\nabla\theta^{n+1} \cdot \nabla c^{n+1} - a(c^n)\nabla\theta^n \cdot \nabla c^n, \zeta^n) \\
&\equiv T_{16}+T_{17}+T_{18}, \qquad (4.76)\\
T_{16} &= (\theta^l, \nabla \cdot (a(c^l)\zeta^l\nabla c^l)) - \left\langle \theta^l, a(c^l)\zeta^l\frac{\partial c^l}{\partial n}\right\rangle \equiv T_{19}+T_{20}, \\
|T_{19}| &\leqq K\|\theta^l\|\|a(c^l)\|_{W^1_\infty}\|\zeta^l\|_1\|c^l\|_{W^2_\infty} \leqq K(\|c\|_{L^\infty(W^2_\infty)}, \|p\|_{L^\infty(H^s)})h^{2s} + \varepsilon\|\zeta^l\|_1^2, \\
|T_{20}| &\leqq K|\theta^l|_{-1/2}|\zeta^l|_{1/2}\left|\frac{\partial c^l}{\partial n}\right|_{1/2+\varepsilon} \leqq K(\|c\|_{L^\infty(H^{2+\varepsilon})}, \|p\|_{L^\infty(H^s)})h^{2s} + \varepsilon\|\zeta^l\|_1^2.
\end{aligned}
$$

Similarly,

$$
(4.77)\qquad |T_{17}| \leqq Kh^{2s} + K\|\zeta^0\|_1^2.
$$

We split T_{18} into three terms,

$$
\begin{aligned}
T_{18} &= \sum_{n=1}^{l-1} ([a(c^{n+1})-a(c^n)]\nabla\theta^{n+1} \cdot \nabla c^{n+1}, \zeta^n) \\
&\quad + \sum_{n=1}^{l-1} (a(c^n)\nabla d_t\theta^n \cdot \nabla c^{n+1}, \zeta^n)\,\Delta t_c \qquad (4.78)\\
&\quad + \sum_{n=1}^{l-1} (a(c^n)\nabla\theta^n \cdot \nabla d_t c^n, \zeta^n)\,\Delta t_c \\
&\equiv T_{21}+T_{22}+T_{23}.
\end{aligned}
$$

Letting $a_n = \int_0^1 (\partial a/\partial c)\,(\alpha_c^{n+1} + (1-\alpha)c^n)\,d\alpha$, so $a(c^{n+1}) - a(c^n) = a_n\, d_t c^n\, \Delta t_c$,

$$
\begin{aligned}
T_{21} &= \sum_{n=1}^{l-1} (a_n d_t c^n \nabla \theta^{n+1} \cdot \nabla c^{n+1}, \zeta^n)\,\Delta t_c \\
&= -\sum_{n=1}^{l-1} (\theta^{n+1}, \nabla \cdot (a_n \zeta^n d_t c^n \nabla c^{n+1}))\,\Delta t_c \\
&\quad + \sum_{n=1}^{l-1} \left\langle \theta^{n+1}, a_n \zeta^n d_t c^n \frac{\partial c^{n+1}}{\partial n} \right\rangle \Delta t_c \\
&\equiv T_{24} + T_{25},
\end{aligned}
\tag{4.79}
$$

and noting that $|\nabla a_n| \leqq |\partial^2 a/\partial c\,\partial x_i|\,|\partial c/\partial x_i|$ is bounded, we have

$$
\begin{aligned}
|T_{24}| &\leqq K \sum_{n=1}^{l-1} \|\theta^{n+1}\| \|\zeta^n\|_1 \|d_t c^n\|_{W^1_\infty} \|c^{n+1}\|_{W^2_\infty}\,\Delta t_c \\
&\leqq K(\|c\|_{L^2(W^1_\infty)}, \|p\|_{L^\infty(H^s)}) h^{2s} + K \sum_{n=1}^{l-1} \|\zeta^n\|_1^2\,\Delta t_c, \\
|T_{25}| &\leqq K \sum_{n=1}^{l-1} |\theta^{n+1}|_{-1/2} |\zeta^n|_{1/2} \|d_t c^n\|_{1+\varepsilon} \left\| \frac{\partial c^{n+1}}{\partial n} \right\|_{1+\varepsilon} \Delta t_c \\
&\leqq K(\|c\|_{L^\infty(H^{2+\varepsilon})}, \|c\|_{W^1_\infty(H^{1+\varepsilon})}, \|p\|_{L^2(H^s)}) h^{2s} + K \sum_{n=1}^{l-1} \|\zeta^n\|_1^2\,\Delta t_c.
\end{aligned}
\tag{4.80}
$$

Also,

$$
\begin{aligned}
T_{22} &= -\sum_{n=1}^{l-1} (d_t \theta^n, \nabla \cdot (a(c^n) \zeta^n \nabla c^{n+1}))\,\Delta t_c \\
&\quad + \sum_{n=1}^{l-1} \left\langle d_t \theta^n, a(c^n) \zeta^n \frac{\partial c^{n+1}}{\partial n} \right\rangle \Delta t_c \equiv T_{26} + T_{27},
\end{aligned}
$$

$$
\begin{aligned}
|T_{26}| &\leqq K \sum_{n=1}^{l-1} \|d_t \theta^n\| \|a(c^n)\|_{W^1_\infty} \|\zeta^n\|_1 \|c^{n+1}\|_{W^2_\infty}\,\Delta t_c \\
&\leqq K(\|c\|_{L^\infty(W^2_\infty)}, \|p\|_{H^1(H^s)}) h^{2s} + K \sum_{n=1}^{l-1} \|\zeta^n\|_1^2\,\Delta t_c, \\
|T_{27}| &\leqq K \sum_{n=1}^{l-1} |d_t \theta^n|_{-1/2} |\zeta^n|_{1/2} \|c^{n+1}\|_{2+\varepsilon}\,\Delta t_c \\
&\leqq K(\|c\|_{L^\infty(H^{2+\varepsilon})}, \|p\|_{H^1(H^s)}) h^{2s} + K \sum_{n=1}^{l-1} \|\zeta^n\|_1^2\,\Delta t_c.
\end{aligned}
\tag{4.81}
$$

Finally,

$$
\begin{aligned}
T_{23} &= -\sum_{n=1}^{l-1} (\theta^n, \nabla \cdot (a(c^n) \zeta^n \nabla d_t c^n))\,\Delta t_c + \sum_{n=1}^{l-1} \left\langle \theta^n, a(c^n) \zeta^n \frac{\partial}{\partial n}(d_t c^n) \right\rangle \Delta t_c \\
&\equiv T_{28} + T_{29},
\end{aligned}
$$

$$
\begin{aligned}
|T_{28}| &\leqq K \sum_{n=1}^{l-1} \|\theta^n\| \|a(c^n)\|_{W^1_\infty} \|\zeta^n\|_1 \|d_t c^n\|_{W^2_\infty}\,\Delta t_c \\
&\leqq K(\|c\|_{H^1(W^2_\infty)}, \|p\|_{L^\infty(H^s)}) h^{2s} + K \sum_{n=1}^{l-1} \|\zeta^n\|_1^2\,\Delta t_c,
\end{aligned}
\tag{4.82}
$$

$$
\begin{aligned}
|T_{29}| &\le K \sum_{n=1}^{l-1} |\theta^n|_{-1/2} |\zeta^n|_{1/2} \|d_t c^n\|_{2+\varepsilon} \Delta t_c \\
&\le K(\|c\|_{W^1_\infty(H^{2+\varepsilon})}, \|p\|_{L^2(H^s)}) h^{2s} + K \sum_{n=1}^{l-1} \|\zeta^n\|_1^2 \Delta t_c.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|S_4| \le\, & K(\|c\|_{L^\infty(H^r)}, \|p\|_{L^2(H^s)}) h^{2r} \\
& + K(\|c\|_{L^\infty(H^r)}, \|c\|_{W^1_\infty(H^{2+\varepsilon})}, \|c\|_{H^1(W^2_\infty)}, \|p\|_{H^1(H^s)}) h^{2s} + K(\Delta t_c)^2 \\
& + K(\|\tilde{p}\|_{W^1_\infty(H^1)})(\Delta t_p^0)^3 + K(\|\tilde{p}\|_{H^2(H^1)})(\Delta t_p)^4 \\
& + K\|P_0 - \bar{P}_0\|_{a_0}^2 \Delta t_p^0 + K \sum_{m=1}^{k-1} \|P_m - \bar{P}_m\|_{a_m}^2 \Delta t_p \qquad (4.83)\\
& + K\|\zeta_0\|^2 \Delta t_p^0 + K \sum_{m=1}^{k-1} \|\zeta_m\|^2 \Delta t_p + K \sum_{n=0}^{l-1} \|\zeta^n\|^2 \Delta t_c \\
& + K \sum_{n=0}^{l-1} \|\zeta^n\|_1^2 \Delta t_c + K\|\zeta^0\|_1^2 + \varepsilon \|\zeta^l\|_1^2 + \varepsilon \sum_{n=0}^{l-1} \|d_t \zeta^n\|^2 \Delta t_c,
\end{aligned}
$$

with the appropriate modification if $r = s = 2$.

Next, we split S_5 by writing

$$
\begin{aligned}
S_5 = & \sum_{n=0}^{l-1} \langle g(t^{n+1}, C^n) - g(t^{n+1}, \tilde{c}^n), \zeta^{n+1} - \zeta^n \rangle \\
& + \sum_{n=0}^{l-1} \langle g(t^{n+1}, \tilde{c}^n) - g(t^{n+1}, \tilde{c}^{n+1}), \zeta^{n+1} - \zeta^n \rangle \qquad (4.84)\\
\equiv\, & T_{30} + T_{31}.
\end{aligned}
$$

For the same reasons as before, we sum by parts in time. For T_{31}, we have

$$
\begin{aligned}
T_{31} = & \langle g(t^l, \tilde{c}^{l-1}) - g(t^l, \tilde{c}^l), \zeta^l \rangle - \langle g(t^1, \tilde{c}^0) - g(t^1, \tilde{c}^1), \zeta^0 \rangle \\
& - \sum_{n=1}^{l-1} \langle [g(t^{n+1}, \tilde{c}^n) - g(t^{n+1}, \tilde{c}^{n+1})] - [g(t^n, \tilde{c}^{n-1}) - g(t^n, \tilde{c}^n)], \zeta^n \rangle \qquad (4.85)\\
\equiv\, & T_{32} + T_{33} + T_{34}.
\end{aligned}
$$

Then

$$
\begin{aligned}
|T_{32}| &\le K\|d_t \tilde{c}^{l-1}\|_1 \|\zeta^l\|_1 \Delta t_c \le K\left(\left\|\frac{\partial \tilde{c}}{\partial t}\right\|_{L^\infty(H^1)}\right)(\Delta t_c)^2 + \varepsilon \|\zeta^l\|_1^2, \\
|T_{33}| &\le K\|d_t \tilde{c}^0\|_1 \|\zeta^0\|_1 \Delta t_c \le K(\Delta t_c)^2 + \|\zeta^0\|_1^2.
\end{aligned} \qquad (4.86)
$$

We then split T_{34} as follows:

$$
\begin{aligned}
-T_{34} = & \sum_{n=1}^{l-1} \langle [g(t^{n+1}, \tilde{c}^n) - g(t^{n+1}, \tilde{c}^{n+1})] \\
& \qquad - [g(t^n, \tilde{c}^n) - g(t^n, \tilde{c}^{n+1})], \zeta^n \rangle \qquad (4.87)\\
& + \sum_{n=1}^{l-1} \langle [g(t^n, \tilde{c}^n) - g(t^n, \tilde{c}^{n+1})] - [g(t^n, \tilde{c}^{n-1}) - g(t^n, \tilde{c}^n)], \zeta^n \rangle \\
\equiv\, & T_{35} + T_{36}.
\end{aligned}
$$

We next define

$$
(4.88)\qquad
\begin{aligned}
g_{1,n}&=\int_0^1 \frac{\partial g}{\partial c}(t^{n+1}, \alpha\tilde{c}^n+(1-\alpha)\tilde{c}^{n+1})\,d\alpha,\\
g_{2,n}&=\int_0^1 \frac{\partial g}{\partial c}(t^{n}, \alpha\tilde{c}^n+(1-\alpha)\tilde{c}^{n+1})\,d\alpha,\\
g_{3,n}&=\int_0^1 \frac{\partial g}{\partial c}(t^{n}, \alpha\tilde{c}^{n-1}+(1-\alpha)\tilde{c}^{n})\,d\alpha,
\end{aligned}
$$

and we note that by an argument like (4.65)–(4.66), we have

$$
(4.89)\qquad
\begin{aligned}
|g_{1,n}-g_{2,n}|&\leqq K\left(\frac{\partial^2 g}{\partial t\,\partial c}\right)\Delta t_c,\\
|g_{2,n}-g_{3,n}|&\leqq K\left(\frac{\partial^2 g}{\partial c^2}\right)(|d_t\tilde{c}^{n-1}|+|d_t\tilde{c}^n|)\,\Delta t_c.
\end{aligned}
$$

We then see that

$$
(4.90)\qquad
\begin{aligned}
|T_{35}|&=\left|\sum_{n=1}^{l-1}\langle d_t\tilde{c}^n(g_{1,n}-g_{2,n}),\zeta^n\rangle\,\Delta t_c\right|\\
&\leqq\sum_{n=1}^{l-1}|d_t\tilde{c}^n|_0|g_{1,n}-g_{2,n}|_{L^\infty}|\zeta^n|_0\,\Delta t_c\\
&\leqq K(\|\tilde{c}\|_{W^1_\infty(H^1)})(\Delta t_c)^2+K\sum_{n=1}^{l-1}\|\zeta^n\|_1^2\,\Delta t_c,
\end{aligned}
$$

and

$$
(4.91)\qquad
\begin{aligned}
|T_{36}|&=\left|\sum_{n=1}^{l-1}\langle g_{2,n}d_t\tilde{c}^n-g_{3,n}d_t\tilde{c}^{n-1},\zeta^n\rangle\,\Delta t_c\right|\\
&=\left|\sum_{n=1}^{l-1}\langle d_t\tilde{c}^n[g_{2,n}-g_{3,n}]+(d_t\tilde{c}^n-d_t\tilde{c}^{n-1})g_{3,n},\zeta^n\rangle\,\Delta t_c\right|\\
&\leqq K(\|\tilde{c}\|_{H^2(H^1)})(\Delta t_c)^2+K\sum_{n=1}^{l-1}\|\zeta^n\|_1^2\,\Delta t_c.
\end{aligned}
$$

To bound T_{30}, we define

$$
(4.92)\qquad g_{4,n}=\int_0^1\frac{\partial g}{\partial c}(t^{n+1},\alpha C^n+(1-\alpha)\tilde{c}^n)\,d\alpha.
$$

We note that $g_{4,n}-g_{4,n-1}$ can be split into two terms and analyzed by an argument like (4.65)–(4.66) to obtain

$$
(4.93)\qquad
\begin{aligned}
|g_{4,n}-g_{4,n-1}|&\leqq K(\Delta t_c+|C^n-C^{n-1}|+|\tilde{c}^n-\tilde{c}^{n-1}|)\\
&\leqq K\,\Delta t_c(1+|d_t\zeta^{n-1}|+|d_t\tilde{c}^{n-1}|).
\end{aligned}
$$

Then

$$
(4.94)\quad
\begin{aligned}
T_{32} &= \sum_{n=0}^{l-1} \langle g_{4,n}\zeta^n, \zeta^{n+1}-\zeta^n\rangle \\
&= \sum_{n=0}^{l-1} \left\langle g_{4,n}\left[\frac{1}{2}(\zeta^{n+1}+\zeta^n)-\frac{1}{2}(\zeta^{n+1}-\zeta^n)\right], \zeta^{n+1}-\zeta^n\right\rangle \\
&= \frac{1}{2}\sum_{n=0}^{l-1} \langle g_{4,n}, (\zeta^{n+1})^2-(\zeta^n)^2\rangle - \frac{1}{2}\sum_{n=0}^{l-1} \langle g_{4,n} d_t\zeta^n, d_t\zeta^n\rangle(\Delta t_c)^2 \\
&\equiv T_{37}+T_{38}.
\end{aligned}
$$

Summing T_{37} by parts, we have

$$
\begin{aligned}
T_{37} &= \frac{1}{2}\langle g_{4,l-1}, (\zeta^l)^2\rangle - \frac{1}{2}\langle g_{4,0}, (\zeta^0)^2\rangle \\
&\quad -\frac{1}{2}\sum_{n=1}^{l-1} \langle g_{4,n}-g_{4,n-1}, (\zeta^n)^2\rangle \equiv T_{39}+T_{40}+T_{41},
\end{aligned}
$$

$$
(4.95)\quad
\begin{aligned}
|T_{39}| &\leqq K|\zeta^l|^2 \leqq K\|\zeta^l\|^2+\varepsilon\|\zeta^l\|_1^2, \\
|T_{40}| &\leqq K\|\zeta^0\|_1^2, \\
|T_{41}| &\leqq K\sum_{n=1}^{l-1} \langle 1+|d_t\zeta^{n-1}|+|d_t\tilde{c}^{n-1}|, (\zeta^n)^2\rangle\,\Delta t_c \\
&\leqq K\sum_{n=1}^{l-1} \langle \zeta^n, \zeta^n\rangle\,\Delta t_c + K\sum_{n=1}^{l-1} \|d_t\zeta^{n-1}\|_{L_\infty}\langle \zeta^n, \zeta^n\rangle\,\Delta t_c \\
&\leqq K\sum_{n=1}^{l-1} (1+\|d_t\zeta^{n-1}\|_{L^\infty})\|\zeta^n\|_1^2\,\Delta t_c, \\
|T_{38}| &\leqq K\sum_{n=0}^{l-2} \langle d_t\zeta^n, d_t\zeta^n\rangle(\Delta t_c)^2 + K\langle \zeta^l-\zeta^{l-1}, \zeta^l-\zeta^{l-1}\rangle \\
&\leqq K\sum_{n=0}^{l-2} \|d_t\zeta^n\|\|d_t\zeta^n\|_1(\Delta t_c)^2 + K\|\zeta^l\|\|\zeta^l\|_1 + K\|\zeta^{l-1}\|\|\zeta^{l-1}\|_1 \\
&\leqq K\sum_{n=0}^{l-2} \|d_t\zeta^n\|^2(\Delta t_c)^2 h^{-1} + K\|\zeta^l\|^2 + K\|\zeta^{l-1}\|^2 + \varepsilon\|\zeta^l\|_1^2 + \varepsilon\|\zeta^{l-1}\|_1^2.
\end{aligned}
$$

We note that we would like to have avoided the evaluation of g on $\tilde{c}^n$ by summing all of S_5 by parts at once. This does not work, since the analogue of T_{36} in T_{30} becomes a term of the form $\sum_{n=1}^{l-1} \langle d_t\zeta^{n-1}, \zeta^n\rangle\,\Delta t_c$, which demands a bound on $\|d_t\zeta^{n-1}\|_1$. We would also like to have replaced C by C^* in S_5, but this would prematurely introduce absolute values into the argument (4.94)–(4.95) and cause the summation by parts to fail. We collect terms in (4.84) through (4.95) to see that

$$
(4.96)\quad
\begin{aligned}
|S_5| &\leqq K(\|\tilde{c}\|_{H^2(H^1)})(\Delta t_c)^2 + K\sum_{n=1}^{l-1} (1+\|d_t\zeta^{n-1}\|_{L^\infty})\|\zeta^n\|_1^2\,\Delta t_c \\
&\quad + K\sum_{n=0}^{l-2} \|d_t\zeta^n\|^2(\Delta t_c)^2 h^{-1} + K\|\zeta^l\|^2 + K\|\zeta^{l-1}\|^2 \\
&\quad + \varepsilon\|\zeta^l\|_1^2 + \varepsilon\|\zeta^{l-1}\|_1^2 + K\|\zeta^0\|_1^2.
\end{aligned}
$$

We note that the evaluations of g on C^n were confined to $0 \leqq n \leqq l-1$. By an induction argument like the one for (4.21), we can show that C^n, $0 \leqq n \leqq l-1$, and $\tilde{c}$ may be assumed to lie in $[-\varepsilon, 1+\varepsilon]$. Then the evaluations of g on these arguments present no difficulties.

Finally, we analyze S_6 and S_7 together, noting that for Δt_c sufficiently small,

$$(4.97)\quad \begin{aligned} |S_6+S_7| &\leqq |||C^1-\bar{C}^1|||_0|||d_t\zeta^0|||_0+\sum_{n=1}^{l-1}|||C^{n+1}-\bar{C}^{n+1}|||_n|||d_t\zeta^n|||_n \\ &\leqq \rho_c'|||\delta C^0|||_0|||d_t\zeta^0|||_0+\sum_{n=1}^{l-1}\rho_c'|||\delta^2C^n|||_n|||d_t\zeta^n|||_n \equiv T_{42}+T_{43}, \\ |T_{42}| &\leqq K(\Delta t_c)^{1/2}(|||\delta\tilde{c}^0|||_0+\Delta t_c|||d_t\zeta^0|||_0)|||d_t\zeta^0|||_0 \\ &\leqq K(\Delta t_c)^{1/2}(|||d_t\zeta^0|||_0+|||d_t\zeta^0|||_0^2)\,\Delta t_c \\ &\leqq \mathrm{K}(\Delta t_c)^2+\varepsilon|||d_t\zeta^0|||_0^2\,\Delta t_c, \end{aligned}$$

and, since $\rho_c' < \frac{1}{5}/(1-\frac{1}{5}) = \frac{1}{4}$ for $n \geqq 2$,

$$(4.98)\quad \begin{aligned} |T_{43}| &\leqq \sum_{n=1}^{l-1}\left(\frac{1}{4}-\varepsilon\right)(|||\delta^2\tilde{c}^n|||_n+|||\delta^2\zeta^n|||_n)|||d_t\zeta^n|||_n \\ &\leqq \left(\frac{1}{4}-\varepsilon\right)\sum_{n=1}^{l-1}\left[K(\Delta t_c)^{3/2}\left\|\frac{\partial^2\tilde{c}}{\partial t^2}\right\|_{L^2(t^{n-1},\,t^{n+1};\,H^1)}\right. \\ &\qquad \left.+\Delta t_c(|||d_t\zeta^n|||_n+|||d_t\zeta^{n-1}|||_n)\right]\cdot|||d_t\zeta^n|||_n \\ &\leqq \left(\frac{1}{4}-\varepsilon\right)\sum_{n=1}^{l-1}\left[K(\Delta t_c)^{1/2}\left\|\frac{\partial^2\tilde{c}}{\partial t^2}\right\|_{L^2(t^{n-1},t^{n+1};H^1)}+|||d_t\zeta^{n-1}|||_n\right]|||d_t\zeta^n|||_n\,\Delta t_c \\ &\qquad +\left(\frac{1}{4}-\varepsilon\right)\sum_{n=1}^{l-1}|||d_t\zeta^n|||_n^2\,\Delta t_c \\ &\leqq K\|\tilde{c}\|_{H^2(H^1)}^2(\Delta t_c)^2+\frac{1}{2}\left(\frac{1}{4}-\varepsilon\right)\sum_{n=1}^{l-1}|||d_t\zeta^{n-1}|||_n^2\,\Delta t_c \\ &\qquad +\frac{3}{2}\left(\frac{1}{4}-\varepsilon\right)\sum_{n=1}^{l-1}|||d_t\zeta^n|||_n^2\,\Delta t_c. \end{aligned}$$

Noting that

$$(4.99)\quad \begin{aligned} |||d_t\zeta^{n-1}|||_n^2 &= |||d_t\zeta^{n-1}|||_{n-1}^2+\Delta t_c([b(C^{*n})-b(C^{*n-1})]\nabla d_t\zeta^{n-1},\nabla d_t\zeta^{n-1}) \\ &\leqq |||d_t\zeta^{n-1}|||_{n-1}^2+K\|C^n-C^{n-1}\|_{L^\infty}\|\nabla d_t\zeta^{n-1}\|^2\,\Delta t_c \\ &\leqq |||d_t\zeta^{n-1}|||_{n-1}^2[1+K\,\Delta t_c(\|d_t\tilde{c}^{n-1}\|_{L^\infty}+\|d_t\zeta^{n-1}\|_{L^\infty})] \\ &\leqq |||d_t\zeta^{n-1}|||_{n-1}^2[1+K\,\Delta t_c(1+\|d_t\zeta^{n-1}\|_{L^\infty})], \end{aligned}$$

we have

$$(4.100)\quad \begin{aligned} |T_{43}| &\leqq K(\|\tilde{c}\|_{H^2(H^1)})(\Delta t_c)^2+\frac{3}{2}\left(\frac{1}{4}-\varepsilon\right)|||d_t\zeta^{l-1}|||_{l-1}^2\,\Delta t_c \\ &\qquad +\frac{1}{2}\left(\frac{1}{4}-\varepsilon\right)|||d_t\zeta^0|||_0^2\,\Delta t_c \\ &\qquad +2\left(\frac{1}{4}-\varepsilon\right)\sum_{n=1}^{l-2}|||d_t\zeta^n|||_n^2\,\Delta t_c[1+K\,\Delta t_c(1+\|d_t\zeta^n\|_{L^\infty})]. \end{aligned}$$

Thus,

$$
\begin{aligned}
|S_6+S_7| &\leqq K(\|\tilde{c}\|_{H^2(H^1)})(\Delta t_c)^2+\left(\frac{3}{8}-\varepsilon\right)|||d_t\zeta^{l-1}|||_{l-1}^2\,\Delta t_c \\
&\quad+\left(\frac{1}{2}-\varepsilon\right)\sum_{n=0}^{l-2}|||d_t\zeta^n|||_n^2\,\Delta t_c[1+K\,\Delta t_c(1+\|d_t\zeta^n\|_{L^\infty})].
\end{aligned} \tag{4.101}
$$

We note that the linearly extrapolated initial guess was essential here, since the factor $(\Delta t_c)^{3/2}$ in the argument (4.98) could not be replaced by Δt_c.

Combining (4.56), (4.57), (4.58), (4.68), (4.83), (4.96), and (4.101), we have shown that

$$
\begin{aligned}
&\frac{1}{2}\sum_{n=0}^{l-1}|||d_t\zeta^n|||_n^2\,\Delta t_c+\frac{1}{2}\sum_{n=0}^{l-1}(\|\zeta^{n+1}\|_{b^n}^2-\|\zeta^n\|_{b^n}^2) \\
&\leqq K(\|c\|_{H^1(H^r)},\|c\|_{W_\infty^1(L^\infty)},\|\tilde{c}\|_{W_\infty^1(W_\infty^1)},\|p\|_{L^2(H^s)},\|\tilde{p}\|_{L^\infty(W_\infty^1)})h^{2r} \\
&\quad+K(\|c\|_{L^\infty(H^r)},\|c\|_{W_\infty^1(H^{2+\varepsilon})},\|p\|_{H^1(H^s)},\|c\|_{H^1(W_\infty^2)})h^{2s} \\
&\quad+K(\|c\|_{L^\infty(W_\infty^1)},\|c\|_{W_\infty^1(L^\infty)},\|c\|_{H^2(H^1)},\|\tilde{c}\|_{W_\infty^1(W_\infty^1)},\|\tilde{c}\|_{H^2(H^1)})(\Delta t_c)^2 \\
&\quad+K(\|\tilde{c}\|_{L^\infty(W_\infty)},\|\tilde{p}\|_{W_\infty(H^1)})(\Delta t_p^0)^3+K(\|\tilde{c}\|_{L^\infty(W_\infty)},\|\tilde{p}\|_{H^2(H^1)})(\Delta t_p)^4 \\
&\quad+\left(\frac{1}{2}-\varepsilon\right)\sum_{n=0}^{l-2}|||d_t\zeta^n|||_n^2\,\Delta t_c[1+K\,\Delta t_c(1+\|d_t\zeta^n\|_{L^\infty})]+\left(\frac{3}{8}-\varepsilon\right)|||d_t\zeta^{l-1}|||_{l-1}^2\,\Delta t_c \\
&\quad+K(\|c\|_{W_\infty^1(L^\infty)},\|\tilde{c}\|_{W_\infty^1(W_\infty^1)})\sum_{n=1}^{l-1}(1+\|d_t\zeta^{n-1}\|_{L^\infty})\|\zeta^n\|_1^2\,\Delta t_c \\
&\quad+K(\|\tilde{c}\|_{L^\infty(W_\infty^1)},\|\tilde{p}\|_{L^\infty(W_\infty^1)})\sum_{m=1}^{k-1}\|\zeta_m\|^2\,\Delta t_p \\
&\quad+K\sum_{n=0}^{l-2}\|d_t\zeta^n\|^2(\Delta t_c)^2h^{-1}+K(\|\tilde{c}\|_{L^\infty(W_\infty^1)})\sum_{m=1}^{k-1}\|P_m-\bar{P}_m\|_{a_m}^2\,\Delta t_p \\
&\quad+\varepsilon\|\zeta^l\|_1^2+\varepsilon\|\zeta^{l-1}\|_1^2+K\|\zeta^l\|^2+K(\|\tilde{c}\|_{L^\infty(W_\infty^1)})\|\zeta^{l-1}\|^2 \\
&\quad+K(\|\tilde{c}\|_{L^\infty(W_\infty)})\|\zeta^0\|_1^2+K(\|\tilde{c}\|_{L^\infty(W_\infty^1)})\|P_0-\bar{P}_0\|_{a_0}^2\,\Delta t_p^0 \\
&\equiv K(h^{2r}+h^{2s}+(\Delta t_c)^2+(\Delta t_p^0)^3+(\Delta t_p)^4)+A_1+A_2+\cdots+A_{12},
\end{aligned} \tag{4.102}
$$

with the appropriate modification if $r=s=2$.

Our next step is to modify the left-hand side of (4.102), intending to obtain a collapsing sum in a norm equivalent to the H^1-norm. To add L^2 terms to the sum, we note that

$$
\begin{aligned}
\|\zeta^{n+1}\|^2-\|\zeta^n\|^2&=(\zeta^{n+1}+\zeta^n,\zeta^{n+1}-\zeta^n) \\
&=(\zeta^{n+1}-\zeta^n,\zeta^{n+1}-\zeta^n)+2(\zeta^n,\zeta^{n+1}-\zeta^n) \\
&=\|d_t\zeta^n\|^2(\Delta t_c)^2+2(\zeta^n,d_t\zeta^n)\,\Delta t_c.
\end{aligned} \tag{4.103}
$$

Summing this from $n=0$ to $n=l-1$ we obtain, for Δt_c sufficiently small,

$$
\begin{aligned}
\|\zeta^l\|^2-\|\zeta^0\|^2&=\sum_{n=0}^{l-1}(\|\zeta^{n+1}\|^2-\|\zeta^n\|^2) \\
&\leqq\Delta t_c\sum_{n=0}^{l-1}\|d_t\zeta^n\|^2\,\Delta t_c+\varepsilon\sum_{n=0}^{l-1}\|d_t\zeta^n\|^2\,\Delta t_c+K\sum_{n=0}^{l-1}\|\zeta^n\|^2\,\Delta t_c \\
&\leqq\varepsilon(A_1+A_2)+A_3+A_{11}.
\end{aligned} \tag{4.104}
$$

This also shows that

$$A_9 \leqq \varepsilon(A_1+A_2)+A_3+A_{11}, \tag{4.105}$$

and by summing from $n=0$ to $n=l-2$ we can similarly estimate A_{10}. To make the sum collapse, we must replace $\|\zeta^n\|_{b^n}^2$ by $\|\zeta^n\|_{b^{n-1}}^2$. An index perturbation argument of the form of (4.99) shows that

$$\|\zeta^n\|_{b^n}^2-\|\zeta^n\|_{b^{n-1}}^2 \leqq K\,\Delta t_c(1+\|d_t\zeta^{n-1}\|_{L^\infty})\|\zeta^n\|_1^2, \tag{4.106}$$

so that

$$\sum_{n=1}^{l-1} (\|\zeta^n\|_{b^n}^2-\|\zeta^n\|_{b^{n-1}}^2) \leqq A_3. \tag{4.107}$$

Adding (4.104) and (4.107) to both sides of (4.102), we have a sum collapsing in the norm $(\|\zeta^n\|^2+\|\zeta^n\|_{b^{n-1}}^2)^{1/2}$, which is equivalent to the H^1-norm.

Next, we make the induction hypothesis that

$$\sum_{n=0}^{l-2} |||d_t\zeta^n|||_n^2\,\Delta t_c \leqq K(h^{2r}+h^{2s}+(\Delta t_c)^2), \tag{4.108}$$

with the appropriate modification if $r=s=2$. For $l=1$ this is vacuous, and we will demonstrate at the end of the argument that it holds if summed through $n=l-1$. As a consequence, we have, for h sufficiently small,

$$\begin{aligned}\sum_{n=0}^{l-2} \|d_t\zeta^n\|_{L^\infty}\,\Delta t_c &\leqq K_0h^{-1}\,\Delta t_c \sum_{n=0}^{l-2} \|d_t\zeta^n\| \\ &= K_0h^{-1}\,\Delta t_c\left[\left(\sum_{n=0}^{l-2} \|d_t\zeta^n\|\right)^2\right]^{1/2} \\ &\leqq K_0h^{-1}\,\Delta t_c\left[(l-1)\sum_{n=0}^{l-2} \|d_t\zeta^n\|^2\right]^{1/2} \\ &\leqq K_0h^{-1}T^{1/2}\left[\sum_{n=0}^{l-2} \|d_t\zeta^n\|^2\,\Delta t_c\right]^{1/2} \leqq Kh^{-1}(h^r+h^s+\Delta t_c) \\ &< \varepsilon,\end{aligned} \tag{4.109}$$

since $\Delta t_c=o(h)$. If $r=s=2$, we have an extra term of the form $h^{-1}(h^2|\log h|)$, which is still small. Thus, in particular,

$$\Delta t_c(1+\|d_t\zeta^n\|_{L^\infty})<\varepsilon, \qquad 0\leqq n\leqq l-2, \tag{4.110}$$

$$\sum_{n=1}^{l-1} (1+\|d_t\zeta^{n-1}\|_{L^\infty})\,\Delta t_c \leqq K. \tag{4.111}$$

We now estimate most of the terms A_1 through A_{12}. By (4.110), A_1 hides on the left-hand side of (4.102). A_2 hides at once. A_3 and A_4 will disappear when the discrete Gronwall lemma is applied in the H^1-norm, at the cost of allowing the other constants to depend on $\|c\|_{W^1_\infty(L^\infty)}$, $\|\tilde{c}\|_{W^1_\infty(W^1_\infty)}$, and $\|\tilde{p}\|_{L^\infty(W^1_\infty)}$. A_5 hides since $\Delta t_c h^{-1}=o(1)$. A_6 is estimated below. A_7 and A_8 hide in the collapsing H^1 sum. A_9 and A_{10} were handled in (4.105). A_{11} and A_{12} are estimated by (3.19) and (3.20), respectively. We note that the treatment of A_3 depended on (4.111), which in turn depended on our ability to estimate $d_t\zeta$, which we could not do in Theorem 4.1. The estimate for $d_t\zeta$ also enabled us to perturb the norm in (4.99) without losing a factor of the form

b^*/b_*, which would have demanded a norm reduction ρ_c of order b_*/b^*. We will also be able to avoid a factor of a_*/a^* in ρ_p by using the $d_t\zeta$ estimate to perturb the a_m-norms. Thus we achieve stability for the method with a number of iterations per time step which is both fixed and small.

It remains to estimate A_6, which will require the a_m-norm perturbation. Using (4.108), we see that for $r \geqq 3$ or $s \geqq 3$,

$$\begin{aligned}\|d_t\zeta^n\|_{L^\infty}\,\Delta t_c &= [\|d_t\zeta^n\|^2_{L^\infty}(\Delta t_c)^2]^{1/2}\\ &\leqq \left[\sum_{n=0}^{l-2}\|d_t\zeta^n\|^2_{L^\infty}(\Delta t_c)^2\right]^{1/2}\\ &\leqq K_0h^{-1}(\Delta t_c)^{1/2}\left[\sum_{n=0}^{l-2}\|d_t\zeta^n\|^2\,\Delta t_c\right]^{1/2}\\ &\leqq Kh^{-1}(\Delta t_c)^{1/2}(h^r+h^s+\Delta t_c), \qquad 0\leqq n\leqq l-2,\end{aligned} \tag{4.112}$$

so that for $1\leqq \tilde m\leqq k-2$, since $\Delta t_p = O((\Delta t_c)^{1/2})$,

$$\begin{aligned}\|d_t\zeta_{\tilde m}\|_{L^\infty}\,\Delta t_p = \|\zeta_{\tilde m+1}-\zeta_{\tilde m}\|_{L^\infty} &\leqq \sum_{m=\tilde m}\|\zeta^{n+1}-\zeta^n\|_{L^\infty}\\ &\leqq \frac{\Delta t_p}{\Delta t_c}(\max_{m=\tilde m}\|d_t\zeta^n\|_{L^\infty}\,\Delta t_c)\\ &\leqq \frac{\Delta t_p}{\Delta t_c}Kh^{-1}(\Delta t_c)^{1/2}(h_r+h^s+\Delta t_c)\\ &\leqq Kh^{-1}(h_r+h^s+\Delta t_c)\\ &<\varepsilon\end{aligned} \tag{4.113}$$

for h sufficiently small. A trivial modification of (4.112)–(4.113) handles the case $r=s=2$. Similarly,

$$\|d_t\zeta_0\|_{L^\infty}\,\Delta t_p^0 \leqq \frac{\Delta t_p^0}{\Delta t_c}Kh^{-1}(\Delta t_c)^{1/2}(h^r+h^s+\Delta t_c)<\varepsilon. \tag{4.114}$$

Then for $2\leqq m\leqq k-1$, we have the norm perturbation

$$\begin{aligned}\|\eta_{m-1}\|^2_{a_m} &= \|\eta_{m-1}\|^2_{a_{m-1}} + ([a(C^*_m)-a(C^*_{m-1})]\nabla\eta_{m-1},\nabla\eta_{m-1})\\ &\leqq \|\eta_{m-1}\|^2_{a_{m-1}} + K\|C_m-C_{m-1}\|_{L^\infty}\|\nabla\eta_{m-1}\|^2\\ &\leqq \|\eta_{m-1}\|^2_{a_{m-1}}[1+K\,\Delta t_p(\|d_t\tilde c_{m-1}\|_{L^\infty}+\|d_t\zeta_{m-1}\|_{L^\infty})]\\ &\leqq \|\eta_{m-1}\|^2_{a_{m-1}}[1+K\,\Delta t_p(1+\|d_t\zeta_{m-1}\|_{L^\infty})]\\ &\leqq \|\eta_{m-1}\|^2_{a_{m-1}}(1+\varepsilon)\end{aligned} \tag{4.115}$$

by (4.113), for Δt_p sufficiently small. A slight extension of this calculation shows that, for $3\leqq m\leqq k-1$,

$$\|\eta_{m-2}\|^2_{a_m}\leqq \|\eta_{m-2}\|^2_{a_{m-2}}(1+\varepsilon). \tag{4.116}$$

Using (4.114), we obtain

$$\begin{aligned}\|\eta_0\|^2_{a_2} &\leqq \|\eta_0\|^2_{a_0}(1+\varepsilon),\\ \|\eta_0\|^2_{a_1} &\leqq \|\eta_0\|^2_{a_0}(1+\varepsilon).\end{aligned} \tag{4.117}$$

We now bound A_6. Considering $m=1$ first, we have

$$\|P_1-\bar{P}_1\|_{a_1}^2 \leqq (\rho_p')^2\|\delta P_0\|_{a_1}^2 \leqq K(\Delta t_p^0)^{1/4}\|\delta P_0\|_{a_1}^2, \tag{4.118}$$

and we proceed exactly as in (4.29) to obtain the proper bound. Since

$$\rho_p' < \frac{1}{3+\sqrt{5}}\Big/\left(1-\frac{1}{3+\sqrt{5}}\right) = \frac{1}{2+\sqrt{5}} \quad \text{for } m \geqq 2,$$

$$\begin{aligned}
\|P_m-\bar{P}_m\|_{a_m}^2 &\leqq (\rho_p')^2\|\delta^2 P_{m-1}\|_{a_m}^2 \\
&\leqq \left(\frac{1}{(2+\sqrt{5})^2}-\varepsilon\right)\|\delta^2\eta_{m-1}\|_{a_m}^2 + K\|\delta^2\tilde{p}_{m-1}\|_{a_{m-1}}^2 \\
&\leqq \left(\frac{1}{(2+\sqrt{5})^2}-\varepsilon\right)(\|\eta_m\|_{a_m}+\|\eta_{m-1}\|_{a_m}+\|\eta_{m-1}\|_{a_m}+\|\eta_{m-2}\|a_m)^2 \\
&\quad + K(\Delta t_p)^3\left\|\frac{\partial^2\tilde{p}}{\partial t^2}\right\|_{L^2(t_{m-2},t_m;H^1)}^2 \\
&\leqq \left(\frac{1}{2+\sqrt{5}}-\varepsilon\right)(\|\eta_m\|_{a_m}^2+2\|\eta_{m-1}\|_{a_m}^2+\|\eta_{m-2}\|_{a_m}^2) \\
&\quad + K(\Delta t_p)^3\left\|\frac{\partial^2\tilde{p}}{\partial t^2}\right\|_{L^2(t_{m-2},t_m;H^1)}^2 \\
&\leqq \left(\frac{1}{2+\sqrt{5}}-\varepsilon\right)(\|\eta_m\|_{a_m}^2+2\|\eta_{m-1}\|_{a_{m-1}}^2+\|\eta_{m-2}\|_{a_{m-2}}^2) \\
&\quad + K(\Delta t_p)^3\left\|\frac{\partial^2\tilde{p}}{\partial t^2}\right\|_{L^2(t_{m-2},t_m;H^1)}^2,
\end{aligned} \tag{4.119}$$

if we use (4.115), (4.116), and (4.117), as $(a+b+c+d)^2 \leqq 4(a^2+b^2+c^2+d^2)$. We are now in exactly the position of (4.33), and we argue as in Theorem 4.1 to find that

$$A_6 \leqq K(h^{2r}+(\Delta t_p)^4)+A_4. \tag{4.120}$$

Combining (4.102) with the analysis of A_1 through A_{12}, we have shown

$$\begin{aligned}
\sum_{n=0}^{l-1} |||d_t\zeta^n|||_n^2\,\Delta t_c + \|\zeta^l\|_1^2 &\leqq K(\|c\|_{W_\infty^1(L^\infty)}, \|\tilde{c}\|_{W_\infty^1(W_\infty^1)}, \|\tilde{p}\|_{L^\infty(W_\infty^1)}) \\
&\quad\cdot [K(\|c\|_{H^1(H^r)}, \|p\|_{L^2(H^s)})h^{2r} \\
&\qquad + K(\|c\|_{L^\infty(H^r)}, \|c\|_{W_\infty^1(W_\infty^1)}, \|p\|_{H^1(H^s)}, \|c\|_{H^1(W_\infty^2)})h^{2s} \\
&\qquad + K(\|c\|_{L^\infty(W_\infty^1)}, \|c\|_{H^2(H^1)}, \|\tilde{c}\|_{H^2(H^1)})(\Delta t_c)^2 \\
&\qquad + K(\|\tilde{p}\|_{W_\infty^1(H^1)})(\Delta t_p^0)^3 + K(\|\tilde{p}\|_{H^2(H^1)})(\Delta t_p)^4],
\end{aligned} \tag{4.121}$$

with a term of the form $K(h^4|\log h|^2)$ if $r=s=2$. With the time step choices

$$\begin{aligned}
\Delta t_p^0 &\approx \left(\frac{\|\tilde{c}\|_{H^2(H^1)}}{\|\tilde{p}\|_{W_\infty^1(H^1)}}\right)^{2/3}(\Delta t_c)^{2/3}, \\
\Delta t_p &\approx \left(\frac{\|\tilde{c}\|_{H^2(H^1)}}{\|\tilde{p}\|_{H^2(H^1)}}\right)^{1/2}(\Delta t_c)^{1/2},
\end{aligned} \tag{4.122}$$

we obtain the desired result (4.55), assuming that the chosen norms dominate.

Finally, we check the induction hypotheses (4.21) and (4.108). We verify (4.21) exactly as in Corollary 4.2, requiring the hypothesis $\Delta t_c = o(h)$. Next, (4.108) is immediately checked by (4.121) and (4.122), and the proof is complete.

The regularity assumptions $c \in L^\infty(H^r)$, $\partial c/\partial t \in L^2(H^r)$, $p \in L^\infty(H^s)$, and $\partial p/\partial t \in L^2(H^s)$ of Theorem 4.3 do not balance if time differentiation is considered equivalent to two space differentiations. For this reason, we would like to weaken the assumptions on the time derivatives to $\partial c/\partial t \in L^2(H^{r-1})$ and $\partial p/\partial t \in L^2(H^{s-1})$. The next result shows that, under certain hypotheses, we can do this and obtain a work estimate intermediate to those of Corollary 4.2 and Theorem 4.3.

COROLLARY 4.4. *Let the hypotheses of Theorem* 4.3 *hold, weakened by requiring* $\partial c/\partial t \in L^2(H^{r-1})$, $\partial p/\partial t \in L^2(H^{s-1})$, *and strengthened by assuming that both* $r \geqq 3$ *and* $s \geqq 3$, Ω *is* H^3*-regular, and* $\rho_c = O(h)$ *for* $n \geqq 2$. *Then, for h sufficiently small,*

$$\begin{aligned} &\sup_n \|C^n - c^n\| \leqq K_{10}(h^r + h^s + \Delta t_c), \\ &\sup_n \|C^n - c^n\|_1 \leqq K_{10}(h^{r-1} + h^{s-1} + \Delta t_c). \end{aligned} \tag{4.123}$$

Proof. We combine our earlier results. The proof of Theorem 4.1, as modified in Corollary 4.2, goes through up to (4.26), where we have

$$\begin{aligned} |T_7| &\leqq K(\rho_c')^2 |||\delta C^0|||_0^2 + \varepsilon |||\zeta^1|||_0^2 \\ &\leqq K(\rho_c')^2(|||\delta \tilde{c}^0|||_0^2 + |||\delta\zeta^0|||_0^2) + \varepsilon|||\zeta^1|||_0^2 \\ &\leqq K(\Delta t_c)^2 + Kh^2 |||d_t\zeta^0|||_0^2\,\Delta t_c + \varepsilon\|\zeta^1\|_\phi^2 + \varepsilon\|\zeta^1\|_{b^0}^2\,\Delta t_c, \end{aligned} \tag{4.124}$$

since $\rho_c' = O(h)$, and

$$\begin{aligned} |T_8| &\leqq \sum_{n=1}^{l-1}\left((\rho_c')^2|||\delta^2 C^n|||_n^2 \frac{1}{\Delta t_c} + |||\zeta^{n+1}|||_n^2\,\Delta t_c\right) \\ &\leqq K(\rho_c')^2 \sum_{n=1}^{l-1}(|||\delta^2\tilde{c}^n|||_n^2 + |||\delta\zeta^n|||_n^2 + |||\delta\zeta^{n-1}|||_n^2)\frac{1}{\Delta t_c} \\ &\quad + \sum_{n=1}^{l-1}\|\zeta^{n+1}\|_\phi^2\,\Delta t_c + \Delta t_c \sum_{n=1}^{l-1}\|\zeta^{n+1}\|_{b^n}^2\,\Delta t_c \\ &\leqq K\|\tilde{c}\|_{H^2(H^1)}^2 \frac{(\Delta t_c)^3}{\Delta t_c} + Kh^2 \sum_{n=0}^{l-1} |||d_t\zeta^n|||_n^2 \frac{(\Delta t_c)^2}{\Delta t_c} + \sum_{n=1}^{l-1}\|\zeta^{n+1}\|_\phi^2\,\Delta t_c \\ &\quad + \varepsilon\sum_{n=1}^{l-1}\|\zeta^{n+1}\|_{b^n}^2\,\Delta t_c, \end{aligned} \tag{4.125}$$

so that

$$\begin{aligned} |S_6 + S_7| &\leqq K(\|\tilde{c}\|_{H^2(H^1)})(\Delta t_c)^2 + Kh^2 \sum_{n=0}^{l-1}|||d_t\zeta^n|||_n^2\,\Delta t_c \\ &\quad + K\sum_{n=1}^{l-1}\|\zeta^{n+1}\|_\phi^2\,\Delta t_c + \varepsilon\sum_{n=0}^{l-1}\|\zeta^{n+1}\|_{b^n}^2\,\Delta t_c + \varepsilon\|\zeta^1\|_\phi^2. \end{aligned} \tag{4.126}$$

Hiding the ε terms, we obtain the estimate (4.28) with the additional term

$$T_0 = Kh^2 \sum_{n=0}^{l-1}|||d_t\zeta^n|||_n^2\,\Delta t_c \tag{4.127}$$

on the right-hand side, and with new dependence in the coefficient of $(\Delta t_c)^2$.

With $r-1$ and $s-1$ playing the roles of r and s, we see that all hypotheses of Theorem 4.3 are satisfied. Examination of the proof of Theorem 4.3 shows that the logarithm in the final result for the case $r-1=s-1=2$ appears only in (4.75), which does not require $\partial c/\partial t \in L^2(H^r)$ or $\partial p/\partial t \in L^2(H^s)$. Thus we may apply Theorem 4.3 and obtain from (4.121) the estimate

$$T_0 \leqq Kh^2(h^{2r-2}+h^{2s-2}+(\Delta t_c)^2) \leqq K(h^{2r}+h^{2s}+(\Delta t_c)^2). \tag{4.128}$$

Theorem 4.3 also gives us the bound $\|\zeta^l\|_1^2 \leqq K(h^{2r-2}+h^{2s-2}+(\Delta t_c)^2)$, which suffices to prove our H^1 result. From here, the argument of Theorem 4.1 carries through, except that A_2 of (4.28) must be estimated by the argument for A_6 of (4.102) in Theorem 4.3.

A special case of important physical interest occurs when $b=b(x)$ is independent of both the concentration and the pressure gradient. A glance at (3.2) and (3.3) reveals that this causes the matrix $L^n(\kappa, \pi)$ to be independent of the time step n. Thus we can suppress the concentration iterative procedure and replace it with a factorization of $\Phi+\Delta t_c B$, done only once, together with a simple backsolve at each time step. We get results corresponding to Corollary 4.2, Theorem 4.3, and Corollary 4.4.

COROLLARY 4.5. *Suppose that $b=b(x)$, (R_1) holds, $c \in L^2(W_\infty^2)$, and $p \in L^\infty(H^s)$. If $r \geqq 3$, assume also that Ω is H^3-regular. Suppose that the relations* (4.51) *and* (4.52) *hold, and that we achieve norm reductions of the form*

$$\begin{aligned} \rho_p' &= O((\Delta t_p^0)^{1/8}), && m=1,\\ \rho_p' &< \left(\frac{1}{8+4\sqrt{5}}\right)^{1/2}\left(\frac{a_*}{a^*}\right)^{1/2}, && m \geqq 2. \end{aligned} \tag{4.129}$$

Then for h sufficiently small,

$$\sup_n (\|C^n-c^n\|+h\|C^n-c^n\|_1) \leqq \begin{cases} K_8(h^r+h^s+\Delta t_c), & r\geqq 3 \text{ or } s \geqq 3,\\ K_8(h^2|\log h|+\Delta t_c), & r=s=2. \end{cases} \tag{4.130}$$

If also (R_3) *holds, then without the assumption of H^3-regularity and with the norm reduction*

$$\rho_p \leqq \delta_p < \frac{1}{3+\sqrt{5}}, \qquad m \geqq 2, \tag{4.131}$$

where δ_p is independent of m, we obtain

$$\begin{aligned} \sup_n \|C^n-c^n\| &\leqq \begin{cases} K_9(h^r+h^s+\Delta t_c), & r\geqq 3 \text{ or } s\geqq 3,\\ K_9(h^2|\log h|+\Delta t_c), & r=s=2, \end{cases}\\ \sup_n \|C^n-c^n\|_1 &\leqq \begin{cases} K_9(h^{r-1}+h^s+\Delta t_c), & r\geqq 3 \text{ or } s\geqq 3,\\ K_9(h|\log h|+\Delta t_c), & r=s=2. \end{cases} \end{aligned} \tag{4.132}$$

If we weaken (R_3) *to require $\partial c/\partial t \in L^2(H^{r-1})$ and $\partial p/\partial t \in L^2(H^{s-1})$ instead of $\partial c/\partial t \in L^2(H^r)$ and $\partial p/\partial t \in L^2(H^s)$, then by demanding H^3-regularity and both $r \geqq 3$ and $s \geqq 3$, we can obtain*

$$\begin{aligned} \sup_n \|C^n-c^n\| &\leqq K_{10}(h^r+h^s+\Delta t_c),\\ \sup_n \|C^n-c^n\|_1 &\leqq K_{10}(h^{r-1}+h^{s-1}+\Delta t_c). \end{aligned} \tag{4.133}$$

We now consider the use of the more efficient test function $\chi = \zeta^{n+1} - \zeta^n = \Delta t_c d_t \zeta^n$ for the case $b = b(x, c, \nabla p)$, aiming for an improvement on the asymptotic work estimate for Theorem 4.1. As in Theorem 4.3, we need a bit more regularity, though not as much as in Theorem 4.3. Some other hypotheses will also have to be strengthened. As before, we will have to use summation by parts in time to avoid the term $\|d_t\zeta\|_1$, and the new dependence of b on ∇p will cause some discrete time derivatives to appear which did not occur in Theorem 4.3. One of these, $\|d_t(\nabla \tilde{p} - E\nabla \tilde{p})\|$, will be estimated by $O(\Delta t_p)$ instead of $O((\Delta t_p)^2)$, and so we will need $\Delta t_p = O(\Delta t_c)$ instead of $\Delta t_p = O((\Delta t_c)^{1/2})$. Careful analysis will show, however, that Δt_p can still be a large constant multiple of Δt_c. Another term, $\|d_t \nabla \eta\|$, will force us to consider the difference $\|d_t \nabla (P - \bar{P})\|$, which will fail to provide us with a factor of Δt_p. Because of this, we will have to require $\rho_p = O(\Delta t_p)$. In relation to Theorem 4.1, the result will be that we transfer the poor iteration count from the concentration to the pressure, which is not computed as often, and that the fixed concentration iteration count will be independent of variations of the coefficients. Thus the asymptotic work estimate is reduced by a factor of $\Delta t_p / \Delta t_c$. Since Δt_p is smaller in this result than in Theorem 4.1, it is unclear which result gives the better work estimate for practical computations. As in Theorem 4.3, the test function $\chi = \zeta^{n+1} - \zeta^n$ will yield a better H^1 error estimate.

As by-products of the altered assumptions above, the connections between the space and time discretizations will be weakened somewhat, the stronger starting estimate (3.21) is needed, and the initial time steps will be slightly different. The first pressure and concentration time steps, denoted by Δt_p^0 and Δt_c^0, will coincide and be of order $O((\Delta t_p)^2)$, and the second concentration step will also be Δt_c^0. The obvious modifications of (2.37) will apply. For economy of notation, we will often write expressions such as

$$\sum_{n=0}^{l-1} \|d_t \zeta^n\|^2 \, \Delta t_c$$

when we actually mean

$$\sum_{n=0}^{1} \|d_t \zeta^n\|^2 \, \Delta t_c^0 + \sum_{n=2}^{l-1} \|d_t \zeta^n\|^2 \, \Delta t_c,$$

where also $d_t \zeta^n = (1/\Delta t_c^0)(\zeta^{n+1} - \zeta^n)$ for $n = 0$ or 1.

THEOREM 4.6. *Suppose that* (3.21) *holds, the above modifications to the initial time steps are made,* $b = b(x, c, \nabla p)$, (R_6) *holds, and* $s \geqq 3$. *Let the space and time discretizations satisfy the relation*

$$\Delta t_c = o(h), \tag{4.134}$$

and assume that the pressure and concentration time steps are related by

$$\Delta t_c^0 = \Delta t_p^0 = O((\Delta t_p)^2), \qquad \Delta t_p = O(D \, \Delta t_c), \tag{4.135}$$

where

$$D = \frac{\|\tilde{c}\|_{H^2(H^1)}}{\|\tilde{p}\|_{H^2(H^1)}}. \tag{4.136}$$

If we achieve norm reductions of the form

$$
\begin{aligned}
&\rho_c < \tfrac{3}{11}, && n = 1,\\
&\rho_c \leqq \delta_c < \tfrac{1}{5}, && n \geqq 2,\\
&\rho_p = O((\Delta t_p^0)^{1/2}), && m = 1,\\
&\rho_p = O(\Delta t_p), && m \geqq 2,
\end{aligned}
\tag{4.137}
$$

where δ_c is independent of n then, for h sufficiently small,

$$
\begin{aligned}
&\sup_n \|C^n - c^n\| \leqq K_{11}(h^r + h^{s-1} + \Delta t_c),\\
&\sup_n \|C^n - c^n\|_1 \leqq K_{11}(h^{r-1} + h^{s-1} + \Delta t_c).
\end{aligned}
\tag{4.138}
$$

Proof. We follow the proof of Theorem 4.3, making changes and additions where necessary. The dependence of b on ∇p changes (4.56)–(4.57) to

$$
\begin{aligned}
&\frac{1}{2}\sum_{n=0}^{l-1} |||d_t\zeta^n|||_n^2 \Delta t_c + \frac{1}{2}\sum_{n=0}^{l-1} (\|\zeta^{n+1}\|_{b^n}^2 - \|\zeta^n\|_{b^n}^2)\\
&\qquad \leqq S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7,
\end{aligned}
\tag{4.139}
$$

where the definitions of the weighted norms have been modified to include the dependence of b on ∇p, and where S_3 and S_7 have been changed to

$$
\begin{aligned}
S_3 &= \sum_{n=0}^{l-1} ([b(c^{n+1}, \nabla p^{n+1}) - b(C^{*n}, E\nabla P^{n+1})]\nabla \tilde{c}^{n+1}, \nabla d_t\zeta^n)\,\Delta t_c,\\
S_7 &= \sum_{n=0}^{l-1} (b(C^{*n}, E\nabla P^{n+1})\nabla(C^{n+1} - \bar{C}^{n+1}), \nabla d_t\zeta^n)\,\Delta t_c.
\end{aligned}
\tag{4.140}
$$

We collect in one place the induction hypotheses which the analysis will require. We assume (4.21), (4.108) with $s-1$ replacing s, and

$$
\sum_{n=0}^{l-2} \|d_t\zeta^n\|_{L^\infty}\,\Delta t_c \leqq \varepsilon,
\tag{4.141}
$$

$$
\|\zeta_m\|^2 \leqq K(h^{2r} + h^{2s-2} + (\Delta t_c)^2), \qquad 0 \leqq m \leqq k-1,
\tag{4.142}
$$

$$
\sum_{m=0}^{k-2} \|d_t\nabla\eta_m\|^2\,\Delta t_p \leqq K(h^{2r} + h^{2s-2} + (\Delta t_c)^2),
\tag{4.143}
$$

where the constants have the same dependences as K_{11}, to be determined later. For $l = 1$, we check (4.21) as in Theorems 4.1 and 4.3. Except for (4.142), the others are vacuous. We immediately check (4.142) from (3.19). At the end of the argument, we will verify (4.108) and (4.141) for the next value of l, and if $t^l = t_k$ we will check the others for the next value of k.

Next, we obtain new bounds for S_3. We write

$$
\begin{aligned}
S_3 = &\sum_{n=0}^{l-1} ([b(c^{n+1}, \nabla p^{n+1}) - b(c^n, \nabla p^{n+1})]\nabla \tilde{c}^{n+1}, \nabla d_t\zeta^n)\, \Delta t_c \\
&+ \sum_{n=0}^{l-1} ([b(c^n, \nabla p^{n+1}) - b(\tilde{c}^n, \nabla p^{n+1})]\nabla \tilde{c}^{n+1}, \nabla d_t\zeta^n)\, \Delta t_c \\
&+ \sum_{n=0}^{l-1} ([b(\tilde{c}^n, \nabla p^{n+1}) - b(C^{*n}, \nabla p^{n+1})]\nabla \tilde{c}^{n+1}, \nabla d_t\zeta^n)\, \Delta t_c \\
&+ \sum_{n=0}^{l-1} ([b(C^{*n}, \nabla p^{n+1}) - b(C^{*n}, \nabla \tilde{p}^{n+1})]\nabla \tilde{c}^{n+1}, \nabla d_t\zeta^n)\, \Delta t_c \\
&+ \sum_{n=0}^{l-1} ([b(C^{*n}, \nabla \tilde{p}^{n+1}) - b(C^{*n}, E\nabla \tilde{p}^{n+1})]\nabla \tilde{c}^{n+1}, \nabla d_t\zeta^n)\, \Delta t_c \\
&+ \sum_{n=0}^{l-1} ([b(C^{*n}, E\nabla \tilde{p}^{n+1}) - b(C^{*n}, E\nabla P^{n+1})]\nabla \tilde{c}^{n+1}, \nabla d_t\zeta^n)\, \Delta t_c \\
\equiv &\, T_1 + T_2 + T_3 + T_4 + T_5 + T_6.
\end{aligned}
\tag{4.144}
$$

For reasons noted earlier, we sum each term by parts in time. The terms T_1, T_2, and T_3 are similar to S_3 of Theorem 4.3. An analogous argument, with extra terms coming from time differences on ∇p, leads to the estimate

$$
\begin{aligned}
|T_1 + T_2 + T_3| \leqq &\, K(\|c\|_{H^1(H^r)})h^{2r} + K(\|c\|_{H^2(L^2)})(\Delta t_c)^2 \\
&+ K \sum_{n=1}^{l-1} (1 + \|d_t\zeta^{n-1}\|_{L^\infty})\|\zeta^n\|_1^2\, \Delta t_c \\
&+ K\|\zeta^{l-1}\|^2 + K\|\zeta^0\|_1^2 + \varepsilon\|\zeta^l\|_1^2.
\end{aligned}
\tag{4.145}
$$

The arguments for the other three terms have a similar but slightly different form. We sum T_4 by parts in time and obtain

$$
\begin{aligned}
T_4 = &\, ([b(C^{*l-1}, \nabla p^l) - b(C^{*l-1}, \nabla \tilde{p}^l)]\nabla \tilde{c}^l, \nabla \zeta^l) \\
&- ([b(C^{*0}, \nabla p^1) - b(C^{*0}, \nabla \tilde{p}^1)]\nabla \tilde{c}^1, \nabla \zeta^0) \\
&- \sum_{n=1}^{l-1} ([b(C^{*n}, \nabla p^{n+1}) - b(C^{*n}, \nabla \tilde{p}^{n+1})]\nabla \tilde{c}^{n+1} \\
&- [b(C^{*n-1}, \nabla p^n) - b(C^{*n-1}, \nabla \tilde{p}^n)]\nabla \tilde{c}^n, \nabla \zeta^n) \\
\equiv &\, T_7 + T_8 - T_9.
\end{aligned}
\tag{4.146}
$$

Then

$$
|T_7| \leqq K\|\nabla \theta^l\|\|\nabla \tilde{c}^l\|_{L^\infty}\|\nabla \zeta^l\| \leqq K(\|p\|_{L^\infty(H^s)})h^{2s-2} + \varepsilon\|\zeta^l\|_1^2,
$$

$$
|T_8| \leqq Kh^{2s-2} + K\|\zeta^0\|_1^2,
$$

$$
\begin{aligned}
T_9 = &\sum_{n=1}^{l=1} ([b(C^{*n}, \nabla p^{n+1}) - b(C^{*n}, \nabla \tilde{p}^{n+1})]\nabla d_t \tilde{c}^n, \nabla \zeta^n)\, \Delta t_c \\
&+ \sum_{n=1}^{l-1} (\{[b(C^{*n}, \nabla p^{n+1}) - b(C^{*n}, \nabla \tilde{p}^{n+1})] \\
&\qquad - [b(C^{*n}, \nabla p^n) - b(C^{*n}, \nabla \tilde{p}^n)]\}\nabla \tilde{c}^n, \nabla \zeta^n)
\end{aligned}
\tag{4.147}
$$

$$+\sum_{n=1}^{l-1}(\{[b(C^{*n},\nabla p^n)-b(C^{*n},\nabla\tilde{p}^n)]$$
$$-[b(C^{*n-1},\nabla p^n)-b(C^{*n-1},\nabla\tilde{p}^n)]\}\nabla\tilde{c}^n,\nabla\zeta^n)$$
$$\equiv T_{10}+T_{11}+T_{12},$$
$$|T_{10}|\leqq K\sum_{n=1}^{l-1}\|\nabla\theta^{n+1}\|\|\nabla d_t\tilde{c}^n\|_{L^\infty}\|\nabla\zeta^n\|\,\Delta t_c\leqq K(\|p\|_{L^2(H^s)})h^{2s-2}+K\sum_{n=1}^{l-1}\|\zeta^n\|^2\,\Delta t_c.$$

To analyze T_{11} and T_{12}, we set

$$\text{(4.148)}\qquad \begin{aligned} b_{3,n}&=\int_0^1\frac{\partial b}{\partial\nabla p}(C^{*n},\alpha\nabla p^{n+1}+(1-\alpha)\nabla\tilde{p}^{n+1})\,d\alpha,\\ b_{4,n}&=\int_0^1\frac{\partial b}{\partial\nabla p}(C^{*n},\alpha\nabla p^{n}+(1-\alpha)\nabla\tilde{p}^{n})\,d\alpha,\end{aligned}$$

and by an argument of the form of (4.65)–(4.66), we see that

$$\text{(4.149)}\qquad \begin{aligned} |b_{3,n}-b_{4,n}|&\leqq K\left(\frac{\partial^2 b}{\partial\nabla p^2}\right)(|\nabla d_t p^n|+|\nabla d_t\tilde{p}^n|)\,\Delta t_c,\\ |b_{4,n}-b_{3,n-1}|&\leqq K\left(\frac{\partial^2 b}{\partial c\,\partial\nabla p}\right)|C^{*n}-C^{*n-1}|\leqq K(|d_t\tilde{c}^{n-1}|+|d_t\zeta^{n-1}|)\,\Delta t_c.\end{aligned}$$

Then

$$T_{11}=\sum_{n=1}^{l-1}([b_{3,n}\nabla\theta^{n+1}-b_{4,n}\nabla\theta^n]\nabla\tilde{c}^n,\nabla\zeta^n)$$
$$=\sum_{n=1}^{l-1}(b_{3,n}\nabla d_t\theta^n\nabla\tilde{c}^n,\nabla\zeta^n)\,\Delta t_c+\sum_{n=1}^{l-1}([b_{3,n}-b_{4,n}]\nabla\theta^n\nabla\tilde{c}^n,\nabla\zeta^n)$$
$$\equiv T_{13}+T_{14},$$

$$\text{(4.150)}\qquad \begin{aligned} |T_{13}|&\leqq K\sum_{n=1}^{l-1}\|\nabla d_t\theta^n\|\|\nabla\tilde{c}^n\|_{L^\infty}\|\nabla\zeta^n\|\,\Delta t_c\\ &\leqq K(\|p\|_{H^1(H^s)})h^{2s-2}+K\sum_{n=1}^{l-1}\|\zeta^n\|_1^2\,\Delta t_c,\\ |T_{14}|&\leqq\sum_{n=1}^{l-1}(\|\nabla d_t p^n\|_{L^\infty}+\|\nabla d_t\tilde{p}^n\|_{L^\infty})\,\Delta t_c\|\nabla\theta^n\|\|\nabla\tilde{c}^n\|_{L^\infty}\|\nabla\zeta^n\|\\ &\leqq K(\|p\|_{L^2(H^s)})h^{2s-2}+K\sum_{n=1}^{l-1}\|\zeta^n\|_1^2\,\Delta t_c,\end{aligned}$$

$$T_{12}=\sum_{n=1}^{l-1}([b_{4,n}-b_{3,n-1}]\nabla\theta^n\nabla\tilde{c}^n,\nabla\zeta^n),$$
$$|T_{12}|\leqq K\sum_{n=1}^{l-1}(\|d_t\tilde{c}^{n-1}\|_{L^\infty}+\|d_t\zeta^{n-1}\|_{L^\infty})\,\Delta t_c\|\nabla\theta^n\|\|\nabla\tilde{c}^n\|_{L^\infty}\|\nabla\zeta^n\|$$
$$\leqq K\sum_{n=1}^{l-1}(1+\|d_t\zeta^{n-1}\|_{L^\infty})\|\nabla\theta^n\|^2\,\Delta t_c$$
$$+\sum_{n=1}^{l-1}(1+\|d_t\zeta^{n-1}\|_{L^\infty})\|\zeta^n\|_1^2\,\Delta t_c$$
$$\leqq K(\|p\|_{L^\infty(H^s)})h^{2s-2}+K\sum_{n=1}^{l-1}(1+\|d_t\zeta^{n-1}\|_{L^\infty})\|\zeta^n\|_1^2\,\Delta t_c,$$

by (4.141). Thus,

$$|T_4| \leqq K(\|p\|_{H^1(H^s)})h^{2s-2} + K\sum_{n=1}^{l-1}(1+\|d_t\zeta^{n-1}\|_{L^\infty})\|\zeta^n\|_1^2\Delta t_c + \varepsilon\|\zeta^l\|_1^2 + K\|\zeta^0\|_1^2. \tag{4.151}$$

Next, we handle T_5 similarly, with $\tilde{p}$, $E\tilde{p}$, and $\tilde{p}-E\tilde{p}$ playing the roles of p, $\tilde{p}$, and θ, respectively. This yields the analogues

$$\begin{aligned}
|T_7'| &\leqq K\|\nabla\tilde{p}^l - E\nabla\tilde{p}^l\|\|\nabla\tilde{c}^l\|_{L^\infty}\|\nabla\zeta^l\| \\
&\leqq K(\Delta t_p)^2 + \varepsilon\|\zeta^l\|_1^2, \\
|T_8'| &\leqq K(\Delta t_c^0)^2 + K\|\zeta^0\|_1^2, \\
|T_{10}'| &\leqq K\sum_{n=1}^{l-1}\|\nabla\tilde{p}^{n+1} - E\nabla\tilde{p}^{n+1}\|\|\nabla d_t\tilde{c}^n\|_{L^\infty}\|\nabla\zeta^n\|\Delta t_c \\
&\leqq K(\Delta t_p)^2 + K\sum_{n=1}^{l-1}\|\zeta^n\|_1^2\Delta t_c, \\
|T_{13}'| &\leqq K\sum_{n=1}^{l-1}\|d_t(\nabla\tilde{p}^n - E\nabla\tilde{p}^n)\|\|\nabla\tilde{c}^n\|_{L^\infty}\|\nabla\zeta^n\|\Delta t_c \\
&\leqq K\|d_t(\nabla\tilde{p}^1 - E\nabla\tilde{p}^1)\|^2\Delta t_c^0 + K\sum_{\substack{n=2\\ t^n>t_m}}^{l-1}\|d_t(\nabla\tilde{p}^n - E\nabla\tilde{p}^n)\|^2\Delta t_c \\
&\quad + K\sum_{\substack{n=2\\ t^n=t_m}}^{l-1}\|d_t(\nabla\tilde{p}^n - E\nabla\tilde{p}^n)\|^2\Delta t_c\left(\frac{\Delta t_c}{\Delta t_p}\right) + K\sum_{n=1}^{l-1}\|\zeta^n\|_1^2\Delta t_c \\
&\quad + K\sum_{m=2}^{k-1}\|\zeta_m\|_1^2\Delta t_p \\
&\equiv F_1 + F_2 + F_3 + G,
\end{aligned} \tag{4.152}$$

where the term F_1 has Δt_c replaced by Δt_c^0 since $n=1$. To bound F_1, we note that

$$\begin{aligned}
&E\nabla\tilde{p}^1 = \nabla\tilde{p}^0, \\
&E\nabla\tilde{p}^2 = 2\nabla\tilde{p}^1 - \nabla\tilde{p}^0, \quad \text{since } t^2 - t^1 = t^1 - t^0 = \Delta t_c^0, \\
&d_t(E\nabla\tilde{p}^1) = \frac{1}{\Delta t_c^0}(2\nabla\tilde{p}^1 - 2\nabla\tilde{p}^0) = 2d_t\nabla\tilde{p}^0, \\
&d_t(\nabla\tilde{p}^1 - E\nabla\tilde{p}^1) = d_t\nabla\tilde{p}^1 - d_t(E\nabla\tilde{p}^1) = d_t\nabla\tilde{p}^1 - 2d_t\nabla\tilde{p}^0, \\
&F_1 \leqq K\Delta t_c^0 \leqq K(\Delta t_p)^2,
\end{aligned} \tag{4.153}$$

since $\Delta t_c^0 = O((\Delta t_p)^2)$. In F_2, let $t^n = t_m + \nu(\Delta t_c)$, $1 \leqq \nu < j$. Then

$$\begin{aligned}
&E\nabla\tilde{p}^n = \left(1+\frac{\nu}{j}\right)\nabla\tilde{p}_m - \frac{\nu}{j}\nabla\tilde{p}_{m-1}, \\
&E\nabla\tilde{p}^{n+1} = \left(1+\frac{\nu+1}{j}\right)\nabla\tilde{p}_m - \frac{\nu+1}{j}\nabla\tilde{p}_{m-1}, \\
&d_t(E\nabla\tilde{p}^n) = \frac{1}{\Delta t_c}\left(\frac{1}{j}\nabla\tilde{p}_m - \frac{1}{j}\nabla\tilde{p}_{m-1}\right) = \frac{1}{\Delta t_p}(\nabla\tilde{p}_m - \nabla\tilde{p}_{m-1}) = d_t\nabla\tilde{p}_{m-1}, \\
&d_t(\nabla\tilde{p}^n - E\nabla\tilde{p}^n) = d_t\nabla\tilde{p}^n - d_t\nabla\tilde{p}_{m-1},
\end{aligned} \tag{4.154}$$

$$\|d_t(\nabla\tilde{p}^n - E\nabla\tilde{p}^n)\|^2 \leqq K\left\|\frac{\partial^2\tilde{p}}{\partial t^2}\right\|^2_{L^2(t_{m-1},t^{n+1};H^1)}\Delta t_p,$$

$$F_2 \leqq K\sum_{n=2}^{l-1}\left\|\frac{\partial^2\tilde{p}}{\partial t^2}\right\|^2_{L^2(t_{m-1},t_{m+1};H^1)}\Delta t_p\Delta t_c$$

$$\leqq K\sum_{m=1}^{k-1}\left\|\frac{\partial^2\tilde{p}}{\partial t^2}\right\|^2_{L^2(t_{m-1},t_{m+1};H^1)}\left(\frac{\Delta t_p}{\Delta t_c}\right)\Delta t_p\Delta t_c \leqq K(\|\tilde{p}\|_{H^2(H^1)})(\Delta t_p)^2.$$

In F_3, we have

$$E\nabla\tilde{p}^n = 2\nabla\tilde{p}_{m-1} - \nabla\tilde{p}_{m-2},$$

$$E\nabla\tilde{p}^{n+1} = \left(1+\frac{1}{j}\right)\nabla\tilde{p}_m - \frac{1}{j}\nabla\tilde{p}_{m-1},$$

$$d_t(E\nabla\tilde{p}^n) = \frac{1}{\Delta t_c}\left[\frac{1}{j}(\nabla\tilde{p}_m - \nabla\tilde{p}_{m-1}) + (\nabla\tilde{p}_m - 2\nabla\tilde{p}_{m-1} + \nabla\tilde{p}_{m-2})\right]$$

$$= d_t\nabla\tilde{p}_{m-1} + \frac{(\Delta t_p)^2}{\Delta t_c}d_t^2\nabla\tilde{p}_{m-1},$$

(4.155)

$$d_t(\nabla\tilde{p}^n - E\nabla\tilde{p}^n) = (d_t\nabla\tilde{p}^n - d_t\nabla\tilde{p}_{m-1}) - \frac{(\Delta t_p)^2}{\Delta t_c}d_t^2\nabla\tilde{p}_{m-1},$$

$$\|d_t(\nabla\tilde{p}^n - E\nabla\tilde{p}^n)\|^2 \leqq K\left\|\frac{\partial^2\tilde{p}}{\partial t^2}\right\|^2_{L^2(t_{m-1},t^{n+1};H^1)}\Delta t_p + \frac{(\Delta t_p)^4}{(\Delta t_c)^2}\left\|\frac{\partial^2\tilde{p}}{\partial t^2}\right\|^2_{L^2(t_{m-2},t_m;H^1)}(\Delta t_p)^{-1},$$

$$F_3 \leqq K\left(\Delta t_p + \frac{(\Delta t_p)^3}{(\Delta t_c)^2}\right)\Delta t_c\left(\frac{\Delta t_c}{\Delta t_p}\right) \leqq K(\|\tilde{p}\|_{H^2(H^1)})(\Delta t_p)^2.$$

Thus

$$|T'_{13}| \leqq K(\|\tilde{p}\|_{H^2(H^1)})(\Delta t_p)^2 + K\sum_{n=1}^{l-1}\|\zeta^n\|_1^2\Delta t_c. \tag{4.156}$$

Next,

(4.157)

$$|T'_{14}| \leqq K\sum_{n=1}^{l-1}(\|d_t\nabla\tilde{p}^n\|_{L^\infty} + \|d_tE\nabla\tilde{p}^n\|_{L^\infty})\Delta t_c\|\nabla\tilde{p}^n - E\nabla\tilde{p}^n\|\|\nabla\tilde{c}^n\|_{L^\infty}\|\nabla\zeta^n\|$$

$$\leqq K\sum_{n=1}^{l-1}(1+\|d_tE\nabla\tilde{p}^n\|_{L^\infty}^2)\|\nabla\tilde{p}^n - E\nabla\tilde{p}^n\|^2\Delta t_c + G.$$

We recall that

(4.158)

$$\|\nabla\tilde{p}^1 - E\nabla\tilde{p}^1\|^2 \leqq K\left\|\frac{\partial\tilde{p}}{\partial t}\right\|^2_{L^\infty(H^1)}(\Delta t_c^0)^2 \leqq K(\Delta t_p)^4,$$

$$\|\nabla\tilde{p}^n - E\nabla\tilde{p}^n\|^2 \leqq K\left\|\frac{\partial^2\tilde{p}}{\partial t^2}\right\|_{L^2(t_{m-1},t^n;H^1)}(\Delta t_p)^3 \quad \text{for } n \geqq 2.$$

To bound $\|d_tE\nabla\tilde{p}^n\|_{L^\infty}^2$, we consider the cases $n=1$, $t^n > t_m$, and $t^n = t_m$ and use the formulas found in (4.153) through (4.155). We have

$$
(4.159)\qquad
\begin{aligned}
&\|d_t E\nabla\tilde{p}^1\|_{L^\infty}=2\|d_t\nabla\tilde{p}^0\|_{L^\infty}\leqq K,\\
&\|d_t E\nabla\tilde{p}^n\|_{L^\infty}=\|d_t\nabla\tilde{p}_{m-1}\|_{L^\infty}\leqq K, \qquad t^n>t_m,\\
&\|d_t E\nabla\tilde{p}^n\|_{L^\infty}\leqq K+\frac{(\Delta t_p)^2}{\Delta t_c}\|d_t^2\nabla\tilde{p}_{m-1}\|_{L^\infty}\\
&\qquad\leqq K+K\left(\left\|\frac{\partial\tilde{p}}{\partial t}\right\|_{L^\infty(W^1_\infty)}\right)\frac{\Delta t_p}{\Delta t_c}, \qquad t^n=t_m.
\end{aligned}
$$

Putting these bounds together, we have

$$
(4.160)\qquad |T'_{14}|\leqq K\left(\frac{\Delta t_p}{\Delta t_c}\right)(\Delta t_p)^3+G\leqq\varepsilon(\Delta t_c)^2+K\sum_{n=1}^{l-1}\|\zeta^n\|_1^2\Delta t_c,
$$

for Δt_c sufficiently small. Next,

$$
(4.161)\qquad
\begin{aligned}
|T'_{12}|&\leqq K\sum_{n=1}^{l-1}(\|d_t\tilde{c}^{n-1}\|_{L^\infty}+\|d_t\zeta^{n-1}\|_{L^\infty})\Delta t_c\|\nabla\tilde{p}^n-E\nabla\tilde{p}^n\|\|\nabla\tilde{c}^n\|_{L^\infty}\|\nabla\zeta^n\|\\
&\leqq K\sum_{n=1}^{l-1}\|\nabla\tilde{p}^n-E\nabla\tilde{p}^n\|^2\Delta t_c+K\sum_{n=1}^{l-1}(1+\|d_t\zeta^{n-1}\|_{L^\infty}^2)\|\zeta^n\|_1^2\Delta t_c\\
&\leqq\varepsilon(\Delta t_c)^2+K\sum_{n=1}^{l-1}(1+\|d_t\zeta^{n-1}\|_{L^\infty}^2)\|\zeta^n\|_1^2\Delta t_c,
\end{aligned}
$$

using (4.158). Collecting, we see that

$$
(4.162)\qquad
\begin{aligned}
|T_5|\leqq{}&K(\|\tilde{p}\|_{H^2(H^1)})(\Delta t_p)^2+\varepsilon(\Delta t_c)^2+K(\Delta t_c^0)^2\\
&+K\sum_{n=1}^{l-1}(1+\|d_t\zeta^{n-1}\|_{L^\infty}^2)\|\zeta^n\|_1^2\Delta t_c+K\sum_{m=2}^{k-1}\|\zeta_m\|_1^2\Delta t_p\\
&+\varepsilon\|\zeta^l\|_1^2+K\|\zeta^0\|_1^2.
\end{aligned}
$$

Finally, we perform the same argument for T_6, with $E\tilde{p}$, EP, and $E\eta$ replacing p, $\tilde{p}$, and θ, respectively. We obtain

$$
\begin{aligned}
&|T''_7|\leqq K\|E\nabla\eta^l\|\|\nabla\tilde{c}^l\|_{L^\infty}\|\nabla\zeta^l\|\leqq K(\|\nabla\eta_{k-1}\|^2+\|\nabla\eta_{k-2}\|^2)+\varepsilon\|\zeta^l\|_1^2,\\
&|T''_8|\leqq K\|\nabla\eta_0\|^2+K\|\zeta^0\|_1^2\leqq K(\|c\|_{L^\infty(H^r)})h^{2r}+K\|\zeta^0\|_1^2,
\end{aligned}
$$

by (4.8), (3.19), and (3.20). Also,

$$
\begin{aligned}
|T''_{10}|&\leqq K\sum_{n=1}^{l-1}\|E\nabla\eta^{n+1}\|\|\nabla d_t\tilde{c}^n\|_{L^\infty}\|\nabla\zeta^n\|\Delta t_c\\
&\leqq K\sum_{n=1}^{l-1}(\|\nabla\eta_m\|^2+\|\nabla\eta_{m-1}\|^2)\Delta t_c+K\sum_{n=1}^{l-1}\|\zeta^n\|_1^2\Delta t_c\\
&\leqq K\sum_{m=0}^{k-1}\|\nabla\eta_m\|^2\Delta t_p+K\sum_{n=1}^{l-1}\|\zeta^n\|_1^2\Delta t_c,
\end{aligned}
$$

$$
(4.163)\qquad
\begin{aligned}
|T''_{13}|&\leqq K\sum_{n=1}^{l-1}\|d_t(E\nabla\eta^n)\|\|\nabla\tilde{c}^n\|_{L^\infty}\|\nabla\zeta^n\|\Delta t_c\\
&\leqq\varepsilon\|d_t(E\nabla\eta^1)\|^2\Delta t_c^0+\varepsilon\sum_{\substack{n=2\\ t^n>t_m}}^{l-1}\|d_t(E\nabla\eta^n)\|^2\Delta t_c
\end{aligned}
$$

$$+\varepsilon\sum_{\substack{n=2\\t^n=t_m}}^{l-1}\|d_t(E\nabla\eta^n)\|^2\Delta t_c\left(\frac{\Delta t_c}{\Delta t_p}\right)+K(\|\tilde{p}\|_{L^\infty(W^1_\infty)})\sum_{n=1}^{l-1}\|\zeta^n\|_1^2\Delta t_c$$

$$+K(\|\tilde{p}\|_{L^\infty(W^1_\infty)})\sum_{m=2}^{k-1}\|\zeta_m\|_1^2\Delta t_p$$

$$\equiv H_1+H_2+H_3+G.$$

The dependence of G on $\|\tilde{p}\|_{L^\infty(W^1_\infty)}$ is required here because the ε terms will lead to a sum of the form $\varepsilon\sum_{m=0}^{k-2}\|d_t\nabla\eta_m\|^2\Delta t_p$, which will later be partly bounded by $\varepsilon(\|\tilde{p}\|_{L^\infty(W^1_\infty)})\sum_{m=0}^{k-2}\|d_t\zeta_m\|^2\Delta t_p$. We must choose ε small enough to hide this term on the left-hand side of (4.139). We now have

$$d_t(E\nabla\eta^1)=2d_t\nabla\eta_0,$$

$$H_1\leqq\varepsilon\|d_t\nabla\eta_0\|^2\Delta t_c^0,$$

$$d_t(E\nabla\eta^n)=d_t\nabla\eta_{m-1},\qquad t^n>t_m,$$

$$H_2\leqq\varepsilon\sum_{n=2}^{l-1}\|d_t\nabla\eta_{m-1}\|^2\Delta t_c\leqq\varepsilon\sum_{m=0}^{k-2}\|d_t\nabla\eta_m\|^2\Delta t_p,$$

$$d_t(E\nabla\eta^n)=d_t\nabla\eta_{m-1}+\frac{(\Delta t_p)^2}{\Delta t_c}d_t^2\nabla\eta_{m-1}\tag{4.164}$$

$$=d_t\nabla\eta_{m-1}+\frac{\Delta t_p}{\Delta t_c}(d_t\nabla\eta_{m-1}-d_t\nabla\eta_{m-2}),\qquad t^n=t_m,$$

$$H_3\leqq\varepsilon\sum_{m=2}^{k-1}\left(\frac{\Delta t_p}{\Delta t_c}\right)^2(\|d_t\nabla\eta_{m-1}\|^2+\|d_t\nabla\eta_{m-2}\|^2)\Delta t_c\left(\frac{\Delta t_c}{\Delta t_p}\right)$$

$$\leqq\varepsilon\sum_{m=0}^{k-2}\|d_t\nabla\eta_m\|^2\Delta t_p,$$

$$|T''_{13}|\leqq\varepsilon\sum_{m=0}^{k-2}\|d_t\nabla\eta_m\|^2\Delta t_p+K\sum_{n=1}^{l-1}\|\zeta^n\|_1^2\Delta t_c+K\sum_{m=2}^{k-1}\|\zeta_m\|_1^2\Delta t_p.$$

Next,

$$|T''_{14}|\leqq K\sum_{n=1}^{l-1}(\|d_t(E\nabla\tilde{p}^n)\|_{L^\infty}+\|d_t(E\nabla P^n)\|_{L^\infty})\Delta t_c\|E\nabla\eta^n\|\|\nabla\tilde{c}^n\|_{L^\infty}\|\nabla\zeta^n\|$$

$$\leqq K\sum_{n=1}^{l-1}(\|d_t(E\nabla\tilde{p}^n)\|_{L^\infty}+\|d_t(E\nabla\eta^n)\|_{L^\infty})\|E\nabla\eta^n\|\|\nabla\zeta^n\|\Delta t_c$$

$$\leqq\varepsilon(1+\|d_t(E\nabla\eta^1)\|_{L^\infty}^2)\|\nabla\eta_0\|^2\Delta t_c^0$$

$$+\varepsilon\sum_{\substack{n=2\\t^n>t_m}}^{l-1}(1+\|d_t(E\nabla\eta^n)\|_{L^\infty}^2)(\|\nabla\eta_m\|^2+\|\nabla\eta_{m-1}\|^2)\Delta t_c\tag{4.165}$$

$$+\varepsilon\sum_{\substack{n=2\\t^n=t_m}}^{l-1}\left(\left(\frac{\Delta t_p}{\Delta t_c}\right)^2+\|d_t(E\nabla\eta^n)\|_{L^\infty}^2\right)(\|\nabla\eta_{m-1}\|^2+\|\nabla\eta_{m-2}\|^2)\Delta t_c\left(\frac{\Delta t_c}{\Delta t_p}\right)$$

$$+K(\|\tilde{p}\|_{L^\infty(W^1_\infty)})\sum_{n=1}^{l-1}\|\zeta^n\|_1^2\Delta t_c+K(\|\tilde{p}\|_{L^\infty(W^1_\infty)})\sum_{m=2}^{k-1}\|\zeta_m\|_1^2\Delta t_p$$

$$\equiv J_1+J_2+J_3+G,$$

where we have used (4.159) and have chosen ε to cancel the effect of $\|\tilde{p}\|_{L^\infty(W^1_\infty)}$. We find that

$$
\begin{aligned}
J_1 &\leqq \varepsilon(1+\|d_t\nabla\eta_0\|^2_{L^\infty})\|\nabla\eta_0\|^2\Delta t_c^0\\
&\leqq \varepsilon\|\nabla\eta_0\|^2\Delta t_c^0+\varepsilon h^{-2}\|d_t\nabla\eta_0\|^2\|\nabla\eta_0\|^2\Delta t_c^0,\\
J_2 &\leqq \varepsilon\sum_{n=2}^{l-1}(1+\|d_t\nabla\eta_{m-1}\|^2_{L^\infty})(\|\nabla\eta_m\|^2+\|\nabla\eta_{m-1}\|^2)\Delta t_c\\
&\leqq \varepsilon\sum_{m=0}^{k-1}\|\nabla\eta_m\|^2\Delta t_p+\varepsilon h^{-2}\sum_{m=1}^{k-1}\|d_t\nabla\eta_{m-1}\|^2(\|\nabla\eta_m\|^2+\|\nabla\eta_{m-1}\|^2)\Delta t_p,\\
J_3 &\leqq \varepsilon\sum_{m=2}^{k-1}\left[\left(\frac{\Delta t_p}{\Delta t_c}\right)^2+\|d_t\nabla\eta_{m-1}\|^2_{L^\infty}+\left(\frac{\Delta t_p}{\Delta t_c}\right)^2(\|d_t\nabla\eta_{m-1}\|^2_{L^\infty}+\|d_t\nabla\eta_{m-2}\|^2_{L^\infty})\right]\\
&\quad\cdot(\|\nabla\eta_{m-1}\|^2+\|\nabla\eta_{m-2}\|^2)\Delta t_c\left(\frac{\Delta t_c}{\Delta t_p}\right)\\
&\leqq \varepsilon\sum_{m=0}^{k-2}\|\nabla\eta_m\|^2\Delta t_p+\varepsilon h^{-2}\sum_{m=1}^{k-2}(\|d_t\nabla\eta_m\|^2+\|d_t\nabla\eta_{m-1}\|^2)\\
&\quad\cdot(\|\nabla\eta_m\|^2+\|\nabla\eta_{m-1}\|^2)\Delta t_p.
\end{aligned}
\tag{4.166}
$$

Thus

$$
\begin{aligned}
|T''_{14}| &\leqq \varepsilon h^{-2}\sum_{m=0}^{k-2}\|d_t\nabla\eta_m\|^2(\|\nabla\eta_{m+1}\|^2+\|\nabla\eta_m\|^2+\|\nabla\eta_{m-1}\|^2)\Delta t_p\\
&\quad+\varepsilon\sum_{m=0}^{k-1}\|\nabla\eta_m\|^2\Delta t_p+K(\|\tilde{p}\|_{L^\infty(W^1_\infty)})\sum_{n=1}^{l-1}\|\zeta^n\|_1^2\Delta t_c\\
&\quad+K(\|\tilde{p}\|_{L^\infty(W^1_\infty)})\sum_{m=2}^{k-1}\|\zeta_m\|_1^2\Delta t_p.
\end{aligned}
\tag{4.167}
$$

Next,

$$
\begin{aligned}
|T''_{12}| &\leqq K\sum_{n=1}^{l-1}(\|d_t\tilde{c}^{n-1}\|_{L^\infty}+\|d_t\zeta^{n-1}\|_{L^\infty})\Delta t_c\|E\nabla\eta^n\|\|\nabla\tilde{c}^n\|_{L^\infty}\|\nabla\zeta^n\|\\
&\leqq K\sum_{n=1}^{l-1}\|E\nabla\eta^n\|^2\Delta t_c+K\sum_{n=1}^{l-1}(1+\|d_t\zeta^{n-1}\|^2_{L^\infty})\|\zeta^n\|_1^2\Delta t_c\\
&\leqq K\sum_{m=0}^{k-1}\|\nabla\eta_m\|^2\Delta t_p+K\sum_{n=1}^{l-1}(1+\|d_t\zeta^{n-1}\|^2_{L^\infty})\|\zeta^n\|_1^2\Delta t_c.
\end{aligned}
\tag{4.168}
$$

Collecting terms, we have

$$
\begin{aligned}
|T_6| &\leqq K(\|c\|_{L^\infty(H^r)})h^{2r}+K\sum_{n=1}^{l-1}(1+\|d_t\zeta^{n-1}\|^2_{L^\infty})\|\zeta^n\|_1^2\Delta t_c\\
&\quad+K\sum_{m=2}^{k-1}\|\zeta^m\|_1^2\Delta t_p+K\sum_{m=0}^{k-1}\|\nabla\eta_m\|^2\Delta t_p+\varepsilon\sum_{m=0}^{k-2}\|d_t\nabla\eta_m\|^2\Delta t_p\\
&\quad+\varepsilon h^{-2}\sum_{m=0}^{k-2}\|d_t\nabla\eta_m\|^2(\|\nabla\eta_{m+1}\|^2+\|\nabla\eta_m\|^2+\|\nabla\eta_{m-1}\|^2)\Delta t_p\\
&\quad+\varepsilon\|\zeta^l\|_1^2+K\|\zeta^0\|_1^2+K\|\nabla\eta_{k-1}\|^2+K\|\nabla\eta_{k-2}\|^2.
\end{aligned}
\tag{4.169}
$$

Collecting the estimates of T_1 through T_6, we have a revised bound for S_3.

The term S_4 from Theorem 4.3 is unchanged, but we use a different bound for the term T_{10} arising in the analysis of S_4, since we have nothing to gain by integrating by parts. We write, using the form of u,

$$
\begin{aligned}
T_{10} &= \sum_{n=0}^{l-1} ([u(c^{n+1}, \nabla p^{n+1}) - u(c^{n+1}, \nabla \tilde{p}^{n+1})] \cdot \nabla \tilde{c}^{n+1}, d_t \zeta^n) \Delta t_c \\
&= \sum_{n=0}^{l-1} (a(c^{n+1}) \nabla \theta^{n+1} \cdot \nabla \tilde{c}^{n+1}, d_t \zeta^n) \Delta t_c, \\
|T_{10}| &\leqq K \sum_{n=0}^{l-1} \|\nabla \theta^{n+1}\| \|\nabla \tilde{c}^{n+1}\|_{L^\infty} \|d_t \zeta^n\| \Delta t_c \\
&\leqq K(\|p\|_{L^2(H^s)}) h^{2s-2} + \varepsilon \sum_{n=0}^{l-1} |||d_t \zeta^n|||_n^2 \Delta t_c.
\end{aligned}
\tag{4.170}
$$

This simpler estimate imposes fewer regularity requirements than the one in Theorem 4.3. This leads, subject to the induction hypothesis (4.21), to a revised estimate for S_4.

Next, we note that the new weighted norm definitions are such that

$$
|S_6 + S_7| \leqq \sum_{n=0}^{l-1} |||C^{n+1} - \bar{C}^{n+1}|||_n |||d_t \zeta^n|||_n
\tag{4.171}
$$

still holds. For $n = 0$ the time step $\Delta t_c^0 = O((\Delta t_p)^2)$ enables us to write, noting $\rho_c' < (\frac{3}{11}/(1-\frac{3}{11})) = \frac{3}{8}$,

$$
\begin{aligned}
|||C^1 - \bar{C}^1|||_0 |||d_t \zeta^0|||_0 &\leqq \rho_c' |||\delta C^0|||_0 |||d_t \zeta^0|||_0 \\
&\leqq (\tfrac{3}{8} - \varepsilon)(|||\delta \tilde{c}^0|||_0 + \Delta t_c^0 |||d_t \zeta^0|||_0) |||d_t \zeta^0|||_0 \\
&\leqq (\tfrac{3}{8} - \varepsilon) |||d_t \zeta^0|||_0^2 \Delta t_c^0 + K |||d_t \zeta^0|||_0 \Delta t_c^0 \\
&\leqq (\tfrac{3}{8} - \varepsilon) |||d_t \zeta^0|||_0^2 \Delta t_c^0 + K \Delta t_c^0 + \varepsilon |||d_t \zeta^0|||_0^2 \Delta t_c^0 \\
&\leqq (\tfrac{3}{8} - \varepsilon) |||d_t \zeta^0|||_0^2 \Delta t_c^0 + K(\|\tilde{c}\|_{W_\infty^1(H^1)}) (\Delta t_p)^2.
\end{aligned}
\tag{4.172}
$$

For $n \geqq 1$, the analysis of (4.98)–(4.100) must be modified to handle the dependence of b on ∇p in the analogue of (4.99). This can be done by an argument of the form of (4.178) below. The estimate of $S_6 + S_7$ in Theorem 4.3 is then unchanged, save for the addition of the term $K(\|\tilde{c}\|_{W_\infty^1(H^1)})(\Delta t_p)^2$.

Combining the new estimates of S_1 through S_7, we have

$$
\begin{aligned}
&\frac{1}{2} \sum_{n=0}^{l-1} |||d_t \zeta^n|||_n^2 \Delta t_c + \frac{1}{2} \sum_{n=0}^{l-1} (\|\zeta^{n+1}\|_{b^n}^2 - \|\zeta^n\|_{b^n}^2) \\
&\quad \leqq K(\|c\|_{H^1(H^r)}, \|\tilde{c}\|_{W_\infty^1(W_\infty^1)}, \|p\|_{H^1(W_\infty^1)}, \|\tilde{p}\|_{L^\infty(W_\infty^1)}) h^{2r} \\
&\qquad + K(\|\tilde{c}\|_{W_\infty^1(W_\infty^1)}, \|p\|_{H^1(H^s)}, \|p\|_{W_\infty^1(W_\infty^1)}, \|\tilde{p}\|_{W_\infty^1(W_\infty^1)}) h^{2s-2} \\
&\qquad + K(\|c\|_{H^2(H^1)}, \|c\|_{W_\infty^1(L^\infty)}, \|\tilde{c}\|_{W_\infty^1(W_\infty^1)}, \|\tilde{c}\|_{H^2(H^1)}, \|p\|_{H^1(H^1)}) (\Delta t_c)^2 \\
&\qquad + K(\|\tilde{c}\|_{W_\infty^1(W_\infty^1)}, \|\tilde{p}\|_{H^2(H^1)}) (\Delta t_p)^2 + A_1 + A_2 \\
&\qquad + K(\|c\|_{W_\infty^1(W_\infty^1)}, \|\tilde{p}\|_{W_\infty^1(W_\infty^1)}, \|p\|_{W_\infty^1(W_\infty^1)}) \sum_{n=1}^{l-1} (1 + \|d_t \zeta^{n-1}\|_{L^\infty}^2) \|\zeta^n\|_1^2 \Delta t_c \\
&\qquad + K(\|\tilde{c}\|_{L^\infty(W_\infty^1)}, \|\tilde{p}\|_{W_\infty^1(W_\infty^1)}) \sum_{m=2}^{k-1} \|\zeta^m\|_1^2 \Delta t_p
\end{aligned}
\tag{4.173}
$$

$$
\begin{aligned}
&+A_5+A_6+A_7+A_8+A_9+A_{10}+A_{11}+A_{12} \\
&+K(\|\tilde{c}\|_{W^1_\infty(W^1_\infty)}) \sum_{m=0}^{k-1} \|\nabla \eta_m\|^2 \Delta t_p + \varepsilon \sum_{m=0}^{k-2} \|d_t \nabla \eta_m\|^2 \Delta t_p \\
&+\varepsilon h^{-2} \sum_{m=0}^{k-2} \|d_t \nabla \eta_m\|^2 (\|\nabla \eta_{m+1}\|^2 + \|\nabla \eta_m\|^2 + \|\nabla \eta_{m-1}\|^2) \Delta t_p \\
&+K(\|\tilde{c}\|_{L^\infty(W^1_\infty)}) \|\nabla \eta_{k-1}\|^2 + K(\|\tilde{c}\|_{L^\infty(W^1_\infty)}) \|\nabla \eta_{k-2}\|^2 \\
\equiv\ & K(h^{2r}+h^{2s-2}+(\Delta t_c)^2+(\Delta t_p)^2) + A_1 + A_2 + \cdots + A_{17},
\end{aligned}
$$

where the unwritten A's are as in Theorem 4.3.

We proceed as in Theorem 4.3 to form a collapsing H^1 sum on the left-hand side of (4.173). The only alteration needed is in the index perturbation (4.106), where the weighted norm definition has changed. We note first that, from the induction hypothesis (4.143) and the assumption that $\Delta t_c = o(h)$, we can make an argument of the form (4.108)–(4.109) to obtain

$$
\sum_{m=0}^{k-2} \|d_t \nabla \eta_m\|_{L^\infty} \Delta t_p \leqq \varepsilon, \tag{4.174}
$$

which at once yields

$$
\sum_{n=0}^{l-1} \|d_t \nabla \eta_{m-1}\|_{L^\infty} \Delta t_c \leqq \sum_{m=0}^{k-2} \|d_t \nabla \eta_m\|_{L^\infty} \left(\frac{\Delta t_p}{\Delta t_c}\right) \Delta t_c \leqq \varepsilon. \tag{4.175}
$$

Now we see that

$$
\begin{aligned}
\sum_{n=1}^{l-1} (\|\zeta^n\|^2_{b^n} - \|\zeta^n\|^2_{b^{n-1}}) &= \sum_{n=1}^{l-1} ([b(C^{*n}, E\nabla P^{n+1}) - b(C^{*n}, E\nabla P^n)] \nabla \zeta^n, \nabla \zeta^n) \\
&\quad + \sum_{n=1}^{l-1} ([b(C^{*n}, E\nabla P^n) - b(C^{*n-1}, E\nabla P^n)] \nabla \zeta^n, \nabla \zeta^n) \qquad (4.176) \\
&\equiv B_1 + B_2.
\end{aligned}
$$

We bound B_2 in a manner analogous to (4.99). To handle B_1, we write

$$
|B_1| \leqq K \sum_{n=1}^{l-1} (\|d_t E \nabla \tilde{p}^n\|_{L^\infty} + \|d_t E \nabla \eta^n\|_{L^\infty}) \|\zeta^n\|_1^2 \Delta t_c \equiv B_3 + B_4. \tag{4.177}
$$

By (4.159),

$$
\begin{aligned}
B_3 &\leqq K \sum_{\substack{n=1 \\ t^n > t_m}}^{l-1} \|\zeta^n\|_1^2 \Delta t_c + K \sum_{\substack{n=1 \\ t^n = t_m}}^{l-1} \|\zeta^n\|_1^2 \left(\frac{\Delta t_p}{\Delta t_c}\right) \Delta t_c \\
&\leqq K \sum_{n=1}^{l-1} \|\zeta^n\|_1^2 \Delta t_c + K \sum_{m=1}^{k-1} \|\zeta_m\|_1^2 \Delta t_p \\
&\leqq A_3 + A_4, \\
B_4 &\leqq K \sum_{\substack{n=1 \\ t^n > t_m}}^{l-1} \|d_t \nabla \eta_{m-1}\|_{L^\infty} \|\zeta^n\|_1^2 \Delta t_c \qquad (4.178) \\
&\quad + K \sum_{\substack{n=1 \\ t^n = t_m}}^{l-1} (\|d_t \nabla \eta_{m-1}\|_{L^\infty} + \|d_t \nabla \eta_{m-2}\|_{L^\infty}) \left(\frac{\Delta t_p}{\Delta t_c}\right) \|\zeta^n\|_1^2 \Delta t_c
\end{aligned}
$$

$$\leqq K \sum_{n=1}^{l-1} \|d_t \nabla \eta_{m-1}\|_{L^\infty} \|\zeta^n\|_1^2 \Delta t_c$$

$$+K \sum_{m=1}^{l-1} (\|d_t \nabla \eta_{m-1}\|_{L^\infty} + \|d_t \nabla \eta_{m-2}\|_{L^\infty}) \|\zeta_m\|_1^2 \Delta t_p,$$

using (4.164). By (4.174) and (4.175), these terms may be considered as part of A_3 and A_4 when the discrete Gronwall lemma is applied.

The argument proceeds unchanged through (4.111), up to the bound for A_6. Instead of estimating A_6 itself, we will bound the larger sum

$$A_6^* = (\Delta t_p)^{-2} \sum_{m=2}^{k-1} \|P_m - \bar{P}_m\|_{a_m}^2 \Delta t_p + (\Delta t_p^0)^{-2} \sum_{m=0}^{1} \|P_m - \bar{P}_m\|_{a_m}^2 \Delta t_p^0. \tag{4.179}$$

This will be crucial in the bound for A_{14}, and also will give us a better bound for $\|\nabla(P - \bar{P})\|$. For $m = 0$, the starting estimate (3.21) yields

$$(\Delta t_p^0)^{-1} \|P_0 - \bar{P}_0\|_{a_0} \leqq (\Delta t_p^0)^{-1} (K h^r \Delta t_p)^2 \leqq K h^{2r}. \tag{4.180}$$

For $m = 1$, since $\rho_p' = O((\Delta t_p^0)^{1/2})$, we have

$$\begin{aligned}(\Delta t_p^0)^{-1} \|P_1 - \bar{P}_1\|_{a_1}^2 &\leqq (\Delta t_p^0)^{-1} (\rho_p')^2 \|\delta P_0\|_{a_1}^2 \\ &\leqq K(\|\delta \eta_0\|_{a_1}^2 + \|\delta \tilde{p}_0\|_{a_1}^2) \\ &\leqq K(\Delta t_p^0)^2 + K(\|\nabla \eta_0\|^2 + \|\nabla \eta_1\|^2) \\ &\leqq K(\Delta t_p^0)^2 + K(\|\zeta_0\|^2 + \|\zeta_1\|^2 + h^{2r} + \|P_1 - \bar{P}_1\|_{a_1}^2 + \|P_0 - \bar{P}_0\|_{a_0}^2) \\ &\leqq K(\Delta t_p^0)^2 + K(\|\zeta_0\|^2 + \|\zeta_1\|^2) + K h^{2r},\end{aligned} \tag{4.181}$$

where we have used (3.16), (4.8), and (3.21), and where we hide $\|P_1 - \bar{P}_1\|_{a_1}^2$ for Δt_p^0 sufficiently small. For $m \geqq 2$, we set

$$F_m = (\Delta t_p)^{-2} \|P_m - \bar{P}_m\|_{a_m}^2, \tag{4.182}$$

and we note that, since $\rho_p' = O(\Delta t_p)$,

$$\begin{aligned}F_m &\leqq (\Delta t_p)^{-2} (\rho_p')^2 \|\delta^2 P_{m-1}\|_{a_m}^2 \\ &\leqq K(\|\delta^2 \eta_{m-1}\|_{a_m}^2 + \|\delta^2 \tilde{p}_{m-1}\|_{a_m}^2) \\ &\leqq K\left(\|\eta_m\|_{a_m}^2 + 2\|\eta_{m-1}\|_{a_{m-1}}^2 + \|\eta_{m-2}\|_{a_{m-2}}^2 + (\Delta t_p)^3 \left\|\frac{\partial^2 \tilde{p}}{\partial t^2}\right\|_{L^2(t_{m-2}, t_m; H^1)}^2\right) \\ &\leqq K(\Delta t_p)^2 (F_m + 2F_{m-1} + F_{m-2}) + K(\|c\|_{L^\infty(H^r)}) h^{2r} \\ &\quad + K(\|\zeta_m\|^2 + \|\zeta_{m-1}\|^2 + \|\zeta_{m-2}\|^2) + K(\Delta t_p)^3 \left\|\frac{\partial^2 \tilde{p}}{\partial t^2}\right\|_{L^2(t_{m-2}, t_m; H^1)}^2.\end{aligned} \tag{4.183}$$

We are now in the position of (4.33)–(4.36) with $R = O((\Delta t_p)^2)$, F_m as above, a factor of $(\Delta t_p)^{-2}$ on G_m, and a factor of $(\Delta t_p)^2$ on $\tilde{K}_0$ and $\tilde{K}_1$. We can apply Lemma 2.5 to see that

$$
\begin{aligned}
(\Delta t_p)^{-2} \sum_{m=2}^{k-1} \|P_m - \bar{P}_m\|_{a_m}^2 \Delta t_p &= \sum_{m=2}^{k-1} F_m \Delta t_p \\
&\leqq K(\|c\|_{L^\infty(H^r)})h^{2r} + K \sum_{m=0}^{k-1} \|\zeta_m\|^2 \Delta t_p + K(\|\tilde{p}\|_{H^2(H^1)})(\Delta t_p)^4, \qquad (4.184)
\end{aligned}
$$

and so $A_6^*$is bounded by the same estimate.

To complete our estimate, we must bound the new terms A_{13} through A_{17}. By (4.8), we have

$$
\begin{aligned}
A_{13} &\leqq K \sum_{m=0}^{k-1} (h^{2r} + \|\zeta_m\|^2 + \|P_m - \bar{P}_m\|_{a_m}^2)\Delta t_p \\
&\leqq K(\|c\|_{L^\infty(H^r)})h^{2r} + A_4 + (\Delta t_p)^2 A_6^*. \qquad (4.185)
\end{aligned}
$$

Using (4.142), we also see that, for $0 \leqq m \leqq k-1$,

$$
\begin{aligned}
h^{-2}\|\nabla \eta_m\|^2 &\leqq K h^{-2}(h^{2r} + \|\zeta_m\|^2 + \|P_m - \bar{P}_m\|_{a_m}^2) \\
&\leqq K h^{2r-2} + K(h^{2r-2} + h^{2s-4} + h^{-2}(\Delta t_c)^2) + K h^{-2} \Delta t_p A_6^*, \qquad (4.186) \\
A_6^* &\leqq K h^{2r} + K \sum_{m=0}^{k-1} \|\zeta^m\|^2 \Delta t_p + K(\Delta t_p)^4 \leqq K(h^{2r} + h^{2s-2} + (\Delta t_c)^2), \\
h^{-2}\|\nabla \eta_m\|^2 &\leqq K(h^{2r-2} + h^{2s-4} + h^{-2}(\Delta t_c)^2) \leqq \varepsilon
\end{aligned}
$$

for h sufficiently small, since $\Delta t_c = o(h)$. Thus A_{15} is bounded by A_{14}, which we handle below. For A_{16}, we have

$$
\begin{aligned}
A_{16} &\leqq K(\|c\|_{L^\infty(H^r)})h^{2r} + K\|\zeta_{k-1}\|^2 + K\|P_{k-1} - \bar{P}_{k-1}\|_{a_{k-1}}^2 \\
&\leqq K(h^{2r} + \|\zeta_{k-1}\|^2 + \Delta t_p A_6^*), \qquad (4.187)
\end{aligned}
$$

where $\|\zeta_{k-1}\|^2$ can be bounded by the same technique as A_9 and A_{10} in (4.103) through (4.105). A_{17} is handled similarly.

To bound A_{14}, we require an a priori estimate of $d_t \nabla \eta_m$, which we obtain as follows. We recall the pressure equation (4.6) at time level t_m, subtract it at t_{m-1}, and divide by Δt_p to obtain

$$
\begin{aligned}
(a(C_m^*)\nabla d_t \eta_{m-1}, \nabla \varphi) &= ([a(C_{m-1}^*) - a(C_m^*)]\nabla \eta_{m-1}, \nabla \varphi)\frac{1}{\Delta t_p} \\
&\quad + ([a(c_m) - a(C_m^*)]\nabla d_t \tilde{p}_{m-1}, \nabla \varphi) \\
&\quad + (\{[a(c_m) - a(C_m^*)] - [a(c_{m-1}) - a(C_{m-1}^*)]\}\nabla \tilde{p}_{m-1}, \nabla \varphi)\frac{1}{\Delta t_p} \\
&\quad + (\{[a(C_m^*)\gamma(C_m^*) - a(c_m)\gamma(c_m)] \qquad (4.188) \\
&\qquad - [a(C_{m-1}^*)\gamma(C_{m-1}^*) - a(c_{m-1})\gamma(c_{m-1})]\}\nabla d, \nabla \varphi)\frac{1}{\Delta t_p} \\
&\quad + (a(C_m^*)\nabla[(P_m - \bar{P}_m) - (P_{m-1} - \bar{P}_{m-1})], \nabla \varphi)\frac{1}{\Delta t_p} \\
&\quad + ([a(C_m^*) - a(C_{m-1}^*)]\nabla(P_{m-1} - \bar{P}_{m-1}), \nabla \varphi)\frac{1}{\Delta t_p}, \qquad \varphi \in \mathcal{N}_h.
\end{aligned}
$$

We choose $\varphi = d_t\eta_{m-1}$ as test function, multiply by Δt_p, and sum from 1 to $k-1$ to obtain

$$\sum_{m=1}^{k-1} \|d_t\eta_{m-1}\|_{a_m}^2 \Delta t_p = D_1 + D_2 + D_3 + D_4 + D_5 + D_6. \tag{4.189}$$

We estimate D_1 through D_6 in turn.

First, we see that

$$\begin{aligned}
|D_1| &\leqq K \sum_{m=1}^{k-1} \|C_{m-1} - C_m\|_{L^\infty} \|\nabla \eta_{m-1}\| \|\nabla d_t\eta_{m-1}\| \\
&\leqq K \sum_{m=1}^{k-1} (\|d_t\tilde{c}_{m-1}\|_{L^\infty}^2 + \|d_t\zeta_{m-1}\|_{L^\infty}^2)\|\nabla \eta_{m-1}\|^2 \Delta t_p \\
&\quad + \varepsilon \sum_{m=1}^{k-1} \|d_t\eta_{m-1}\|_{a_m}^2 \Delta t_p \\
&\leqq KA_{13} + K(\|c\|_{L^\infty(H^r)})h^{2r} \sum_{m=1}^{k-1} \|d_t\zeta_{m-1}\|_{L^\infty}^2 \Delta t_p \\
&\quad + K \sum_{m=1}^{k-1} \|d_t\zeta_{m-1}\|_{L^\infty}^2 \|\zeta_{m-1}\|^2 \Delta t_p \\
&\quad + K\Delta t_p A_6^* \sum_{m=1}^{k-1} \|d_t\zeta_{m-1}\|_{L^\infty}^2 \Delta t_p + \varepsilon \sum_{m=1}^{k-1} \|d_t\eta_{m-1}\|_{a_m}^2 \Delta t_p,
\end{aligned} \tag{4.190}$$

and we need a bound for $\sum_{m=1}^{k-1} \|d_t\zeta_{m-1}\|_{L^\infty}^2 \Delta t_p$. By (4.108), we know that

$$d_t\zeta_{\tilde{m}} = \frac{1}{\Delta t_p}(\zeta_{\tilde{m}+1} - \zeta_{\tilde{m}}) = \frac{\Delta t_c}{\Delta t_p}\frac{1}{\Delta t_c} \sum_{m=\tilde{m}} (\zeta^{n+1} - \zeta^n) = \frac{\Delta t_c}{\Delta t_p} \sum_{m=\tilde{m}} d_t\zeta^n,$$

$$|d_t\zeta_{\tilde{m}}|^2 = \left(\frac{\Delta t_c}{\Delta t_p}\right)^2 \left| \sum_{m=\tilde{m}} d_t\zeta^n \right|^2 \leqq \left(\frac{\Delta t_c}{\Delta t_p}\right)^2 \left(\frac{\Delta t_p}{\Delta t_c}\right) \sum_{m=\tilde{m}} |d_t\zeta^n|^2,$$

$$\begin{aligned}
&\|d_t\zeta_{\tilde{m}}\|^2 \Delta t_p \leqq \sum_{m=\tilde{m}} \|d_t\zeta^n\|^2 \Delta t_c, \\
&\sum_{m=1}^{k-1} \|d_t\zeta_{m-1}\|_{L^\infty}^2 \Delta t_p \leqq K_0 h^{-2} \sum_{m=1}^{k-1} \|d_t\zeta_{m-1}\|^2 \Delta t_p \leqq K_0 h^{-2} \sum_{n=0}^{l-2} \|d_t\zeta^n\|^2 \Delta t_c \\
&\qquad \leqq Kh^{-2}(h^{2r} + h^{2s-2} + (\Delta t_c)^2) \\
&\qquad \leqq \varepsilon,
\end{aligned} \tag{4.191}$$

for h sufficiently small. Then

$$\begin{aligned}
|D_1| &\leqq KA_{13} + \varepsilon h^{2r} + K \sum_{m=1}^{k-1} \|d_t\zeta_{m-1}\|_{L^\infty}^2 \|\zeta_{m-1}\|^2 \Delta t_p \\
&\quad + \varepsilon A_6^* + \varepsilon \sum_{m=1}^{k-1} \|d_t\eta_{m-1}\|_{a_m}^2 \Delta t_p.
\end{aligned} \tag{4.192}$$

By (4.191), the discrete Gronwall lemma will handle the third term.

Next, we write

$$
\begin{aligned}
|D_2| &\leqq K \sum_{m=1}^{k-1} (\|\xi_m\| + \|\zeta_m\|) \|\nabla d_t \tilde{p}_{m-1}\|_{L^\infty} \|\nabla d_t \eta_{m-1}\| \Delta t_p \\
&\leqq K(\|c\|_{L^2(H^r)}) h^{2r} + K \sum_{m=1}^{k-1} \|\zeta_m\|^2 \Delta t_p + \varepsilon \sum_{m=1}^{k-1} \|d_t \eta_{m-1}\|_{a_m}^2 \Delta t_p \qquad (4.193) \\
&\leqq K(\|c\|_{L^2(H^r)}) h^{2r} + A_4 + \varepsilon \sum_{m=1}^{k-1} \|d_t \eta_{m-1}\|_{a_m}^2 \Delta t_p.
\end{aligned}
$$

To bound A_3, we set

$$
\begin{aligned}
a_1 &= \int_0^1 \frac{\partial a}{\partial c}(\alpha c_m + (1-\alpha) c_{m-1}) \, d\alpha, \\
a_2 &= \int_0^1 \frac{\partial a}{\partial c}(\alpha C_m^* + (1-\alpha) C_{m-1}^*) \, d\alpha,
\end{aligned}
\qquad (4.194)
$$

and by an argument of the form of (4.65)–(4.66), we have

$$
|a_1 - a_2| \leqq K\left(\frac{\partial^2 a}{\partial c^2}\right)(|\xi_m| + |\zeta_m| + |\xi_{m-1}| + |\zeta_{m-1}|). \qquad (4.195)
$$

Then

$$
\begin{aligned}
D_3 &= \sum_{m=1}^{k-1} ([a_1(c_m - c_{m-1}) - a_2(C_m^* - C_{m-1}^*)] \nabla \tilde{p}_{m-1}, \nabla d_t \eta_{m-1}) \\
&= \sum_{m=1}^{k-1} ([(a_1 - a_2) d_t c_{m-1} + a_2(d_t(c - C^*)_{m-1})] \nabla \tilde{p}_{m-1}, \nabla d_t \eta_{m-1}) \Delta t_p,
\end{aligned}
$$

$$
\begin{aligned}
|D_3| &\leqq K \sum_{m=1}^{k-1} (\|\xi_m\| + \|\zeta_m\| + \|\xi_{m-1}\| + \|\zeta_{m-1}\|) \|d_t c_{m-1}\|_{L^\infty} \|\nabla \tilde{p}_{m-1}\|_{L^\infty} \|\nabla d_t \eta_{m-1}\| \Delta t_p \\
&\quad + K \sum_{m=1}^{k-1} (\|d_t \xi_{m-1}\| + \|d_t \zeta_{m-1}\|) \|\nabla \tilde{p}_{m-1}\|_{L^\infty} \|\nabla d_t \eta_{m-1}\| \Delta t_p \qquad (4.196) \\
&\leqq K(\|c\|_{H^1(H^r)}) h^{2r} + K \sum_{m=0}^{k-1} \|\zeta_m\|^2 \Delta t_p + K(\|\tilde{p}\|_{L^\infty(W_\infty^1)}) \sum_{m=1}^{k-1} \|d_t \zeta_{m-1}\|^2 \Delta t_p \\
&\quad + \varepsilon \sum_{m=1}^{k-1} \|d_t \eta_{m-1}\|_{a_m}^2 \Delta t_p,
\end{aligned}
$$

and we recall that ε in (4.163) was chosen sufficiently small so that the fourth term hides in A_1. The argument (4.191) shows that the fourth term may be converted from a sum on m to a sum on n.

Next, D_4 is bounded in the same way as D_3, using a bound for $\|\nabla d\|_{L^\infty}$. To handle D_5, we see that

$$
\begin{aligned}
|D_5| &\leqq K \sum_{m=1}^{k-1} (\|P_m - \bar{P}_m\|_{a_m} + \|P_{m-1} - \bar{P}_{m-1}\|_{a_{m-1}}) \|\nabla d_t \eta_{m-1}\| \\
&\leqq K(\Delta t_p)^{-2} \sum_{m=0}^{k-1} \|P_m - \bar{P}_m\|_{a_m}^2 \Delta t_p + \varepsilon \sum_{m=1}^{k-1} \|\nabla d_t \eta_{m-1}\|^2 \Delta t_p \qquad (4.197) \\
&\leqq A_6^* + \varepsilon \sum_{m=1}^{k-1} \|d_t \eta_{m-1}\|_{a_m}^2 \Delta t_p.
\end{aligned}
$$

This is where the norm reduction $\rho_p = O(\Delta t_p)$ leading to the estimate of A_6^* is essential. Finally, D_6 is handled by the analysis of D_1, since $\|\nabla(P_{m-1}-\bar{P}_{m-1})\|^2$ is one of the terms used to bound $\|\nabla \eta_{m-1}\|^2$.

Combining the estimates of D_1 through D_6, we have shown that

$$A_{14} \leqq Kh^{2r} + K\sum_{m=1}^{k-1} \|d_t\zeta_{m-1}\|_{L^\infty}^2 \|\zeta_{m-1}\|^2 \Delta t_p + KA_4 + \varepsilon A_1 + A_6^* + A_{13}. \tag{4.198}$$

Assembling (4.173) and the subsequent modifications and estimates, we have

$$\begin{aligned} \sum_{n=0}^{l-1} |||d_t\zeta^n|||_n^2 \Delta t_c + \|\zeta^l\|_1^2 \\ &\leqq K(\|c\|_{L^\infty(W^1_\infty)}, \|c\|_{W^1_\infty(L^\infty)}, \|\tilde{c}\|_{W^1_\infty(W^1_\infty)}, \|p\|_{W^1_\infty(W^1_\infty)}, \|\tilde{p}\|_{W^1_\infty(W^1_\infty)}) \\ &\quad \cdot [K(\|c\|_{H^1(H^r)})h^{2r} + K(\|p\|_{H^1(H^s)})h^{2s-2} \\ &\quad + K(\|c\|_{H^2(H^1)}, \|\tilde{c}\|_{H^2(H^1)}, \|p\|_{H^2(H^1)})(\Delta t_c)^2 + K(\|\tilde{p}\|_{H^2(H^1)})(\Delta t_p)^2]. \end{aligned} \tag{4.199}$$

With the time step choice

$$\Delta t_p \approx \left(\frac{\|\tilde{c}\|_{H^2(H^1)}}{\|\tilde{p}\|_{H^2(H^1)}}\right)\Delta t_c$$

we obtain

$$\sum_{n=0}^{l-1} |||d_t\zeta^n|||_n^2 \Delta t_c + \|\zeta^l\|_1^2 \leqq K(h^{2r} + h^{2s-2} + (\Delta t_c)^2, \tag{4.200}$$

which yields the desired result as in Theorem 4.3.

It remains to check the induction hypotheses. The argument for (4.21) is easier, because (4.184) allows us to replace (4.42) by

$$\|P_k - \bar{P}_k\|_{a_k}^2 \leqq K\Delta t_p(h^{2r} + h^{2s-2}) + K(\Delta t_p)^5, \tag{4.201}$$

so that (4.43) becomes

$$\begin{aligned} \|\nabla P_k\|_{L^\infty}^2 &\leqq 2K_5^2 + Kh^{-2}(h^{2r} + h^{2s-2} + (\Delta t_c)^2) \\ &\leqq 4K_5^2 \end{aligned} \tag{4.202}$$

for h sufficiently small, requiring that $\Delta t_c = o(h)$. We verify (4.108) as in Theorem 4.3, and (4.142) is immediate. With (4.108) in hand, we can check (4.141) by an argument of the form of (4.109). To check (4.143), we follow the argument (4.188) through (4.198) which bounded A_{14}, summing from $m=1$ to k and dropping the multiplier ε. From (4.198), we obtain

$$\begin{aligned} \sum_{m=1}^{k} \|d_t\nabla \eta_{m-1}\|^2 \Delta t_p &\leqq K(h^{2r} + (\Delta t_p)^2) + K\sum_{m=1}^{k} \|d_t\zeta_{m-1}\|_{L^\infty}^2 \|\zeta_{m-1}\|^2 \Delta t_p \\ &\quad + K\sum_{m=0}^{k} \|\zeta_m\|^2 \Delta t_p + K\sum_{m=1}^{k} \|d_t\zeta_{m-1}\|^2 \Delta t_p + A_6^* + A_{13}, \end{aligned} \tag{4.203}$$

where A_6^* and A_{13} are now summed through $m=k$. Since we now have (4.108) at the advanced time level, we can obtain (4.191) summed through $m=k$. Then (4.108), (4.142), and (4.191) handle the summations in (4.203). Finally, we can bound A_6^* and A_{13} by the arguments (4.179) through (4.185), yielding a term of the form $\sum_{m=0}^k \|\zeta^m\|^2 \Delta t_p$ which is estimated using (4.142). Thus (4.143) is verified and the proof is complete.

We remark that if linear extrapolation is not used in the initial guess for the pressure iteration, we obtain (4.184) with time truncation term $K\|\partial\tilde{p}/\partial t\|^2_{L^2(H^1)}(\Delta t_p)^2$, and this suffices for the rest of the argument. Thus we may use the last pressure as an initial guess for this theorem.

The time truncation error $O(\Delta t_p)$ might suggest the use of a first-order method, instead of linear extrapolation, to evaluate the pressure arguments of the concentration coefficients. A glance at (4.154) shows that this will not work, since $d_t(E\nabla\tilde{p}^n)$ will be replaced by zero and $d_t(\nabla\tilde{p}^n - E\nabla\tilde{p}^n)$ by $d_t\nabla\tilde{p}^n$, which is bounded but is not of order Δt_p. Linear extrapolation gives order Δt_p, and higher-order extrapolation does no better.

We now consider the possible benefits of refactoring the matrices L^n and A_m every $O((\Delta t_c)^{-1/2})$ and $O((\Delta t_p)^{-1/2})$ time steps, respectively. As noted in [6], [13], if L^n is so factored and used as an updated preconditioner, we achieve the norm reduction $O((\Delta t_c)^{1/2})$ with one iteration and $O(\Delta t_c)$ with two. Similar statements hold for A_m. We obtain the following corollary.

COROLLARY 4.7. *If we refactor L^n every $O((\Delta t_c)^{-1/2})$ time steps, then we obtain the results of Theorem* 4.1 *and Corollary* 4.2 *with a fixed number of pressure iterations per time step, and with two concentration iterations per step. If $\Delta t_c = O(h^2)$, then we obtain the result of Corollary* 4.4 *with one concentration iteration per time step. If we refactor A_m every $O((\Delta t_p)^{-1/2})$ time steps, then we obtain the result of Theorem* 4.6 *with a fixed number of concentration iterations and two pressure iterations per time step.*

Next, we consider the alternative of extrapolating the Darcy velocity u, instead of its pressure argument, in the concentration equation (3.17a). We replace (3.17a) by

$$\begin{aligned}(4.204)\qquad &\left(\phi\frac{\bar{C}^{n+1}-C^n}{\Delta t_c},\chi\right)+(b(EU^{n+1})\nabla\bar{C}^{n+1},\nabla\chi)\\ &\quad= -(EU^{n+1}\cdot\nabla C^n,\chi)+\langle g(t^{n+1},C^{*n}),\chi\rangle,\qquad \chi\in\mathcal{M}_h,\end{aligned}$$

where

$$(4.205)\qquad \begin{aligned}&EU^{n+1}=u(C_0^*,\nabla P_0), && \text{if } m=0,\\ &EU^{n+1}=\left(1+\frac{\nu}{j}\right)u(C_m^*,\nabla P_m)-\frac{\nu}{j}u(C_{m-1}^*,\nabla P_{m-1}), && \text{if } m\geqq 1,\end{aligned}$$

where $\nu\in\{1,2,\cdots,j\}$ is chosen so that

$$(4.206)\qquad t^{n+1}=t_m+\nu\Delta t_c.$$

We obtain the following corollary.

COROLLARY 4.8. *If* (3.17a) *is replaced by* (4.204), *then all previous error estimates still hold.*

The proof uses mostly the same ideas already presented. Details of the arguments can be found in [22].

Since the physical problem modeled by our coupled system is three-dimensional, it is of interest to see how the results are altered if $d=3$. The error estimates for each of the theorems and corollaries hold for $d=3$. Dimensionality was used only in those parts of the arguments involving inverse inequalities and imbedding properties. The assumptions in (I) relating L^∞- and L^2-norms must lose $h^{-3/2}$ instead of h^{-1}, and c will have to lie in $W^1_\infty(H^3)$ in order to bound the W^1_∞-norm of the elliptic projection $\tilde{c}$. Each of the theorems must require $\Delta t_c = o(h^{3/2})$ instead of $o(h)$ in order to verify various induction assumptions, such as (4.21).

Finally, we consider the application of our analysis to a single quasilinear parabolic equation of the form

$$
\text{(1.2')}\quad
\begin{aligned}
&\text{(c)} && \phi\frac{\partial c}{\partial t}-\nabla\cdot(b(x,c)\nabla c)+u(x,c)\cdot\nabla c=f(x,t,c), && x\in\Omega, \quad t\in J,\\
&\text{(d)} && b(x,c)\frac{\partial c}{\partial n}=g(x,t,c), && x\in\partial\Omega, \quad t\in J,\\
&\text{(e)} && c(x,0)=c_0(x), && x\in\Omega.
\end{aligned}
$$

We can modify our numerical scheme (3.17) by eliminating the pressure equation and all dependence of the concentration equation on the pressure, and by using a uniform time step Δt. With the resulting scheme, we can argue as in Theorem 4.3 to obtain the following theorem.

THEOREM 4.9. *Suppose that* (R_3), *as applied to c, holds, that* $\Delta t=o(h)$, *and that we achieve norm reductions of the form*

$$
\text{(4.207)}\quad
\begin{aligned}
&\rho=O((\Delta t)^{1/2}), && n=1,\\
&\rho\leqq\delta<1/5, && n=2,
\end{aligned}
$$

with δ *independent of n. Then, for h sufficiently small,*

$$
\text{(4.208)}\quad
\begin{aligned}
&\sup_n \|C^n-c^n\|\leqq K_9(h^r+\Delta t),\\
&\sup_n \|C^n-c^n\|_1\leqq K_9(h^{r-1}+\Delta t).
\end{aligned}
$$

Proof. Since u no longer depends on p, the term T_{10} in (4.70) does not appear. Thus the form of u is never used, and the term $h^2|\log h|$ in the case $r=s=2$ is gone. We still need $\Delta t=o(h)$ at several places in the argument.

We note that by arguing as in Theorem 4.1, we can reduce the regularity on $\partial c/\partial t$ to $\partial c/\partial t\in L^2(H^{r-1})$ and eliminate the mesh restriction $\Delta t=o(h)$, at the cost of assuming H^3-regularity of Ω and a norm reduction $\rho=O(h)$. Details of this type of analysis together with analysis of a Crank–Nicolson version of (4.208) which yields L^2 error estimates of $O(h^r+(\Delta t)^2)$ under weaker initial conditions can be found in [12].

5. Computational considerations. In this section we obtain some estimates for the work involved in our preconditioned conjugate gradient (PCG) method. We will see that the operation counts are optimal or nearly optimal.

We consider the case $d=2$. We note that the analysis will permit the use of distinct mesh parameters h_c and h_p for the two equations. Assume that h_c and h_p are chosen so as to balance the corresponding error terms; in practice, since p is smoother than c, this will usually mean that $h_p>h_c$. Assume also that Δt_c and Δt_p are chosen, as in the theorems, to balance their error terms. Then Theorem 4.1 yields an L^2 error estimate of the form

$$
\text{(5.1)}\quad E_1=O(h_c^r+\Delta t_c)=O(M_c^{-r/2}+N_c^{-1}),
$$

where $M_c=\dim\mathcal{M}_h=O(h_c^{-2})$ and $N_c=T/\Delta t_c=O((\Delta t_c)^{-1})$. We correspondingly define M_p and N_p for the pressure.

We now examine the work W_1 required to achieve this estimate. By (3.13) and (4.3), we know that a fixed number of PCG iterations per pressure time step and $O(\log N_c)$ iterations per concentration step are needed to stabilize the procedure. Assume that the work necessary to factor the preconditioners L_0 and A_0 is $O(M_c^{3/2})$

and $O(M_p^{3/2})$, respectively; the nested dissection process of George [16] can achieve this in the case of a rectangular mesh on a rectangle, and the work of Hoffman, Martin and Rose [17] shows that it cannot be improved. Given the factorizations, the work required for the backsolves in each PCG iteration is $O(M_c \log M_c)$ and $O(M_p \log M_p)$, respectively. Then the total work estimate is

$$(5.2)\qquad \begin{aligned} W_1 &= O(M_c^{3/2} + N_c(M_c \log M_c)(\log N_c) + M_p^{3/2} + N_p(M_p \log M_p)) \\ &= O(M_c^{(r/2)+1} (\log M_c)^2), \end{aligned}$$

where we have assumed that $M_c > M_p$, $N_c > N_p$, and $N_c = O(M_c^{r/2})$. An optimal work estimate would be the number of parameters in the solution, or

$$(5.3)\qquad W = O(M_cN_c + M_pN_p) = O(M_cN_c) = O(M_c^{1+r/2}),$$

so we see that our estimate is nearly optimal.

This is a large improvement over the estimate for the standard backward difference-Galerkin procedure, which has the form

$$(5.4)\qquad \begin{aligned} W_s &= O(N_c(M_c^{3/2} + M_c \log M_c) + N_p(M_p^{3/2} + M_p \log M_p)) \\ &= O(M_c^{(r/2)+(3/2)}), \end{aligned}$$

is dominated by the work of refactorization, and is far from optimal. For example, if $r = 2$ and the errors are balanced, the PCG procedure obtains L^2 error $O(h_c^2)$ with work $O(h_c^{-4} |\log h_c|^2)$, while the standard method requires $O(h_c^{-5})$.

In Theorem 4.3, we can stabilize the procedure in the case $b = b(x, c)$ with a fixed number of iterations for each equation. This replaces $(\log M_c)^2$ by $\log M_c$ in (5.2). In Theorem 4.6, at least asymptotically, the pressure work dominates, having the form

$$(5.5)\qquad W_6 = O(N_p(M_p \log M_p) \log N_p).$$

For practical values of the mesh parameters, this domination may not take place, and it is not clear which of Theorems 4.1 and 4.6 provides the better work estimate for the case $b = b(x, c, \nabla p)$.

In certain cases, we may improve the work estimates still further. If the elements being used are such that the matrices L^0 and A_0 are comparable with their diagonals D^0 and D_0, or with band matrices M^0 and M_0 with bandwidths independent of h, we may use D^0 and D_0, or M^0 and M_0, as preconditioners instead of L^0 and A_0. The work required for a backsolve in a PCG iteration is then $O(M)$ instead of $O(M \log M)$. This will eliminate one power of the logarithm in each of the preceding work estimates. In particular, we obtain optimal estimates in Theorem 4.3.

The procedure outlined in Corollary 4.7 may also be helpful. If we refactor every $(\Delta t_c)^{-1/2} \approx N_c^{1/2}$ time steps in the concentration equation, the factorizations will require work of order $O(M_c^{3/2}N_c^{1/2}) = O(M_c^{3/2+r/4}) = O(M_c^{1+r/2})$, which (5.3) already contains. Then the backsolve will again be $O(M_c \log M_c)$, but a fixed number of PCG iterations will suffice, and one power of the logarithm in (5.2) will be eliminated. Similarly, we can refactor the pressure matrix every $(\Delta t_p)^{-1/2} \approx N_p^{1/2}$ steps in Theorem 4.6, erasing a logarithm in (5.5).

It should be emphasized that the iteration counts supported by our theorems are pessimistic in practice. Rather than iterating a predetermined number of times in a computer program, one can monitor the norm reduction actually produced at each step of the iteration. Thus the process can be stopped when a sufficient norm reduction is achieved. Additional stopping criteria can be built into the monitoring process. A discussion of such criteria for related problems appears in [6].

Finally, we briefly consider the case $d=3$. It is conjectured that optimal factorization and backsolve estimates are $O(M^2)$ and $O(M^{4/3})$, respectively, for a space of dimension M. An optimal estimate for the work in our problem is

$$W=O(M_cN_c+M_pN_p)=O(M_c^{1+r/3}), \tag{5.6}$$

since $N_c=O((\Delta t_c)^{-1})=O(h_c^{-r})=O(M_c^{r/3})$. The work required by our method in Theorem 4.1 would be

$$\begin{aligned} W_1&=O(M_c^2+N_cM_c^{4/3}\log N_c+M_p^2+N_pM_p^{4/3})\\ &=O(M_c^{(4/3)+(r/3)}\log M_c), \end{aligned} \tag{5.7}$$

with similar estimates for the other results. A standard method would need

$$\begin{aligned} W_s&=O(N_c(M_c^2+M_c^{4/3})+N_p(M_p^2+M_p^{4/3}))\\ &=O(M_c^{2+r/3}), \end{aligned} \tag{5.8}$$

a much larger work requirement.

All of these observations can be applied appropriately to the single quasilinear parabolic equation outlined in Theorem 4.9.

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