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EFFICIENT MULTISTEP PROCEDURES FOR NONLINEAR PARABOLIC PROBLEMS WITH NONLINEAR NEUMANN BOUNDARY CONDITIONS

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ABSTRACT

Efficient multistep procedures for time-stepping Galerkin methods for nonlinear parabolic partial differential equations with nonlinear Neumann boundary conditions are presented and analyzed. The procedures involve using a preconditioned iterative method for approximately solving the different linear equations arising at each time step in a discrete time Galerkin method. Optimal order convergence rates are obtained for the iterative methods. Work estimates of almost optimal order are obtained. to p. -B-

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SIGNIFICANCE AND EXPLANATION

Many mathematical models for heat flow or fluid flow involve the specification of a flow rate across the boundary of a region which may depend in a nonlinear fashion upon the unknown variable (e.g. temperature). Formulation and analysis of efficient numerical procedures for approximating the solutions of such problems are studied. Previously, finite element methods used for modeling these physical problems ~~have been~~ were at most second order correct in the time-discretization error. ~~We produce~~ Methods are produced which are second, third, and fourth order correct in time and which convert the nonlinear problems into solution of large systems of linear equations via an extremely stable algorithm with essentially no restrictions between sizes of time and space discretizations.

The basic multistep methods presented produce different systems of linear equations at each time step. A preconditioned iterative stabilization procedure is presented and analyzed which allows for the factorization of only one large matrix to be used at each time level in the solution process. Optimal order error estimates are obtained. The paper also contains work estimates which show the large computational savings of the preconditioned iterative stabilization technique. Almost optimal order work estimates are obtained.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

EFFICIENT MULTISTEP PROCEDURES FOR NONLINEAR PARABOLIC PROBLEMS WITH NONLINEAR NEUMANN BOUNDARY CONDITIONS

Richard E. Ewing[†]

I. Introduction. We shall consider the numerical solution of nonlinear parabolic partial differential equations with nonlinear Neumann boundary conditions of the form

$$
\begin{aligned}
&\text{a)}\quad c(x,u)\frac{\partial u}{\partial t} - \nabla\cdot[a(x,u)\nabla u + b(x,u)] = f(x,t,u), \quad x\in\Omega,\ t\in J\ ,\\
&\text{b)}\quad a(x,u)\frac{\partial u}{\partial \nu} + b(x,u)\cdot\nu = g(x,t,u), \quad x\in\partial\Omega,\ t\in J\ ,\qquad (1.1)\\
&\text{c)}\quad u(x,0) = u_0(x), \quad x\in\Omega\ ,
\end{aligned}
$$

where Ω is a bounded domain in $\mathbb{R}^d$, $d \le 3$, with boundary $\partial\Omega$, ν is the outward unit normal to $\partial\Omega$, $J \equiv (0,T]$, and c, a, b, f, g, and u_0 are prescribed. We shall use a Galerkin approximation in the space variable and high-order, efficient, multistep time-stepping procedures. We first present basic multistep time-stepping procedures which produce a different linear system of equations to be solved at each time step. We then modify the basic procedures by using a preconditioned iterative method to approximate the solution of the linear equations. The use of a time-independent preconditioning matrix eliminates the need to refactor a new matrix at each time step, while the iterative procedure stabilizes the resulting algorithm. Using this modification, we obtain the same order error estimates as for the base scheme with greatly reduced computational requirements. We obtain very nearly optimal possible work estimates for our procedure.

Galerkin procedures for parabolic problems with nonlinear Neumann boundary conditions were first considered by Douglas and Dupont in [8]. Then, in [17], Luskin extended this work of [8] to quasilinear equations similar to those considered here. Luskin used Crank-Nicolson time-stepping methods which are second

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order correct in the time discretization. In [12], the author used the iterative stabilization techniques developed in [9, 10] to present computationally efficient variants of the methods of Luskin and extended these methods to treat coupled systems of nonlinear partial differential equations with nonlinear boundary conditions. In this paper, we present time-stepping procedures which are higher-order in time than those analyzed in [8-13, 17]. These time-stepping schemes are based on the backward differentiation multistep schemes [cf. 15, 14, 19]. They have been presented and analyzed for quasilinear parabolic equations by Bramble and Sammon in [2, 7]. Very efficient alternating direction variants for use on rectangular domains will appear in [4, 5].

The efficient time-stepping techniques presented here can also be used to analyze approximation procedures for initial boundary value problems for many other types of nonlinear partial differential equations. The author has applied iterative stabilization techniques to equations of Sobolev type (in [10]) which have applications in thermodynamics, fluid flow in fissured rock, and shearing of second order fluids. In [11, 12], the methods are applied to coupled systems of equations which model miscible displacement in porous media. Also the author has used iterative methods successfully for second order in time equations (in [13]) which have applications in vibrational problems and nonlinear viscoelasticity.

In Section 2 we introduce certain notational preliminaries and present the base time-stepping Galerkin schemes. In Section 3 we present our iterative modifications of the base methods and analyze the effect of the iterative approximation on a single time step. In Section 4 we obtain global error estimates for a particular multistep method. Section 5 contains a brief discussion of the computational complexity of the methods presented.

II. Preliminaries and Description of Galerkin Methods. Let $(\varphi,\psi) = \int_\Omega \varphi\psi dx$, $\|\psi\|^2 = (\psi,\psi)$, $\langle\varphi,\psi\rangle = \int_{\partial\Omega}\varphi\psi ds$, and $|\psi|^2 = \langle\psi,\psi\rangle$. Let $W_s^k(\Omega)$ be the Sobolev space on Ω with norm

$$\|\psi\|_{W_s^k} = \left(\sum_{|\alpha|\leq k}\left\|\frac{\partial^\alpha\psi}{\partial x^\alpha}\right\|_{L^s(\Omega)}^s\right)^{1/s} \tag{2.1}$$

with the usual modification for $s = \infty$. When $s = 2$, let $\|\psi\|_{W_2^k} = \|\psi\|_{H^k} = \|\psi\|_k$ and $|\psi|_{W_2^k} = |\psi|_{H^k} = |\psi|_k$. If $\nabla F = (F_1,F_2)$, write $\|\nabla F\|_{W_s^k}$ in place of $\left(\|F_1\|_{W_s^k}^s + \|F_2\|_{W_s^k}^s\right)^{1/s}$. For definitions of corresponding fractional order spaces, see [16].

Let $\{M_h\}$ be a family of finite-dimensional subspaces of $H^1(\Omega)$ with the following property:

For $p = 2$ or $p = \infty$, there exist an integer $r \geq 2$ and a constant K_0 such that, for $1 \leq q \leq r$ and $\psi \in W_p^q(\Omega)$,

$$\inf_{\chi\in M_h}\{\|\psi-\chi\|_{W_p^0} + h\|\psi-\chi\|_{W_p^1}\} \le K_0\|\psi\|_{W_p^q} h^q . \tag{2.2}$$

We also assume that $\{M_h\}$ satisfies the following so-called "inverse assumptions":

if $\psi \in M_h$,

$$\begin{aligned} &\text{a)} \quad \|\psi\|_1 \le h^{-1}K_0\|\psi\| , \\ &\text{b)} \quad |\psi| \le h^{-1/2}K_0\|\psi\| , \\ &\text{c)} \quad \|\psi\|_{L^\infty(\Omega)} + h\|\nabla\psi\|_{L^\infty(\Omega)} \le K_0 h^{-\frac{d}{2}}\|\psi\| . \end{aligned} \tag{2.3}$$

Restrict Ω as follows (with (S) denoting the collection of restrictions):

1) Ω is H^2-regular.

(S) : 2) $\partial\Omega$ is Lipschitz.

3) There exists a constant K_0 such that

$$|\varphi|^2 \le K_0\|\varphi\| \, \|\varphi\|_1 . \tag{2.4}$$

If X is a normed space on Ω with norm $\|\cdot\|_X$ and $\varphi : [0,T] \to X$, then we define

$$\begin{aligned} &\text{a)} \quad \|\varphi\|_{L^s(J;X)} = \left[\int_0^T \|\varphi(t)\|_X^s dt\right]^{1/s}, \quad 1 \le s < \infty , \\ &\text{b)} \quad \|\varphi\|_{L^\infty(J;X)} = \sup_{t\in[0,T]} \|\varphi(t)\|_X . \end{aligned} \tag{2.5}$$

Throughout the paper we shall assume that a and c are bounded above and below by positive constants and that a, b, c, and g are smooth functions of their arguments. We shall also assume that the solution u is sufficiently smooth for our arguments to hold. For typical explicit smoothness assumptions on u and the coefficients, see [8-12, 17].

As in [18], we shall introduce an auxiliary elliptic problem to aid in our analysis. Let $\lambda > 0$ be chosen sufficiently large that the bilinear form

$$N(\psi;\varphi,\chi) \equiv (a(\psi)\nabla\varphi,\nabla\chi) + \lambda(\varphi,\chi) - (g(t,\varphi),\chi)$$

satisfies

$$N(\psi;\varphi,\varphi) \ge K_0\|\varphi\|_1^2, \quad \varphi,\psi \in M_h .$$

Let $W \in M_h$ be the projection of u into M_h, defined, for each $t \in J$, by

$$N(u(\cdot,t);\, W(\cdot,t),\chi) = N(u(\cdot,t);\, u(\cdot,t),\chi) \tag{2.6}$$
$$= -\left(c(u)\frac{\partial u}{\partial t},\chi\right) + (b(u),\nabla\chi) + (f(u),\chi) + \lambda(u,\chi), \quad \chi \in M_h \ .$$

Then, as in [8, 9, 12, 18], we can obtain the following lemma.

Lemma 2.1. There exists a constant $K_1 = K_1(u)$ such that if $\eta = u - W$, $s = 0$ or $s = 1$, and $2 \le q \le r$,

$$\text{a)}\quad \|\eta\|_{L^\infty(J;H^s)} \le K_1 h^{q-s}\|u\|_{L^\infty(J;H^q)} \tag{2.7}$$
$$\text{b)}\quad \left\|\frac{\partial\eta}{\partial t}\right\|_{L^2(J;H^s)} \le K_1 h^{q-s}\left\{\|u\|_{L^2(J;H^q)} + \left\|\frac{\partial u}{\partial t}\right\|_{L^2(J;H^q)}\right\} \ .$$

In order to require weak smoothness assumptions on $\frac{\partial u}{\partial t}$, we shall need to use some duality theory and obtain some approximation theory results in negative-indexed norms. For these results, assume that Ω, a, b, c, and g are sufficiently smooth [16] that for each $t \in J$, if

$$\text{a)}\quad -\nabla\cdot[a(x,u)\nabla u] + \lambda_1 u = \psi_1, \qquad x \in \Omega \ , \tag{2.8}$$
$$\text{b)}\qquad a(x,u)\frac{\partial u}{\partial \nu} = \psi_2, \qquad x \in \partial\Omega \ ,$$

then

$$\|u\|_{k+2} \le K(u)\{\|\psi_1\|_k + |\psi_2|_{k+\frac{1}{2}}\} \ . \tag{2.9}$$

If (2.8)-(2.9) holds, we shall say that Ω is H^{k+2}-regular. Next, define for $k \ge 0$,

$$\text{a)}\quad \|\psi\|_{-k} \equiv \sup\{(\psi,\varphi) : \|\varphi\|_k = 1\} \ , \tag{2.10}$$
$$\text{b)}\quad |\psi|_{-k} \equiv \sup\{\langle\psi,\varphi\rangle : |\varphi|_k = 1\} \ .$$

Lemma 2.2. If Ω is H^{k+2}-regular for $k \le 1$, there exists a constant $K(u)$ such that for $1 \le q \le r$ and $t \in J$,

$$\|\eta\|_{-k} + |\eta|_{-(k+\frac{1}{2})} + \left\|\frac{\partial\eta}{\partial t}\right\|_{-k} \le K(u)h^{q+k}\left\{\|u\|_q + \left\|\frac{\partial u}{\partial t}\right\|_q\right\} \ . \tag{2.11}$$

Proof: See [12].

We also make the assumption on $\{M_h\}$ and u that there exists a constant K_2 such that

$$\|w\|_{L^\infty(J;L^\infty)} + \|\nabla w\|_{L^\infty(J;L^\infty)} + \left\|\frac{\partial w}{\partial t}\right\|_{L^\infty(J;L^\infty)} + \left\|\nabla\frac{\partial w}{\partial t}\right\|_{L^2(J;L^\infty)}$$

$$+ \left\|\frac{\partial^2 w}{\partial t^2}\right\|_{L^\infty(J;H^1)} + \left\|\frac{\partial^3 w}{\partial t^3}\right\|_{L^\infty(J;H^1(\partial\Omega))} + \left\|\frac{\partial^4 w}{\partial t^3}\right\|_{L^\infty(J;H^1)} \leq K_2 \;.$$

Sufficient conditions for the above to hold can be found in [9, 10, 18].

We next consider discrete-time Galerkin approximations. Let $\Delta t > 0$, $N = T/\Delta t \in \mathbb{Z}$ and $t^\sigma = \sigma\Delta t$, $\sigma \in \mathbb{R}$. Also let $\psi^n \equiv \psi^n(x) \equiv \psi(x,t^n)$, and

$$\text{a)}\quad d_t\psi^{n+1} = \frac{\psi^{n+1}-\psi^n}{\Delta t}$$

$$\text{b)}\quad \delta\psi^{n+1} = \psi^{n+1} - \psi^n$$

$$\text{c)}\quad \delta^2\psi^{n+1} = \psi^{n+1} - 2\psi^n + \psi^{n-1} \tag{2.12}$$

$$\text{d)}\quad \delta^3\psi^{n+1} = \psi^{n+1} - 3\psi^n + 3\psi^{n-1} - \psi^{n-2}$$

$$\text{e)}\quad \delta^4\psi^{n+1} = \psi^{n+1} - 4\psi^n + 6\psi^{n-1} - 4\psi^{n-2} + \psi^{n-3} \;.$$

We next define a family of extrapolated coefficient backwards differentiation multi-step discrete time methods.

Let $U : \{t_0,\dots,t_N\} \to M_h$ be an approximate solution of (1.1). Assume that U^k are known for $k \leq n$. Then, given certain choices of parameters β, α_1, α_2, α_3, and α_4 and an extrapolation $\hat{U}^{n+1}$, we determine U^{n+1} to satisfy

$$\left(c(\hat{U}^{n+1})\,\frac{U^{n+1}-U^n}{\Delta t}, \chi\right) + \beta\left(a(\hat{U}^{n+1})\nabla U^{n+1}, \nabla\chi\right)$$

$$= \beta\left(g(t^{n+1}, \hat{U}^{n+1}), \chi\right) + \left(c(\hat{U}^{n+1})\,\frac{1}{\Delta t}\,[\alpha_1 U^n + \alpha_2 U^{n-1} + \alpha_3 U^{n-2} + \alpha_4 U^{n-2}], \chi\right) \tag{2.13}$$

$$- \beta\left(b(\hat{U}^{n+1}), \nabla\chi\right) + \beta\left(f(t^{n+1}, \hat{U}^{n+1}), \chi\right), \quad \chi \in M_h \;.$$

A particular example from this family of methods is the choice $\hat{U}^{n+1} = U^n$, $\beta = 1$ and $\alpha_i = 0$, $i = 1,2,3,4$. This choice is the the well-known backward Euler method with lagged coefficients which is known to have time-discretization error of order Δt. Other choices of the parameters and extrapolation in the coefficients yield temporal errors of order $(\Delta t)^2$, $(\Delta t)^3$, and $(\Delta t)^4$.

We present these special choices in the following table.

Table 1: Selected Multistep Methods

Extrapolation $\hat{U}^{n+1}$	β	α_1	α_2	α_3	α_4	Time-discretization Error $(\Delta t)^{\mu}$
$U^{n+1}-\delta U^{n+1}$	1	0	0	0	0	Δt
$U^{n+1}-\delta^2 U^{n+1}$	2/3	1/3	-1/3	0	0	$(\Delta t)^2$
$U^{n+1}-\delta^3 U^{n+1}$	6/11	7/11	-9/11	2/11	0	$(\Delta t)^3$
$U^{n+1}-\delta^4 U^{n+1}$	12/25	23/25	-36/25	16/25	-3/25	$(\Delta t)^4$

We note that by extrapolating the coefficients in (2.13), we have reduced each of the above problems to the solution of a different set of linear equations at each time step.

III. Iterative Stabilization Procedures. In this section we consider efficient methods for solving the linear equations arising from (2.13). We note that the coefficient matrices from (2.13) change with each time step. In order to avoid the factorization of different matrices at each time step to solve the different systems of linear equations, we shall present an iterative method for approximating their solution to sufficient accuracy.

Let $\{\varphi_i\}_{i=1}^M$ be a basis for M_h and let U^m from (2.13) be written as

$$U^m = \sum_{i=1}^{M} \xi_i^m \varphi_i \quad . \tag{3.1}$$

Using (3.1), (2.13) can be written as

$$L^n(\xi)(\xi^{n+1}-\xi^n) \equiv [C^n(\xi)\{\sum_{i=1}^{4} \alpha_i \xi^{n+1-i}\} + \Delta t F_1^n(\xi)] \tag{3.2}$$

$$\equiv F^n(\xi)$$

where the matrices and vectors are of the form

$$
\begin{aligned}
&\text{a)} \quad L^n(\xi) = C^n + \Delta t \beta A^n \quad , \\
&\text{b)} \quad C^n(\xi) = ((c(\widehat{\sum_{k=1}^{M} \xi_k^n \varphi_k})\varphi_j, \varphi_i)) \quad , \\
&\text{c)} \quad A^n(\xi) = ((a(\widehat{\sum_{k=1}^{M} \xi_k^n \varphi_k})\nabla \varphi_j, \nabla \varphi_i)) \quad , \\
&\text{d)} \quad F_1^n(\xi) = \beta((g(t^{n+1}, \widehat{\sum_{k=1}^{M} \xi_k^n \varphi_k}), \varphi_i) - (b(\widehat{\sum_{k=1}^{M} \xi_k^n \varphi_k}), \nabla \varphi_i) \\
&\qquad\qquad + (f(t^{n+1}, \widehat{\sum_{k=1}^{M} \xi_k^n \varphi_k}), \varphi_i)) \quad ,
\end{aligned}
\tag{3.3}
$$

for $i, j = 1, \ldots, M$.

Instead of solving (3.2) exactly, we shall approximate its solution by using an iterative procedure which has been preconditioned by L^0, the associated matrix with coefficients evaluated at $t = 0$, for each time step. The preconditioning process eliminates the need for factoring new matrices at each time step, while the iterative procedure stabilizes the resulting problem. The stabilization process requires iteration only until a predetermined norm reduction is achieved.

Denote by

$$
V^m = \sum_{k=1}^{M} \gamma_k^m \varphi_k \tag{3.4}
$$

the approximation to U^m produced by only approximately solving (3.2). An iterative procedure for obtaining the necessary V^k starting values using the iterative procedure described here will appear in [3]. We assume such a starting procedure has been used to obtain sufficiently accurate (see (4.7)) starting values. Thus assume $V^0, \ldots, V^n$ have been determined. We shall determine the M-dimensional vector γ^{n+1} (and thus V^{n+1}) using a preconditioned iterative method to approximate ξ^{n+1} from (3.2). As an initial guess for $\xi^{n+1} - \xi^n$ we shall extrapolate from previously determined values. Specifically, for a particular method having time-truncation error $(\Delta t)^\mu$, we shall use as the initialization for our iterative procedure

$$
X_0 = (\gamma^{n+1} - \gamma^n) - \delta^{\mu+1}\gamma^{n+1} \quad , \tag{3.5}
$$

where the m^{th} backward difference operator δ^m is defined in (2.12) for $m = 1, \ldots, 4$. Since we are using previously determined γ^i in the coefficient matrices to determine γ^{n+1}, our errors accumulate.

In order to estimate the cumulative error, we first consider the single step error. We define $\bar{\gamma}^{n+1}$ to satisfy

$$L^n(\gamma)(\bar{\gamma}^{n+1} - \gamma^n) = F^n(\gamma), \quad n \geq \mu \quad . \tag{3.6}$$

We can use any preconditioned iterative method which yields norm reductions of the form

$$\|L^n(\gamma)^{1/2}(\bar{\gamma}^{n+1} - \gamma^{n+1})\|_e \leq \rho_n \|L^n(\gamma)^{1/2}(\bar{\gamma}^{n+1} - \gamma^{n+1} + \delta^{\mu+1}\gamma^{n+1})\|_e \tag{3.7}$$

where $0 < \rho_n < 1$ and the subscript e denotes the Euclidean norm of the vector. A specific iterative procedure for obtaining (3.7) is the preconditioned conjugate gradient method analyzed in [1, 9, 10].

Let

$$\begin{aligned}
&\text{a)} \quad \|\varphi\|^2_{c^n} \equiv (c(\hat{V}^{n+1})\varphi,\varphi) \quad , \\
&\text{b)} \quad \|\varphi\|^2_{a^n} = (a(\hat{V}^{n+1})\nabla\varphi,\nabla\varphi) \quad , \\
&\text{c)} \quad |||\varphi|||_n = \|\varphi\|_{c^n} + (\Delta t)^{1/2}\|\varphi\|_{a^n}
\end{aligned} \tag{3.8}$$

be special norms and seminorms. Note that $\|\cdot\|_{c^n}$ and $\|\cdot\|_{a^n}$ are uniformly equivalent to $\|\cdot\|$ and $\|\nabla\cdot\|$, respectively. Then letting

$$\bar{V}^m = \sum_{i=1}^{M} \bar{\gamma}_i^m \varphi_i \quad , \tag{3.9}$$

with $\bar{\gamma}^m$ defined in (3.6), we see that $\bar{V}^{n+1}$ satisfies

$$\begin{aligned}
&(c(\hat{V}^{n+1})\ \frac{\bar{V}^{n+1}-V^n}{\Delta t},\chi) + \beta(a(\hat{V}^{n+1})\nabla\bar{V}^{n+1},\nabla\chi) + \beta(b(\hat{V}^{n+1}),\nabla\chi) \\
&= \beta(g(t^{n+1},\hat{V}^{n+1}),\chi) + \beta(f(t^{n+1},\hat{V}^{n+1}),\chi) + (c(\hat{V}^{n+1})\ \frac{1}{\Delta t}\sum_{i=1}^{4}\alpha_i V^{n+1-i},\chi), \quad \chi \in M_h \quad .
\end{aligned} \tag{3.10}$$

Also using (3.8), our single-step error (3.7) becomes

$$|||\bar{V}^{n+1}-V^{n+1}|||_n \leq \frac{\rho_n}{1-\rho_n}\ |||\delta^{\mu+1}V^{n+1}|||_n, \quad n \geq \mu + 1 \quad . \tag{3.11}$$

We note that as in [6, 12], there is a Q depending upon bounds for the coefficients, such that

a) $\rho_n \leq 2Q^{\kappa}$, with $0 < Q < 1$, and

$$\text{b)} \quad \frac{\rho_n}{1+\rho_n} \equiv \rho'_n \leq n\Delta t, \quad n \geq 1 \quad . \tag{3.12}$$

IV. A Priori Error Estimates. In this section we develop a priori bounds for the errors $V^n - u^n$ for the procedures defined in (3.10) using the base schemes defince in (2.13). The techniques for treating the nonlinearities in the coefficients of a, b, and f are tedious and appear in [7, 9, 12]. Therefore, for simplicity of exposition, we shall consider the simplified problem

$$\text{a)} \quad c(x,t)\frac{\partial u}{\partial t} - \nabla \cdot [a(x,t)\nabla u] = 0 \quad , \quad x \in \Omega,\ t \in J \quad ,$$

$$\text{b)} \quad a(x,t)\frac{\partial u}{\partial \nu} = g(x,t,u) \quad , \quad x \in \partial\Omega,\ t \in J \quad , \tag{4.1}$$

$$\text{c)} \quad u(x,0) = u_0(x) \quad , \quad x \in \Omega \quad .$$

We can thus examine the higher-order efficient time-stepping procedures without the added complexity of nonlinearities, except in the Neumann boundary condition.

Also, for simplicity, we shall present the details for the particular method whose choice of parameters yields time-discretization error of order $(\Delta t)^{\mu}$ where $\mu = 3$. Proofs of stability and convergence for the other methods follow similarly and can be derived from the proofs of similar problems which appear in [7].

For $\mu = 3$, the base approximation scheme for (4.1) from (2.13) can be written as

$$\begin{aligned} &(c_{n+1}\delta U^{n+1}, X) + \frac{6}{11}\Delta t(a_{n+1}\nabla U^{n+1}, \nabla X) \\ &= \frac{6}{11}\Delta t(g(t^{n+1}, \hat{U}^{n+1}), X) + (c_{n+1}[\frac{7}{11}\delta U^n - \frac{2}{11}\delta U^{n-1}], X), \ X \in M_h \quad , \end{aligned} \tag{4.2}$$

where $c_{n+1} \equiv c(x,t^{n+1})$, $a_{n+1} \equiv a(x,t^{n+1})$, and $\hat{U}^{n+1} = 3U^n - 3U^{n-1} + U^{n-2}$. Let $\eta^n = u^n - W^n$ and $\zeta^n = V^n - W^n$. We know from Lemma 2.1 that W is a function in M_h which is sufficiently close to u. We next estimate how close V and W are. From (2.6), (4.1), and (4.2), we obtain the following error equation

$$
\begin{aligned}
&(c_{n+1}\delta\zeta^{n+1},\chi) + \frac{6\Delta t}{11}(a_{n+1}\nabla\zeta^{n+1},\nabla\chi) \\
&\quad = (c_{n+1}[\frac{7}{11}\,\delta\zeta^{n} - \frac{2}{11}\,\delta\zeta^{n-1}],\chi) + [\frac{6}{11}\,\lambda\Delta t(\eta^{n+1},\chi) \\
&\qquad + \Delta t(c_{n+1}[d_t\eta^{n+1} - \frac{7}{11}\,d_t\eta^{n} + \frac{2}{11}\,d_t\eta^{n-1}],\chi)] \\
&\qquad + (c_{n+1}[\frac{6}{11}\,\Delta t\,\frac{\partial u^{n+1}}{\partial t} - \delta u^{n+1} + \frac{7}{11}\,\delta u^{n} - \frac{2}{11}\,\delta u^{n-1}],\chi) \\
&\qquad + \frac{6}{11}\,\Delta t\langle g(t^{n+1},\hat{V}^{n+1}) - g(t^{n+1},W^{n+1}),\chi\rangle \\
&\qquad + [(c_{n+1}(V^{n+1} - \bar{V}^{n+1}),\chi) + \frac{6}{11}\,\Delta t(a_{n+1}\nabla(V^{n+1} - \bar{V}^{n+1}),\nabla\chi)] \\
&\quad \equiv (c_{n+1}[\frac{7}{11}\,\delta\zeta^{n} - \frac{2}{11}\,\delta\zeta^{n-1}],\chi) + T_1^{n+1}(\chi) + T_2^{n+1}(\chi) \\
&\qquad + T_3^{n+1}(\chi) + T_4^{n+1}(\chi), \quad \chi \in M_h \ .
\end{aligned}
\tag{4.3}
$$

Term T_1 enters because we are comparing V to W instead of directly to u. Term T_2 measures how well the multistep scheme approximates $\frac{\partial u}{\partial t}$ and term T_3 arises from the nonlinearity of g. Finally, the single-step error made by using the iterative procedure to approximately solve the linear equations appears in term T_4.

We shall first present a few lemmas which will help separate the various parts of our analysis. First we note that the parameters $\beta(\mu)$ and $\alpha_i(\mu)$, $i = 1,\ldots,4$, are chosen in (2.13) to insure the following consistency result.

<u>Lemma 4.1</u>. For each $\mu = 1,2,3,4$, the choice of parameters $\beta(\mu)$ and $\alpha_i(\mu)$, $i = 1,2,3,4$ given in Table 1 yields

$$
\|\beta(\mu)\Delta t\,\frac{\partial u^{n+1}}{\partial t} - [\delta u^{n+1} - \sum_{i=1}^{4}\alpha_i(\mu)u^{n+1-i}]\| \le K_3(\Delta t)^{\mu+1} \ . \tag{4.4}
$$

We next consider the following lemma which will provide the estimates for the basic stability of our methods.

<u>Lemma 4.2</u>. Assume that Z^n satisfies, for $m \geq 2$,

$$
\begin{aligned}
&\sum_{n=m}^{\ell-1}[(c_{n+1}\delta Z^{n+1},\chi) + \frac{6}{11}\,\Delta t(a_{n+1}\nabla Z^{n+1},\nabla\chi)] \\
&\quad = \sum_{n=m}^{\ell-1}[(c_{n+1}[\frac{7}{11}\,\delta Z^{n} - \frac{2}{11}\,\delta Z^{n-1}],\chi) + (F^{n+1},\chi)], \quad \chi \in M_h \ .
\end{aligned}
\tag{4.5}
$$

Then there exist constants K_4, K_5 and K_6 such that setting $\chi = Z^{n+1}$ yields

$$\|Z^\ell\|^2 + \sum_{n=m}^{\ell-1}[\|\delta Z^{n+1}\|^2 + \|Z^{n+1}\|^2_{a^{n+1}}\Delta t] \tag{4.6}$$
$$\leq K_4[\|Z^m\|^2 + \sum_{n=m-2}^{\ell-1}\|\delta Z^{n+1}\|^2 + \sum_{n=m-2}^{\ell-1}\|Z^{n+1}\|^2\Delta t + |\sum_{n=m}^{\ell-1}(F^{n+1},Z^{n+1})|] \quad ;$$

setting $\chi = \delta Z^{n+1}$ yields

$$\sum_{n=m}^{\ell-1}|||\delta Z^{n+1}|||^2_n + \Delta t\|Z^\ell\|_1^2 \leq K_5[\Delta t\|Z^m\|_1^2 \tag{4.7}$$
$$+ \sum_{n=m-2}^{m-1}|||\delta Z^{n+1}|||_n^2 + \sum_{n=m-2}^{\ell-1}\|Z^{n+1}\|_1^2(\Delta t)^2 + |\sum_{n=m}^{\ell-1}(F^{n+1},\delta Z^{n+1})|] \quad ;$$

also, setting $\chi = (n+1)\delta Z^{n+1}$ yields

$$\sum_{n=m}^{\ell-1}(n+1)|||\delta Z^{n+1}|||^2_n + \ell\Delta t\|Z^\ell\|_1^2 \leq K_6[m\Delta t \ \|Z^m\|_1^2 + \sum_{n=m-2}^{\ell-1}\|\delta Z^n\|^2 \tag{4.8}$$
$$+ \sum_{n=m}^{\ell-1}\frac{6\Delta t(1+\Delta t)}{22}\|Z^{n+1}\|^2_{a^{n+1}} + |\sum_{n=m}^{\ell-1}(n+1)(F^{n+1},\delta Z^{n+1})|] \quad .$$

Proof: See [7].

The following version of the discrete Gronwall lemma is trivial.

Lemma 4.3. Let $f_j \geq 0$, $\beta_j \geq 0$, and $\gamma > 0$. Assume that for $n = 1,\ldots,\ell$,

$$f_n \leq \sum_{j=m}^{n-1}\beta_j f_j\Delta t + \gamma$$

and

$$\sum_{j=m}^{n-1}\beta_j\Delta t \leq M \quad .$$

Then, $f_n \leq \gamma \exp M$, $n = m-1,\ldots,\ell$.

We shall assume that an efficient start-up procedure using the same preconditioned iterative methods as described in Section 3 has been used to determine initial approximations satisfying

$$\sum_{i=0}^{3}|||\zeta^i|||_i^2 + \sum_{i=1}^{3}|||\delta\zeta^i|||^2_{i-1} \leq K[h^{2r} + (\Delta t)^6] \quad . \tag{4.9}$$

For the description of such a start-up procedure and proof of the given estimates, see [3].

We next state the major result of the paper.

Theorem 4.1. Let u and U satisfy (4.1) and (4.2), respectively. Let V be the iterative variant of U satisfying (4.9), (3.10), and (3.11) with ρ_n satisfying (4.21) below. Let $u \in L^2(J;H^r) \cap W^4_\infty(J;W^1_3)$ and either

a) $\int_0^T t\left\|\frac{\partial u}{\partial t}(\cdot,t)\right\|_r^2 dt \leq K$ when Ω is H^3-regular and $h^2 \leq C\Delta t$, or

b) $\frac{\partial u}{\partial t} \in L^2(J;H^r)$.

Then there exist constants $K_8(u)$, depending upon the norms of u, and h_0 and τ_0 such that if $r > d/2$, $\Delta t \leq \min\{\tau_0, h^{d/6}\}$, and $h \leq h_0$,

$$\sup_n \|u^n - V^n\| \leq K_8(u)\,[h^r + (\Delta t)^3] \quad .$$

Proof: Letting $\chi = \zeta^{n+1}$ in (4.3) with $m = 3$ and using (4.6), we obtain

$$\begin{aligned}\|\zeta^\ell\|^2 &+ \sum_{n=3}^{\ell-1}[\|\delta\zeta^{n+1}\|^2 + \Delta t\|\zeta^{n+1}\|^2_{a^{n+1}}] \\ &\leq K_4[\|\zeta^3\|^2 + \sum_{n=1}^{\ell-1}\{\|\delta\zeta^{n+1}\|^2 + \|\zeta^{n+1}\|^2\} + |\sum_{n=3}^{\ell-1}\sum_{i=1}^{4} T_i^{n+1}(\zeta^{n+1})|] \quad .\end{aligned} \tag{4.10}$$

Next, we see that from (2.7) and (2.11),

$$\begin{aligned}\sum_{n=3}^{\ell-1}|T_1^{n+1}(\zeta^{n+1})| &\leq K\sum_{n=3}^{\ell-1}[\|\eta^{n+1}\|\,\|\zeta^{n+1}\| + \sum_{j=0}^{2}\|d_t\eta^{n+1-j}\|_{-1}\|\zeta^{n+1}\|_1]\Delta t \\ &\leq K_9(u)\ h^{2r} + \frac{1}{8}\sum_{n=3}^{\ell-1}\|\zeta^{n+1}\|^2_{a^{n+1}}\Delta t \quad ,\end{aligned} \tag{4.11}$$

where $K_9 = K_9(\|u\|_{L^2(J;H^r)} + \left\|\frac{\partial u}{\partial t}\right\|_{L^2(J;H^{r-1})})$. We note that use of (2.7b) instead of Lemma (2.2), would have required the assumption $\frac{\partial u}{\partial t} \in L^2(J;H^r)$, a much stronger smoothness assumption. From Lemma (4.1) we see that

$$\sum_{n=3}^{\ell-1}|T_2^{n+1}(\zeta^{n+1})| \leq K(u)(\Delta t)^6 + \frac{1}{8}\sum_{n=3}^{\ell-1}\|\zeta^{n+1}\|^2_{a^{n+1}}\Delta t \quad . \tag{4.12}$$

We next use (2.4) and smoothness of W to obtain the bound

$$\sum_{n=3}^{\ell-1} |T_3^{n+1}(\zeta^{n+1})| \le K \sum_{n=3}^{\ell-1} (|\zeta^{n+1}| + K_2(\Delta t)^3 + \sum_{j=0}^{2} |\delta\zeta^{n+1-j}|)|\zeta^{n+1}|\Delta t$$

$$\le K(u)\{(\Delta t)^6 + \sum_{n=1}^{\ell-1} [\|\zeta^{n+1}\|^2 + \|\delta\zeta^{n+1}\|_1^2]\Delta t\} + \frac{1}{8}\sum_{n=3}^{\ell-1} \|\zeta^{n+1}\|^2_{a^{n+1}}\Delta t \quad . \tag{4.13}$$

Using (3.8), (3.11) and (3.12) we see that

$$\sum_{n=3}^{\ell-1} |T_4^{n+1}(\zeta^{n+1})| \le \sum_{n=3}^{\ell-1} |||v^{n+1} - \bar{v}^{n+1}|||_n |||\zeta^{n+1}|||_n$$

$$\le \sum_{n=3}^{\ell-1} \rho'_{n+1} |||\delta^4 v^{n+1}|||_n |||\zeta^{n+1}|||_n$$

$$\le \sum_{n=3}^{\ell-1} K(u)\rho'_{n+1}\{\sum_{i=0}^{3} |||\delta\zeta^{n+1-i}|||_{n-i} + (\Delta t)^4\} |||\zeta^{n+1}|||_n \tag{4.14}$$

$$\le K(u)[(\Delta t)^6 + \sum_{n=3}^{\ell-1}\|\zeta^n\|^2\Delta t] + \frac{1}{8}\sum_{n=3}^{\ell-1}\|\zeta^n\|^2_{a^{n+1}}\Delta t$$

$$+ \sum_{n=0}^{\ell-1} \frac{\rho'_{n+1}}{16\Delta t} |||\delta\zeta^{n+1}|||_n^2 \quad .$$

Noting that the multiplier in the last term on the right side of (4.14) is bounded by $(n+1)/16$ using (3.12), we combine (4.10) - (4.14) and use (4.9) to obtain

$$\|\zeta^\ell\|^2 + \frac{1}{2}\sum_{n=3}^{\ell-1}[\|\delta\zeta^{n+1}\|^2 + \|\zeta^{n+1}\|^2_{a^{n+1}}\Delta t]$$

$$\le K(u)[h^{2r} + (\Delta t)^6 + \sum_{n=3}^{\ell-1}\|\zeta^n\|^2\Delta t] \tag{4.15}$$

$$+ K_4 \sum_{n=3}^{\ell-1} |||\delta\zeta^{n+1}|||_n^2 + \sum_{n=3}^{\ell-1} \frac{(n+1)}{16} |||\delta\zeta^{n+1}|||_n^2 \quad .$$

We note that if we can bound the last two terms on the right of (4.15), we can then use the discrete Gronwall Lemma to obtain our result. In order to bound the next to the last term on the right side of (4.15) we let $\chi = \delta\zeta^{n+1}$ in (4.3) and use (4.7) to obtain

$$\sum_{n=3}^{\ell-1} |||\delta\zeta^{n+1}|||_n^2 + \Delta t\|\zeta^\ell\|_1^2 \le K_5[\Delta t\|\zeta^3\|_1^2 + |||\delta\zeta^2|||_1^2 + |||\delta\zeta^3|||_2^2$$

$$+ \sum_{n=1}^{\ell-1}\|\zeta^{n+1}\|_1^2(\Delta t)^2 + |\sum_{n=3}^{\ell-1}\sum_{i=1}^{4} T_i^{n+1}(\delta\zeta^{n+1})|] \quad . \tag{4.16}$$

As in (4.11) we use (2.7) and (2.11) to obtain

$$\sum_{n=3}^{\ell-1} |T_1^{n+1}(\delta\zeta^{n+1})| \le K(u)h^{2r} + \frac{1}{8}\sum_{n=3}^{\ell-1} |||\delta\zeta^{n+1}|||_n^2 \quad . \tag{4.17}$$

Similarly we see that

$$\sum_{n=3}^{\ell-1} |T_2^{n+1}(\delta\zeta^{n+1})| \le K(u)(\Delta t)^6 + \frac{1}{8}\sum_{n=3}^{\ell-1} |||\delta\zeta^{n+1}|||_n^2 \quad . \tag{4.18}$$

Using (2.4) we then see that

$$\begin{aligned}\sum_{n=3}^{\ell-1} |T_3^{n+1}(\delta\zeta^{n+1})| &\le K \sum_{n=3}^{\ell-1} (|\zeta^{n+1}| + K_2(\Delta t)^3 + \sum_{j=0}^{2} |\delta\zeta^{n+1-j}|)|\delta\zeta^{n+1}|\Delta t \\ &\le \frac{1}{8}\sum_{n=1}^{\ell-1} |||\delta\zeta^{n+1}|||_n^2 + K(u)\{(\Delta t)^6 + \sum_{n=1}^{\ell-1} [\|\zeta^{n+1}\|^2 \Delta t + \|\zeta^{n+1}\|_1^2 (\Delta t)^2]\} \quad .\end{aligned} \tag{4.19}$$

Then, as (4.14), we use (3.8), (3.11), and (3.12) to obtain

$$\begin{aligned}\sum_{n=3}^{\ell-1} |T_4^{n+1}(\delta\zeta^{n+1})| &\le \sum_{n=3}^{\ell-1} \rho_n' |||\delta^4 V^{n+1}|||_n |||\delta\zeta^{n+1}|||_n \\ &\le \sum_{n=3}^{\ell-1} 3K_{10,n}\rho_n'\{\sum_{i=0}^{3} |||\delta\zeta^{n+1-i}|||_{n-i} + (\Delta t)^4\}|||\delta\zeta^{n+1}|||_n \\ &\le K(u)(\Delta t)^3 + 12\sum_{n=0}^{\ell-1} \rho_n' K_{10,n} |||\delta\zeta^{n+1}|||_n^2\end{aligned} \tag{4.20}$$

where $K_{10,n}$ depends upon local upper and lower bounds for the coefficients a_n and c_n (see (4.21)). Then iterating on the preconditioned iterative procedure sufficiently often that

$$\rho_n \le (48K_{10,n})^{-1} \equiv \frac{\min\{a(t^j) : j=n+1,n,n-1,n-2\}}{48\sup\{a(t^j) : j=n+1,n,n-1,n-2\}} \quad , \tag{4.21}$$

combining (4.16) - (4.20), and using (4.9) we see that

$$\begin{aligned}&\sum_{n=3}^{\ell-1} |||\delta\zeta^{n+1}|||_n^2 + \Delta t\|\zeta^\ell\|_1^2 \\ &\le K(u)[h^{2r} + (\Delta t)^6 + \sum_{n=3}^{\ell-1}\{\|\zeta^{n+1}\|^2 \Delta t + \|\zeta^{n+1}\|_1^2 (\Delta t)^2\}] \quad .\end{aligned} \tag{4.22}$$

In order to bound the last term on the right side of (4.15), we let $\chi = (n+1)\delta\zeta^{n+1}$ and use (4.8) to obtain

$$\sum_{n=3}^{\ell-1}(n+1)\left|\left|\left|\delta\zeta^{n+1}\right|\right|\right|_n^2 + \ell\Delta t\|\zeta^\ell\|_1^2 \le K_6[3\Delta t\|\zeta^3\|_1^2 + \sum_{n=1}^{\ell-1}\|\delta\zeta^n\|^2$$

$$+ \sum_{n=3}^{\ell-1}\frac{7\Delta t}{22}\|\zeta^{n+1}\|^2_{a^{n+1}} + \left|\sum_{n=3}^{\ell-1}\sum_{i=1}^{4}T_i^{n+1}((n+1)\delta\zeta^{n+1})\right|] \;. \tag{4.23}$$

We note that (4.9) and (4.22) can be used to bound the first term on the right side of (4.23). We next obtain

$$\left|\sum_{n=3}^{\ell-1}(n+1)T_1^{n+1}(\delta\zeta^{n+1})\right| \le K\sum_{n=3}^{\ell-1}[\|\eta^{n+1}\|\;\|\delta\zeta^{n+1}\| + \sum_{j=0}^{2}\|d_t\eta^{n+1-j}\|_{-1}\|\delta\zeta^{n+1}\|_1](n+1)\Delta t$$

$$\le \frac{1}{16}\sum_{n=3}^{\ell-1}(n+1)\left|\left|\left|\delta\zeta^{n+1}\right|\right|\right|_n^2 + K\sum_{n=3}^{\ell-1}\|\eta^{n+1}\|^2\Delta t + K\sum_{n=1}^{\ell-1}\|d_t\eta^{n+1}\|_{-1}^2(n+1)\Delta t \;. \tag{4.24}$$

If $u \in L^2(J;H^r)$, we have from (2.7) that

$$\sum_{n=1}^{\ell-1}\|\eta^{n+1}\|^2\Delta t \le K(u)h^{2r} \;. \tag{4.25}$$

Then, using (2.11) we have, if $h^2 \le \Delta t$,

$$\sum_{n=1}^{\ell-1}\|d_t\eta^{n+1}\|_{-1}^2(n+1)\Delta t \le K\sum_{n=1}^{\ell-1}[\|u^{n+1}\|_r^2 + \left\|\frac{\partial u}{\partial t}^{n+1}\right\|_r^2]h^{2r+2}(n+1)\Delta t$$

$$\le K\left(\int_0^T t[\|u(\cdot,t)\|_r^2 + \left\|\frac{\partial u}{\partial t}(\cdot,t)\right\|_r^2]dt\right)h^{2r} \;. \tag{4.26}$$

Note that $h^2 \le \Delta t$ is not a strong restriction for these high order time-stepping methods. The constant on the right of (4.26) determines the smoothness assumptions we need on u and $\frac{\partial u}{\partial t}$ for this argument. We note that for linear, time-dependent problems the assumption

$$\int_0^T t\left\|\frac{\partial u}{\partial t}(\cdot,t)\right\|_r^2 dt \le K \tag{4.27}$$

is roughly equivalent to $\frac{\partial u}{\partial t} \in L^2(J;H^{r-1})$, the assumption needed for (4.11), and much weaker than the assumption $\frac{\partial u}{\partial t} \in L^2(J;H^r)$ which has been made in [7, 11, 17] for similar estimates. Using (4.4) we see that

$$\sum_{n=3}^{\ell-1}\left|T_2^{n+1}((n+1)\delta\zeta^{n+1})\right| \le \frac{1}{16}\sum_{n=3}^{\ell-1}(n+1)\left|\left|\left|\delta\zeta^{n+1}\right|\right|\right|_n^2 + K(u)(\Delta t)^6 \;. \tag{4.28}$$

We next consider the T_3 term from (4.23). Note that

$$|\sum_{n=3}^{\ell-1} T_3^{n+1}((n+1)\delta\zeta^{n+1})| \le |\sum_{n=3}^{\ell-1} \langle \frac{\partial g}{\partial u}\, \delta^3 W^{n+1}, \delta\zeta^{n+1}\rangle (n+1)\Delta t|$$

$$+ |\sum_{n=3}^{\ell-1} \langle \frac{\partial g}{\partial u}\, \delta^3 \zeta^{n+1}, \delta\zeta^{n+1}\rangle (n+1)\Delta t| + |\sum_{n=3}^{\ell-1} \langle \frac{\partial g}{\partial u}\, \zeta^{n+1}, \delta\zeta^{n+1}\rangle (n+1)\Delta t| \tag{4.29}$$

$$\equiv T_5 + T_6 + T_7 \quad .$$

T_6 can be bounded, using (2.4), as follows

$$T_6 \le K \sum_{n=3}^{\ell-1} \sum_{j=0}^{2} |\delta\zeta^{n+1-j}|\ |\delta\zeta^{n+1}|(n+1)\Delta t$$

$$\le \frac{1}{16} \sum_{n=1}^{\ell-1} \|\delta\zeta^{n+1}\|^2 (n+1) + K \sum_{n=1}^{\ell-1} \|\delta\zeta^{n+1}\|_1^2 (n+1)(\Delta t)^2 \quad . \tag{4.30}$$

Then using a technical summation by parts argument and estimates like those used in (4.30) we can obtain (see [12, p. 27-29] for details)

$$T_5 + T_7 \le \frac{1}{16} \sum_{n=1}^{\ell-1} \{\|\delta\zeta^{n+1}\|^2 (n+1) + \|\zeta^{n+1}\|^2_{a^{n+1}} \Delta t\} + \frac{1}{22} \|\zeta^\ell\|_1^2 \ell\Delta t$$

$$+ K\{\|\zeta^3\|_1^2 \Delta t + (\Delta t)^6 + \sum_{n=1}^{\ell-1} [\|\delta\zeta^n\|^2_{L^\infty} n + \Delta t]\ [\|\zeta^{n+1}\|^2 + \|\zeta^{n+1}\|_1^2 (n+1)\Delta t] \tag{4.31}$$

$$+ K_{12} \|\zeta^\ell\|^2 \ell\Delta t \quad .$$

As in (4.20), we see that

$$\sum_{n=3}^{\ell-1} |T_4^{n+1}((n+1)\delta\zeta^{n+1})| \le \sum_{n=3}^{\ell-1} K_{10,n}\rho_n'\{\sum_{i=0}^{3} |||\delta\zeta^{n+1-i}|||_{n-i} + (\Delta t)^4\} |||\delta\zeta^{n+1}|||_n (n+1)$$

$$\le K(u)(\Delta t)^6 + 4K_{10,n} \sum_{n=0}^{\ell-1} \rho_n' K_{10,n} |||\delta\zeta^{n+1}|||_n^2 (n+1) \quad . \tag{4.32}$$

Next, by iterating sufficiently often to satisfy (4.21), combining (4.23) - (4.32), and using (4.9), we obtain

$$\frac{1}{2}\left\{\sum_{n=1}^{\ell-1}(n+1)|||\delta\zeta^{n+1}|||_n^2 + \ell\Delta t\|\zeta^\ell\|_1^2\right\} \leq \frac{4}{11}\sum_{n=3}^{\ell-1}\|\zeta^{n+1}\|_{a^{n+1}}^2\Delta t$$

$$+ K(u)[h^{2r} + (\Delta t)^6] + K_{12}\|\zeta^\ell\|^2\ell\Delta t \tag{4.33}$$

$$+ K(u)\sum_{n=1}^{\ell-1}\{\|\zeta^{n+1}\|^2 + \|\zeta^{n+1}\|_1^2(n+1)\Delta t\}\{\|\delta\zeta^n\|_{L^\infty}^2 n + \Delta t\} \quad .$$

Now adding inequalities (4.15) and (4.33) to K_4 times inequality (4.22) and simplifying we obtain

$$\|\zeta^\ell\|^2 + \|\zeta^\ell\|_1^2\ell\Delta t + \sum_{n=3}^{\ell-1}\{|||\delta\zeta^{n+1}|||_n^2(n+1) + \|\zeta^{n+1}\|_{a^{n+1}}^2\Delta t\}$$

$$\leq \bar{K}(u)\{h^{2r} + (\Delta t)^6\} + 4K_{12}\|\zeta^\ell\|^2\ell\Delta t \tag{4.34}$$

$$+ \bar{\bar{K}}(u)\sum_{n=1}^{\ell-1}\{\|\zeta^{n+1}\|^2 + \|\zeta^{n+1}\|_1^2 n\Delta t\}\{\|\delta\zeta^n\|_{L^\infty}^2 n + \Delta t\} \quad .$$

We next indicate how to treat the term multiplied by $4K_{12}$ on the right side of (4.34). Note that for some $\epsilon_1 > 0$,

$$\|\zeta^{n+1}\|^2(n+1)\Delta t - \|\zeta^n\|^2 n\Delta t - \|\zeta^{n+1}\|^2\Delta t$$
$$\leq \epsilon_1(n+1)\|\delta\zeta^{n+1}\|^2 + K\|\zeta^n\|^2\Delta t \quad . \tag{4.35}$$

We sum (4.35) from $n = 3$ to $n = \ell - 1$, multiply the results by $4K_{12} + \frac{1}{2}$, and add the final inequality to (4.34). Then take $\epsilon_1 \leq (8K_{12} + 1)^{-1}$. Next, we make the induction hypothesis that

$$\sum_{n=1}^{\ell-1}\|\delta\zeta^n\|_{L^\infty}^2 n < 1 \quad . \tag{4.36}$$

Then it follows from (4.34) - (4.36) and Lemma 4.3 that

$$\sum_{n=1}^{\ell-1}\|\delta\zeta^n\|^2 n \leq 2\exp\{(1+T)\bar{\bar{K}}(u)\}\bar{K}(u)[h^{2r} + (\Delta t)^6] \quad . \tag{4.37}$$

It then follows from (4.37) and the inverse hypothesis (2.3.c) that

$$\sum_{n=1}^{\ell-1}\|\delta\zeta^n\|_{L^\infty}^2 n \leq K_0 h^{-d}\sum_{n=1}^{\ell-1}\|\delta\zeta^n\|^2 n \leq Kh^{-d}[h^{2r} + (\Delta t)^6] \quad . \tag{4.38}$$

We note that the right hand side of (4.38) tends to zero as h tends to zero if

$$r > \frac{d}{2} \quad \text{and} \quad \Delta t < h^{\frac{d}{6}} \tag{4.39}$$

which justifies the induction hypothesis. Since this implies

$$\|\zeta^\ell\|^2 + \|\zeta^\ell\|_1^2 \ell\Delta t \le K[h^{2r} + (\Delta t)^6] \quad , \tag{4.40}$$

the result follows from (4.40), Lemma 2.1, and the triangle inequality.

We note that similar theorems hold for the original nonlinear problem and for the other various multistep methods presented. Also, if Ω is a rectangle, rectangular solid, or unions of these regions, alternating direction variants of the multistep methods presented here are even more computationally efficient. See [4, 5] for these results.

V. Computational Considerations. In this section we shall consider some rough operation counts to estimate the computational complexity of the methods presented here. We shall see that the preconditioned iterative methods allow us to obtain very nearly optimal order work estimates and are thus very efficient computationally.

We shall give estimates for $d = 2$. The procedures of setting up and factoring L^n requires $O(M^{3/2})$ operations, where $M = \dim M_h$. The solution of (3.2), given the factorization, requires $O(M \log M)$ operations. Such bounds have been shown to be minimal. If we conjecture the validity of the above estimates for our problem and refactor L^n and solve (3.2) at each time step, the total amount of work done is

$$O(N\{M^{3/2} + M \log M\}) = O(NM^{3/2}) \quad , \tag{5.1}$$

where N is the total number of time steps $(N \approx (\Delta t)^{-1})$. Note that the work of factorization dominates the estimate.

Using the preconditioned iterative procedure presented here, only the preconditioner, L^0, must be factored. Let κ_n be the number of iterations needed to achieve the necessary norm reductions in (3.11) and (3.12). We note that κ_n can be bounded by a fixed constant κ which is independent of h, n, and Δt. Using this method the total work done is

$$O(M^{3/2} + N\kappa M \log M) \quad . \tag{5.2}$$

Since balancing the spatial and temporal errors yields

$$N \approx (\Delta t)^{-1} \approx h^{-\frac{r}{\mu}} = O(M^{r/2\mu})$$

we note that for $r \geq \mu$, the work of solving dominates the estimate, while for $r < \mu$ the amount of work of solving is even less than the work to factor one matrix, a necessary piece of work. Clearly, in any case, (5.2) is much preferable to (5.1). Also, since the total number of unknowns in the problem is

$$O(NM) \quad ,$$

(5.2) represents a nearly optimal order work estimate when the work is at least as much as factoring one matrix. If alternating direction variants of these methods can be used, the $\log M$ term can be removed from (5.2) and optimal order work estimates are obtained (see [4, 5]).

It is computationally wasteful to iterate exactly κ times at each time step in order to achieve the pessimistic bounds on ρ_n given in (4.21). Instead, one can monitor the norm reduction actually produced at each time step of the iteration and stop iterating when sufficient norm reduction is achieved. Additional stopping criteria can be imposed in this monitoring process. See [9] for a discussion of stopping criteria for related methods.

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