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The Cauchy Problem for a Linear Parabolic Partial Differential Equation

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Numerical approximation of the solution of the Cauchy problem for the linear parabolic partial differential equation is considered. The problem: $(p(x)u_x)_x - q(x)u = \rho(x)u_t$, $0 < x < 1$, $0 < t \leqslant T$; $u(0, t) = f_1(t)$, $0 < t \leqslant T$; $u(1, t) = f_2(t)$, $0 < t \leqslant T$; $p(0)\, u_x(0, t) = g(t)$, $0 < t_0 \leqslant t \leqslant T$, is ill-posed in the sense of Hadamard. Complex variable and Dirichlet series techniques are used to establish Hölder continuous dependence of the solution upon the data under the additional assumption of a known uniform bound for $|u(x, t)|$ when $0 \leqslant x \leqslant 1$ and $0 \leqslant t \leqslant T$. Numerical results are obtained for the problem where the data f_1, f_2 and g are known only approximately.

1. INTRODUCTION

Consider the numerical approximation of the solution $u = u(x, t)$ of the problem

$$\begin{aligned}
&\text{(a)} \quad \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) - q(x)\,u = \rho(x)\frac{\partial u}{\partial t}, && 0 < x < 1, \quad 0 < t \leqslant T,\\
&\text{(b)} \quad u(0, t) = f_1(t), && 0 < t \leqslant T,\\
&\text{(c)} \quad u(1, t) = f_2(t), && 0 < t \leqslant T,\\
&\text{(d)} \quad p(0)\frac{\partial u}{\partial x}(0, t) = g(t) && 0 < t_0 \leqslant t \leqslant T,
\end{aligned} \tag{1.1}$$

where the data f_1, f_2, and g are known only approximately as f_1^*, f_2^*, and g^* such that

$$\begin{aligned}
&\text{(a)} \quad \|f_1 - f_1^*\|_{[0,T]} < \epsilon_0,\\
&\text{(b)} \quad \|f_2 - f_2^*\|_{[0,T]} < \epsilon_0,\\
&\text{(c)} \quad \|g - g^*\|_{[t_0,T]} < \epsilon_0,
\end{aligned} \tag{1.2}$$

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with $\epsilon_0 > 0$ and where for any function $h = h(t)$

$$\| h \|_{[a,b]} = \sup_{a \leqslant t \leqslant b} | h(t)| .$$

The Cauchy problem (1.1) is not well-posed in the sense of Hadamard [5, 7, 10, 15–17, 23] since the solution does not depend continuously upon the data. We shall show that under some reasonable additional assumptions, when the solution exists, it depends continuously upon the data f_1, f_2, and g. We make the following assumptions:

1. f_1, f_2, g, p, q, and ρ are such that a classical solution u to (1.1) exists.
2. There exists a positive constant $M > 0$ such that

$$\sup_{\substack{0 \leqslant x \leqslant 1 \\ 0 \leqslant t \leqslant T}} | u(x, t)| \leqslant M. \tag{1.3}$$

3. The functions p, p', q and ρ are uniformly Hölder continuous in $0 \leqslant x \leqslant 1$ and satisfy

(a) $0 < p_* \leqslant p(x) \leqslant p^*$,

(b) $0 < \rho_* \leqslant \rho(x) \leqslant \rho^*$,

(c) $0 \leqslant q_* \leqslant q(x) \leqslant q^*$,

(d) $| p'(x)| \leqslant p'^*$.

(We shall denote these constants collectively as $\mathscr{D}$.)

4. f_1 and f_2 are continuously differentiable and a constant $K_1 > 0$ exists such that

$$\| f_1 \|_{[0,T]} + \| f_1' \|_{[0,T]} + \| f_2 \|_{[0,T]} + \| f_2' \|_{[0,T]} + \| g \|_{[t_0,T]} \leqslant K_1 . \tag{1.5}$$

The study of ill-posed problems with approximate data often divides naturally into two tasks: firstly, establishment of *a priori* stability estimates which assure continuous dependence on data with the prescribed bound, and secondly, development of adequate computational methods. This paper addresses both tasks.

Carlo Pucci studied the Cauchy problem for a linear parabolic partial differential equation in [23]. Under the additional assumption of positivity of the solutions, he demonstrated the continuous dependence of the solution upon the bounds for the solution and its first derivatives at a certain portion of the boundary. He obtained no estimate of the degree of the continuous dependence. In [16], Ginsberg considered the Cauchy problem for the heat equation, $U_{xx} = U_t$, and obtained Hölder continuous dependence upon the data. He produced a numerical treatment where $g(t) \equiv 0$. In [5], Cannon presented estimates for Hölder continuity for the heat equation and with (1.1.c) replaced by $u(x, 0) = 0$,

$0 < x < 1$. He then reduced the problem of numerical approximation to that of mathematical programming techniques for solving Volterra integral equations of the first kind. In [7], Cannon and Douglas considered the Cauchy problem for the heat equation with the data specified on a curve $x = s(t)$ and a known initial condition. Hölder continuous dependence was derived and applications to the Inverse Stefan Problem were given. In [10], Cannon and the author presented a direct numerical method for the Cauchy problem for the heat equation in which a Taylor series expansion for the data is numerically approximated.

Related types or problems have also been considered in the control theory literature [19, 21, 24]. Coupling results like those of Seidman, MacCamy and Mizel [19, 21, 24] on well-posedness of boundary controllability with results like those of the author, Showalter, and Miller [13, 20, 25] on well-posedness of the backward heat equation, one could obtain continuous dependence results (without estimates of the degree of continuity) for problems related to (1.1) with (1.1.c) replaced by

$$p(1)\frac{\partial u}{\partial x}(1, t) = g_2(t), \qquad 0 < t_0 \leqslant t \leqslant T. \tag{1.6}$$

A numerical approximation of this different problem utilizing the backward heat equation approximation would be much more difficult than the method presented here ([13, 20]). Slight variations of the methods developed here could also directly treat the problem with 1.1.c) replaced by (1.6).

In Section 2, we use the linearity of the partial differential operator to split the problem (1.1) into two simpler problems. We then state the continuous dependence results which are known for one of these simpler problems and present some preliminary estimates for the rest of the paper. In Section 3, we use complex variable and Dirichlet series techniques to show the continuous dependence upon data properties of the second of the problems defined in Section 2. In Section 4, we obtain an asymptotic estimate for the degree of continuity for the results of Section 3. We obtain Hölder continuous dependence upon the data. Finally, in Section 5, we discuss numerical procedures in our approximations. We again use linearity to split our problem. The first part is then treated by standard finite difference or finite element techniques. Linear programming methods are then used to obtain both *a priori* and *a posteriori* error estimates for the second part.

2. Preliminaries

From the linearity of our parabolic operator we see that the solution u of (1.1) and (1.3) can be written as

$$u = w + z \tag{2.1}$$

where w satisfies

$$\begin{array}{llll}
\text{(a)} & \dfrac{\partial}{\partial x}\left(p\,\dfrac{\partial w}{\partial x}\right) - qw = \rho\,\dfrac{\partial w}{\partial t}, & 0 < x < 1, \quad 0 < t \leqslant T, & \\
\text{(b)} & w(0, t) = f_1(t), & 0 < t \leqslant T, & \\
\text{(c)} & w(1, t) = f_2(t), & 0 < t \leqslant T, & (2.2)\\
\text{(d)} & w(x, 0) = 0, & 0 < x < 1, &
\end{array}$$

and z satisfies

$$\begin{array}{llll}
\text{(a)} & \dfrac{\partial}{\partial x}\left(p\,\dfrac{\partial z}{\partial x}\right) - qz = \rho\,\dfrac{\partial z}{\partial t}, & 0 < x < 1, \quad 0 < t \leqslant T, & \\
\text{(b)} & z(0, t) = 0, & 0 < t \leqslant T, & \\
\text{(c)} & z(1, t) = 0, & 0 < t \leqslant T, & (2.3)\\
\text{(d)} & z(x, 0) = u(x, 0), & 0 < x < 1, & \\
\text{(e)} & p(0)\,\dfrac{\partial z}{\partial x}(0, t) = G(t), & 0 < t_0 < t \leqslant T, &
\end{array}$$

with

$$G(t) = g(t) - p(0)\,\frac{\partial w}{\partial x}(0, t). \tag{2.4}$$

Elemantary potential theoretic representations [15] and maximum principle arguments [15, 18] show that there exists a positive constant $K_2 = K_2(T, \mathscr{D})$ such that for $0 \leqslant x \leqslant 1$ and $0 < t_0 \leqslant t \leqslant T$

$$\begin{array}{ll}
\text{(a)} & |\, w(x, t)| \leqslant K_2\{\| f_1 \|_{[0,T]} + \| f_2 \|_{[0,T]}\}, \\
\text{(b)} & \left| \dfrac{\partial w}{\partial x}(x, t) \right| \leqslant K_2\{\| f_1 \|_{[0,T]} + \| f_1' \|_{[0,T]} + \| f_2 \|_{[0,T]} + \| f_2' \|_{[0,T]}\}.
\end{array} \tag{2.5}$$

Thus, since w and $\partial w/\partial x$ depend continuously upon the data, it suffices to consider the dependence of z upon the data. Also from (2.4) and (2.5.b), we see that if

$$\| G \|_{[t_0,T]} \equiv \eta, \tag{2.6}$$

then

$$\eta \leqslant \| g \|_{[0,T]} + p^* K_2\{\| f_1 \|_{[0,T]} + \| f_1' \|_{[0,T]} + \| f_2 \|_{[0,T]} + \| f_2' \|_{[0,T]}\} \tag{2.7}$$

is a measure of our data f_1, f_2, and g.

We next consider the Sturm–Liouville problem which is associated with our differential operator. Let λ_n and φ_n be the eigenvalues and corresponding normalized eigenfunctions for the Sturm–Liouville problem

$$\begin{aligned} &\text{(a)} \quad (p\varphi_n')' - q\varphi_n + \rho\lambda_n\varphi_n = 0, \qquad 0 < x < 1, \\ &\text{(b)} \qquad \varphi_n(0) = \varphi_n(1) = 0. \end{aligned} \tag{2.8}$$

From our assumption (1.4) and as in [6, 22] we obtain the following estimates on our eigenvalues and eigenfunctions:

$$\begin{aligned} &\text{(a)} \qquad \lambda_{n*} \equiv (\rho^*)^{-1}(p_*\pi^2n^2 + q_*) \leqslant \lambda_n \leqslant (\rho_*)^{-1}(p^*\pi^2n^2 + q^*) \\ &\qquad\qquad \equiv \lambda_n^*, \\ &\text{(b)} \qquad |\varphi_n(x)| \leqslant p_*^{-1/2}\lambda_n^{1/2} \leqslant p_*^{-1/2}\lambda_n^{*1/2} \equiv \mu_{1n}, \qquad 0 \leqslant x \leqslant 1, \\ &\text{(c)} \quad |p(0)\,\varphi_n'(0)| \leqslant (p'^*\lambda_n)^{1/2} + \rho_*^{-1/2}(\lambda_n\rho^* + q^*) \\ &\qquad\qquad \leqslant (p'^*\lambda_n^*)^{1/2} + \rho_*^{-1/2}(\lambda_n^*\rho^* + q^*) \equiv \mu_{2n}, \\ &\text{(d)} \qquad |\varphi_n'(x)| \leqslant p_*^{-1}(p'^*\lambda_n)^{1/2} + 2p_*^{-1}\rho_*^{-1/2}(\lambda_n\rho^* + q^*) \\ &\qquad\qquad \leqslant p_*^{-1}(p'^*\lambda_n^*)^{1/2} + 2p_*^{-1}\rho_*^{-1/2}(\lambda_n^*\rho^* + q^*) \equiv \mu_{3n}, \\ &\qquad\qquad\qquad 0 \leqslant x \leqslant 1. \end{aligned} \tag{2.9}$$

3. Continuous Dependence of z and $\partial z/\partial x$ upon the Data

The assumed smoothness of z from (2.3) allows us to use the eigenvalues and eigenfunctions from (2.8) z into the formal series representation

$$z(x, t) = \sum_{n=1}^{\infty} a_n \exp\{-\lambda_n t\}\,\varphi_n(x) \tag{3.1}$$

where

$$a_n = \int_0^1 \rho(x)\,z(x, 0)\,\varphi_n(x)\,dx, \qquad n = 1, 2,..., \tag{3.2}$$

are the Fourier coefficients of $z(x, 0)$. Since the eigenfunctions are normalized, from (1.3) and Schwarz's inequality, we have

$$|a_n| < \rho^{*1/2}M. \tag{3.3}$$

The bounds in (2.9) and (3.3) show that the series representation for z converges absolutely and uniformly and its partial derivative with respect to x can be obtained by differentiating term by term. We see that this yields

$$F(t) = p(0)\frac{\partial z}{\partial x}(0, t) = \sum_{n=1}^{\infty} a_n \exp\{-\lambda_n t\}\, \varphi_n'(0)\, p(0) \\ = \sum_{n=1}^{\infty} c_n \exp\{-\lambda_n t\} \tag{3.4}$$

where

$$c_n = a_n \varphi_n'(0)\, p(0), \qquad n = 1, 2, \ldots. \tag{3.5}$$

From (2.3.e) and (2.6), we see that

$$|F(t)| \leqslant \eta, \qquad 0 < t_0 \leqslant t \leqslant T. \tag{3.6}$$

Let $\zeta = t + i\tau$. Clearly, $F(\zeta)$ is an analytic function in the complex domain $\operatorname{Re} \zeta \geqslant t_0$. Moreover, there exists a positive constant $K_3 = K_3(\mathscr{D}, M)$ such that for all ζ with $\operatorname{Re} \zeta \geqslant t_0$,

$$|F(\zeta)| \leqslant K_3. \tag{3.7}$$

Using logarithmic convexity arguments as in [5–12], we can show there exists a computable constant $\alpha = \alpha(t_0, T, \tau^*)$, $0 < \alpha < 1$, and a positive constant $K_4 = K_4(\mathscr{D}, M)$ such that for all ζ satisfying

$$\begin{aligned} &\text{(a)} \quad \operatorname{Re} \zeta \in \mathscr{T} \equiv [\tfrac{3}{4}t_0 + \tfrac{1}{4}T, \tfrac{1}{4}t_0 + \tfrac{3}{4}T] \\ &\text{(b)} \quad |\operatorname{Im} \zeta| \leqslant 2\tau^*, \end{aligned} \tag{3.8}$$

we have

$$|F(\zeta)| \leqslant K_4 \eta^{\alpha}. \tag{3.9}$$

A straightforward application of a lemma of Binmore [2, 4] yields, for $t^* \in \mathscr{T}$ from (3.8),

$$|c_n \exp\{-\lambda_n t^*\}| \leqslant H(\lambda_n)\, K_4 \eta^{\alpha} \tag{3.10}$$

where

$$H(\lambda_n) = \left\{\prod_{j=1}^{n-1} \cos\left(\frac{\pi}{2}\frac{\lambda_j}{\lambda_n}\right) \prod_{j=n+1}^{\infty} \cos\left(\frac{\pi}{2}\frac{\lambda_n}{\lambda_j}\right)\right\}^{-1} \tag{3.11}$$

and $\tau^* = \tau^*(\mathscr{D})$ is chosen to exceed every

$$\tau_n = \frac{\pi}{2}\left\{\frac{n}{\lambda_n} + \sum_{j=n+1}^{\infty} \lambda_j^{-1}\right\}, \qquad n = 1, 2,.... \tag{3.12}$$

We note that the above choice of τ^* is possible since (2.9.a) implies that τ_n is a decreasing sequence after at most a few terms.

Applying Gronwall's lemma as in [11, 12], we can obtain positive computable constants $K_6 = K_6(\mathscr{D})$ and $K_7 = K_7(\mathscr{D})$ such that

$$|\varphi_n'(0)| \geqslant \rho^{*1/2} \exp\{-(K_6 n^2 + K_7)\}, \qquad n = 1, 2,.... \tag{3.13}$$

Then using (1.4), (3.4), (3.5), (3.10), (3.11) and (3.13), we have for $n = 1, 2,...,$

$$\begin{aligned} |a_n| &\leqslant [\varphi_n'(0)\, p(0)]^{-1} \exp\{\lambda_n t\}\, H(\lambda_n)\, K_4 \eta^\alpha \\ &\leqslant p_*^{-1} \rho^{*1/2} \exp\{\lambda_n t^* + K_6 n^2 + K_7\}\, H(\lambda_n)\, K_4 \eta^\alpha. \end{aligned}$$

Finally, we use (3.14) to bound the a_n, $n = 1, 2,..., N$, and (3.3) to bound a_n for $n > N$. Then, using (2.9), we obtain, for $0 \leqslant x \leqslant 1$ and $t_0 \leqslant t \leqslant T$,

$$\begin{aligned} &\text{(a)} \quad |z(x, t)| \leqslant \eta^\alpha K_4 p_*^{-1} \rho^{*1/2} \sum_{n=1}^{N} \mu_{1n} H(\lambda_n) \exp\{\lambda_n(t^* - t) + K_6 n^2 + K_7\} \\ &\qquad\qquad + \rho^{*1/2} M \sum_{n=N+1}^{\infty} \mu_{1n} \exp\{-\lambda_{n*} t_0\}, \\ &\text{(b)} \quad \left|\frac{\partial z}{\partial x}(x, t)\right| \leqslant \eta^\alpha K_4 p_*^{-1} \rho^{*1/2} \sum_{n=1}^{N} \mu_{3n} H(\lambda_n) \exp\{\lambda_n(t^* - t) + K_6 n^2 + K_7\} \\ &\qquad\qquad + \rho^{*1/2} M \sum_{n=N+1}^{\infty} \mu_{3n} \exp\{-\lambda_{n*} t_0\}. \end{aligned} \tag{3.15}$$

We see that (3.15) is in the form

$$\begin{aligned} &\text{(a)} \quad |z(x, t)| \leqslant A_{1N} \eta^\alpha + B_{1N}, \\ &\text{(b)} \quad \left|\frac{\partial z}{\partial x}(x, t)\right| \leqslant A_{2N} \eta^\alpha + B_{2N}, \end{aligned} \tag{3.16}$$

where $\lim_{N\to\infty} A_{iN} = \infty$ and $\lim_{N\to\infty} B_{iN} = 0$, $i = 1, 2$. We thus have shown the following theorem.

THEOREM 1. *If assumptions 1–4 hold, then for each $N > 0$, there exist constants A_{iN} and B_{iN}, $i = 1, 2$, such that $\lim_{N\to\infty} A_{iN} = \infty$ and $\lim_{N\to\infty} B_{iN} = 0$ and there exists a constant α, $0 < \alpha < 1$, such that for $0 \leqslant x \leqslant 1$ and $t_0 \leqslant t \leqslant T$,*

$$\begin{aligned} &\text{(a)} \quad | z(x, t)| \leqslant A_{1N}\eta^{\alpha} + B_{1N}, \\ &\text{(b)} \quad \left| \frac{\partial z}{\partial x}(x, t) \right| \leqslant A_{2N}\eta^{\alpha} + B_{2N}. \end{aligned} \tag{3.17}$$

We see that for $0 \leqslant x \leqslant 1$ and $t_0 \leqslant t \leqslant T$, our theorem implies that

$$\lim_{\eta\to 0} | z(x, t)| = \lim_{\eta\to 0} \left| \frac{\partial z}{\partial x}(x, t) \right| = 0, \tag{3.18}$$

and z and $\partial z/\partial x$, and thus u and $\partial u/\partial x$, from (2.1) and (2.5), depend continuously upon the data. In the next section, we shall determine the rate of convergence in (3.18).

4. AN ASYMPTOTIC ESTIMATE AS $\eta \to 0$

An ill-posed problem in the sense of Hadamard is computationally feasible only if the dependence of the solution upon the data is either Hölder ($O(\eta^{\beta})$, $0 < \beta \leqslant 1$) or logarithmic, ($O(\log 1/\eta)^{-\beta}$, $0 < \beta \leqslant 1$). In this section, we shall obtain estimates that show that for our problem, the solution depends Hölder continuously upon the data.

Since (2.9) shows that $A_{1N} \leqslant A_{2N}$ and $B_{1N} \leqslant B_{2N}$ in (3.15) and (3.16), it suffices to consider A_{2N} and B_{2N}. We first consider A_{2N}. From (3.15) we see that we must estimate $H(\lambda_n)$, given in (3.11). We note that if $\lambda_k > 2\lambda_n$, we have

$$0 \leqslant \frac{\pi}{2} \frac{\lambda_n}{\lambda_k} \leqslant \frac{\pi}{4}. \tag{4.1}$$

From elementary calculations one can show that there exist positive constants $K_8 = K_8(\mathscr{D})$ and $K_9 = K_9(\mathscr{D})$ so that for

$$k \geqslant N_n = (K_8 n^2 + K_9)^{1/2}. \tag{4.2}$$

then (4.1) holds. Next, we estimate

$$\prod_{k=N_n}^{\infty} \cos\left(\frac{\pi}{2} \frac{\lambda_n}{\lambda_k}\right) = \exp\left\{ \sum_{k=N_n}^{\infty} \log \cos \frac{\pi}{2} \frac{\lambda_n}{\lambda_k} \right\} \tag{4.3}$$

by noting that

$$\log \cos \frac{\pi}{2} \frac{\lambda_n}{\lambda_k} = - \int_0^{\pi/2(\lambda_n/\lambda_k)} \tan \xi \, d\xi \geqslant - \frac{\pi}{2} \left(\frac{\lambda_n}{\lambda_k}\right)^2 \tag{4.4}$$

and using (2.9.a) to obtain

$$\prod_{k=N_n}^{\infty} \cos \frac{\pi}{2} \frac{\lambda_n}{\lambda_k} \geqslant \exp\{-K_{10}n^2 - K_{11}\} \tag{4.5}$$

where $K_{10} = K_{10}(\mathscr{D})$ and $K_{11} = K_{11}(\mathscr{D})$. At this point we must make an additional assumption upon the eigenvalues of our Sturm–Liouville problem. Let

$$d \equiv \inf_n (\lambda_{n+1} - \lambda_n) \tag{4.6}$$

and assume that $d > 0$. Then, noting that

$$\prod_{k=1}^{n-1} \cos\left(\frac{\pi}{2} \frac{\lambda_k}{\lambda_n}\right) \prod_{k=n+1}^{N_n-1} \cos\left(\frac{\pi}{2} \frac{\lambda_n}{\lambda_k}\right) \geqslant \left\{\min\left(\cos \frac{\pi}{2} \frac{\lambda_n - d}{\lambda_n}, \cos \frac{\pi}{2} \frac{\lambda_n}{\lambda_n + d}\right)\right\}^{N_n}, \tag{4.7}$$

we can finish the estimate for $H(\lambda_n)$. We thus obtain $K_{12} = K_{12}(\mathscr{D})$ and $K_{13} = K_{13}(\mathscr{D})$ such that

$$H(\lambda_n) \leqslant d^{-N_n}(K_{12}n^2 + K_{13})^{N_n} \exp\{K_{10}n^2 + K_{11}\}. \tag{4.8}$$

Then elementary estimates yield the existence of $K_{14} = K_{14}(\mathscr{D}, d)$ and $K_{15} = K_{15}(\mathscr{D}, d)$ such that

$$.H(\lambda_n) \leqslant \exp\{K_{14}n^2 + K_{15}\}. \tag{4.9}$$

Finally, from (3.15) and (3.16) we see that

$$A_{1N} \leqslant A_{2N} \leqslant \exp\{K_{16}N^2 + K_{17}\} \tag{4.10}$$

where $K_{16} = K_{16}(\mathscr{D}, t_0, T, M, d)$ and $K_{17} = K_{17}(\mathscr{D}, t_0, T, M, d)$.

We next see from (2.9), (3.15) and (3.16), that

$$B_{1N} \leqslant B_{2N} \leqslant \exp\{-K_{18}N^2 + K_{19}\} \tag{4.11}$$

where $K_{18} = K_{18}(\mathscr{D}, t_0, M)$ and $K_{19} = K_{19}(\mathscr{D}, t_0, M)$. Combining (3.16), (4.10) and (4.11), we have

$$\begin{aligned} &\text{(a)} \quad | z(x, t)| \leqslant \exp\{K_{16}N^2 + K_{17}\} \eta^\alpha + \exp\{-K_{18}N^2 + K_{19}\}, \\ &\text{(b)} \quad \left| \frac{\partial z}{\partial x}(x, t) \right| \leqslant \exp\{K_{16}N^2 + K_{17}\} \eta^\alpha + \exp\{-K_{18}N^2 + K_{19}\}. \end{aligned} \tag{4.12}$$

Setting

$$\exp\{K_{16}N^2 + K_{17}\}\,\eta^\alpha = \exp\{-K_{18}N^2 + K_{19}\} \tag{4.13}$$

to balance the effect of the two terms in the bound in (4.12), we see that by picking N such that

$$N^2 = \left[\!\left[\frac{-\alpha}{K_{16} + K_{18}} \log \eta + \frac{K_{19} - K_{17}}{K_{16} + K_{18}}\right]\!\right], \tag{4.14}$$

we obtain, for $0 \leqslant x \leqslant 1$, $t_0 \leqslant t \leqslant T$,

$$\text{(a)} \qquad |\, z(x, t)| \leqslant 2 \exp\left\{\frac{K_{19}}{K_{16} + K_{18}} + \max[K_{17}, K_{19}]\right\} \eta^{\alpha K_{18}/(K_{16}+K_{18})}$$

$$\equiv K_{20}\eta^\nu,$$

$$\text{(b)} \qquad \left|\frac{\partial z}{\partial x}(x, t)\right| \leqslant K_{20}\eta^\nu,$$

where $0 < \nu < 1$. We can thus establish the following result by combining (2.5), (2.7) and (4.15).

THEOREM 2. *If our assumptions* 1–4 *hold and if there exists a minimum positive separation between the eigenvalues of the Sturm–Liouville problem* (2.8), *then the norm of the solution of* (1.1) *defined by*

$$\sup_{\substack{0\leqslant x\leqslant 1\\ t_0\leqslant t\leqslant T}} |\, u(x, t)| + \sup_{\substack{0\leqslant x\leqslant 1\\ t_0\leqslant t\leqslant T}} \left|\frac{\partial u}{\partial x}(x, t)\right|, \tag{4.16}$$

depends Hölder continuously upon the uniform norm of the data given by (2.6) *and* (2.7).

5. Numerical Procedures

In this section we consider the problem of numerically approximating (1.1) subject to the restriction (1.2). The restriction (1.2) comes from the fact that data measurement is, in general, accurate only to within some measurement tolerance ϵ_0. From the linearity of the operator, we note that the solution w of (2.2) can be written as

$$w = w_1 + w_2\,. \tag{5.1}$$

Here w_1 satisfies

$$\begin{aligned}
&\text{(a)} \quad Lw_1 \equiv \frac{\partial}{\partial x}\left(p(x)\frac{\partial w_1}{\partial x}\right) - q(x)\, w_1 - \rho(x)\frac{\partial w_1}{\partial t} = 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad 0 < x < 1, \quad 0 < t \leqslant T, \\
&\text{(b)} \quad w_1(0, t) = f_1 - f_1^*, \qquad\qquad 0 < t \leqslant T, \\
&\text{(c)} \quad w_1(1, t) = f_2 - f_2^*, \qquad\qquad 0 < t \leqslant T, \\
&\text{(d)} \quad w_1(x, 0) = 0, \qquad\qquad 0 < x < 1,
\end{aligned} \tag{5.2}$$

and w_2 satisfies

$$\begin{aligned}
&\text{(a)} \quad Lw_2 = 0, \qquad 0 < x < 1, \quad 0 < t \leqslant T, \\
&\text{(b)} \quad w_2(0, t) = f_1^*, \qquad 0 < t \leqslant T, \\
&\text{(c)} \quad w_2(1, t) = f_2^*, \qquad 0 < t \leqslant T, \\
&\text{(d)} \quad w_2(x, 0) = 0, \qquad 0 < x < 1.
\end{aligned} \tag{5.3}$$

At this point we make the somewhat restrictive assumption upon the data measurements f_1^* and f_2^*, that

$$\begin{aligned}
&\text{(a)} \quad \left\| \frac{d}{dt}(f_1 - f_1^*) \right\|_{[0,T]} \leqslant \epsilon_0 , \\
&\text{(b)} \quad \left\| \frac{d}{dt}(f_2 - f_2^*) \right\|_{[0,T]} \leqslant \epsilon_0 .
\end{aligned} \tag{5.4}$$

From (5.2), (1.2) and (5.4), we use the same arguments as were employed to derive (2.5) to obtain the existence of a constant K_{21} such that for $0 \leqslant x \leqslant 1$ and $0 < t_0 \leqslant t \leqslant T$,

$$\begin{aligned}
&\text{(a)} \quad | w_1(x, t)| \leqslant K_{21}\epsilon_0 , \\
&\text{(b)} \quad \left| \frac{\partial w_1}{\partial x}(x, t) \right| \leqslant K_{21}\epsilon_0 .
\end{aligned} \tag{5.5}$$

Standard finite difference or finite element techniques [1, 3, 14, 26] will yield a numerical approximation $w^*(x, t)$ which is sufficiently accurate that we have for $0 \leqslant x \leqslant 1$ and $0 < t_0 \leqslant t \leqslant T$ and $\epsilon_1 > 0$

$$\begin{aligned}
&\text{(a)} \quad | w_2(x, t) - w^*(x, t)| \leqslant \epsilon_1 , \\
&\text{(b)} \quad \left| \frac{\partial w_2}{\partial x}(x, t) - \frac{\partial w^*}{\partial x}(x, t) \right| \leqslant \epsilon_1 .
\end{aligned} \tag{5.6}$$

Next we consider the numerical approximation of z from (2.3) under the assumption of approximate data. Let z_1 satisfy

$$\begin{aligned}
&\text{(a)} \qquad Lz_1 = 0, && 0 < x < 1, \quad 0 < t < T, \\
&\text{(b)} \qquad z_1(0, t) = z_1(1, t) = 0, && 0 < t \leqslant T, \\
&\text{(c)} \quad p(0) \frac{\partial z_1}{\partial x}(0, t) = g^*(t) - p(0) \frac{\partial w^*}{\partial x}(0, t), && 0 < t_0 \leqslant t \leqslant T.
\end{aligned} \tag{5.7}$$

Recall that from (1.3) and (2.3.d), for $0 < x < 1$,

$$| z(x, 0)| = | u(x, 0)| \leqslant M. \tag{5.8}$$

Also note that from (2.3.e), (5.5), (5.6), and (5.7), for $0 < t_0 \leqslant t \leqslant T$,

$$\begin{aligned}
&\left| p(0) \frac{\partial z}{\partial x}(0, t) - p(0) \frac{\partial z_1}{\partial x}(0, t) \right| \\
&\quad = \left| g(t) - p(0) \frac{\partial w}{\partial x}(0, t) - \left\{ g^*(t) - p(0) \frac{\partial w^*}{\partial x}(0, t) \right\} \right| \\
&\quad \leqslant | g(t) - g^*(t)| + p(0) \left| \frac{\partial w}{\partial x}(0, t) - \frac{\partial w^*}{\partial x}(0, t) \right| \\
&\quad \leqslant \epsilon_0 + p(0) \left[K_{21}\epsilon_0 + \epsilon_1 \right] \equiv K_{22}\epsilon_2 .
\end{aligned} \tag{5.9}$$

Then, from (5.9) and Theorem 2, we have the existence of computable constants $\alpha = \alpha(t_0, T, \tau^*, M)$ and $K_{23} = K_{23}(t_0, T, \tau^*, M, \mathscr{D}, K_{22})$ such that, for $0 \leqslant x \leqslant 1$ and $t_0 \leqslant t \leqslant T$,

$$\begin{aligned}
&\text{(a)} \qquad | z(x, t) - z_1(x, t)| \leqslant K_{23}\epsilon_2^{\alpha}, \\
&\text{(b)} \quad \left| \frac{\partial z}{\partial x}(x, t) - \frac{\partial z_1}{\partial x}(x, t) \right| \leqslant K_{23}\epsilon_2^{\alpha}.
\end{aligned} \tag{5.10}$$

We now pick $N > 0$ such that the tails of the series for $z(x, t)$, $(\partial z/\partial x)(x, t)$, and $p(0)(\partial z/\partial x)(0, t)$ are small simultaneously. If we let

$$l_n = \max\{\mu_{1n}, \mu_{2n}, \mu_{3n}\}, \qquad n = 1, 2, \ldots, \tag{5.11}$$

from (2.9) and choose $N > 0$ such that for some $\epsilon_3 > 0$

$$\sum_{n=N+1}^{\infty} \rho^{*1/2} M_1 \exp\{-\lambda_{n*} t_0\}\, l_n \leqslant \epsilon_3 , \tag{5.12}$$

then we have for $t_0 \leqslant t \leqslant T$,

$$\text{(a)} \qquad \left| \sum_{n=N+1}^{\infty} a_n \exp\{-\lambda_n t\} \varphi_n(x) \right| \leqslant \epsilon_3, \qquad 0 \leqslant x \leqslant 1,$$

$$\text{(b)} \qquad \left| \sum_{n=N+1}^{\infty} a_n \exp\{-\lambda_n t\} \varphi_n'(x) \right| \leqslant \epsilon_3, \qquad 0 \leqslant x \leqslant 1, \qquad (5.13)$$

$$\text{(c)} \quad \left| \sum_{n=N+1}^{\infty} a_n \exp\{-\lambda_n t\} p(0) \varphi_n'(0) \right| \leqslant \epsilon_3 .$$

We note that integral estimates for the sum in (5.12) could be used in practice to aid in the choice of N. Also, in practice, ϵ_3 would probably be chosen as

$$\text{(a)} \quad \epsilon_3 = \epsilon_0 \qquad \text{from (1.2), or}$$

$$\text{(b)} \quad \epsilon_3 = K_{22}\epsilon_2 \qquad \text{from (5.9), or} \qquad (5.14)$$

$$\text{(c)} \quad \epsilon_3 = K_{23}{\epsilon_2}^{\alpha} \qquad \text{from (5.10).}$$

We next use available methods [1, 14, 26] to numerically approximate the eigenvalues and eigenfunctions of our operator. For $n = 1, 2,..., N$, $\epsilon_4 > 0$, and $\epsilon_5 > 0$, determine $Q_{1n}(x, t)$, $Q_{2n}(x, t)$, and $Q_{3n}(t)$ such that for $0 \leqslant x \leqslant 1$ and $t_0 \leqslant t \leqslant T$,

$$\text{(a)} \qquad |Q_{1n}(x, t) - \exp\{-\lambda_n t\} \varphi_n(x)| \leqslant \epsilon_4[NM\rho^{*1/2}]^{-1},$$

$$\text{(b)} \qquad |Q_{2n}(x, t) - \exp\{-\lambda_n t\} \varphi_n'(x)| \leqslant \epsilon_4[NM\rho^{*1/2}]^{-1}, \qquad (5.15)$$

$$\text{(c)} \quad |Q_{3n}(t) - \exp\{-\lambda_n t\} p(0) \varphi_n'(0)| \leqslant \epsilon_5[NM\rho^{*1/2}]^{-1}.$$

(Presumably ϵ_5 could be much smaller than ϵ_4 with comparable work.)

We shall use a linear programming problem to determine A_n^*, $n = 1, 2,..., N$, which approximate the a_n, $n = 1, 2,..., N$ from (3.1) and (3.2) well. We shall then define our numerical approximations to z and $\partial z/\partial x$ from (2.3) by

$$\text{(a)} \qquad z_N^*(x, t) = \sum_{n=1}^{N} A_n^* Q_{1n}(x, t),$$

$$(5.16)$$

$$\text{(b)} \quad \frac{\partial z^*}{\partial z_N}(x, t) = \sum_{n=1}^{N} A_N^* Q_{2n}(x, t),$$

for $0 \leqslant x \leqslant 1$ and $t_0 \leqslant t \leqslant T$. Combining (5.10), (5.13), (5.15), and (5.16), we obtain

$$\begin{aligned}
| z(x,t) - z_N^*(x,t)| &\leqslant | z(x,t) - z_1(x,t)| + | z_1(x,t) - z_N^*(x,t)| \\
&\leqslant | z(x,t) - z_1(x,t)| + \left| \sum_{n=N+1}^{\infty} a_n \exp\{-\lambda_n t\}\, \varphi_n(x) \right| \\
&\quad + \left| \sum_{n=1}^{N} a_n[\exp\{-\lambda_n t\}\, \varphi_n(x) - Q_{1n}(x,t)] \right| \\
&\quad + \left| \sum_{n=1}^{N} [a_n - A_n^*]\, Q_{1n}(x,t) \right| \\
&\leqslant K_{23}\epsilon_2^{\alpha} + \epsilon_3 + \epsilon_4 + \left| \sum_{n=1}^{N} [a_n - A_n^*]\, Q_{1n}(x,t) \right|. \qquad (5.17)
\end{aligned}$$

Similarly, we obtain

$$\left| \frac{\partial z}{\partial x}(x,t) - \frac{\partial z^*}{\partial x_N}(x,t) \right| \leqslant K_{23}\epsilon_2^{\alpha} + \epsilon_3 + \epsilon_4 + \left| \sum_{n=1}^{N} [a_n - A_n^*]\, Q_{2n}(x,t) \right|. \qquad (5.18)$$

Therefore, in order to get good numerical approximation of z and $\partial z/\partial x$, and thus u and $\partial u/\partial x$, we must find a set of A_n^*, $n = 1, 2, \ldots, N$, which approximate the corresponding a_n well.

Since we cannot measure z or $\partial z/\partial x$ directly (even approximately), we cannot use the last terms in (5.17) or (5.18) in a linear programming problem to estimate the size of the terms directly. We can measure $p(0)\,(\partial z/\partial x)\,(0,t)$ (approximately), so we will define our linear programming problem in terms of this measurement. Note that, from (5.15.c),

$$\sum_{n=1}^{N} A_n^* Q_{3n}(t) \qquad (5.19)$$

is our numerical approximation for this term while

$$g^*(t) - p(0)\,\frac{\partial w^*}{\partial x}(0,t) \qquad (5.20)$$

is the approximation we measure (or compute from measurements in the case of $(\partial w^*/\partial x)\,(0,t)$). Therefore (5.19) and (5.20) should be close to each other.

Suppose that there exists a set of A_n^*, $|A_n^*| \leqslant \rho^{*1/2}M$, $n = 1, 2, \dots, N$, such that for $t^* \in \mathscr{T}$ from (3.8)

$$\left| g^*(t^*) - p(0) \frac{\partial w^*}{\partial x}(0, t^*) - \sum_{n=1}^{N} A_n^* Q_{3n}(t^*) \right| \leqslant \sigma_1 . \tag{5.21}$$

Then from (3.4), (5.7.c) we obtain the following estimate.

$$\begin{aligned}
&\left| \sum_{n=1}^{\infty} a_n \exp\{-\lambda_n t^*\} \varphi_n'(0) p(0) - \sum_{n=1}^{N} A_n^* \exp\{-\lambda_n t^*\} \varphi_n'(0) p(0) \right| \\
&\quad \leqslant \left| p(0) \frac{\partial z}{\partial x}(0, t^*) - p(0) \frac{\partial z_1}{\partial x}(0, t^*) \right| \\
&\quad\quad + \left| g^*(t^*) - p(0) \frac{\partial w^*}{\partial x}(0, t^*) - \sum_{n=1}^{N} A_n^* Q_{3n}(t^*) \right| \\
&\quad\quad + \left| \sum_{n=1}^{N} A_n^* Q_{3n}(t^*) - \sum_{n=1}^{N} A_n^* \exp\{-\lambda_n t^*\} \varphi_n'(0) p(0) \right| \\
&\quad \leqslant p^* K_{23} \epsilon_2^{\alpha} + \epsilon_5 + \sigma_1 .
\end{aligned} \tag{5.22}$$

Then, if we define $A_n^* = 0$, $n \geqslant N + 1$, we can use (4.9) and the techniques of Section 3 to obtain computable constants ω, K_{24}, and K_{25} (now depending upon $2M$ where the corresponding constants in the earlier sections depended upon M) such that $0 < \omega < 1$ and for $n = 1, 2, \dots, N$

$$| a_n - A_n^* | \leqslant \exp\{K_{24} n^2 + K_{25}\} (p^* K_{23} \epsilon_2^{\alpha} + \epsilon_5 + \sigma_1)^{\omega}. \tag{5.23}$$

Next we note from (2.9) and (5.15) that for $n = 1, 2, \dots, N$,

$$\begin{aligned}
&\text{(a)} \quad |Q_{1n}(x, t)| \leqslant \mu_{1n} + \epsilon_4 [N M \rho^{*1/2}]^{-1} \equiv \theta_1 , \\
&\text{(b)} \quad |Q_{2n}(x, t)| \leqslant p^* \mu_{3n} + \epsilon_4 [N M \rho^{*1/2}]^{-1} \equiv \theta_2 .
\end{aligned} \tag{5.24}$$

Finally combining (5.17), (5.18), (5.23), and (5.24) we obtain the following *a posteriori* estimates,

LEMMA 1. *If* A_n^*, $|A_n^*| \leqslant (\rho^*)^{1/2} M$, $n = 1, 2, \dots, N$, *satisfy* (5.21), *then for* $0 \leqslant x \leqslant 1$ *and* $t_0 \leqslant t \leqslant T$,

$$\begin{aligned}
&\text{(a)} \quad | z(x, t) - z_N^*(x, t)| \leqslant K_{23} \epsilon_2^{\alpha} + \epsilon_3 + \epsilon_4 + \theta_1 S, \\
&\text{(b)} \quad \left| \frac{\partial z}{\partial x}(x, t) - \frac{\partial z^*}{\partial x_N}(x, t) \right| \leqslant K_{23} \epsilon_2^{\alpha} + \epsilon_3 + \epsilon_4 + \theta_2 S,
\end{aligned} \tag{5.25}$$

where we define

$$S \equiv N \exp\{K_{24}N^2 + K_{25}\} (p^* K_{23}\epsilon_2{}^\alpha + \epsilon_5 + \sigma_1)^\omega. \tag{5.26}$$

By improving the data measurements and by numerically approximating w_2, $\partial w_2/\partial x$, λ_n, φ_n, and φ_n more accurately, it follows theoretically that ϵ_i, $i = 1, 2,..., 5$ can be made arbitrarily small. We shall now show that there exists a set of A_n^*, $n = 1,..., N$, such that σ_1 from (5.21) is bounded in terms of $\epsilon_1,..., \epsilon_5$.

Consider

$$G(A) = \left| g^*(t^*) - p(0) \frac{\partial w^*}{\partial x} (0, t^*) - \sum_{n=1}^{N} A_n Q_{3n}(t^*) \right|, \tag{5.27}$$

where $A = (A_1,..., A_n)$ and $|A_n| \leqslant (\rho^*)^{1/2} M$, $n = 1, 2,..., N$. Since $G(A)$ is a continuous function defined on a compact set, it follows that $G(A)$ assumes its minimum for some A^* in the set.

LEMMA 2.

$$\inf_{\substack{A=(A_1,\ldots,A_n) \\ |A_n| \leqslant (\rho^*)^{1/2} M,\ n=1,2,\ldots,N}} G(A) \leqslant K_{23}\epsilon_2{}^\alpha + \epsilon_3 + \epsilon_5. \tag{5.28}$$

Proof. From (5.10)

$$\begin{aligned} G(A) \leqslant & \left| \frac{\partial z_1}{\partial x} (0, t^*) - \frac{\partial z}{\partial x} (0, t^*) \right| + \left| \sum_{n=N+1}^{\infty} a_n \exp\{-\lambda_n t^*\} \varphi_n'(0) p(0) \right| \\ & + \left| \sum_{n=1}^{N} a_n [\exp\{-\lambda_n t^*\} \varphi_n'(0) p(0) - Q_{3n}(t^*)] \right| \\ & + \left| \sum_{n=1}^{N} [a_n - A_n] Q_{3n}(t^*) \right| \\ \leqslant & K_{23}\epsilon_2{}^\alpha + \epsilon_3 + \epsilon_5 + \left| \sum_{n=1}^{N} [a_n - A_n] Q_{3n}(t^*) \right|. \end{aligned} \tag{5.29}$$

Since the set of Fourier coefficients $a = (a_1,..., a_N)$ from (3.1)–(3.3) satisfy $|a_n| \leqslant (\rho^*)^{1/2} M$, $n = 1, 2,..., N$, and are thus candidates for A, it follows that by setting $a = A$,

$$G(a) \leqslant K_{23}\epsilon_2{}^\alpha + \epsilon_3 + \epsilon_5. \tag{5.30}$$

Since

$$\inf_{\substack{A=(A_1,\ldots,A_n)\\ |A_n|\leqslant(\rho^*)^{1/2}M,\ n=1,2,\ldots,N}} G(A) \leqslant G(a), \tag{5.31}$$

the result (5.28) follows from (5.30).

Applying the results of Lemmas 1 and 2 with (2.1), (5.5), (5.6) and (5.21) we obtain the following *a priori* estimate.

THEOREM 3. *There exist numerical approximations* $w^*(x, t)$, $(\partial w^*/\partial x)(x, t)$, $Q_{1n}(x, t)$, *and* $Q_{2n}(x, t)$ *and a set* $A_1^*, A_2^*, \ldots, A_N^*$, $|A_n^*| \leqslant (\rho^*)^{1/2} M$, $n = 1, 2, \ldots, N$, such that for $0 \leqslant x \leqslant 1$ and $t_0 \leqslant t \leqslant T$,

$$\begin{aligned}
&\text{(a)}\quad \left| u(x, t) - w^*(x, t) - \sum_{n=1}^{N} A_n^* Q_{1n}(x, t) \right| \\
&\qquad \leqslant K_{21}\epsilon_0 + \epsilon_1 + K_{23}\epsilon_2{}^{\alpha} + \epsilon_3 + \epsilon_4 + \theta_1 S_1 \\
&\text{(b)}\quad \left| \frac{\partial u}{\partial x}(x, t) - \frac{\partial w^*}{\partial x}(x, t) - \sum_{n=1}^{N} A_n^* Q_{2n}(x, t) \right| \\
&\qquad \leqslant K_{21}\epsilon_0 + \epsilon_1 + K_{23}\epsilon_2{}^{\alpha} + \epsilon_3 + \epsilon_4 + \theta_2 S_1
\end{aligned} \tag{5.32}$$

where

$$S_1 = N \exp\{K_{24}N^2 + K_{25}\} \left[(1 + p^*)\, K_{23}\epsilon_2{}^{\alpha} + \epsilon_3 + 2\epsilon_5\right] \tag{5.33}$$

and $\epsilon_0, \ldots, \epsilon_5$ are defined above.

Since we now know that a set $A_1^*, A_2^*, \ldots, A_N^*$ exists which yields (5.32), a natural question is how to obtain such a set numerically for use in z_N^* and $\partial z^*/\partial x_N$. We shall use the method of linear programming to determine such a set. Let t_j, $j = 1, \ldots, K$ be a set of times distributed between t_0 and T. Consider the inequalities

$$\begin{aligned}
&\text{(a)}\quad |A_n| \leqslant (\rho^*)^{1/2} M, \qquad n = 1, 2, \ldots, N, \\
&\text{(b)}\quad \left| g^*(t_j) - p(0)\frac{\partial w^*}{\partial x}(0, t_j) - \sum_{n=1}^{N} A_n Q_{3n}(t_j) \right| \leqslant \gamma, \qquad j = 1, 2, \ldots, K, \\
&\text{(c)}\quad \gamma \geqslant 0.
\end{aligned} \tag{5.34}$$

The linear programming problem is the minimization of the linear function

$$L(A, \gamma) = \gamma \tag{5.35}$$

subject to the linear inequalities defined by (5.34). The linear programming problem (5.34)–(5.35) is equivalent to finding

$$\inf_{\substack{|A_n|\leqslant(\rho^*)^{1/2}M\\ n=1,2,\ldots,N}} \max_{j=1,2,\ldots,K} \left| g^*(t_j) - p(0)\frac{\partial w^*}{\partial x}(0, t_j) - \sum_{n=1}^{N} A_n Q_{3n}(t_j) \right|. \tag{5.36}$$

A large K will bring more information from the data to bear upon the problem through (5.34.b) and should thus yield a more accurate set of coefficients. However a larger K would mean a correspondingly more complex linear programming problem. K should be chosen to balance the greater accuracy with the greater difficulty. Denote the infimum in (5.36) by γ_0. Then the continuous function

$$R(A_1,\ldots, A_N) = \max_{1\leqslant j\leqslant k} \left| g^*(t_j) - p(0)\frac{\partial w^*}{\partial x}(0, t_j) - \sum_{n=1}^{N} A_n Q_{3n}(t_j) \right| \tag{5.37}$$

over the compact set $X = \{A = (A_1,\ldots, A_N): | A_n | \leqslant (\rho^*)^{1/2} M\}$ assumes its minimum value at some point in X. Hence there is an $A^* = (A_1^*,\ldots, A_N^*)$ in X such that

$$\gamma_0 = R(A_1^*,\ldots, A_N^*). \tag{5.38}$$

Consequently, the linear programming problem defined by (5.34) and (5.35) is feasible. Standard techniques for solving linear programming problems are available. Suppose that γ^* and $A_1^{**},\ldots, A_N^{**}$ are a solution to the problem (5.34) and (5.35). Then, in conclusion, the following *a posteriori* estimate of error can be given.

THEOREM 4. *Let*

$$\begin{aligned} &\text{(a)} \quad z_N^{**}(x, t) \equiv \sum_{n=1}^{N} A_n^{**} Q_{1n}(x, t),\\ &\text{(b)} \quad \frac{\partial z^{**}}{\partial x_N}(x, t) \equiv \sum_{n=1}^{N} A_n^{**} Q_{2n}(x, t), \end{aligned} \tag{5.39}$$

where A_n^{**}, $n = 1, 2,\ldots, N$ *are obtained from the linear programming problem defined by* (5.34) *and* (5.35). *Then for* $w^*(x, t)$ *and* $(\partial w^*/\partial x\ (w, t)$ *as defined above, we have*

$$\begin{aligned} &\text{(a)} \quad | u(x, t) - w^*(x, t) - z_N^{**}(x, t)|\\ &\qquad \leqslant K_{21}\epsilon_0 + \epsilon_1 + K_{23}\epsilon_2^{\alpha} + \epsilon_3 + \epsilon_4 + \theta_1 S_2,\\ &\text{(b)} \quad \left| \frac{\partial u}{\partial x}(x, t) - \frac{\partial w^*}{\partial x}(x, t) - \frac{\partial z^{**}}{\partial x_N}(x, t) \right|\\ &\qquad \leqslant K_{21}\epsilon_0 + \epsilon_1 + K_{23}\epsilon_2^{\alpha} + \epsilon_3 + \epsilon_4 + \theta_2 S_2, \end{aligned} \tag{5.40}$$

where S_2 *is* S *from* (5.26) *with* σ_1 *replaced by* γ^*.

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