SIAM J. NUMER. ANAL.
Vol. 15, No. 6, December 1978

0036-1429/78/1506-0005 $01.00/0

TIME-STEPPING GALERKIN METHODS FOR NONLINEAR SOBOLEV PARTIAL DIFFERENTIAL EQUATIONS*

RICHARD E. EWING†

Abstract. Three cases for the nonlinear Sobolev equation $c(x, u)(\partial u/\partial t)-\nabla\cdot[a(x, u)\nabla u+b(x, u, \nabla u)\nabla(\partial u/\partial t)]=f(x, t, u, \nabla u)$ are studied. In case I, the coefficients a and b have uniform positive lower bounds in a neighborhood of the solution; in case II, $b=b(x, u)$ is allowed to take zero values and possibly cause the Sobolev equation to degenerate to a parabolic equation; in case III we only require a bound of the form $|a(x, u)|<K$ with a positive lower bound on $b=b(x, u, \nabla u)$. A Crank–Nicolson–Galerkin approximation with extrapolated coefficients is presented for all cases along with a conjugate gradient iterative procedure which can be used efficiently to solve the different linear systems of algebraic equations arising at each step from the Galerkin method. A priori error estimates are derived for each approximation. Optimal order H^1-error estimates are obtained in each case.

1. Introduction. We consider questions arising from approximate solution by Galerkin methods of the nonlinear Sobolev equation

$$c(x, u)\frac{\partial u}{\partial t}-\nabla\cdot\left[a(x, u)\nabla u+b(x, u, \nabla u)\nabla\frac{\partial u}{\partial t}\right]=f(x, t, u, \nabla u) \tag{1.1}$$

for $x\in\Omega$, $t\in J=(0, T]$, where Ω is a bounded domain in $\mathbb{R}^d$, $d\leqq 3$, with boundary $\partial\Omega$. We consider the Neumann boundary conditions for u. In particular, we assume that $u\in C^1(\bar{\Omega}\times[0, T])$ satisfies

$$\begin{aligned} &\text{a)}\ a(x, u)\frac{\partial u}{\partial \nu}+b(x, u, \nabla u)\frac{\partial^2 u}{\partial \nu\,\partial t}=g(x, t), \qquad x\in\partial\Omega, \quad t\in J,\\ &\text{b)}\ u(x, 0)=u_0(x), \qquad x\in\Omega, \end{aligned} \tag{1.2}$$

where $\partial u/(\partial \nu)$ is the normal derivative on the boundary of Ω. We note here that all the following analysis can be carried over immediately to the problem with zero Dirichlet boundary conditions replacing the Neumann conditions provided that appropriate finite-dimensional function spaces are used. Inhomogeneous Dirichlet conditions will be considered elsewhere.

Problems of the form (1.1) arise in the flow of fluids through fissured rock [3], [17], thermodynamics [7], [33], shear in second order fluids [32], consolidation of clay [31], and other applications. See especially [3] for the boundary value problem discussed here. For a discussion of existence and uniqueness results, see [8], [17], [20], [25], [26], [28], [30]. Several applications of the nonlinear problem can be found in [8], [20]. Various numerical treatments of semilinear problems can be found in [16], [19], [34]. Also, Ford presents a predictor-corrector-Galerkin method for (1.1) where there is no dependence on u in $c(x, u)$ and $b(x, u, \nabla u)$ in [18]. For an extensive treatement of equations of the form (1.1) and a comprehensive list of references to the existing literature in the area, see chapter 3 of [6].

In § 2 we present our smoothness assumptions on u and the coefficients in (1.1) and the basic terminology of the paper. We then define a continuous-time-Galerkin

* Received by the editors September 15, 1977, and in revised form March 16, 1978. This work was supported in part by the National Science Foundation through an energy-related postdoctoral fellowship at the University of Chicago.

† Department of Mathematics, Ohio State University, Columbus, Ohio 43210.

approximation and an extrapolated form of a Crank–Nicolson–Galerkin approximation to (1.1). Since time-stepping with the Crank–Nicolson–Galerkin approximations requires the solution of a very different set of linear equations at each time step, we then define a conjugate gradient iteration technique which requires only one matrix to be factored once for all time steps.

In § 3, in the case of uniform positive lower bounds for $a(x, u)$ and $b(x, u, \nabla u)$, we present a priori estimates for both the extrapolated Crank–Nicolson–Galerkin approximation and the conjugate gradient iterative approximation of (1.1). We use approximation theory results of § 2 to derive optimal order H^1-norm rates of convergence (see [9], [14], [24], [34], [35]). The L^2-norm rates obtained are not optimal.

In § 4, we consider some degenerate cases of (1.1). First, we allow $b(x, u)$ to be zero, in which case (1.1) and (1.2) degenerate into the standard nonlinear parabolic Neumann problem. Some applications are discussed. Then we assume a positive lower bound on $b(x, u, \nabla u)$ while allowing the lower bound on $a(x, u)$ to be possibly negative. If $a(x, u) \equiv 0$, we have the equation for a model for long waves considered in [4], [5], [29]. We discuss comparisons with a numerical treatment of this problem presented by Wahlbin in [34]. In both cases, we give error estimates for both the extrapolated Crank–Nicolson–Galerkin scheme and the conjugate gradient iterative variants.

In § 5, we present some rough counts of the number of arithmetic operations required for computation of the various methods. We emphasize the great reduction in operation counts obtained by using the conjugate gradient iterative variants rather than solving the linear systems directly. We also present some machine-oriented stopping procedures for the iterations which are improvements over the theoretical bounds established in §§ 3 and 4.

2. Preliminaries and notation. Let $(u, v) = \int_\Omega uv\,dx$, $\|u\|^2 = (u, u)$, $\langle u, v\rangle = \int_{\partial\Omega} uv\,d\sigma$, and $|u|^2 = \langle u, u\rangle$. Let $W_s^k(\Omega) = W_s^k$ be the Sobolev spaces on Ω with norms

$$\|\varphi\|_{W_s^k} = \left(\sum_{|\alpha| \leqq k} \left\| \frac{\partial^\alpha \varphi}{\partial x^\alpha} \right\|_{L^s(\Omega)}^s \right)^{1/s}.$$

When $s = 2$, denote $\|\varphi\|_{W_2^k} \equiv \|\varphi\|_{H^k} \equiv \|\varphi\|_k$. Let $\{\mathcal{M}_h\}$ be a family of finite-dimensional subspaces of $H^1(\Omega)$, parameterized by h, with the following property:

for $r \in \mathbb{Z}$, $r \geqq 3$, and $3 \leqq p \leqq r$, there exists a constant $K_0 > 0$ such that for $\varphi \in H^p$,

$$\begin{aligned} &\inf_{\chi \in \mathcal{M}_h} \{\|\varphi - \chi\| + h\|\varphi - \chi\|_1 + h^{d/2}[\|\varphi - \chi\|_{L^\infty} + h\|\nabla(\varphi - \chi)\|_{L^\infty}]\} \\ &\qquad \leqq K_0 \|\varphi\|_p h^p. \end{aligned} \tag{2.1}$$

Also assume that our family $\{\mathcal{M}_h\}$ satisfies the following so-called "inverse hypotheses": there exists a constant K_0, independent of h, such that for all $\varphi \in \mathcal{M}_h$,

$$\begin{aligned} &\text{a) } \|\varphi\|_1 \leqq K_0 h^{-1} \|\varphi\| \quad \text{and} \\ &\text{b) } \|\varphi\|_{L^\infty} \leqq K_0 h^{-d/2} \|\varphi\|. \end{aligned} \tag{2.2}$$

The various regularity assumptions on a, b, c, f and u from (1.1) are catalogued as follows:

Q: 1) There exist uniform constants such that for all $x \in \Omega$, $t \in J$ and $-\infty < q_1, q_2 < \infty$,

$$
\begin{aligned}
&\text{a) } |f(x,t,q_1,q_2)| \leqq K_1,\\
&\text{b) } 0 < c_* \leqq c(x,q_1) \leqq K_1,\\
&\text{c) case I: i) } 0 < a_* \leqq a(x,q_1) \leqq K_1,\\
&\qquad\qquad \text{ii) } 0 < b_* \leqq b(x,q_1,q_2) \leqq K_1,\\
&\quad\ \text{case II: i) } 0 < a_* \leqq a(x,q_1) \leqq K_1,\\
&\qquad\qquad \text{ii) } 0 \leqq b_* \leqq b(x,q_1) \leqq K_1,\\
&\quad\ \text{case III: i) } a_* \leqq a(x,q_1) \leqq K_1, \qquad (a_* \leqq 0),\\
&\qquad\qquad \text{ii) } 0 < b_* \leqq b(x,q_1,q_2) \leqq K_1.
\end{aligned}
\tag{2.3}
$$

2) Let a, b, c, and f be continuously differentiable with respect to each variable and assume uniform bounds for $x \in \Omega$, $t \in [0, T]$, and $-\infty < q_1, q_2 < \infty$,

$$
\begin{aligned}
&\text{a) } \left|\frac{\partial a}{\partial q_1}\right|, \left|\frac{\partial b}{\partial q_1}\right|, \left|\frac{\partial b}{\partial q_2}\right|, \left|\frac{\partial c}{\partial q_1}\right|, \left|\frac{\partial f}{\partial q_1}\right|, \left|\frac{\partial f}{\partial q_2}\right| \leqq K_1,\\
&\text{b) } \left|\frac{\partial^2 a}{\partial q_1^2}\right|, \left|\frac{\partial^2 b}{\partial q_1^2}\right|, \left|\frac{\partial^3 b}{\partial q_1^3}\right| \leqq K_1.
\end{aligned}
\tag{2.4}
$$

We note that under the hypotheses of the theorems and corollaries to follow, our approximations converge uniformly to u; thus (2.3) and (2.4) actually need hold only in a neighborhood of the solution.

Let

$$
\|\varphi\|_{L^p((a,b);X)} \equiv \| \, \|\varphi(\cdot\,,t)\|_{X(\Omega)}\|_{L^p((a,b))}. \tag{2.5}
$$

R: For u, the solution of (1.1) and (1.2), and r from (2.1), assume

$$
\begin{aligned}
&\text{a) } \|u\|_{L^\infty(J;W_3^1)} + \left\|\frac{\partial u}{\partial t}\right\|_{L^\infty(J;W_3^1)} \leqq K_2,\\
&\text{b) } \|u\|_{L^2(J;H^r)} + \left\|\frac{\partial u}{\partial t}\right\|_{L^2(J;H^r)} \leqq K_2,\\
&\text{c) } \left\|\frac{\partial^2 u}{\partial t^2}\right\|_{L^2(J;H^1)} + \left\|\frac{\partial^3 u}{\partial t^3}\right\|_{L^2(J;H^1)} \leqq K_2.
\end{aligned}
\tag{2.6}
$$

Let $\Delta t > 0$, $N = T/\Delta t \in \mathbb{Z}$, and $t^\sigma = \sigma\Delta t$, $\sigma \in \mathbb{R}$. Also, for integer n, let $\varphi^n \equiv \varphi^n(x) \equiv \varphi(x, t^n)$, $\varphi^{n+1/2} \equiv (\varphi^{n+1} + \varphi^n)/2$, and denote $d_t\varphi^n \equiv (\varphi^{n+1} - \varphi^n)/\Delta t$ and $d_t^2\varphi^n = (\varphi^{n+1} - 2\varphi^n + \varphi^{n-1})/(\Delta t)^2$.

We shall use the method of comparison with the solution of an auxiliary elliptic problem used by Wheeler [35] (see also [9], [14], [24]). Define W and W_b in $\mathcal{M}_h$ to be the unique functions which, for $t \in [0, T]$, satisfy respectively

$$
\begin{aligned}
&\text{a) } (\nabla[W(\cdot\,,t) - u(\cdot\,,t)], \nabla y) + (W(\cdot\,,t) - u(\cdot,t), y) = 0, \qquad y \in \mathcal{M}_h,\\
&\text{b) } (b(\cdot\,,u(\cdot\,,t))\nabla[W_b(\cdot\,,t) - u(\cdot\,,t)], \nabla y)\\
&\qquad\qquad + (W_b(\cdot\,,t) - u(\cdot\,,t), y) = 0, \qquad y \in \mathcal{M}_h.
\end{aligned}
\tag{2.7}
$$

Clearly W and W_b are weighted H^1-projections of u, the solution of (1.1)–(1.2). Let

$\{\mathcal{M}_h\}$ satisfy (2.1) and (2.2). Restrict Ω such that the Neumann problem for $-\Delta+\mathrm{I}$ on Ω is H^2-regular and $\partial\Omega$ is Lipschitz. Then as in [14] we apply a lemma of Nitsche [23] to obtain the following result.

LEMMA 2.1. *Let W and W_b be defined by* (2.7) *an let u be the solution of* (1.1)–(1.2). *Let $k=2$ or $k=\infty$. For some p satisfying $2\leqq p\leqq r$, let $u\in L^k(J;H^p)$ and $\partial u/(\partial t)\in L^k(J;H^p)$. Then there exists a constant K_3, dependent upon Ω and K_0, such that for $\eta=u-W$, $s=0$ or $s=1$, and $2\leqq p\leqq r$,*

$$\text{(2.8)}\qquad \begin{aligned} &\text{a)}\ \|\eta\|_{L^k(J;H^s)}\leqq K_3h^{p-s}\|u\|_{L^k(J;H^p)},\\ &\text{b)}\ \|d_t\eta\|_{L^k(J;H^s)}\leqq K_3h^{p-s}\left\{\|u\|_{L^k(J;H^p)}+\left\|\frac{\partial u}{\partial t}\right\|_{L^k(J;H^p)}\right\}.\end{aligned}$$

Note that (2.8) also holds for $\eta=u-W_b$ and $s=1$ with K_3 depending also on b_*, K_1, and a bound for $\|\partial u/\partial t\|_{L^\infty(J;L^\infty)}$. Also if $\partial^2u/(\partial t^2)\in L^k(J;H^p)$, then

$$\|d_t^2\eta\|_{L^k(J;H^1)}\leqq K_3h^{p-1}\left\{\|u\|_{L^k(J;H^p)}+\left\|\frac{\partial u}{\partial t}\right\|_{L^k(J;H^p)}+\left\|\frac{\partial^2 u}{\partial t^2}\right\|_{L^k(J;H^p)}\right\}.$$

We shall now present a set of lemmas which will provide needed regularity properties for W and W_b. See [12] for typical proofs.

LEMMA 2.2. *Regularity assumption* R *and the assumption* (2.1) *with $r\geqq 3$ yield the existence of a constant $K_3=K_3(K_0,K_2)$ such that*

$$\text{(2.9)}\qquad \left\|\frac{\partial W}{\partial t}\right\|_{L^2(J;L^\infty)}+\left\|\frac{\partial^2 W}{\partial t^2}\right\|_{L^2(J;H^1)}+\left\|\frac{\partial^3 W}{\partial t^3}\right\|_{L^2(J;H^1)}\leqq K_3.$$

LEMMA 2.3. *In addition to regularity assumption* R, (2.1), *and* (2.4), *assume for some $K_4>0$,*

$$\|u\|_{L^\infty(J;H^3)}+\left\|\frac{\partial u}{\partial t}\right\|_{L^\infty(J;H^2)}\leqq K_4.$$

Then there is a constant $K_5=K_5(K_0,K_1,K_2,K_4)$ such that

$$\text{(2.10)}\qquad \left\|\frac{\partial W_b}{\partial t}\right\|_{L^\infty(J;W_3^1)}+\left\|\frac{\partial^2 W_b}{\partial t^2}\right\|_{L^2(J;H^1)}+\left\|\frac{\partial^3 W_b}{\partial t^3}\right\|_{L^2(J;H^1)}\leqq K_5.$$

LEMMA 2.4. *Assume* (2.1) *and that, for some $K_4>0$,*

$$\left\|\frac{\partial u}{\partial t}\right\|_{L^\infty(J;H^3)}\leqq K_4.$$

Then there is a constant $K_3=K_3(K_0,K_2,K_4)$ such that

$$\text{(2.11)}\qquad \left\|\frac{\partial W}{\partial t}\right\|_{L^\infty(J;W_\infty^1)}\leqq K_3.$$

We now consider the continuous-time-Galerkin approximation of the solution of (1.1)–(1.2). Let $u_h:[0,T]\to\mathcal{M}_h$ be determined by

$$\text{(2.12)}\qquad \begin{aligned} &\text{a)}\ \left(c(\cdot,u_h)\frac{\partial u_h}{\partial t},y\right)+(a(\cdot,u_h)\nabla u_h,\nabla y)+\left(b(\cdot,u_h,\nabla u_h)\nabla\frac{\partial u_h}{\partial t},\nabla y\right)\\ &\qquad=(f(\cdot,t,u_h,\nabla u_h),y)+\langle g(\cdot,t),y\rangle,\qquad y\in\mathcal{M}_h,\\ &\text{b)}\ (a(\cdot,u(\cdot,0))\nabla[u(\cdot,0)-u_h(\cdot,0)],\nabla y)=0,\qquad y\in\mathcal{M}_h.\end{aligned}$$

A standard Crank–Nicolson–Galerkin approximation to the solution of (2.12) would result in a time discretization error of the order $(\Delta t)^2$, but would require the solution of a different nonlinear system of algebraic equations at each time step. (For examples of this property for associated parabolic equations, see [10], [24].)

We shall extrapolate the nonlinear coefficients (see [10], [24]) to obtain the following form of the Crank–Nicolson–Galerkin scheme. Let U: $\{0=t_0, t_1, \cdots, t_N = T\} \to \mathcal{M}_h$, where $t_j - t_{j-1} = \Delta t$, satisfy (using the notation $c(u)=c(x, u(x, t))$ etc.)

$$(2.13)\quad \begin{aligned} &(c(EU^n)d_tU^n, y)+(a(EU^n)\nabla U^{n+1/2}, \nabla y)+(b(EU^n, \nabla EU^n)\nabla d_tU^n, \nabla y)\\ &\qquad =(f(t^{n+1/2}, EU^n, \nabla EU^n), y)+\langle g(t^{n+1/2}), y\rangle, \qquad y \in \mathcal{M}_h, \end{aligned}$$

for $n=1, 2, \cdots, N-1$, where $EU^n = \frac{3}{2}U^n - \frac{1}{2}U^{n-1}$. With this definition of EU^n, we see that information is required at two preceding time levels to advance in time. Thus we need a starting procedure which will retain the overall accuracy of the method. A predictor-corrector starting method (for example see [10], [18], [24]) will suffice. We note that the method given by (2.13) requires only the solution of *one linear* system of algebraic equations at each time step. (By comparison, the predictor-corrector Galerkin approximation described by Ford in [18] for the simpler problem in which c and b do not depend upon the solution, requires the solution of two linear systems of algebraic equations per time step.) However, the solution of (2.13) requires that a different system of linear equations be solved at each time step. We shall also consider a modification of (2.13) which allows us to solve only equations associated with one fixed matrix at all time levels.

Let $M = \dim \mathcal{M}_h$. Let $\{\varphi_l\}_{l=1}^M$ be a basis for $\mathcal{M}_h$ and, at the mth time step, let a solution of (2.13) be given by

$$(2.14)\qquad U^m = \sum_{l=1}^{M} \xi_l^m \varphi_l.$$

Then define the following matrices and vectors:

$$(2.15)\quad \begin{aligned} &\text{a)}\ C^m(\theta)=(c_{ij}^m(\theta)),\ c_{ij}^m(\theta)=\Big(c\Big(\cdot, E\sum_{l=1}^{M}\theta_l^m\varphi_l\Big)\varphi_j, \varphi_i\Big),\\ &\text{b)}\ A^m(\theta)=(a_{ij}^m(\theta)),\ a_{ij}^m(\theta)=\Big(a\Big(\cdot, E\sum_{l=1}^{M}\theta_l^m\varphi_l\Big)\nabla\varphi_j, \nabla\varphi_i\Big),\\ &\text{c)}\ B^m(\theta)=(b_{ij}^m(\theta)),\ b_{ij}^m(\theta)=\Big(b\Big(\cdot, E\sum_{l=1}^{M}\theta_l^m\varphi_l, \nabla E\sum_{l=1}^{M}\theta_l^m\varphi_l\Big)\nabla\varphi_j, \nabla\varphi_i\Big),\\ &\text{d)}\ F^m(\theta)=(f_i^m(\theta)),\ f_i^m(\theta)=\Big(f\Big(\cdot, t^{m+1/2}, E\sum_{l=1}^{M}\theta_l^m\varphi_l, \nabla E\sum_{l=1}^{M}\theta_l^m\varphi_l\Big), \varphi_i\Big)\\ &\text{e)}\ G^m(\theta)=(g_i^m(\theta)),\ g_i^m(\theta)=\langle g(\cdot, t^{m+1/2}), \varphi_i\rangle,\\ &\text{f)}\ C_0=((c_0\varphi_j, \varphi_i)) \quad\text{and}\quad A_0=((a_0\nabla\varphi_j, \nabla\varphi_i)), \end{aligned}$$

for $i=1, \cdots, M$ and $j=1, \cdots, M$, where a_0 and c_0 are any fixed values of a and c.

Using the above notation, we can write (2.13) in the form

$$(2.16)\quad \begin{aligned} L^n(\xi)(\xi^{n+1}-\xi^n) &\equiv \Big(C^n(\xi)+B^n(\xi)+\frac{\Delta t}{2}A^n(\xi)\Big)(\xi^{n+1}-\xi^n)\\ &= -\Delta t A^n(\xi)\xi^n + \Delta t[F^n(\xi)+G^n(\xi)]. \end{aligned}$$

We see from (2.3.I) and (2.15) that L^n is positive definite for each n.

We next present a conjugate gradient iterative scheme which will greatly reduce the work involved in solving (2.16) repeatedly with different coefficients. We use the conjugate gradient method to solve for coefficients γ_l^m, $l=1,2,\cdots,M$, to replace the ξ_l^m in (2.16) and then use

$$V^m=\sum_{l=1}^{M}\gamma_l^m\varphi_l \tag{2.17}$$

as an approximation to U^n, the solution of (2.13). A predictor-corrector method for determining V^0 and V^1 can be obtained using the conjugate gradient method as in [12]. Thus, assume $V^0, V^1,\cdots,V^n$ are known by some method. We find γ^{n+1} (and thus V^{n+1}) using a conjugate gradient iteration (see [1], [2], [11], [12]) as follows:

With

$$L_0\equiv C_0+\left(1+\frac{\Delta t}{2}\right)A_0, \tag{2.18}$$

let

$$\begin{aligned}&\text{a) } x_0=x_0^2=\gamma^1-\gamma^0, \qquad n=1,\\ &\phantom{\text{a) }} x_0=x_0^{n+1}=2\gamma^n-3\gamma^{n-1}+\gamma^{n-2}, \qquad n\geqq 2,\\ &\text{b) } q_0=s_0=L^n(\gamma)x_0+\Delta tA^n(\gamma)\gamma^n-\Delta t[F^n(\gamma)+G^n(\gamma)].\end{aligned} \tag{2.19}$$

Then for μ, to be determined later, and $k=1,2,\cdot,\cdots,\mu$, set

$$\begin{aligned}&\text{a) } x_{k+1}=x_k+\alpha_k s_k, \quad \text{where } \alpha_k=-\frac{(L_0^{-1}q_k,q_k)}{(s_k,L^n(\gamma)s_k)},\\ &\text{b) } q_{k+1}=q_k+\alpha_k L^n(\gamma)s_k,\\ &\text{c) } s_{k+1}=L_0^{-1}q_{k+1}+\beta_k s_k, \quad \text{where } \beta_k=\frac{(L_0^{-1}q_{k+1},q_{k+1})}{(L_0^{-1}q_k,q_k)}.\end{aligned} \tag{2.20}$$

Finally, set

$$\gamma^{n+1}=\gamma^n+x_\mu \tag{2.21}$$

where μ is to be chosen independently of n. We define $\bar\gamma^{n+1}$ to be the exact solution for one time step of (2.16) where the ξ^n have been replaced by the γ^n computed by the above scheme. We thus define $\bar\gamma^{n+1}$ to satisfy

$$L^n(\gamma)(\bar\gamma^{n+1}-\gamma^n)=-\Delta tA^n(\gamma)\gamma^n+\Delta t[F^n(\gamma)+G^n(\gamma)]. \tag{2.22}$$

We know from the theory of conjugate gradient methods [1], [2] that there exists a constant $\rho<1$ such that we have a norm reduction of the following form

$$\begin{aligned}&\text{a) } \|L^1(\gamma)^{1/2}(\bar\gamma^2-\gamma^2)\|_l\leqq\rho\|L^1(\gamma)^{1/2}(\bar\gamma^2-2\gamma^1+\gamma^0)\|_l,\\ &\text{b) } \|L^n(\gamma)^{1/2}(\bar\gamma^{n+1}-\gamma^{n+1})\|_l\leqq\rho\|L^n(\gamma)^{1/2}(\bar\gamma^{n+1}-3\gamma^n+3\gamma^{n-1}-\gamma^{n-2})\|_l, \quad n\geqq 2,\end{aligned} \tag{2.23}$$

where the norm symbol refers to the Euclidean norm of the vector. For example, if we let $a_0\equiv 1$ and $c_0\equiv 1$ in (2.15), then from (2.3.I) we obtain

$$\begin{aligned}&\text{a) } 0<\psi_0\leqq\frac{x^TL^n(\gamma)x}{x^TL_0x}\leqq\psi_1, \qquad 0\neq x\in\mathbb{R}^M\\ &\text{b) } \psi_0\equiv\min\left\{\frac{a_*\Delta t}{2}+b_*,c_*\right\}>0,\\ &\text{c) } \psi_1\equiv\left(1+\frac{\Delta t}{2}\right)K_1\geqq\psi_0.\end{aligned} \tag{2.24}$$

Then from Axelsson [1], [2], we have

$$\rho<2\left\{\frac{\psi_1^{1/2}-\psi_0^{1/2}}{\psi_1^{1/2}+\psi_0^{1/2}}\right\}^{\mu}\equiv 2\omega^{\mu}. \tag{2.25}$$

We note that a choice of coefficients other than $a_0=c_0\equiv 1$ in the definition of A_0 and C_0 would slightly alter (2.24) and (2.25). A better choice for computational purposes could be $a_0=a(x, u_0(x))$ and $c_0=c(x, u_0(x))$. Next, letting

$$\bar{V}^{n+1}=\sum_{i=1}^{M}\bar{\gamma}_i^{n+1}\varphi_i, \tag{2.26}$$

and using (2.17) and (2.26), we see that V^n and $\bar{V}^{n+1}$ satisfy

$$\begin{aligned}\left(c(\cdot\,,EV^n)\frac{\bar{V}^{n+1}-V^n}{\Delta t}, y\right)&+\left(a(\cdot\,,EV^n)\nabla\frac{\bar{V}^{n+1}-V^n}{2}, \nabla y\right)\\ &+\left(b(\cdot\,,EV^n, \nabla EV^n)\nabla\frac{\bar{V}^{n+1}-V^n}{\Delta t}, \nabla y\right)\\ &=(f(\cdot\,,t^{n+1/2}, EV^n, \nabla EV^n), y)+\langle g(\cdot\,,t^{n+1/2}), y\rangle, \qquad y\in\mathcal{M}_h.\end{aligned} \tag{2.27}$$

Define

$$\begin{aligned}&\text{a) } \|\varphi\|_{c^n}^2\equiv(c(\cdot\,,EV^n)\varphi, \varphi),\\ &\text{b) } \|\varphi\|_{a^n}^2=(\tfrac{1}{2}a(\cdot\,,EV^n)\nabla\varphi, \nabla\varphi),\\ &\text{c) } \|\varphi\|_{b^n}^2=(b(\cdot\,,EV^n, \nabla EV^n)\nabla\varphi, \nabla\varphi).\end{aligned} \tag{2.28}$$

We shall abuse the notation of (2.28) in § 3 by replacing EV^n by EU^n in the coefficients with the same notation. From (2.3), we note that for each n, $\|\cdot\|_{c^n}$ is equivalent to $\|\cdot\|$, and $\|\cdot\|_{a^n}$ and $\|\cdot\|_{b^n}$ are equivalent to $\|\nabla\cdot\|$. In the notation of (2.28), (2.23) and the triangle inequality yield

$$\begin{aligned}&\text{a) } \|\bar{V}^2-V^2\|_1\leqq\rho K\|\delta^2V^1\|_1,\\ &\text{b) } \|\bar{V}^{n+1}-V^{n+1}\|_1\leqq\rho K\|\delta^3V^n\|_1, \qquad n\geqq 2,\end{aligned} \tag{2.29}$$

where K depends upon the constants from the norm and semi-norm equivalences and an upper bound on Δt and we define

$$\begin{aligned}&\text{a) } \delta^2V^n\equiv V^{n+1}-2V^n+V^{n-1},\\ &\text{b) } \delta^3V^n\equiv V^{n+1}-3V^n+3V^{n-1}-V^{n-2}.\end{aligned} \tag{2.30}$$

We note from (2.25) that if $\alpha>0$ and

$$\mu\geqq\left(\log(2K)+\alpha\log\left(\frac{1}{\Delta t}\right)\right)\Big/\log\left(\frac{1}{\omega}\right), \tag{2.31}$$

then

$$pK<(\Delta t)^{\alpha}. \tag{2.32}$$

In particular, if μ is chosen from (2.31) with $\alpha=1$ for $n=1$, (2.29a) yields

$$\|\bar{V}^2-V^2\|_1\leqq\Delta t\|\delta^2V^1\|_1. \tag{2.33}$$

We shall use (2.29b) and (2.33) in the next section to obtain a priori bounds on the difference $\zeta^n=V^n-W^n$. We emphasize that the conjugate gradient method outlined above is only one way to obtain the necessary norm reduction. The remaining theory depends only on the inequalities (2.29) and (2.33) and not on the method of obtaining them.

3. A priori H^1 error estimates. In this section for case I, we develop a priori bounds for the error $U^n - u^n$ and then for the error in the conjugate gradient modifications, $V^n - u^n$, defined in § 2. We then use the weighted H^1-projection W_b defined in (2.7b) and the notion of a negative-indexed norm to reduce the smoothness assumptions on $\partial u/(\partial t)$ when spaces $\mathcal{M}_h$ with large r are used. In each case of the conjugate gradient semi-iterative method, the error at each time step need only be reduced by a fixed (sufficiently small) factor that is independent of n, Δt and h. We obtain optimal order H^1-estimates.

We shall first state the following lemma which can be proved using the techniques which follow in this section.

LEMMA 3.1. *Let case* I *of* (2.3), (2.4a), R *and the restrictions on* $\{\mathcal{M}_h\}$ *of* § 2 *be satisfied. Let* W *be defined in* (2.7a) *and* u *be the solution of* (1.1)–(1.2). *Define* U^0 *to be the* L^2*-projection of* $u_0(x)$ *into* $\mathcal{M}_h$. *Then a predictor-corrector Crank–Nicolson–Galerkin method can be defined for* U^1 (*as in* [10], [12], [24]) *which satisfies for* $\Delta t \leqq \tau_0$,

$$\|U^0 - W^0\|_1 + \|U^1 - W^1\|_1 + (\Delta t)^{1/2}\|d_t(U^0 - W^0)\|_1 \leqq C_1\{(\Delta t)^2 + h^{r-1}\} \tag{3.1}$$

where τ_0 *and* C_1 *are positive constants with* C_1 *depending on* a_*, b_*, c_*, K_0, K_1, K_2, K_3, *and* $\|u_0\|_r$.

Using Lemma 3.1, we prove the following error estimates for the solution of (2.13).

THEOREM 3.1. *Let case* I *of* (2.3), (2.4a), R *and the restriction on* $\{\mathcal{M}_h\}$ *of* § 2 *be satisfied. Let*

$$\left\|\frac{\partial u}{\partial t}\right\|_{L^\infty(J;W^1_\infty)} \leqq K_2 \tag{3.2}$$

and let (3.1) *hold. Then there exist positive constants* τ_0 *and* C_2, *with* $C_2 = C_2(Q, K_1, K_2, K_3, C_1)$ *such that if* $h \leqq \tau_0$, $\Delta t \leqq h^{d/3}$, *and* $r \geqq 3$,

$$\sup_{t^n} \|U - u\|_1 \leqq C_2\{(\Delta t)^2 + h^{r-1}\}. \tag{3.3}$$

Proof. Letting $\eta^n = u^n - W^n$ and $Z^n = U^n - W^n$, we see that

$$\begin{aligned}
&(c(EU^n)\, d_t Z^n, y) + (a(EU^n)\nabla Z^{n+1/2}, \nabla y) + (b(EU^n, \nabla EU^n)\nabla\, d_t Z^n, \nabla y)\\
&\quad = (c(EU^n)\, d_t\eta^n, y) - \left(\left[c(EU^n)\, d_t u^n - c(u(t^{n+1/2}))\frac{\partial u}{\partial t}(t^{n+1/2})\right], y\right)\\
&\qquad + (a(EU^n)\nabla\eta^{n+1/2}, \nabla y) - ([a(EU^n)\nabla u^{n+1/2} - a(u(t^{n+1/2}))\nabla u(t^{n+1/2})], \nabla y)\\
&\qquad + (b(EU^n, \nabla EU^n)\nabla d_t\eta^n, \nabla y) - \left(\left[b(EU^n, \nabla EU^n)\nabla d_t u^n\right.\right.\\
&\qquad\qquad \left.\left. - b(u(t^{n+1/2}), \nabla u(t^{n+1/2}))\nabla \frac{\partial u}{\partial t}(t^{n+1/2})\right], \nabla y\right)\\
&\qquad + ([f(t^{n+1/2}, EU^n, \nabla EU^n) - f(t^{n+1/2}, u(t^{n+1/2}), \nabla u(t^{n+1/2}))], y), \qquad y \in \mathcal{M}_h.
\end{aligned} \tag{3.4}$$

We shall obtain estimates on the H^1-norm by using $y = Z^{n+1} - Z^n \equiv \delta Z^n = \Delta t d_t Z^n$ as a test function. Using (2.28), we obtain

$$\begin{aligned}&(c(EU^n)d_tZ^n, \delta Z^n)+(a(EU^n)\nabla Z^{n+1/2}, \nabla\delta Z^n)+(b(EU^n, \nabla EU^n)\nabla d_tZ^n, \nabla\delta Z^n)\\ &\quad = \Delta t\|d_tZ^n\|^2_{c^n}+\Delta t\|d_tZ^n\|^2_{b^n}+\{\|Z^{n+1}\|^2_{a^n}-\|Z^n\|^2_{a^n}\}\\ &\quad \geqq \beta\Delta t\|d_tZ^n\|^2_1+\{\|Z^{n+1}\|^2_{a^n}-\|Z^n\|^2_{a^n}\}\end{aligned} \tag{3.5}$$

where $\beta=\min\{b_*, c_*\}$ from case I of (2.3). We shall use Hölder's inequality and the inequality

$$\nu_1\nu_2 \leqq \left(\frac{\nu_1^2}{\varepsilon}+\varepsilon\nu_2^2\right)\Big/2, \qquad \varepsilon>0, \tag{3.6}$$

to split all the terms on the right side of (3.4). We choose ε in (3.6) so as to have small quantities multiplying each of the contributions from the test function. For example, we use a generalized Hölder inequality and the fact that $H^1\subset L^6$ continuously for $d\leqq 3$ to obtain

$$\begin{aligned}&|(a(EU^n)\nabla u^{n+1/2}, \nabla d_tZ^n)\Delta t-(a(u(t^{n+1/2}))\nabla u(t^{n+1/2}), \nabla d_tZ^n)\Delta t|\\ &\quad\leqq|\Delta t(a(EU^n)[\nabla u^{n+1/2}-\nabla u(t^{n+1/2})], \nabla d_tZ^n)|\\ &\qquad+|\Delta t([a(EU^n)-a(EW^n)]\nabla u(t^{n+1/2}), \nabla d_tZ^n)|\\ &\qquad+|\Delta t([a(EW^n)-a(Eu^n)+a(Eu^n)\\ &\qquad\qquad -a(u(t^{n+1/2}))]\nabla u(t^{n+1/2}), \nabla d_tZ^n)|\\ &\quad\leqq C\Delta t\{\|u^{n+1/2}-u(t^{n+1/2})\|_1+[\|EZ^n\|_{L^6(\Omega)}+\|E\eta^n\|_{L^6(\Omega)}\\ &\qquad+\|Eu^n-u(t^{n+1/2})\|_{L^6(\Omega)}]\|u\|_{W^1_3}\}\|d_tZ^n\|_1\\ &\quad\leqq\frac{\beta\Delta t}{16}\|d_tZ^n\|_1^2+C\Delta t\{\|\eta^n\|_1^2+\|\eta^{n-1}\|_1^2+\|Z^n\|_1^2+\|Z^{n-1}\|_1^2\}+C\sigma_{1,n}(\Delta t)^4\end{aligned} \tag{3.7}$$

where

$$\sigma_{1,n}=\int_{t^{n-1}}^{t^{n+1}}\left\|\frac{\partial^2u}{\partial t^2}(\cdot, s)\right\|_1^2 ds \tag{3.8}$$

and the last constants in (3.7) depend upon K_2 from (2.6 a). Similarly, we treat the sixth term on the right of (3.4) as follows:

$$\begin{aligned}&\left|(b(EU^n, \nabla EU^n)\nabla d_tu^n, \nabla d_tZ^n)\Delta t-\left(b(u(t^{n+1/2}), \nabla u(t^{n+1/2}))\nabla\frac{\partial u}{\partial t}, \nabla d_tZ^n\right)\Delta t\right|\\ &\quad\leqq C\Delta t\left\{\left\|d_tu^n-\frac{\partial u}{\partial t}\right\|_1\right.\\ &\qquad\left.+\left[\|EZ^n\|_1+\|E\eta_1{}^n\|+\|Eu^n-u(t^{n+1/2})\|_1\right]\left\|\frac{\partial u}{\partial t}\right\|_{W^1_\infty}\right\}\|d_tZ^n\|_1\\ &\quad\leqq\frac{\beta\Delta t}{16}\|d_tZ^n\|_1^2+C\Delta t\{\|\eta^n\|_1^2+\|\eta^{n-1}\|_1^2+\|Z^n\|_1^2+\|Z^{n-1}\|_1^2\}\\ &\qquad+C[\sigma_{1,n}+\sigma_{2,n}](\Delta t)^4\end{aligned} \tag{3.9}$$

where

$$\sigma_{2,n}=\int_{t^n}^{t^{n+1}}\left\|\frac{\partial^3u}{\partial t^3}(\cdot, s)\right\|_1^2 ds \tag{3.10}$$

and the last constants in (3.9) depend upon K_2 from (3.2). Clearly, the last term in (3.4) can be estimated as above. In a similar manner, we treat the first, third, and fifth terms on the right of (3.4) as follows

$$\begin{aligned}&\Delta t|(c(EU^n)d_t\eta^n, d_tZ^n)+(a(EU^n)\nabla\eta^{n+1/2}, \nabla d_tZ^n)+(b(EU^n, \nabla EU^n)\nabla d_t\eta^n, \nabla d_tZ^n)|\\ &\qquad \leqq \frac{\beta\Delta t}{16}\|d_tZ^n\|_1^2+C\Delta t\{\|\eta^n\|_1^2+\|\eta^{n+1}\|_1^2+\|d_t\eta^n\|_1^2\}.\end{aligned}\tag{3.11}$$

Using similar estimates, combining the results, and subtracting the small multiples of $\beta\Delta t\|d_tZ^n\|_1^2$ from both sides, we sum on $n=1,\cdots,l-1$ to obtain

$$\begin{aligned}&\sum_{n=1}^{l-1}\{\tfrac{3}{4}\beta\Delta t\|d_tZ^n\|_1^2+\|Z^{n+1}\|_{a^n}^2-\|Z^n\|_{a^n}^2\}\\ &\qquad\leqq C\Delta t\sum_{n=0}^{l-1}[\|\eta^n\|_1^2+\|d_t\eta^n\|_1^2]+C\Delta t\sum_{n=0}^{l-1}\|Z^n\|_1^2+C(\Delta t)^4\sum_{n=1}^{l-1}[\sigma_{1,n}+\sigma_{2,n}].\end{aligned}\tag{3.12}$$

We note that assumption (2.6c) implies that

$$\sum_{n=1}^{N-1}[\sigma_{1,n}+\sigma_{2,n}]\leqq 2K_2.\tag{3.13}$$

Then from (2.8) and (3.1) we have that

$$\begin{aligned}&\sum_{n=1}^{l-1}\{\tfrac{3}{4}\beta\Delta t\|d_tZ^n\|_1^2+\|Z^{n+1}\|_{a^n}^2-\|Z^n\|_{a^n}^2\}\\ &\qquad\leqq C\{(\Delta t)^4+h^{2r-2}\}+C\Delta t\sum_{n=1}^{l-1}\|Z^n\|_1^2.\end{aligned}\tag{3.14}$$

In order to use the discrete Gronwall lemma in (3.14), we need a telescoping sum in the $\|\cdot\|_{a^n}$ semi-norms. We use an idea of Rachford [24] to establish comparabilities $\|Z^n\|_{a^n}^2$ and $\|Z^n\|_{a^{n-1}}^2$ to obtain this telescoping sum. Note that

$$\begin{aligned}\|Z^n\|_{a^n}^2&=\|Z^n\|_{a^{n-1}}^2+([a(EU^n)-a(EU^{n-1})]\nabla Z^n,\nabla Z^n)\\ &=\|Z^n\|_{a^{n-1}}^2+\left(\left[\frac{\partial a}{\partial u}E(Z^n-Z^{n-1})+\frac{\partial a}{\partial u}E(W^n-W^{n-1})\right]\nabla Z^n,\nabla Z^n\right)\\ &\leqq\|Z^n\|_{a^{n-1}}^2+C\{\sigma_{3,n}+\|\delta Z^{n-1}\|_{L^\infty(\Omega)}+\|\delta Z^{n-2}\|_{L^\infty(\Omega)}\}\|Z^n\|_1^2\end{aligned}\tag{3.15}$$

where

$$\sigma_{3,n}=\int_{t^{n-2}}^{t^n}\left\|\frac{\partial W}{\partial t}(\cdot,s)\right\|_{L^\infty(\Omega)}ds.\tag{3.16}$$

Thus, as in [12], [22], we have

$$\begin{aligned}-\sum_{n=2}^{l-1}\|Z^n\|_{a^{n-1}}^2&\leqq-\sum_{n=2}^{l-1}\|Z^n\|_{a^n}^2\\ &\quad+C\sum_{n=2}^{l-1}\{\sigma_{3,n}+\|\delta Z^{n-1}\|_{L^\infty(\Omega)}+\|\delta Z^{n-2}\|_{L^\infty(\Omega)}\}\|Z^n\|_1^2.\end{aligned}\tag{3.17}$$

Then, using (3.17) in (3.14) and (3.1), we use the telescoping property of the sum to obtain

$$\frac{3}{4}\sum_{n=1}^{l-1}\beta\Delta t\|d_tZ^n\|_1^2+\|Z^l\|_{a^{l-1}}^2$$
$$\leqq C\{(\Delta t)^4+h^{2r-2}\}+C\Delta t\sum_{n=1}^{l-1}\|Z^n\|_1^2 \tag{3.18}$$
$$+C\sum_{n=2}^{l-1}\{\sigma_{3,n}+\|\delta Z^{n-1}\|_{L^\infty(\Omega)}+\|\delta Z^{n-2}\|_{L^\infty(\Omega)}\}\|Z^n\|_1^2.$$

Next we note that we must introduce an L^2-component on the left side of (3.18) to combine with the $\|\cdot\|_{a^{l-1}}$ term to obtain terms equivalent to the H^1-norm to apply the discrete Gronwall lemma. We also note that if $\Delta t\leqq\beta/(8a_*)$, then

$$\begin{aligned}a_*\{\|Z^{n+1}\|^2-\|Z^n\|^2\}&=2a_*\Delta t(d_tZ^n, Z^n)+a_*(\Delta t)^2\|d_tZ^n\|^2\\&\leqq\frac{\beta\Delta t}{4}\|d_tZ^n\|_1^2+C\Delta t\|Z^n\|_1^2.\end{aligned}\tag{3.19}$$

Summing (3.19) from $n=1$ to $n=l-1$, adding the result to (3.18), and multiplying the result by $\max\{1, 2/\beta, 1/a_*\}$, we obtain

$$\begin{aligned}\sum_{n=1}^{l-1}\Delta t\|d_tZ^n\|_1^2+\|Z^l\|_1^2&\leqq C_3\{(\Delta t)^4+h^{2r-2}\}+C_4\Delta t\sum_{n=1}^{l-1}\|Z^n\|_1^2\\&\quad+C_5\sum_{n=2}^{l-1}\{\sigma_{3,n}+\|\delta Z^{n-1}\|_{L^\infty(\Omega)}+\|\delta Z^{n-2}\|_{L^\infty(\Omega)}\}\|Z^n\|_1^2.\end{aligned}\tag{3.20}$$

Since $\sum_{n=2}^{l-1}\sigma_{3,n}\leqq 2K_3$ from (2.9), in order to apply a version of the discrete Gronwall lemma, we must show that for some $C_6>0$,

$$\sum_{n=0}^{l-2}\|\delta Z^n\|_{L^\infty(\Omega)}\leqq C_6. \tag{3.21}$$

We shall use an induction argument as in [12], [22], [24] to yield (3.21) with the summation starting at $n=1$. For $l=2$, the inequality (3.20) and the estimate (3.1) yield

$$\begin{aligned}\Delta t\|d_tZ^0\|_1^2+\Delta t\|d_tZ^1\|_1^2&\leqq C_7\{(\Delta t)^4+h^{2r-2}\}+C_4\Delta t\|Z^1\|_1^2\\&\leqq C_8\{(\Delta t)^4+h^{2r-2}\}.\end{aligned}\tag{3.22}$$

We note that if

$$\begin{aligned}&\text{a) } h\leqq(2C_8)^{-3/d},\\&\text{b) } \Delta t\leqq h^{d/3},\\&\text{c) } r\geqq 3\geqq\tfrac{2}{3}d+1,\end{aligned}\tag{3.23}$$

then

$$\Delta t\|d_tZ^0\|_1^2+\Delta t\|d_tZ^1\|_1^2\leqq h^d. \tag{3.24}$$

Assume the following induction hypothesis:

For h sufficiently small,

$$\Delta t \sum_{n=0}^{k} \|\nabla d_t Z^n\|^2 \leqq h^d, \quad \text{for } 1 \leqq k \leqq l-2. \tag{3.25}$$

If we use the inverse hypothesis (2.2b), (3.25), and the fact that $N\Delta t = T$, we see that

$$\begin{aligned}\sum_{n=0}^{l-2} \|\delta Z^n\|_{L^\infty(\Omega)} &\leqq (l-1)^{1/2}\left(\sum_{n=0}^{l-2} \|\delta Z^n\|^2_{L^\infty(\Omega)}\right)^{1/2} \\ &\leqq N^{1/2} K_0 h^{-d/2}\left(\sum_{n=0}^{l-2} \|\delta Z^n\|^2\right)^{1/2} \\ &\leqq (\Delta t)^{-1/2} T^{1/2} K_0 h^{-d/2} (\Delta t)^{1/2}\left(\Delta t \sum_{n=0}^{l-2} \|\nabla d_t Z^n\|^2\right)^{1/2} \\ &\leqq T^{1/2} K_0.\end{aligned} \tag{3.26}$$

Then, with $C_6 = T^{1/2} K_0$, we apply the discrete Gronwall lemma in (3.20) to obtain

$$\sum_{n=1}^{l-1} \Delta t \|d_t Z^n\|_1^2 + \|Z^l\|_1^2 \leqq C_9\{(\Delta t)^4 + h^{2r-2}\} \tag{3.27}$$

where

$$C_9 \leqq C_3 \exp\{C_4 T + 2C_5[K_3 + T^{1/2} K_0]\}. \tag{3.28}$$

Then, with C_8 in (3.23a) replaced by C_9, we see that the induction argument is completed. Since (3.27) holds for each $l = 1$ to $l = N$, we have

$$\sup_{t^n} \|U - W\|_1 \leqq C_{10}\{(\Delta t)^2 + h^{r-1}\}. \tag{3.29}$$

Then (2.8), (3.1), (3.29) and the triangle inequality yield the desired result.

COROLLARY 3.1. *If b in case* I *of* (2.3) *is independent of* ∇u (*i.e.*, $b = b(x, u)$) *then the results of Theorem* 3.1 *will hold with* (3.2) *replaced by*

$$\left\|\frac{\partial u}{\partial t}\right\|_{L^\infty(J; W_3^1)} \leqq K_2. \tag{3.30}$$

Proof. The norm $\|\partial u/\partial t\|_{W^1_\infty}$ in line 2 of (3.9) can be replaced by $\|\partial u/\partial t\|_{W^1_3}$ by using the generalized Hölder inequality as in (3.7) and the fact that $H^1 \subset L^6$ continuously.

The results of § 2 will now be applied to develop a priori estimates for the error for incomplete iterative approximation. Let $\zeta^n = V^n - W^n$.

THEOREM 3.2. *Let the assumptions of Theorem* 3.1 *be satisfied. Assume V^0 and V^1 are determined to satisfy*

$$\|V^0 - W^0\|_1 + \|V^1 - W^1\|_1 + (\Delta t)^{1/2}\|d_t(V^0 - W^0)\|_1 \leqq C_{11}\{(\Delta t)^2 + h^{r-1}\} \tag{3.31}$$

and V^2 satisfies (2.33). *Then there exist positive constants C_{12}, C_{13} and τ_0 such that if μ is chosen independently of n, h and Δt for $n \geqq 2$ such that $\rho K \leqq C_{12}$ in* (2.29b), *and* $r \geqq 3$,

$$\sup_{t^n} \|V - u\|_1 \leqq C_{13}\{(\Delta t)^2 + h^{r-1}\}. \tag{3.32}$$

The constants C_{11} and C_{13} have the same dependencies as C_2.

Proof. A predictor-corrector version of the conjugate gradient method presented in § 2 and similar to that discussed in [12] will yield (3.31). From (2.7), we note that W^n satisfies an analog to (3.4) with EU^n replaced by EV^n in the coefficients. The analog of (3.4) for $\zeta^n = V^n - W^n$ is

$$
\begin{aligned}
&(c(EV^n)d_t\zeta^n, y)+(a(EV^n)\nabla\zeta^{n+1/2}, \nabla y)+(b(EV^n, \nabla EV^n)\nabla d_t\zeta^n, \nabla y)\\
&\quad=(c(EV^n)d_t\eta^n, y)-\left(\left[c(EV^n)d_tu^n-c(u(t^{n+1/2}))\frac{\partial u}{\partial t}(t^{n+1/2})\right], y\right)\\
&\qquad+(a(EV^n)\nabla\eta^{n+1/2}, \nabla y)\\
&\qquad-([a(EV^n)\nabla u^{n+1/2}-a(u(t^{n+1/2}))\nabla u(t^{n+1/2})], \nabla y)\\
&\qquad+(b(EV^n, \nabla EV^n)\nabla d_t\eta^n, \nabla y)\\
&\qquad-\left(\left[b(EV^n, \nabla EV^n)\nabla d_tu^n-b(u(t^{n+1/2}), \nabla u(t^{n+1/2}))\nabla\frac{\partial u}{\partial t}\right], \nabla y\right)\\
&\qquad+([f(t^{n+1/2}, EV^n, \nabla EV^n)-f(t^{n+1/2}, u(t^{n+1/2}), \nabla u(t^{n+1/2}))], y)\\
&\qquad+\left\{\left(c(EV^n)\frac{V^{n+1}-\bar V^{n+1}}{\Delta t}, y\right)+\left(a(EV^n)\nabla\left(\frac{V^{n+1}-\bar V^{n+1}}{2}\right), \nabla y\right)\right.\\
&\qquad\left.+\left(b(EV^n, \nabla EV^n)\nabla\left(\frac{V^{n+1}-\bar V^{n+1}}{\Delta t}\right), \nabla y\right)\right\}, \qquad y\in\mathcal{M}_h.
\end{aligned}
\tag{3.33}
$$

By comparing (3.4) with (3.33), we see that the only significant difference in the proof of this result is the method of treating the last term in brackets in (3.33). Using (2.33), (3.31), and the equivalence of norms, we have

$$
\begin{aligned}
&\left|\left(c(EV^1)\frac{V^2-\bar V^2}{\Delta t}, \Delta t d_t\zeta^1\right)+\left(a(EV^1)\nabla\left(\frac{V^2-\bar V^2}{2}\right), \nabla(\Delta t d_t\zeta^1)\right)\right.\\
&\qquad\left.+\left(b(EV^1, \nabla EV^1)\nabla\frac{V^2-\bar V^2}{\Delta t}, \nabla(d_t\zeta^1\Delta t)\right)\right|\\
&\quad\leqq C\|V^2-\bar V^2\|_1\,\|d_t\zeta^1\|_1\\
&\quad\leqq C\Delta t\|\delta^2V^1\|_1\,\|d_t\zeta^1\|_1\\
&\quad\leqq C\Delta t\|\delta^2\zeta^1+\delta^2W^1\|_1\,\|d_t\zeta^1\|_1\\
&\quad\leqq C\Delta t\left\{\|d_t\zeta^1\|_1\Delta t+\|d_t\zeta^0\|_1\Delta t\right.\\
&\qquad\left.+C\Delta t\int_{t^0}^{t^2}\left\|\frac{\partial^2W}{\partial t^2}(\cdot, s)\right\|_1 ds\right\}\|d_t\zeta^1\|_1\\
&\quad\leqq\tfrac{1}{32}\beta\Delta t\|d_t\zeta^1\|_1^2+C((\Delta t)^4+h^{2r-2})
\end{aligned}
\tag{3.34}
$$

if $\Delta t<\beta/64$, since by the Cauchy inequality and (2.9)

$$
\left(\Delta t\int_{t^0}^{t^2}\left\|\frac{\partial^2W}{\partial t^2}(\cdot, s)\right\|_1 ds\right)^2\leqq C(\Delta t)^3\int_{t^0}^{t^2}\left\|\frac{\partial^2W}{\partial t^2}(\cdot, s)\right\|_1^2 ds\leqq CK_3(\Delta t)^3.
\tag{3.35}
$$

For $n \geqq 2$, by (2.29b) and the equivalence of norms, we have

$$\begin{aligned}
&\Big|(c(EV^n)(V^{n+1}-\bar{V}^{n+1}), d_t\zeta^n)+\Big(a(EV^n)\nabla\Big(\frac{V^{n+1}-\bar{V}^{n+1}}{2}\Big), \nabla d_t\zeta^n\Big)\Delta t \\
&\qquad\qquad +(b(EV^n, \nabla EV^n)\nabla(V^{n+1}-\bar{V}^{n+1}), \nabla d_t\zeta^n)\Big| \\
&\quad \leqq C_{14}\|V^{n+1}-\bar{V}^{n+1}\|_1\,\|d_t\zeta^n\|_1 \\
&\quad \leqq C_{14}K\rho\|\delta^3 V^n\|_1\,\|d_t\zeta^n\|_1 \\
&\quad \leqq C_{14}K\rho\{\|\delta^2 d_t\zeta^n\|_1\Delta t+\|\delta^3 W^n\|_1\}\|d_t\zeta^n\|_1 \\
&\quad \leqq \Delta t C_{14}K\rho\|d_t\zeta^n\|_1\,\{\|d_t\zeta^n\|_1+2\|d_t\zeta^{n-1}\|_1+\|d_t\zeta^{n-2}\|_1+C\Delta t\sigma_{4,n}\}
\end{aligned} \tag{3.36}$$

where

$$\sigma_{4,n}=\int_{t^{n-2}}^{t^{n+1}}\left\|\frac{\partial^3 W}{\partial t^3}(\cdot, s)\right\|_1 ds. \tag{3.37}$$

Note that from (2.9) and as in (3.35), $\sum_{n=1}^{N-1}\sigma_{4,n}^2 \leqq CK_3\Delta t$. Since we have seven terms involving $\|d_t\zeta^n\|_1$ for various n in (3.36), we want to iterate sufficiently many times so that

$$\rho K<\beta/(16C_{14})\equiv C_{12}. \tag{3.38}$$

We note that this number of iterations is independent of n, Δt and h. We then see that we can group the terms involving $\|d_t\zeta^n\|_1$ after summation, use (3.31), and absorb them all in the left side of (3.33) as before. Then doing exactly as we did in the proof of Theorem 3.1, we obtain the desired result. □

If b in case I of (2.3) is independent of ∇u, we can determine W_b as in (2.7b) and obtain the results of Theorems 3.1 and 3.2 with weaker smoothness assumptions on $\partial u/(\partial t)$. Note that, from (2.7b), we have

$$d_t[b(u^n)\nabla u^n]=d_t[b(u^n)\nabla W_b^n]-d_t\eta^n. \tag{3.39}$$

Also, we have

$$d_t[b(u^n)\nabla u^n]=(d_t b(u^n))\nabla u^{n+1/2}+(b(u))^{n+1/2}\nabla d_t u^n. \tag{3.40}$$

Combining (3.39) and (3.40), we have

$$(b(u))^{n+1/2}\nabla d_t u^n=(b(u))^{n+1/2}\nabla d_t W_b^n-(d_t b(u^n))\nabla\eta^{n+1/2}-d_t\eta^n. \tag{3.41}$$

The following notion of a dual norm is useful; define

$$\|\varphi\|_{H^{-1}(\Omega)}=\sup\left\{\int_\Omega \varphi\psi\,dx : \|\psi\|_1=1\right\}. \tag{3.42}$$

Assume that the Neumann problem for $-\Delta+I$ on Ω is H^3-regular. Then, as in [13], for each $t\in[0, T]$,

$$\|d_t\eta^n\|_{H^{-1}(\Omega)}\leqq Ch\|d_t\eta^n\|\leqq Ch^{r-1}\left\{\|u\|_{r-2}+\left\|\frac{\partial u}{\partial t}\right\|_{r-2}\right\}. \tag{3.43}$$

We now state the following corollary.

COROLLARY 3.2. *Let the assumptions of Theorem* 3.1 *hold with* (2.6b) *replaced by*

$$\|u\|_{L^2(J;H^r)}+\left\|\frac{\partial u}{\partial t}\right\|_{L^2(J;H^{r-2})}\leqq K_2. \tag{3.44}$$

In addition, assume that $\partial u/(\partial t)\in L^\infty(J;H^2)$ boundedly and the Neumann problem for $-\Delta+I$ on Ω is H^3-regular. If b is independent of ∇u, then the results of Theorem 3.1 *hold.*

Proof. With W_b defined by (2.7 b) and satisfying (3.41), equation (3.4) becomes

$$
\begin{aligned}
&(c(EU^n)d_tZ^n, y)+(a(EU^n)\nabla Z^{n+1/2}, \nabla y)+(b(EU^n)\nabla d_tZ^n, \nabla y)\\
&\quad=(c(EU^n)d_t\eta^n, y)-\left(\left[c(EU^n)d_tu^n-c(u(t^{n+1/2}))\nabla\frac{\partial u}{\partial t}(t^{n+1/2})\right], y\right)\\
&\qquad+(a(EU^n)\nabla\eta^{n+1/2}, \nabla y)\\
&\qquad-([a(EU^n)\nabla u^{n+1/2}-a(u(t^{n+1/2}))\nabla u(t^{n+1/2})], \nabla y)\\
&\qquad+([b(EU^n)-(b(u))^{n+1/2}]\nabla d_tW_b^n, \nabla y)+(d_t\eta^n, y)\\
&\qquad+(d_t(b(u^n))\nabla\eta^{n+1/2}, \nabla y)\\
&\qquad+\left(\left[(b(u))^{n+1/2}\nabla d_tu^n-b(u(t^{n+1/2}))\nabla\frac{\partial u}{\partial t}(t^{n+1/2})\right], \nabla y\right)\\
&\qquad+([f(t^{n+1/2}, EU^n, \nabla EU^n)\\
&\qquad\qquad-f(t^{n+1/2}, u(t^{n+1/2}), \nabla u(t^{n+1/2}))], y), \qquad y\in\mathcal{M}_h.
\end{aligned}
\tag{3.45}
$$

Using the test function $y=Z^{n+1}-Z^n=d_tZ^n\Delta t$, the analysis follows exactly as before except for the terms involving $d_t\eta^n$ and the coefficients b. Using (3.42) and (3.43), we treat the first and sixth terms on the right of (3.45) as follows

$$
\begin{aligned}
&\Delta t|(c(EU^n)d_t\eta^n, d_tZ^n)+(d_t\eta^n, d_tZ^n)|\\
&\quad\leqq\tfrac{1}{16}\beta\Delta t\|d_tZ^n\|_1^2+C\Delta t\|d_t\eta^n\|_{H^{-1}(\Omega)}^2\\
&\quad\leqq\tfrac{1}{16}\beta\Delta t\|d_tZ^n\|_1^2+C\Delta th^{2r-2}\left\{\|u\|_{r-2}^2+\left\|\frac{\partial u}{\partial t}\right\|_{r-2}^2\right\}.
\end{aligned}
\tag{3.46}
$$

The fifth term on the right of (3.45) is bounded by

$$
\begin{aligned}
&|([b(EU^n)-b(EW_b^n)+b(EW_b^n)-b(Eu^n)+b(Eu^n)-(b(u))^{n+1/2}]\nabla d_tW_b^n, \nabla d_tZ^n)\Delta t|\\
&\quad\leqq\tfrac{1}{16}\beta\Delta t\|d_tZ^n\|_1^2+C\Delta t\{\|Z^n\|_1^2+\|Z^{n-1}\|_1^2+\|\eta^n\|_1^2+\|\eta^{n-1}\|_1^2+\sigma_{1,n}\}.
\end{aligned}
\tag{3.47}
$$

We note that the constant C in (3.47) depends upon K_1 and a bound on $\|\partial W_b/\partial t\|_{L^\infty(J;W_3^1)}$ from (2.10). The rest of the proof follows as before. □

COROLLARY 3.3. *Let the assumptions of Corollary* 3.2 *and* (2.4) *hold. If we also assume $u\in L^\infty(J;H^3)$ boundedly, the results of Theorem* 3.2 *hold.*

Proof. The proof of Theorem 3.2 is modified using an analog of (3.45) as above. □

4. Extensions to other cases. In this section we relax some of the assumptions of case I of Q and thus treat a wider class of applications of (1.1). We emphasize that, in case II, b is independent of ∇u but, in case III, we again have $b=(bx, u, \nabla u)$.

First we shall treat case II of Q. We see that by letting b have the value zero, we are allowing our Sobolev equation to degenerate to a nonlinear parabolic equation for some set in Ω. We note that the Neumann conditions for (1.1) degenerate at the same time to standard Neumann conditions for the parabolic problem. In terms of the model of fluid flow in a fissured medium introduced by [3], this would allow the size of the blocks in the fissuring to tend to zero and approach the standard model for fluid flow in a porous medium, without fissures.

Mathematically, the assumption of case II of (2.3) will still force $L^n(\xi)$ from (2.16) to be positive definite for each n, so we have no problems with the existence of a solution to (2.13) in this case. Similarly, we can define our conjugate gradient iteration just as before to obtain the approximation V^n, $n=0, 1, \cdots, N$. However, we used the fact that we had a positive multiple of a term equivalent to $\|d_tZ^n\|_1^2$ (or $\|d_t\zeta^n\|_1^2$) on the left side of (3.4) and (3.33) in the proofs of Theorem 3.1 and Theorem 3.2 respectively, in a very crucial manner. We recall that we grouped small multiples of terms of this type which arose from bounds of the form (3.7) and subtracted them from the positive term mentioned above in our proofs. We must therefore modify our proofs in this case to account for the lack of these important terms.

THEOREM 4.1. *Let case* II *of* (2.3), (2.4), R *and the restriction on* $\{\mathcal{M}_h\}$ *of* § 2 *be satisfied. Let* $r \geqq 3$ *and an analog of* (3.1) *for case* II *hold. Assume that*

$$\|u\|_{H^2(J;H^r)} + \left\|\frac{\partial^4 u}{\partial t^4}\right\|_{L^1(J;H^1)} + \|u\|_{W^3_\infty(J;H^1)} \leqq K_2. \tag{4.1}$$

Then there exist positive constants τ_0 *and* C_{16} *such that if* $h \leqq \tau_0$ *and* $\Delta t \leqq h^{d/3}$,

$$\sup_{t^n} \|U-u\|_1 \leqq C_{16}\{(\Delta t)^2 + h^{r-1}\}. \tag{4.2}$$

The constant C_{16} *has the same dependencies as* C_2.

Proof. We first note that if the same test function is used as above and if $\eta^n = u^n - W^n$, then estimates of all terms on the right of (3.4) will follow as before except for the terms involving ∇y. These terms are contributed by the coefficients a and b and must be treated differently. We note that, as in (3.7),

$$\begin{aligned}
&(a(EU^n)\nabla u^{n+1/2}, \nabla y) - (a(u(t^{n+1/2}))\nabla u(t^{n+1/2}), \nabla y) \\
&\quad = ([a(EU^n) - a(u(t^{n+1/2}))]\nabla u^{n+1/2}, \nabla y) \\
&\qquad + (a(u(t^{n+1/2}))[\nabla u^{n+1/2} - \nabla u(t^{n+1/2})], \nabla y) \\
&\quad \equiv A_{1,n} + A_{2,n}.
\end{aligned} \tag{4.3}$$

Similarly, we see that

$$\begin{aligned}
&(b(EU^n)\nabla d_t u^n, \nabla y) - \left(b(u(t^{n+1/2}))\nabla \frac{\partial u}{\partial t}(t^{n+1/2}), \nabla y\right) \\
&\quad = ([b(EU^n) - b(u(t^{n+1/2}))]\nabla d_t u^n, \nabla y) \\
&\qquad + \left(b(u(t^{n+1/2}))\left[\nabla d_t u^n - \nabla \frac{\partial u}{\partial t}(t^{n+1/2})\right], \nabla y\right) \\
&\quad \equiv B_{1,n} + B_{2,n}.
\end{aligned} \tag{4.4}$$

We shall use summation by parts in time to bound each of $A_{i,n}$, $B_{i,n}$, $i=1, 2$. As typical examples, we shall give estimates for $B_{1,n}$ and $B_{2,n}$. With $y = Z^{n+1} - Z^n$, we sum by parts to obtain

$$\left|\sum_{n=1}^{l-1} B_{2,n}\right| \leqq \sum_{n=2}^{l-1} \left|\Delta t\left(\frac{\left\{b(u(t^{n+1/2}))\left[\nabla d_t u^n - \nabla \frac{\partial u}{\partial t}\right] - b(u(t^{n-1/2}))\left[\nabla d_t u^{n-1} - \nabla \frac{\partial u}{\partial t}\right]\right\}}{\Delta t}, \nabla Z^n\right)\right|$$

$$+\left|\left(b(u(t^{l-1/2}))\nabla\left(d_t u^{l-1} - \frac{\partial u}{\partial t}(t^{l-1/2})\right), \nabla Z^l\right)\right|$$

$$+\left|\left(b(u(t^{3/2}))\nabla\left(d_t u^{1} - \frac{\partial u}{\partial t}(t^{3/2})\right), \nabla Z^1\right)\right|$$

$$\leqq \sum_{n=2}^{l-1}\left|\left(\left\{\frac{b(u(t^{n+1/2})) - b(u(t^{n-1/2}))}{\Delta t}\right\}\nabla\left[d_t u^n - \frac{\partial u}{\partial t}\right]\Delta t\right.\right. \tag{4.5}$$

$$\left.\left. + b(u(t^{n-1/2}))\nabla\left\{\frac{d_t u^n - \frac{\partial u}{\partial t}(t^{n+1/2}) - \left[d_t u^{n-1} - \frac{\partial u}{\partial t}(t^{n-1/2})\right]}{\Delta t}\right\}\Delta t, \nabla Z^n\right)\right|$$

$$+\tfrac{1}{32}\|Z^l\|^2_{a^{l-1}} + \|Z^1\|_1^2 + C_{17}(\Delta t)^4$$

$$\leqq \tfrac{1}{32}\|Z^l\|^2_{a^{l-1}} + \|Z^1\|_1^2 + C_{18}(\Delta t)^4 + C_{19}\sum_{n=1}^{l-1}[\Delta t + \sigma_{5,n}]\|Z^n\|_1^2,$$

where

$$\sigma_{5,n} = \int_{t^{n-1}}^{t^{n+1}} \left\|\frac{\partial^4 u}{\partial t^4}(\cdot, s)\right\|_1 ds. \tag{4.6}$$

We note that C_{17} depends upon the bound K_2 for $\|\partial^3 u/\partial t^3\|_{L^\infty(J;H^1)}$ from (4.1) while C_{18} also depends upon the bound for $\|\partial^4 u/\partial t^4\|_{L^1(J;H^1)}$. To estimate $B_{1,n}$, sum by parts in time again to obtain

$$\left|\sum_{n=1}^{l-1} B_{1,n}\right| \leqq \left|\sum_{n=2}^{l-1} (\{[b(EU^n) - b(u(t^{n+1/2}))]\nabla d_t u^n\right.$$

$$\left. - [b(EU^{n-1}) - b(u(t^{n-1/2}))]\nabla d_t u^{n-1}\}, \nabla Z^n)\right|$$

$$+ |([b(EU^{l-1}) - b(u(t^{l-1/2}))]\nabla d_t u^{l-1}, \nabla Z^l)|$$

$$+ |([b(EU^{1}) - b(u(t^{3/2}))]\nabla d_t u^{1}, \nabla Z^1)| \tag{4.7}$$

$$\leqq C_{20}\{\|Z^1\|_1^2 + \|Z^0\|^2 + \|Z^{l-2}\|^2 + \|Z^{l-1}\|^2 + \|\eta^0\|^2$$

$$+ \|\eta^1\|^2 + \|\eta^{l-2}\|^2 + \|\eta^{l-1}\|^2 + (\Delta t)^4\}$$

$$+ \tfrac{1}{20}\|Z^l\|^2_{a^{l-1}} + \left|\sum_{n=2}^{l-1}([b(EU^n)\right.$$

$$\left. - b(u(t^{n+1/2}))][\nabla d_t u^n - \nabla d_t u^{n-1}], \nabla Z^n)\right|$$

$$+ \left|\sum_{n=2}^{l-1}(\nabla d_t u^{n-1}[b(EU^n) - b(u(t^{n+1/2}))\right.$$

$$\left. - \{b(EU^{n-1}) - b(u(t^{n-1/2}))\}], \nabla Z^n)\right|.$$

Note that C_{20} depends upon the bound K_2 from (4.1). We bound the next to the last term on the right of (4.7) as follows:

$$\left|\sum_{n=2}^{l-1}([b(EU^n)-b(u(t^{n+1/2}))][\nabla d_t u^n-\nabla d_t u^{n-1}],\nabla Z^n)\right| \tag{4.8}$$
$$\leqq C_{21}\sum_{n=1}^{l-1}\sigma_{6,n}[\|Z^n\|_1^2+\|\eta^n\|^2+(\Delta t)^4]$$

where

$$\sigma_{6,n}=\int_{t^{n-1}}^{t^{n+2}}\left\|\nabla\frac{\partial^2 u}{\partial t^2}(\cdot,s)\right\|_{L^3(\Omega)}ds,\qquad n=1,\cdots,l-2, \tag{4.9}$$

and $\sigma_{6,l-1}$ is as in (4.9) with t^{n+2} replaced by t^{n+1}. Then $\sum_{n=1}^{N-1}\sigma_{6,n}\leqq CK_2$ by (4.1). Next define

$$\text{a) } b'_{1,n}(x)=\int_0^1\frac{\partial b}{\partial u}(x,\theta EU^n+(1-\theta)EU^{n-1})\,d\theta, \tag{4.10}$$
$$\text{b) } b'_{2,n}(x)=\int_0^1\frac{\partial b}{\partial u}(x,\theta u(t^{n+1/2})+(1-\theta)u(t^{n-1/2}))\,d\theta.$$

We now treat the last term on the right of (4.7) as follows

$$\left|\sum_{n=2}^{l-1}(\nabla d_t u^{n-1}[b(EU^n)-b(u(t^{n+1/2}))-\{b(EU^{n-1})-b(u(t^{n-1/2}))\}],\nabla Z^n)\right|$$
$$=\left|\sum_{n=2}^{l-1}(\nabla d_t u^{n-1}[b'_{1,n}\{EU^n-EU^{n-1}\}-b'_{2,n}\{u(t^{n+1/2})-u(t^{n-1/2})\}],\nabla Z^n)\right|$$
$$\leqq\sum_{n=2}^{l-1}\left|\left(\left\{\nabla d_t u^{n-1}[b'_{1,n}-b'_{2,n}]\left[\frac{u(t^{n+1/2})-u(t^{n-1/2})}{\Delta t}\right]\Delta t\right.\right.\right. \tag{4.11}$$
$$+\nabla d_t u^{n-1}b'_{1,n}[E\delta Z^{n-1}-E\delta\eta^{n-1}+E\delta u^{n-1}-\{u(t^{n+1/2})$$
$$\left.\left.\left.-u(t^{n-1/2})\}]\right\},\nabla Z^n\right)\right|$$
$$\leqq C_{22}\Delta t\sum_{n=0}^{l-1}\{\|Z^n\|_1^2+\|\eta^n\|^2+\|d_t\eta^n\|^2\}$$
$$+\tfrac{1}{32}c_*\Delta t\sum_{n=1}^{l-1}\|d_tZ^n\|^2+C_{23}\Delta t\|d_tZ^0\|^2+C_{24}(\Delta t)^4.$$

One can easily see that corresponding bounds can be obtained in exactly the same way for $A_{1,n}$ and $A_{2,n}$. Similarly, using summation by parts, we see that

$$\left|\sum_{n=1}^{l-1}(b(EU^n)\nabla d_t\eta^n,\nabla(Z^{n+1}-Z^n))\right|$$
$$\leqq\left|\sum_{n=2}^{l-1}([b(EU^n)-b(EU^{n-1})]\nabla d_t\eta^n,\nabla Z^n)\right| \tag{4.12}$$
$$+\left|\sum_{n=2}^{l-1}(b(EU^{n-1})[\nabla d_t\eta^n-\nabla d_t\eta^{n-1}],\nabla Z^n)\right|$$
$$+|(b(EU^{l-1})\nabla d_t\eta^{l-1},\nabla Z^l)|+|(b(EU^1)\nabla d_t\eta^1,\nabla Z^1)|.$$

We use the fact that $\|\nabla\partial\eta/\partial t\|_{L^\infty(J;L^\infty)}\leqq C\|u\|_{W^1_\infty(J;H^3)}$ to be bound the first term on the right of (4.12) as follows:

$$
(4.13)\quad
\begin{aligned}
&\left|\sum_{n=2}^{l-1}([b(EU^n)-b(EU^{n-1})]\nabla d_t\eta^n, \nabla Z^n)\right|\\
&\quad=\left|\sum_{n=2}^{l-1}\left(\frac{\partial b}{\partial u}[E\delta Z^{n-1}+E\delta W^{n-1}]\nabla d_t\eta^n, \nabla Z^n\right)\right|\\
&\quad\leqq \sum_{n=2}^{l-1} C_{25}[\|\nabla d_t\eta^n\|_{L^\infty(\Omega)}\|E\delta Z^{n-1}\|+\|E\delta W^{n-1}\|_{L^\infty(\Omega)}\|d_t\eta^n\|_1]\|Z^n\|_1\\
&\quad\leqq C_{26}\Delta t\|d_tZ^0\|^2+\tfrac{1}{32}c_*\Delta t\sum_{n=1}^{l-1}\|d_tZ^n\|^2+C_{27}\Delta t\sum_{n=2}^{l-1}[\|d_t\eta^n\|_1^2+\|Z^n\|_1^2].
\end{aligned}
$$

This yields a bound for (4.12) of the form

$$
\begin{aligned}
&\left|\sum_{n=1}^{l-1}(b(EU^n)\nabla d_t\eta^n, \nabla(Z^{n+1}-Z^n))\right|\\
(4.14)\quad &\leqq \tfrac{1}{20}\|Z^l\|_{a^{l-1}}^2+\tfrac{1}{32}c_*\Delta t\sum_{n=1}^{l-1}\|d_tZ^n\|^2+C_{28}\{\|Z^1\|_1^2+\|d_tZ^0\|^2\Delta t+\|d_t\eta^{l-1}\|_1^2+\|d_t\eta^1\|_1^2\}\\
&\quad+C_{29}\Delta t\sum_{n=2}^{l-1}[\|Z^n\|_1^2+\|d_t\eta^n\|_1^2+\|d_t^2\eta^n\|_1^2].
\end{aligned}
$$

We note here that if a projection W_b of the form (2.7b) could be defined, then, as in the proof of Corollary 3.2, we would not have to treat the term bounded in (4.14) and an assumption on u of the form $u\in H^1(J;H^r)$ instead of the stronger $u\in H^2(J;H^r)$ would suffice for this result. However, since b is allowed to take on the value zero, the projection W_b from (2.7b) is not well-defined. The third term on the right of (3.4) can be bounded using similar techniques. Combining the above estimates with those obtained in the proof of Theorem 3.1, we obtain

$$
(4.15)\quad
\begin{aligned}
&\tfrac{3}{4}c_*\Delta t\sum_{n=1}^{l-1}\|d_tZ^n\|^2+\Delta t\sum_{n=1}^{l-1}\|d_tZ^n\|_{b^n}^2+\sum_{n=1}^{l-1}\{\|Z^{n+1}\|_{a^n}^2-\|Z^n\|_{a^n}^2\}\\
&\quad\leqq \tfrac14\|Z^l\|_{a^{l-1}}^2+C_{30}\{\|Z^1\|_1^2+\|Z^0\|_1^2+\Delta t\|d_tZ^0\|\}+C_{31}\{\|Z^{l-1}\|^2+\|Z^{l-2}\|^2\}\\
&\qquad+C_{32}\{\|\eta^0\|^2+\|\eta^1\|^2+\|\eta^{l-2}\|^2+\|\eta^{l-1}\|^2+\|d_t\eta^1\|_1^2+\|d_t\eta^{l-1}\|_1^2+(\Delta t)^4\}\\
&\qquad+C_{33}\sum_{n=1}^{l-1}[\Delta t+\sigma_{6,n}][\|\eta^n\|_1^2+\|d_t\eta^n\|_1^2+\|d_t^2\eta^n\|_1^2]\\
&\qquad+C_{34}\sum_{n=1}^{l-1}[\Delta t+\sigma_{5,n}+\sigma_{6,n}]\|Z^n\|_1^2.
\end{aligned}
$$

We next introduce an L^2-norm of Z to the left side of (4.15) in order to apply the discrete Gronwall lemma in the H^1-norm and simultaneously treat the term multiplied by C_{31} as follows:

$$
(4.16)\quad
\begin{aligned}
(C_{31}+\tfrac12)\{\|Z^{n+1}\|^2-\|Z^n\|^2\}&=2(C_{31}+\tfrac12)\Delta t(d_tZ^n, Z^n)+(C_{31}+\tfrac12)(\Delta t)^2\|d_tZ^n\|^2\\
&\leqq \tfrac{1}{12}c_*\Delta t\|d_tZ^n\|^2+C_{35}\|Z^n\|^2\Delta t
\end{aligned}
$$

if $\Delta t< c_*[16(C_{31}+\frac12)]^{-1}$. Sum the above inequality from $n=1$ to $l-1$ (with $C_{31}=0$), from $n=1$ to $l-2$, and then from $n=1$ to $l-3$. Then use the telescoping properties of the sums and add the results to (4.15) to obtain

$$\tag{4.17}\begin{aligned}&\tfrac{1}{2}c^*\Delta t\sum_{n=1}^{l-1}\|d_tZ^n\|+\tfrac{1}{2}\|Z^l\|^2+\sum_{n=1}^{l-1}\{\|Z^{n+1}\|_{a^n}^2-\|Z^n\|_{a^{n-1}}^2\}+\Delta t\sum_{n=1}^{l-1}\|d_tZ^n\|_{b^n}^2\\&\leqq\tfrac{1}{4}\|Z^l\|_{a^{l-1}}^2+C_{36}((\Delta t)^4+h^{2r-2})+C_{37}\sum_{n=1}^{l-1}[\Delta t+\sigma_{5,n}+\sigma_{6,n}]\|Z^n\|_1^2\end{aligned}$$

We have used Lemma 2.1 and the result for case II corresponding to (3.1) to obtain the bound appearing in the second term on the right of (4.17). We then use the ideas of (3.15)–(3.28), the telescoping sum, the comparability of $\|\cdot\|_{a^n}+\|\cdot\|$ with $\|\cdot\|_1$, and the discrete Gronwall lemma to obtain a result similar to (3.29). The final result of the theorem follows from (2.8), the inequalities analogous to (3.1) and (3.29), and the triangle inequality. □

We note here that the assumption of a positive uniform lower bound for $b(x, q)$ was used very strongly in the proof of Theorem 3.2. Since assumption (2.3) case II does not allow this, we must again modify our proof and Theorem 4.2 will be weaker than Theorem 3.2. By making the "inverse assumption" (2.2a) on $\mathcal{M}_h$ and making the additional assumption that, for some constant C_{39}, independent of h, we have

$$\tag{4.18}\Delta t\leqq C_{39}h^2,$$

we can use the results of Theorem 4.1 to obtain the following result.

THEOREM 4.2. *Assume the hypotheses of Theorem* 4.1 *and* (4.18) *hold. Assume that* V^0 *and* V^1 *satisfy*

$$\tag{4.19}\|V^0-W^0\|_1+\|V^1-W^1\|_1+(\Delta t)^{1/2}\|d_t(V^0-W^0)\|\leqq C_{40}\{(\Delta t)^2+h^{r-1}\}.$$

For $n\geqq 1$, *let* V^n *satisfy* (2.29) *with* ρK *satisfying* (2.32) *with* $\alpha=1$. *Then there exist positive constants* C_{41}, C_{42} *and* τ_0 *such that if* $h\leqq\tau_0$ *and* $\Delta t\leqq C_{39}h^2$,

$$\sup_{t^n}\|V-u\|_1\leqq C_{42}\{(\Delta t)^2+h^{r-1}\}.$$

The constant C_{42} *has the same dependencies as* C_{40}.

Proof. The proof of this theorem follows from Theorem 4.1 as in Theorem 3.2, except we need new estimates for (3.34) and (3.36) to fit into the analysis of Theorem 4.1. From (2.29) we obtain the following to replace (3.36). The replacement for (3.34) is determined analogously. For $n\geqq 2$, we use (2.2a) and (4.18) to obtain

$$\tag{4.20}\begin{aligned}\Delta t\Big|&\Big(c(EV^n)\frac{V^{n+1}-\bar V^{n+1}}{\Delta t},d_t\zeta^n\Big)+(\tfrac{1}{2}a(EV^n)\nabla(V^{n+1}-\bar V^{n+1}),\nabla d_t\zeta^n)\\&\qquad+\Big(b(EV^n)\nabla\Big(\frac{V^{n+1}-\bar V^{n+1}}{\Delta t}\Big),\nabla d_t\zeta^n\Big)\Big|\\&\leqq C_{43}\|V^{n+1}-\bar V^{n+1}\|_1\,\|d_t\zeta^n\|_1\\&\leqq\rho KC_{43}\|\delta^3V^n\|_1\,\|d_t\zeta^n\|_1\\&\leqq\rho KC_{43}\{\Delta t[\|d_t\zeta^n\|_1+\|d_t\zeta^{n-1}\|_1+\|d_t\zeta^{n-1}\|_1]+C_{44}(\Delta t)^3\}\|d_t\zeta^n\|_1\\&\leqq\rho KC_{43}\Big\{\frac{(\Delta t)^{1/2}K_0}{h}[\|d_t\zeta^n\|+\|d_t\zeta^{n-1}\|]+\|d_t\zeta^{n-2}\|]\\&\qquad+C_{44}(\Delta t)^{5/2}\Big\}\frac{(\Delta t)^{1/2}K_0}{h}\|d_t\zeta^n\|\\&\leqq\rho KC_{43}\{K_0C_{39}^{1/2}[\|d_t\zeta^n\|+\|d_t\zeta^{n-1}\|+\|d_t\zeta^{n-2}\|]+C_{44}(\Delta t)^{5/2}\}K_0C_{39}^{1/2}\|d_t\zeta^n\|\\&\leqq\tfrac{1}{8}c_*\Delta t\|d_t\zeta^n\|^2+\tfrac{1}{24}c_*\Delta t[\|d_t\zeta^{n-1}\|^2+\|d_t\zeta^{n-2}\|^2]+C_{45}(\Delta t)^6,\end{aligned}$$

if we choose μ such that we have

$$\rho < c_* \Delta t / (16 K C_{39} C_{43} K_0^2). \tag{4.21}$$

The rest of the proof follows as before. □

COROLLARY 4.1. *If $b \equiv 0$, then we can weaken the assumptions of Theorems 4.1 and 4.2 to*

$$\|u\|_{H^1(J;H^r)} + \|u\|_{W^1_\infty(J;H^2)} + \|u\|_{L^\infty(J;H^3)} + \|u\|_{H^3(J;H^1)} \leqq K_2 \tag{4.22}$$

with $r \geqq 2$. The results of Theorem 4.1 as well as optimal order L^2-estimates can be obtained. A conjugate gradient iterative process with (4.21) replaced by a fixed constant which is independent of n, h, and Δt yields optimal order L^2 and H^1 error estimates.

Proof. (See [12]).

We shall now consider a different weakening of the assumption of case I of Q. The new assumption is given by case III of Q and allows a_* to be less than or equal to zero. We note that in case $a(x, q) \equiv 0$, we have a form of the model used in [4], [5], [29], [34] to model long waves. Wahlbin considered this equation when $\Omega = \mathbb{R}$ in one dimension and b and c were independent of q. He achieved optimal order L^2, H^1, and L^∞ estimates in [34]. If $a_* < 0$, the equation can model backward time problems and has been used in [15], [27] to approximate unstable backward parabolic problems. For numerical methods for backward in time quasilinear problems, see [16], [19].

Mathematically, the assumption (2.3c.III) will allow $L^n(\xi)$ from (2.16) to lose its positive definite character unless $a_* = 0$. If $a_* < 0$, we can see that by making the further restriction on Δt, that for some $0 < \varepsilon < b_*$,

$$0 < \Delta t \leqq \frac{2b_* - \varepsilon}{|a_*|}, \tag{4.23}$$

then $L^n(\xi)$ will again be positive definite for each n and we again have the existence of a solution to (2.13). We can define the conjugate gradient iteration as before to obtain the approximation V^n, $n = 0, 1, \cdots, N$. We note that we no longer have the telescoping sequence of semi-norms $\|\cdot\|_{a^n}$ as in (3.6). We must therefore use a different test function in (3.5). Define

$$\bar{a} = \max\{a^*, |a_*|\}. \tag{4.24}$$

THEOREM 4.3. *Let case* III *of* (2.3), (2.4a), (3.1), R *and the restrictions on* $\{\mathcal{M}_h\}$ *of* § 2 *be satisfied. Let Δt satisfy* (4.23), $r \geqq 3$, *and*

$$\left\|\frac{\partial u}{\partial t}\right\|_{L^\infty(J;W^1_\infty)} \leqq K_2 \tag{4.25}$$

hold. Then there exist positive constants τ_0 and C_{46} such that if $h \leqq \tau_0$ and $\Delta t \leqq h^{d/3}$,

$$\sup_{t^n} \|U - u\|_1 \leqq C_{46}\{(\Delta t)^2 + h^{r-1}\}. \tag{4.26}$$

The constant C_{46} has the same dependencies as C_2.

Proof. First consider the test function $y = d_t Z^n \Delta t$ as before. Due to case III of (2.3) we shall treat the second term on the left of (3.4) with the terms on the right. We obtain

$$|(a(EU^n)\nabla Z^{n+1/2}, \nabla d_t Z^n \Delta t)| \leqq \tfrac{1}{4}\beta \Delta t \|d_t Z^n\|_1^2 + C_{47}\Delta t[\|Z^{n+1}\|_1^2 + \|Z^n\|_1^2]. \tag{4.27}$$

Letting $\Delta t \leqq (4C_{47})^{-1}\beta$ and using (4.27) we obtain as in (3.14)

$$\frac{\beta}{2}\Delta t \sum_{n=1}^{l-1} \|d_t Z^n\|_1^2 \leqq \frac{\beta}{4}\|Z^l\|_1^2 + C\Delta t \sum_{n=1}^{l-1} \|Z^n\|_1^2 + C\{(\Delta t)^4 + h^{2r-2}\}. \tag{4.28}$$

Since this does not give the estimate we need, we also use the test function $y = Z^{n+1/2}$ in (3.4) and treat the second term on the left of (3.4) with those on the right using (4.24). After multiplying by $2\Delta t$ and obtaining bounds as above, we use (2.28) to see that

$$\begin{aligned}\sum_{n=1}^{l-1} \{\|Z^{n+1}\|_{c^n}^2 - \|Z^n\|_{c^n}^2 + \|Z^{n+1}\|_{b^n}^2 + \|Z^n\|_{b^n}^2\} \\ \leqq \frac{\beta}{4}\|Z^l\|_1^2 + C\Delta t \sum_{n=1}^{l-1} \|Z^n\|_1^2 + C\{(\Delta t)^4 + h^2 r^{-2}\}.\end{aligned} \tag{4.29}$$

We shall now use our comparability of norms argument to make the sums on the left side of (4.29) telescope. Since $b = b(x, u, \nabla u)$, the argument is slightly more complicated than before. As in (3.14)–(3.18) we obtain from (4.29)

$$\begin{aligned}\beta\|Z^l\|_1^2 \leqq \frac{\beta}{4}\|Z^l\|_1^2 + C\Delta t \sum_{n=1}^{l-1} \|Z^n\|_1^2 + C\{(\Delta t)^4 + h^{2r-2}\} \\ + C \sum_{n=2}^{l-1} \{\sigma_{7,n} + \|\delta Z^{n-1}\|_{W_\infty^1} + \|\delta Z^{n-2}\|_{W_\infty^1}\}\|Z^n\|_1^2\end{aligned} \tag{4.30}$$

where

$$\sigma_{7,n} = \int_{t^{n-2}}^{t^{n+1}} \left\|\frac{\partial W}{\partial t}(\cdot, s)\right\|_{W_\infty^1} ds. \tag{4.31}$$

Note that $\sum_{n=2}^{l-1} \sigma_{7,n} \leqq CK_2$ since $\partial u/(\partial t) \in L^2(J; H^3)$ from (2.6b). Then adding (4.28) and (4.30), we obtain

$$\begin{aligned}\tfrac{1}{2}\beta\Delta t \sum_{n=1}^{l-1} \|d_t Z^n\|_1^2 + \tfrac{1}{2}\beta\|Z^l\|_1^2 \\ \leqq C\{(\Delta t)^4 + h^{2r-2}\} + C\Delta t \sum_{n=1}^{l-1} \|Z^n\|_1^2 \\ + C \sum_{n=2}^{l-1} \{\sigma_{7,n} + \|\delta Z^{n-1}\|_{W_\infty^1} + \|\delta Z^{n-2}\|_{W_\infty^1}\}\|Z^n\|_1^2.\end{aligned} \tag{4.32}$$

The same induction argument as used in the proof of Theorem 3.1 will then yield

$$\sum_{n=0}^{l-2} \|\delta Z^n\|_{W_\infty^1} \leqq C, \tag{4.33}$$

and the discrete Gronwall lemma will establish our result. □

THEOREM 4.4. *Let the assumptions of Theorem* 4.3 *be satisfied. Assume* V^0 *and* V^1 *satisfy* (3.31) *and* V^2 *satisfies* (2.33). *Then there exist positive constants* C_{48} *and* τ_0 *such that if* μ *is chosen independently of n, h, and* Δt *for* $n \geqq 2$ *such that* $\rho K \leqq \frac{1}{2}C_{12}$ *in* (2.29b), *if* $h \leqq \tau_0$ *and* $\Delta t \leqq h^{d/3}$,

$$\sup_{t^n} \|V - u\|_1 \leqq C_{48}\{(\Delta t)^2 + h^{r-1}\}. \tag{4.34}$$

Proof. Recall that to use the analysis of Theorem 4.3, we must use (3.34) and (3.36) as well as corresponding estimates for the test function $y = \zeta^{n+1/2}$. For example, after multiplying by Δt, (3.36) with the new test function yields

$$\begin{aligned}
&\Big|(c(EV^n)(V^{n+1}-\bar{V}^{n+1}),\zeta^{n+1/2})+\Big(a(EV^n)\nabla\frac{V^{n+1}-\bar{V}^{n+1}}{2},\nabla\zeta^{n+1/2}\Big)\Delta t\\
&\qquad\qquad+(b(EV^n)\nabla(V^{n+1}-\bar{V}^{n+1}),\nabla\zeta^{n+1/2})\Big|\\
&\leqq C_{49}\|V^{n+1}-\bar{V}^{n+1}\|_1\,\|\zeta^{n+1/2}\|_1\\
&\leqq \Delta t\rho K C_{49}\{\|d_t\zeta^n\|_1+2\|d_t\zeta^{n-1}\|_1+\|d_t\zeta^{n-2}\|_1+C\Delta t\sigma_{4,n}\}\{\|\zeta^{n+1}\|_1+\|\zeta^n\|_1\}\\
&\leqq \frac{\beta\Delta t}{32}\{\|d_t\zeta^n\|_1^2+\|d_t\zeta^{n-1}\|_1^2+\|d_t\zeta^{n-2}\|_1^2\}+C_{50}\Delta t\{\|\zeta^{n+1}\|_1^2+\|\zeta^n\|_1^2\}+C(\Delta t)^4.
\end{aligned}$$

We see that this estimate will fit into the analysis together with (3.34) and (3.36) and the techniques of the proof of Theorem 4.3 will yield the desired result. □

5. Computational considerations. In this section we consider some computational aspects of the extrapolated Crank–Nicolson–Galerkin (ECNG) method and its preconditioned conjugate gradient (PCG) iterative variant. We compare some rough operation counts for the two methods. Then we consider some machine-oriented stopping procedures which, while preserving the error bound, would stop the conjugate gradient iteration earlier than the pessimistic theoretical bounds given by (2.31) for case II and by (4.21) for case I and case III.

For estimating the arithmetic operation counts, we restrict our attention to spaces of piecewise polynomials over quasi-regular meshes. Although the heuristic arguments presented below can currently be made precise only in cases in which the meshes have very special structure (such as uniform mesh on a square), numerical experiments indicate that the assumptions are more generally valid.

First consider $d=2$. Let M be the dimension of $\mathcal{M}_h$ and N be the number of time steps. Assume that the work to factor a matrix with the structure of L^n or L_0 from (2.16) or (2.18), respectively, is $F\approx M^{3/2}$. For a rectangular mesh on a rectangle this order work estimate can be achieved optimally by the nested dissection process of George [21]. We next assume that the amount of work to solve the system once the matrix has been factored is essentially the number of nonzero elements, or $S\approx M\log M$.

Combining the above ideas, since the ECNG method requires a factorization at each time level, the total work for the ECNG method for cases I, II or III is of the order

$$N(F+S)=O(N(M^{3/2}+M\log M))=O(NM^{3/2}). \tag{5.1}$$

For cases I and III, we note that we need only a fixed number of iterations of the PCG method at each step with only one factorization. Thus the work estimate for the PCG method for cases I and III is

$$F+NS=O(M^{3/2}+NM\log M)=O(NM\log M). \tag{5.2}$$

For case II, where a norm reduction factor of $Q(\Delta t)$ is necessary, $O(\log(1/\Delta t))=O(\log N)$ iterations are required at each time step and the total work is

$$F+NS=O(M^{3/2}+N\log N(M\log M))=O(NM\log N\log M). \tag{5.3}$$

If $d=3$, the best conjectures we know of say that $F\approx M^2$ and $S\approx M^{4/3}$. We use these F and S in (5.1)–(5.3) to obtain work estimates for $d=3$. We define an "optimal" work estimate as one which is proportional to the number of unknowns in the problem—MN.

We summarize the above remarks in the following table of total work estimates:

	$d=2$	$d=3$
"Optimal"	NM	NM
Cases I and III of PCG	$NM \log M$	$NM^{4/3}$
Case II of PCG	$NM \log N \log M$	$NM^{4/3} \log N$
ECNG	$NM^{3/2}$	NM^2

One can clearly see that the PCG incomplete iterative methods yield work estimates that are "close to optimal" and which are much better than the ECNG methods for large M and N. For some slight modifications of the PCG methods with comparable work estimates see [12].

We now consider alternative stopping procedures for the iterations in the PCG methods. Let V_k^{n+1} correspond to x_k of (2.20). (2.24) then yields the comparability

$$0<\psi_0 \leqq \|V_k^{n+1} - \bar{V}^{n+1}\|_1^2/(L_0^{-1}q_k, q_k)_l \leqq \psi_1. \tag{5.4}$$

Since we compute $(L_0^{-1}q_k, q_k)_l$ during the PCG procedure, we can easily estimate the size of the error $\|V_k^{n+1} - \bar{V}^{n+1}\|_1$.

If we had

$$\|V_k^{n+1} - \bar{V}^{n+1}\|_1^2 \leqq \lambda, \tag{5.5}$$

then the left-hand side of (4.20) would be bounded by

$$C_{43}\|V_k^{n+1} - \bar{V}^{n+1}\|_1 \, \|d_t\zeta^n\|_1 \leqq C_{43}(\Delta t)^{-1/2}\|V_k^{n+1} - \bar{V}^{n+1}\|_1 \frac{(\Delta t)^{1/2}K_0}{h}\|d_t\zeta^n\|$$
$$\leqq \varepsilon\,\Delta t c_*\|d_t\zeta^n\|^2 + C\lambda(\Delta t)^{-2}.$$

Thus at each step if either a norm reduction factor of $O(\Delta t)$ is achieved or

$$\lambda(\Delta t)^{-2} \leqq C((\Delta t)^4 + h^{2r-2})\Delta t, \tag{5.6}$$

then the results of Theorem 4.2 will hold. Thus, in the program, one could set a parameter $\kappa = O((\Delta t)^3((\Delta t)^4 + h^{2r-2}))$ and stop iterating if

$$(L_0^{-1}q_k, q_k)_l \leqq \kappa \tag{5.7}$$

due to (5.4).

Also note that by comparing $(L_0^{-1}q_0, q_o)_l$ and $(L_0^{-1}q_k, q_k)_l$ we can observe the actual factor by which the norm is reduced. Another stopping procedure could be to define another parameter $\rho_1 = O(\Delta t)$ for case II, for example, and stop the iteration if

$$(L_0^{-1}q_k, q_k)_l/(L_0^{-1}q_0, q_0)_l \leqq \rho_1^2. \tag{5.8}$$

The results of Theorem 4.2 will still hold.

For each of the cases, there corresponds appropriate choices of κ and ρ_1 to retain the error bounds. The following table summarizes these choices.

Result	$\kappa \leqq$	$\rho_1 \leqq$
Theorem 3.2	$C(\Delta t)^2((\Delta t)^4 + h^{2r-2})$	$(C_{12}/K)(\psi_1/\psi_0)^{1/2}$
Theorem 4.2	$C(\Delta t)^3((\Delta t)^4 + h^{2r-2})$	$C\Delta t$
Theorem 4.4	$C(\Delta t)^2((\Delta t)^4 + h^{2r-2})$	$(C_{12}/K)(\psi_1/\psi_0)^{1/2}$

Acknowledgment. I would like to thank Professors Jim Douglas, Jr., and Todd Dupont for several suggestions and stimulating conversations.

REFERENCES

[1] O. AXELSSON, *On preconditioning and convergence acceleration in sparse matrix problems*, CERN European Organization for Nuclear Research, Geneva, 1974.

[2] ———, *A class of iterative methods for finite element equations*, Report 75, 03R Dept. of Computer Science, Chalmers University of Technology, Göteborg, Sweden, 1975.

[3] G. I. BARENBLATT, I. P. ZHELTOV AND I. N. KOCHINA, *Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks*, J. Appl. Math. Mech., 24 (1960), pp. 1286–1303.

[4] T. B. BENJAMIN, J. L. BONA AND J. J. MAHONEY, *Model equations for long waves in nonlinear dispersive systems*, Philos. Trans. Roy. Soc. London, Ser. A., 272 (1972), pp. 47–78.

[5] J. L. BONA AND P. J. BRYANT, *A mathematical model for long waves generated by wavemakers in non-linear dispersive systems*, Proc. Cambridge Philos. Soc., (1973), pp. 391–405.

[6] R. W. CARROLL AND R. E. SHOWALTER, *Singular and Degenerate Cauchy Problems*, Mathematics in Science and Engineering, vol. 127, Academic Press, New York, 1976.

[7] P. J. CHEN AND M. E. GURTIN, *On a theory of heat conduction involving two temperatures*, Z. Angew. Math. Phys., 19 (1968), pp. 614–627.

[8] P. L. DAVIS, *A quasilinear parabolic and a related third order problem*, J. Math. Anal. Appl., 49 (1970), pp. 327–335.

[9] J. E. DENDY, JR., *An analysis of some Galerkin schemes for the solution of nonlinear time-dependent problems*, this Journal, 12 (1975), pp. 541–565.

[10] J. DOUGLAS, JR. AND T. DUPONT, *Galerkin methods for parabolic equations*, this Journal 7 (1970), pp. 575–626.

[11] ———, *Preconditioned conjugate gradient iteration applied to Galerkin methods for a mildly nonlinear Dirichlet problem*, Sparse Matrix Computations, J. R. Bunch and D. J. Rose, eds., Academic Press, New York, 1976, pp. 333–348.

[12] J. DOUGLAS, JR., T. DUPONT AND R. E. EWING, *Incomplete iteration for time-stepping a nonlinear parabolic Galerkin method*, to appear.

[13] T. DUPONT, *L_2 error estimates for projection methods for parabolic equations in approximating domains*, Mathematical Aspects of Finite Elements in Partial Differential Equations, Academic Press, New York, 1974, pp. 313–352.

[14] T. DUPONT, G. FAIRWEATHER AND P. J. JOHNSON, *Three-level Galerkin methods for parabolic equations*, this Journal, 11 (1974), pp. 392–410.

[15] R. E. EWING, *The approximation of certain parabolic equations backward in time by Sobolev equations*, SIAM J. Math. Anal., 6 (1975), pp. 283–294.

[16] ———, *Numerical solution of Sobolev partial differential equations*, this Journal, 12 (1975), pp. 345–363.

[17] ———, *A coupled non-linear hyperbolic-Sobolev system*, Ann. Mat. Pura Appl., Ser. IV, 114 (1977), pp. 331–349.

[18] W. H. FORD, *Galerkin approximations to non-linear pseudo-parabolic partial differential equations*, Aequationes Math., 14 (1976), pp. 271–291.

[19] W. H. FORD AND T. W. TING, *Uniform error estimates for difference approximations to nonlinear pseudo-parabolic partial differential equations*, this Journal, 11 (1974), pp. 155–169.

[20] H GAJEWSKI AND K. ZACHARIAS, *Zur starken konvergenz partieller differentialgleichungen*, Math. Nachr., 47 (1970), pp. 365–376.

[21] J. A. GEORGE, *Nested dissection on a regular finite element mesh*, this Journal, 10 (1973), pp. 345–363.

[22] M. LUSKIN, *A Galerkin method for nonlinear parabolic equations with nonlinear boundary conditions*, to appear.

[23] J. NITSCHE, *Lineare spline-funktionen und die methoden von Ritz für elliptische randwert probleme*, Arch. Rational Mech. Anal., 36 (1970), pp. 348–355.

[24] H. H. RACHFORD, JR., *Two-level discrete-time Galerkin approximations for second order nonlinear parabolic partial differential equations*, this Journal, 10 (1973), pp. 1010–1026.

[25] R. E. SHOWALTER, *Well-posed problems for a partial differential equation of order $2m+1$*, SIAM J. Math. Anal., 1 (1970), pp. 214–231.

[26] ———, *Existence and representation theorems for a semi-linear Sobolev equation in Banach space*, SIAM J. Math. Anal., 3 (1972), pp. 527–543.

[27] ———, *The final value problem for evolution equations*, J. Math. Anal. Appl., 47 (1974), pp. 563–572.

[28] ———, *Nonlinear degenerate evolution equations and partial differential equations of mixed type*, SIAM J. Math. Anal., 6 (1975), pp. 25–42.

[29] ———, *A Sobolev equation for long waves in nonlinear dispersive systems*, SIAM J. Appl. Math., to appear.

[30] R. E. SHOWALTER AND T. W. TING, *Pseudo-parabolic partial differential equations*, SIAM J. Math. Anal., 1 (1970), pp. 1–26.

[31] D. TAYLOR, *Research on Consolidation of Clays*, Massachusetts Institute of Technology Press, Cambridge, MA, 1952.

[32] T. W. TING, *Certain non-steady flows of second-order fluids*, Arch. Rational Mech. Anal., 14 (1963), pp. 1–26.

[33] ———, *A cooling process according to two-temperature theory of heat conduction*, J. Math. Anal. Appl., 45 (1974), pp. 23–31.

[34] L. WAHLBIN, *Error estimates for a Galerkin method for a class of model equations for long waves*, Numer. Math. 23 (1975), pp. 289–303.

[35] M. F. WHEELER, *A priori L^2-error estimates for Galerkin approximations to parabolic partial differential equations*, this Journal, 10 (1973), pp. 723–759.