A Coupled Non-Linear Hyperbolic-Sobolev System (*).

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Summary. – *A boundary-initial value problem for a quasilinear hyperbolic system in one space variable is coupled to a boundary-initial value problem for a quasilinear equation of Sobolev type in two space variables of the form* $Mu_t(x, t) + L(t)u(x, t) = f(x, t, u(x, t))$ *where* M *and* $L(t)$ *are second order elliptic spacial operators. The coupling occurs through one of the boundary conditions for the hyperbolic system and the source term in the equation of Sobolev type. Such a coupling can arise in the consideration of oil flowing in a fissured medium and out of that medium via a pipe. Barenblatt, Zheltov, and Kochina* [2] *have modeled flow in a fissured medium via a special case of the above equation. A local existence and uniqueness theorem is demonstrated. The proof involves the method of characteristics, some applications of results of R. Showalter and the contraction mapping theorem.*

1. – Introduction.

To model subsonic flow in a pipe, it is standard practice to use a one-dimensional version of Euler's equations of motion which includes the friction between the fluid and the pipe. This system can be reduced to a standard hyperbolic system via a change of variables [7]. In [2], BARENBLATT, ZHELTOV and KOCHINA have modeled fluid flow in fissured rocks by an equation of Sobolev type. Since fluids are extracted from fissured rocks through pipes, we shall combine these models into one system. By consideration as a volumetric flow rate per unit area, the fluid velocity at the end of the pipe is used as part of the sink term in the non-linear partial differential equation of Sobolev type. The sink term is used to model the removal of fluids from the fissured medium. The coupling is completed by requiring that the density of the fluid in the pipe at its end be equal to the density of the fluid in the fissured medium at the end of the pipe. The density of the fluid in the medium is related in a non-linear fashion to the pressure which is modeled by the equation of Sobolev type [2]. In [3], CANNON and the author have discussed a similar coupled system involving flow in a porous medium modeled by a parabolic equation instead of flow in the fissured medium. See [3] for some similar results.

The preceding considerations motivate the study of the mathematical problem of determining real-valued functions $p = p(z, t)$, $q = q(z, t)$ and $w = w(x, t)$ such

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that the triple (p, q, w) satisfies

$$
(1.1)\quad
\begin{array}{lll}
a) & \dfrac{\partial p}{\partial t} + \lambda_1(z, t, p, q)\dfrac{\partial p}{\partial z} = R_1(z, t, p, q)\,, & 0 < z < 1\,,\ 0 < t \leq T\,,\\
b) & \dfrac{\partial q}{\partial t} + \lambda_2(z, t, p, q)\dfrac{\partial q}{\partial z} = R_2(z, t, p, q)\,, & 0 < z < 1,\ 0 < t \leq T\,,\\
c) & q(1, t) = G\big(t, p(1)\big)\,, & 0 < t \leq T\,,\\
d) & p(z, 0) = p_0(z)\,, & 0 < z < 1\,,\\
e) & q(z, 0) = q_0(z)\,, & 0 < z < 1\,,\\
f) & S(t)w = f\big(x, t, p(0, t), q(0, t), w(x, t)\big)\,, & x \in \Omega\,,\quad 0 < t \leq T\,,\\
g) & B(w) = 0\,, & x \in \partial\Omega\,,\quad 0 < t \leq T\,,\\
h) & w(x, 0) = \varphi(x)\,, & x \in \Omega\,,\\
i) & p(0, t) = \zeta\big(t, q(0, t), w(0, t)\big)\,, & 0 < t \leq T\,,
\end{array}
$$

where $x = (x_1, x_2) \in \boldsymbol{R}^2$, Ω is a domain in $\boldsymbol{R}^2$ which contains the origin, $\partial\Omega$ is the boundary of Ω, $S(t)$ is a time-dependent partial differential operator of Sobolev type to be discussed below, B denotes a boundary operator, and $\lambda_1, \lambda_2, R_1, R_2, G, p_0, q_0, \varphi, f$, and ζ are known functions of their respective arguments. In our application Ω represents the fissured medium, $\{0 < z < 1\}$ represents the pipe and $z = 0$ represents the end of the pipe in the fissured medium at a point which is taken to be the origin of the coordinate system for the fissured medium.

We shall now describe the operators $S(t)$ and B. To generalize the model used by [2] we shall consider the operator

$$
(1.2)\qquad S(t)w \equiv M\frac{\partial w}{\partial t}(x, t) + L(t)w(x, t)
$$

where M and, for each $t \in [0, T]$, $L(t)$ are elliptic differential operators of order 2 which satisfy restrictions to be specified in section 2. We note that in [2] $L(t)$ was a constant times the negative of the Laplacian operator and M was the identity operator minus a constant times the Laplacian operator.

We consider a « no-flow » condition at the boundary of Ω [2] which means that the conormal derivatives on $\partial\Omega$ which are determined by the operator M [1, p. 146, 2, 11, p. 263] are set equal to zero on $\partial\Omega$.

We shall consider the problem described in (1.1 f), (1.1 g) and (1.2) in terms of a generalized problem involving vector-valued functions $w(t)$ which map $t \in [0, T]$ to various Banach spaces of functions on Ω. This reformulation will be described in detail in section 2. In this formulation, the boundary condition (1.1 g) is a « natural » or « variational » condition which arises from restriction of the notion of a solution to certain types of Banach spaces [1, p. 146, 4, 11, p. 263].

The basic aim of this paper is to demonstrate that for T sufficiently small there exists a unique solution of (1.1). In section 2 we give a weaker formulation of the Sobolev part of (1.1) and define the notion of a strong solution to this part. In section 3 we give a definition of weak solution of (1.1) using the strong solution of the Sobolev part and a reformulation of the hyperbolic part of (1.1) into integral equations via the characteristics of the hyperbolic equations (1.1*a*) and (1.1*b*). In section 3 we also formulate a mapping $\mathcal{T}$. A fixed point of $\mathcal{T}$ will yield a weak solution of (1.1). We develop a priori estimates in sections 4, 5, and 6 which allow preservation of the function classes under the mapping $\mathcal{T}$. We obtain a priori estimates on the characteristics in section 4, on the solutions of the hyperbolic part in section 5, and on the solutions of the Sobolev part in section 6. In section 7 we use the results of sections 4, 5, and 6 to demonstrate the preservation of function classes, continuity, and contraction properties of the mapping $\mathcal{T}$. The statement of the main result of the paper is given at the end of section 7.

2. – A weaker formulation of the Sobolev part.

We shall adopt some notation and results of [10]. The space of continuous linear operators from the normed linear space X to the normed linear space Y will be denoted by $L(X, Y)$. Let W be a reflexive and separable Banach space with norm $|w|_W$, let W' be its dual, and let $\langle f, v\rangle$ be the $W'-W$ duality, i.e. the value of $f \in W'$ on $v \in W$.

The Sobolev space $H^k(\Omega) \equiv H^k$ is the Hilbert space of (equivalence classes of) real-valued functions in $L^2(\Omega) \equiv H$, all of whose distributional derivatives through order k belong to H. The inner product and norm are given, respectively, by

$$(u, v)_k = \sum\left\{\int_\Omega D^\alpha u D^\alpha v\, dx\colon |\alpha| \leqslant k\right\} \tag{2.1}$$

and $|u|_k = \sqrt{(u, u)_k}$. Let W be a dense subset of H such that the injection $W \hookrightarrow H$ is continuous. For our application we shall choose $W \equiv H^1$. Then for some C_1, we have

$$|w|_H \leqslant C_1 |w|_W . \tag{2.2}$$

Let $T > 0$ and $I_T \equiv [0, T]$. Let $m(\cdot,\cdot)$ and, for each $t \in I_T$, $l(t;\cdot,\cdot)$ be continuous bilinear forms on W. These forms define the operator $\mathcal{M} \in L(W, W')$ and the family of operators $\mathcal{L}(t) \in L(W, W')$ by the identities

$$\langle \mathcal{M}u, v\rangle = m(u, v)\,, \qquad u, v \in W\,, \tag{2.3}$$

and

$$\langle \mathcal{L}(t)u, v\rangle = l(t; u, v)\,, \qquad u, v \in W,\ t \in I_T\,. \tag{2.4}$$

Let M and $L(t)$ denote the respective restrictions of $\mathcal{M}$ and $\mathcal{L}(t)$ to H. These restrictions are unbounded operators on H with respective domains given by

$$D(M) = \{u \in W: m(u, v) = (Mu, v)_H\,,\ v \in W\} \tag{2.5}$$

and for each $t \in I_T$,

$$D\big(L(t)\big) = \{u \in W: l(t; u, v) = \big(L(t)u, v\big)_H\,,\ v \in W\}\,. \tag{2.6}$$

Assume that we are given a function $f: I_T \times W \to W'$.

DEFINITION. – A function $w: I_T \to W$ is a *weak solution* of

$$\mathcal{M}w'(t) + \mathcal{L}(t)w(t) = f\big(t, w(t)\big)\,, \qquad w(0) = \varphi\,, \tag{2.7}$$

if it is continuously differentiable on I_T and (2.7) is satisfied (in W') on I_T.

DEFINITION. – A *strong solution* of (2.7) is a weak solution for which each term of the equation is in H on I_T. Thus, in H,

$$\text{(2.8)} \qquad \begin{aligned} &a) \quad Mw'(t) + L(t)w(t) = f\big(t, w(t)\big)\,, \qquad t \in I_T\,,\\ &b) \quad w(0) = \varphi\,. \end{aligned}$$

We note that (2.8) is the equation in terms of vector valued functions which corresponds to (1.1 f) with the Sobolev operator defined as in (1.2) and the dependency on $p(0, t)$ and $q(0, t)$ temporarily suppressed. We shall specify some strong coercivity assumptions in section 6 on $m(\cdot,\cdot)$ and $l(t;\cdot,\cdot)$ which will insure that M and $L(t)$ are second-order elliptic operators [1]. If we choose $W \equiv H^1$, then the regularity theory for elliptic operators [1] will give that

$$\text{(2.9)} \qquad \begin{aligned} &a) \quad D(M) \quad\ = \{u \in H^2: m(u, v) = (Mu, v)_H\,,\ v \in H^1\}\\ &b) \quad D\big(L(t)\big) = \{u \in H^2: l(t; u, v) = \big(L(t)u, v\big)_H\,,\ v \in H^1\}\,. \end{aligned}$$

If $\partial\Omega$ is smooth enough for the divergence theorem to apply, the condition in (2.9) yields the « variational » or « natural » boundary condition [1, 4, 11] which was described in section 1. Thus the boundary condition (1.1 g) is built into our choice of W in this case.

We shall now consider the weaker formulation of the Sobolev part of (1.1) described above. We shall denote by (S) *the Sobolev part of* (1.1):

$$\text{(2.10)} \qquad \begin{aligned} &a) \quad Mw'(t) + L(t)w(t) = f\big(t, p(0, t), q(0, t), w(t)\big) \qquad \text{in } H,\ t \in I_T\,,\\ &b) \quad w(0) = \varphi\,. \end{aligned}$$

For further examples of the Sobolev equations, the concepts and terminology defined in this section, and references to their applications, see [10, 11, 12]. We note here that the definitions of weak and strong solutions in this section differ from the corresponding definitions in [10, 11]. C^1 solutions are obtained here under stronger assumptions than are made to obtain absolutely continuous solutions in [10, 11].

3. – A weaker formulation of the hyperbolic part.

Since (1.1) arises from physical considerations, it is natural to assume that the λ_i and R_i are smooth bounded functions that are defined on

$$Q_T = \{(z, t, p, q): 0 \leqslant z \leqslant 1,\ 0 \leqslant t \leqslant T,\ -\infty < p < \infty,\ -\infty < q < \infty\}.$$

It is also natural to assume that there exists a constant $\delta > 0$ such that, uniformly in Q_T,

$$\lambda_2 < -\delta < 0 < \delta < \lambda_1 . \tag{3.1}$$

If a classical smooth solution (p, q, w) of (1.1) exists, we can define the characteristics

$$z_i = z_i(\tau; z, t), \qquad \max(0, t_i) \leqslant \tau \leqslant t, \qquad i = 1, 2 \tag{3.2}$$

as solutions of the initial value problems

$$\begin{aligned} &a)\ \frac{dz_i}{d\tau} = \lambda_i\big(z_i, \tau, p(z_i, \tau), q(z_i, \tau)\big), \qquad \max(0, t_i) \leqslant \tau \leqslant t,\ \ i = 1, 2 \\ &b)\ z_i(t) = z, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2 . \end{aligned} \tag{3.3}$$

Here we define

$$t_i = t_i(z, t), \qquad i = 1, 2 \tag{3.4}$$

to be the unique time at which the characteristic z_i assumes the value $z = i - 1$. From the bounds on λ_i in (3.1), we can take T sufficiently small that if t_1 or t_2 is positive, the characteristic emanating from $(0, t_1)$ or $(1, t_2)$ using λ_2 or respectively λ_1 does not strike the opposite boundary for positive t. In other words, we can restrict T so that we have at most one bounce of a characteristic to consider.

If we integrate (1.1a) and (1.1b) along their respective characteristics as in [5, 6, 9], we see that for $0 \leqslant t \leqslant T$, any classical solution of (1.1) must satisfy (H) *the hyperbolic* *part of* (1.1):

$$\text{(3.5)}\quad \begin{aligned} &a)\quad p(z,t) = p_0\big(z_1(0;z,t)\big) + \int_0^t R_1\Big(z_1(\tau;z,t),\tau, p\big(z_1(\tau;z,t),\tau\big), q\big(z_1(\tau;z,t),\tau\big)\Big)\,d\tau \\ &b)\quad q(z,t) = q_0\big(z_2(0;z,t)\big) + \int_0^t R_2\Big(z_2(\tau;z,t),\tau, p\big(z_2(\tau;z,t),\tau\big), q\big(z_2(\tau;z,t),\tau\big)\Big)\,d\tau \end{aligned}$$

or (3.5*b*) and

$$\text{(3.6)}\quad \begin{aligned} p(z,t) = {}&\zeta\Big(t_1(z,t), q\big(0,t_1(z,t)\big), w\big(0,t_1(z,t)\big)\Big) \\ &+ \int_{t_1(z,t)}^t R_1\Big(z_1(\tau;z,t),\tau, p\big(z_1(\tau;z,t),\tau\big), q\big(z_1(\tau;z,t),\tau\big)\Big)\,d\tau \end{aligned}$$

where $q\big(0,t_1(z,t)\big)$ is computed by replacing z and t in (3.5*b*) by 0 and $t_1(z,t)$ respectively, or (3.5*a*) and

$$\text{(3.7)}\quad \begin{aligned} q(z,t) = {}&G\Big(t_2(z,t), p\big(1,t_2(z,t)\big)\Big) \\ &+ \int_{t_2(z,t)}^t R_2\Big(z_2(\tau;z,t),\tau, p\big(z_2(\tau;z,t),\tau\big), q\big(z_2(\tau;z,t),\tau\big)\Big)\,d\tau \end{aligned}$$

where $p\big(1,t_2(z,t)\big)$ is computed from (3.5*a*) by the replacement of z and t by 1 and $t_2(z,t)$ respectively.

DEFINITION. – A *weak solution* of (1.1) is any triple of functions (p,q,w) such that p and q are continuous for $0\leqslant z\leqslant 1$ and $0\leqslant t\leqslant T$ and satisfy (H) the hyperbolic part of (1.1), which is described by (3.5*a*), (3.5*b*), (3.6) and (3.7), and w is a strong solution of (S) the Sobolev part of (1.1) described by (2.10).

We shall now describe the mapping which will furnish our solution via an application of the contraction mapping theorem. We take $w(0,t)$ in (3.6) and replace it by a function $v=v(t)$. After solving the hyperbolic part of (1.1) for p and q, we substitute $p(0,t)$ and $q(0,t)$ into (1.1*f*) and solve the Sobolev part of (1.1) for w. The mapping $\mathcal{C}$ is obtained by setting

$$\text{(3.8)}\qquad w(0,t) = \mathcal{C}v(t)\,.$$

In order to demonstrate that $\mathcal{C}$ is a contraction for T sufficiently small, we need to obtain some a priori estimates.

4. – A priori estimates on the solutions of the characteristic equations.

We shall use a lemma from the theory of ordinary differential equations.

LEMMA 4.1 [8]. – Let y and Y be two functions satisfying

$$(4.1)\qquad \begin{aligned} &a)\quad y' = f(x, y)\,, && |x-a| \leq h\,,\\ &b)\quad y(a) = \alpha\,, &&\\ &c)\quad Y' = F(x, Y)\,, && |x-a|\leq h\,,\\ &d)\quad Y(a) = \beta\,. && \end{aligned}$$

Then, for $|x-a|\leq h$,

$$(4.2)\qquad |Y(x)-y(x)|\leq \exp\{Ph\}\big[|\alpha-\beta| + \sup|f-F|\big]\,,$$

where h is a positive constant and P is the maximum of the uniform Lipschitz constants on f and F.

We assume that $p = p(z, t)$ and $q = q(z, t)$ are uniformly Lipschitz continuous in z and t with Lipschitz constant $K > 1$. Let $C > 1$ denote a constant which bounds λ_i, $i = 1, 2$ and all their first derivatives in absolute value. We then make a simple application of the mean value theorem to obtain

$$(4.3)\qquad |\lambda_i(z^*, \tau, p(z^*, \tau), q(z^*, \tau)) - \lambda_i(z_*, \tau, p(z_*, \tau), q(z_*, \tau))| \leq 3CK|z^* - z_*|\,,\qquad i = 1, 2\,.$$

A similar estimate holds for the τ variable. Recalling (3.4), we let

$$(4.4)\qquad t_i^{(j)} = t_i(z^{(j)}, t)\qquad j = 1, 2$$

denote the times that the characteristics z_i, $i = 1, 2$, emanating from the points $(z^{(j)}, t)$, strike the boundary $z = i-1$. We then obtain from Lemma 4.1 and (4.3) the following lemma.

LEMMA 4.2. – For $\max\{0, t_i^{(1)}, t_i^{(2)}\}\leq\tau\leq t\leq T$,

$$(4.5)\qquad |z_i(\tau; z^{(1)}, t) - z_i(\tau; z^{(2)}, t)| \leq \exp\{3CKt\}|z^{(1)} - z^{(2)}|\,.$$

By integrating (3.3) along the characteristics we obtain

$$(4.6)\qquad z^{(j)} = (i-1) + \int\limits_{t_1^{(j)}}^{t} \lambda_i\Big(z_i(\tau; z^{(j)}, t), \tau, p\big(z_i(\tau; z^{(j)}, t), \tau\big), q\big(z_i(\tau; z^{(j)}, t), \tau\big)\Big)\, d\tau\qquad j = 1, 2,\ i = 1, 2\,.$$

Subtracting $j=2$ of (4.6) from $j=1$, we obtain

$$z^{(1)}-z^{(2)}=\int\limits_{\max(t_i^{(1)},\,t_i^{(2)})}^{t}[\lambda_i^{(1)}-\lambda_i^{(2)}]\,d\tau\pm\int\limits_{\min(t_i^{(1)},\,t_i^{(2)})}^{\max(t_i^{(1)},\,t_i^{(2)})}\lambda_i\,d\tau\,,\tag{4.7}$$

where the choice of the sign and argument for the second term depends upon the $t_i^{(j)}$, $j=1,2$. Solving for the second term on the right hand side of (4.7), recalling (3.1) and using (4.3) and Lemma 4.2, we obtain the following result.

LEMMA 4.3. – For $i=1,2$,

$$|t_i(z^{(1)},t)-t_i(z^{(2)},t)|\leqslant\delta^{-1}\big(1+3CKt\exp\{3CKt\}\big)|z^{(1)}-z^{(2)}|\,.\tag{4.8}$$

Integrating along the characteristics emanating from $(z,t^{(1)})$ and $(z,t^{(2)})$ and using the techniques of the last proof coupled with Gronwall's Lemma we obtain the following result.

LEMMA 4.4. – For $0\leqslant t^{(j)}\leqslant T$, $j=1,2$, and $\max(0,t_i^{(1)},t_i^{(2)})\leqslant\tau\leqslant\min(t^{(1)},t^{(2)})$,

$$|z_i(\tau;z,t^{(1)})-z_i(\tau;z,t^{(2)})|\leqslant C\exp\{3CKT\}|t^{(1)}-t^{(2)}|\tag{4.9}$$

where $t_i^{(j)}$ here denotes the time that the characteristic z_i emanating from $(z,t^{(j)})$ strikes the boundary $z=i-1$.

An argument similar to those of the two preceding results yields our last estimate.

LEMMA 4.5. For $0\leqslant t^{(j)}\leqslant T$, $j=1,2$,

$$|t_i(z,t^{(1)})-t_i(z,t^{(2)})|\leqslant\delta^{-1}C\big(1+3CKT\exp\{3CKT\}\big)|t^{(1)}-t^{(2)}|\,.\tag{4.10}$$

We note that if we restrict

$$T<(3CK)^{-1}\,,\tag{4.11}$$

we can simplify our previous results.

LEMMA 4.6. – For $i=1,2$, $0\leqslant t^{(j)}\leqslant T$, $j=1,2$,

$$\begin{aligned}&a)\quad |t_i(z^{(1)},t)-t_i(z^{(2)},t)|\leqslant4\delta^{-1}|z^{(1)}-z^{(2)}|\\&b)\quad |t_i(z,t^{(1)})-t_i(z,t^{(2)})|\leqslant4C\delta^{-1}|t^{(1)}-t^{(2)}|\,.\end{aligned}\tag{4.12}$$

Finally, noting the results of Lemma 4.2, Lemma 4.4 and Lemma 4.6 we see that with the restriction given in (4.11)

$$C_2=4C\delta^{-1}\tag{4.13}$$

can be used as a uniform Lipschitz constant for z_i and t_i. Without loss of generality we can assume that $C_2 \geq 1$.

5. – A priori estimates for the solution of the hyperbolic part of (1.1).

Since the hyperbolic part of (1.1) consists of integral equations which involve the characteristic equations (2.3), we shall need the estimates of section 4 in the estimates derived below for $p = p(z, t)$ and $q = q(z, t)$. We first state the assumptions we shall make on the data functions R_1, R_2, ζ, G, p_0 and q_0.

We assume that R_1, R_2, and their first derivatives are bounded in absolute value by C in Q_T. We assume that $\zeta = \zeta(t, q, w)$, $G = G(t, p)$, and their first derivatives are bounded in absolute value by C in $\{(t, q, w): t \in I_T, -\infty < q < \infty, -\infty < w < \infty\}$ and $\{(t, p): t \in I_T, -\infty < p < \infty\}$ respectively. Next we assume that $p_0 = p_0(z)$, $q_0 = q_0(z)$, and their first derivatives are bounded in absolute value by C on $0 < z < 1$. Finally, in order to obtain continuous functions p and q we must satisfy the following compatibility conditions upon the data:

$$
(5.1) \qquad \begin{aligned} &a) \quad p_0(0) = \zeta\big(0, q_0(0), \varphi(0)\big), \\ &b) \quad q_0(1) = G\big(0, p_0(1)\big). \end{aligned}
$$

The estimates from this section will be used to show that the mapping described in section 3 will retain certain function classes. As in section 4, we shall assume that $p = p(z, t)$ and $q = q(z, t)$ are uniformly Lipschitz continuous in z and t with Lipschitz constant $K > 1$. In the mapping, we replace $w(0, t)$ in (3.6) by a function $v = v(t)$. We shall assume that v is Lipschitz continuous with constant $V > 1$ and that the absolute value of v is bounded above by C.

In the estimates below we obtain an estimate for the Lipschitz constant for p and q after having assumed the one above. This is because the mapping of section 3 requires the solution of an auxiliary hyperbolic system which has a mapping of its own [5, 6, 9]. Thus, our estimates must reflect the retention of the various function classes through that mapping as well.

From (3.5)-(3.7) we easily obtain the first estimate.

LEMMA 5.1. – For $0 \leq z \leq 1$ and $t \in I_T$,

$$
(5.2) \qquad \begin{aligned} &a) \quad |p(z, t)| \leq (1 + t)\, C \\ &b) \quad |q(z, t)| \leq (1 + t)\, C. \end{aligned}
$$

In order to estimate the Lipschitz constants for p and q from (3.5)-(3.7) we must consider three basic cases for characteristics $z_i^{(j)}$, $j = 1, 2$ emanating respectively from $(z^{(1)}, t^{(1)})$ and $(z^{(2)}, t^{(2)})$.

Case I: Neither characteristic $z_i^{(j)}$, $j=1,2$, hits a lateral boundary before hitting the base $t=0$.

Case II: Both characteristics $z_i^{(j)}$, $j=1,2$, hit the same lateral boundary prior to $t=0$.

Case III: Only one of the characteristics $z_i^{(j)}$, $j=1,2$, hits a lateral boundary while the other hits the base $t=0$.

With no loss of generality, we restrict our consideration to the lateral boundary $z=0$ and the behavior of $z_1^{(j)}$, $j=1,2$. The analysis for the boundary $z=1$ is similar except for the omission of the effect of the function $v=v(t)$.

For Case I, we obtain the following estimate.

LEMMA 5.2. – When neither characteristic hits the lateral boundary prior to $t=0$ and when $0\leq t^{(j)}\leq t\leq T$, $0\leq z^{(j)}\leq 1$, $j=1,2$,

$$(5.4)\qquad \begin{aligned} &a)\quad |p(z^{(1)},t^{(1)})-p(z^{(2)},t^{(2)})|\leq C_2(2C+3CKt)\{|z^{(1)}-z^{(2)}|+|t^{(1)}-t^{(2)}|\}\\ &b)\quad |q(z^{(1)},t^{(1)})-q(z^{(2)},t^{(2)})|\leq C_2(2C+3CKt)\{|z^{(1)}-z^{(2)}|+|t^{(1)}-t^{(2)}|\}\,. \end{aligned}$$

PROOF. – It suffices to consider z_1 and (3.5a) since a similar argument will hold for z_2 and (3.5b). Substituting $(z^{(j)},t^{(j)})$, $j=1,2$ in (3.5a) and subtracting, we obtain

$$(5.5)\qquad \begin{aligned} p(z^{(1)},t^{(1)})-p(z^{(2)},t^{(2)}) &= p_0\big(z_1(0;z^{(1)},t^{(1)})\big)-p_0\big(z_1(0;z^{(2)},t^{(2)})\big)\\ &+\int\limits_0^{\min(t^{(1)},t^{(2)})}\Big\{R_1\Big(z_1(\tau;z^{(1)},t^{(1)}),\tau,p\big(z_1(\tau;z^{(1)},t^{(1)}),\tau\big),q\big(z_1(\tau;z^{(1)},t^{(1)}),\tau\big)\Big)\\ &-R_1\Big(z_1(\tau;z^{(2)},t^{(2)}),\tau,p\big(z_1(\tau;z^{(2)},t^{(2)}),\tau\big),q\big(z_1(\tau;z^{(2)},t^{(2)}),\tau\big)\Big)\Big\}d\tau\\ &+\int\limits_{\min(t^{(1)},t^{(2)})}^{\max(t^{(1)},t^{(2)})}R_1\,d\tau\,. \end{aligned}$$

Then from the mean value theorem and the results of section 4, we obtain

$$(5.6)\qquad \begin{aligned} &|p(z^{(1)},t^{(1)})-p(z^{(2)},t^{(2)})|\leq\\ &\quad\leq\{C_2C+3CKC_2\min(t^{(1)},t^{(2)})\}\{|z^{(1)}-z^{(2)}|+|t^{(1)}-t^{(2)}|\}+C|t^{(1)}-t^{(2)}| \end{aligned}$$

from which (5.4) follows.

For Case II, from (3.6) we see that the difference in $p(z^{(1)},t^{(1)})$ and $p(z^{(2)},t^{(2)})$ will involve a difference in the ζ terms, which will introduce the Lipschitz constant V and several integrals similar to the ones treated above. An application of the mean value theorem and use of Lemma 3.6 yields the following estimate.

LEMMA 5.3. – When both characteristics hit a lateral boundary prior to $t=0$ and when $0\leqslant z^{(j)}\leqslant 1$, $0\leqslant t^{(j)}\leqslant t$, $j=1,2$,

$$(5.7)\quad \begin{aligned} &a)\quad |p(z^{(1)},t^{(1)})-p(z^{(2)},t^{(2)})|\leqslant CC_2\{5C+5CKt+V\}\{|z^{(1)}-z^{(2)}|+|t^{(1)}-t^{(2)}|\},\\ &b)\quad |q(z^{(1)},t^{(1)})-q(z^{(2)},t^{(2)})|\leqslant CC_2\{5C+5CKt\}\{|z^{(1)}-z^{(2)}|+|t^{(1)}-t^{(2)}|\}.\end{aligned}$$

Finally we consider Case III. From the theory of ordinary differential equations [8], we know there is a unique characteristic z_1 with characteristic direction λ_1 passing through the origin. Uniqueness also prevents intersections of this characteristic with $z_1(\tau; z^{(1)}, t^{(1)})$ and $z_1(\tau; z^{(2)}, t^{(2)})$. Then using the Jordan Curve Theorem we can split this case up into a combination of the two cases described above. Combining the cases above we obtain the following result which holds for all cases.

LEMMA 5.4. – For Cases I, II, and III, $0\leqslant z^{(j)}\leqslant 1$, and $0\leqslant t^{(j)}\leqslant t$, $j=1,2$,

$$(5.8)\quad \begin{aligned} &a)\quad |p(z^{(1)},t^{(1)})-p(z^{(2)},t^{(2)})|\leqslant CC_2\{7C+8CKt+V\}\{|z^{(1)}-z^{(2)}|+|t^{(1)}-t^{(2)}|\},\\ &b)\quad |q(z^{(1)},t^{(1)})-q(z^{(2)},t^{(2)})|\leqslant CC_2\{7C+8CKt\}\{|z^{(1)}-z^{(2)}|+|t^{(1)}-t^{(2)}|\}.\end{aligned}$$

6. – A priori estimates for the solution of the Sobolev part of (1.1).

We shall list for future reference some basic assumptions we shall make on the Sobolev part (S) from (2.10).

(I) $m(\cdot,\cdot)$ and $l(t;\cdot,\cdot)$ are uniformly strongly coercive over W. Thus there are constants k_m and k_l such that

$$(6.1)\quad \begin{aligned} &a)\quad |m(u,u)|\geqslant k_m|u|_W^2 \qquad \text{for } u\in W,\\ &b)\quad |l(t;u,u)|\geqslant k_l|u|_W^2 \qquad \text{for } u\in W,\ t\in I_T.\end{aligned}$$

(II) For each pair $u, v\in W$, the function $t\to l(t;u,v)$ is continuous, so there is a constant K_l with

$$(6.2)\quad |l(t;u,v)|\leqslant K_l|u|_W|v|_W,\qquad u,v\in W,\ t\in I_T.$$

(III) For each $t\in I_T$, $D(L(t))=D(L)$ for a fixed $D(L)$. M is «stronger» than $L(t)$ for each $t\in I_T$; i.e., $D(M)\subseteq D(L)$. Also, there is a constant K_1 such that

$$(6.3)\quad |L(t)w|_H\leqslant K_1|Mw|_H.$$

(IV) There are constants K_2, K_3, K_4 and K_5 such that for $f\colon I_T\times\mathbf{R}^2\times W\to W'$, we have

$$(6.4)\quad |f(t,p(0,t),q(0,t),w(t))|_H\leqslant K_2|w(t)|_H,$$

and

$$\text{(6.5)}\quad \begin{aligned} &a)\ \ |f(t, p(0,t),\ q(0,t),\ u) - f(t, p(0,t),\ q(0,t),\ v)|_H \leq K_3|u-v|_{H^2}\,,\\ &b)\ \ |f(t, p_1(0,t), q(0,t),\ w) - f(t, p_2(0,t), q(0,t),\ w)|_H \leq K_4|p_1(0,t) - p_2(0,t)|\,,\\ &c)\ \ |f(t, p(0,t),\ q_1(0,t), w) - f(t, p(0,t),\ q_2(0,t), w)|_H \leq K_5|q_1(0,t) - q_2(0,t)|\,. \end{aligned}$$

Also $f\colon I_T \times \mathbf{R}^2 \times W \to W'$ is continuous.

Then if we choose $W \equiv H^1$, under the above assumptions, we have the following result due to Showalter.

THEOREM 6.1. [10]. – If $\varphi \in D(M)$, there exists a unique strong solution of (S) which has an integral representation given by

$$\text{(6.6)}\qquad w(t) = G(t,0)\varphi + \int_0^t G(t,s)\, M^{-1} f\big(s, p(0,s), q(0,s), w(s)\big)\, ds\,.$$

In the statement above, $G(t,s)$ is the linear propagator [4, 10]. We note that since the integrand in (6.6) is continuous (see section 7), we can use the Riemann integral in (6.6) to obtain a C^1 solution instead of the Bochner integral which yields an absolutely continuous solution as in [10].

Since we are working with supremum norms for the hyperbolic part and we wish to put our results together via a mapping, we must obtain estimates for the Sobolev part in terms of the supremum or L^∞-norm. We need L^∞ bounds on $w'(t)$, where $w(t)$ is the solution of (S), to use as Lipschitz constants for the mapping $\mathcal{C}$. From the Sobolev Lemma [4] we know that each $w'(t) \in H^2(\Omega) \equiv H^2$ (has a unique representative which) is an absolutely continuous function on Ω, and we have an estimate of the form

$$\text{(6.7)}\qquad |w'(t)|_{L^\infty(\Omega)} \leq K_6 |w'(t)|_{H^2(\Omega)}$$

after identifying each such w with this representative. We must then obtain H^2-norm estimates on our results.

From Assumption (I) (6.1) and elliptic operator theory [1] we know there exists a constant K_7 such that for each $t \in I_T$,

$$\text{(6.8)}\qquad |w'(t)|_{H^2} \leq K_7 |Mw'(t)|_H\,.$$

Thus from above we shall obtain estimates on $|w'(t)|_{L^\infty}$ from bounds on $|Mw'(t)|_H$.

We first differentiate $|Mw(t)|_H^2$ with respect to t, use the Schwarz Inequality and the triangle inequality on (2.8*a*) to obtain

$$\text{(6.9)}\qquad \begin{aligned} \frac{d}{dt}|Mw(t)|_H^2 &= 2\big(Mw'(t), Mw(t)\big)_H\\ &\leq 2|Mw'(t)|_H |Mw(t)|_H\\ &\leq 2\big(|L(t)w(t)|_H + |f|_H\big)|Mw(t)|_H\,. \end{aligned}$$

Using the fact that $W \hookrightarrow H$ (is norm-imbedded) coupled with (2.5), (6.1) and the Schwarz inequality, we obtain for some K_8,

$$
\begin{aligned}
(6.10) \qquad |w(t)|_H^2 &\leq K_8 |w(t)|_W^2 \\
&\leq K_8 k_m^{-1} |m(w(t), w(t))| \\
&= K_8 k_m^{-1} |(Mw(t), w(t))_H| \\
&\leq K_8 k_m^{-1} |Mw(t)|_H |w(t)|_H .
\end{aligned}
$$

Then combining (6.4) with (6.10) we obtain

$$
(6.11) \qquad |f(t, p(0,t), q(0,t), w(t))|_H \leq K_2 K_8 k_m^{-1} |Mw(t)|_H .
$$

Next by combining (6.3), (6.9) and (6.11), we have

$$
\begin{aligned}
(6.12) \qquad \frac{d}{dt} |Mw(t)|_H^2 &\leq 2(K_1 + K_2 K_8 k_m^{-1}) |Mw(t)|_H^2 \\
&\equiv K_9^2 |Mw(t)|_H^2 ,
\end{aligned}
$$

where (6.12) defines K_9. Then by Gronwall's Lemma, if $\varphi \in D(M)$, we have that for some positive constant K_{10},

$$
(6.13) \qquad |Mw(t)|_H \leq |Mw(0)|_H \exp\{K_9 T\} \equiv K_{10} .
$$

We can now use (2.10*a*), (6.3), (6.11), (6.13) and the triangle inequality to obtain

$$
\begin{aligned}
(6.14) \qquad |Mw'(t)|_H &\leq |L(t) w(t)|_H + |f(t, p(0,t), q(0,t), w(t))|_H \\
&\leq (K_1 + K_2 K_8 k_m^{-1}) |Mw(t)|_H \\
&\leq (K_1 + K_2 K_8 k_m^{-1}) K_{10} \equiv K_{11} .
\end{aligned}
$$

Finally, combining (6.7), (6.8) and (6.14), we have

$$
(6.15) \qquad |w'(t)|_{L^\infty(\Omega)} \leq K_6 K_7 K_{11} \equiv K_{12} .
$$

Since this constant K_{12} is independent of t, we have

$$
(6.16) \qquad \sup_{0 \leq t \leq T} |w'(t)|_{L^\infty(\Omega)} \leq K_{12} .
$$

We have just proved the following lemma.

LEMMA 6.1. – There exists a positive constant K_{12} which is independent of the Lipschitz constant K of the functions p and q to be determined in section 7 such that

$$\sup_{0\leqslant t\leqslant T}\sup_{x\in\Omega}\left|\frac{\partial w}{\partial t}(x,t)\right|\leqslant K_{12}\,. \tag{6.17}$$

Note. We shall note at this point that the Lipschitz constant V for $v=v(t)$ can be set equal to K_{12} and will be independent of the Lipschitz constant K for p and q.

7. – Preservation of function classes and continuity of $\mathcal{T}$.

We shall consider the mapping described by (3.8) and show that if $v=v(t)$ is in a certain function class, then for t sufficiently small, $w=w(0,t)$ is in the same class. We first consider the hyperbolic part of the mapping $\mathcal{T}$. Since Lemma 5.1 gives a uniform bound on p and q, we can restrict our attention to the Lipschitz constants for p and q. If we restrict T as in (4.11) to simplify the form of some constants, that will incorporate the $T<C^{-1}$ restriction that was necessary to restrict our consideration of one bounce of a characteristic off a lateral boundary which we assumed in section 3.

We know that V, the Lipschitz constant for v, can be set equal to K_{12} from (6.17) which is independent of K, the Lipschitz constant for p and q. From (5.8) we see that for preservation of the Lipschitz classes for p and q we must select K such that for T suitably restricted

$$7C^2C_2+CC_2V+8C^2C_2KT<K\,. \tag{7.1}$$

Let

$$K=7C^2C_2+CC_2V+1\,. \tag{7.2}$$

Substituting (7.2) into (7.1) leads to the restriction

$$T<(56C^4C_2^2+8C^3C_2^2V+8C^2C_2)^{-1} \tag{7.3}$$

while the substitution of (7.2) into (4.11) yields

$$T<(21C^3C_2+3C^2C_2V+3C)^{-1}\,. \tag{7.4}$$

Note that (7.4) is automatically satisfied when (7.3) holds. Thus under the restriction (7.3), the Lipschitz classes for p and q are preserved under the mapping when K is chosen as in (7.2). Thus the mapping $\mathcal{T}$ takes a compact and convex subset of the Banach space of continuous functions on $[0,T]$ with the uniform norm topology into itself. We now shall show that this mapping is continuous.

First define

$$\|f\|_t = \sup_{\substack{0\leq z\leq 1\\ 0\leq \tau\leq t}} |f(z,\tau)| \tag{7.5}$$

and

$$|||v|||_t = \sup_{0\leq\tau\leq t} |v(\tau)| . \tag{7.6}$$

Let (p_j, q_j), $j = 1, 2$ denote solutions of the hyperbolic part of (1.1) which correspond respectively to the functions v_j, $j = 1, 2$. The techniques of section 4 and Lemma 4.1 allow us to obtain the following lemma.

LEMMA 7.1. – For $0\leq z\leq 1$ and $\max(0, t_i)\leq\tau\leq t\leq T$,

$$|z_i^{(1)}(\tau; z, t) - z_i^{(2)}(\tau; z, t)| \leq C\exp\{3CKt\}t\{\|p_1-p_2\|_t + \|q_1-q_2\|_t\}, \qquad i = 1, 2 , \tag{7.7}$$

where $z_i^{(j)}$ corresponds to the i-th characteristic emanating from (p_j, q_j), $j = 1, 2$.

Integrating along the characteristics, we use standard estimates as in the proof of Lemma 4.3 to obtain the next result.

LEMMA 7.2. – For $0\leq z\leq 1$ and $0\leq\min(t_i^{(1)}, t_i^{(2)})\leq t\leq T$,

$$\begin{aligned}|t_i^{(1)}(z,t) - t_i^{(2)}(z,t)| \leq \delta^{-1}[(C+2CK)tC\exp\{3CKt\} + C]\cdot\\ \cdot t\{\|p_1-p_2\|_t + \|q_1-q_2\|_t\}, \qquad i = 1, 2 .\end{aligned} \tag{7.8}$$

We use Lemma 7.1 and Lemma 7.2 in the estimates of the differences of formulas (3.5)-(3.7) into which we substitute, p_j, q_j, and v_j, $j = 1, 2$. Tedious but elementary estimations involving the three cases of section 5 yield the following estimate.

LEMMA 7.3. – There exists a constant C_3 which depends only upon C, K, and V such that for $0\leq t\leq T$,

$$\|p_1-p_2\|_t + \|q_1-q_2\|_t \leq C_3 t\{\|p_1-p_2\|_t + \|q_1-q_2\|_t\} + C|||v_1-v_2|||_t . \tag{7.9}$$

Also, if we restrict T to satisfy

$$T < (2C_3)^{-1} , \tag{7.10}$$

we have

$$\|p_1-p_2\|_t + \|q_1-q_2\|_t \leq 2C|||v_1-v_2|||_t . \tag{7.11}$$

We shall now use the representation of the solution of (2.10) given in (6.6) to estimate the continuous dependence of w upon p and q. First we shall derive some necessary estimates on the norms of the operators in (6.6). We will need the following lemma.

LEMMA 7.4. – Let X_i, $i = 1, 2$, be Banach spaces with respective norms $|\cdot|_i$. Let Y_i be a subset of X_i which is a Banach space with norm $\|\cdot\|_i$ and assume $|y|_i \leqslant c_i\|y\|_i$ when y belongs to Y_i and c_i are positive constants. Let A be a bounded linear transformation from X_1 to X_2 such that A maps Y_1 to Y_2. Then A is bounded from Y_1 to Y_2.

PROOF. – See [12].

From (6.1*a*), one can show that $\mathcal{M}^{-1}\colon W' \to W$ with

$$\|\mathcal{M}^{-1}\|_{L(W',W)} \leqslant k_m^{-1}. \tag{7.12}$$

From [4] and (2.9*a*) we know that $M^{-1}\colon H \to D(M) \subset H^2$. Since $H \hookrightarrow W'$ and $H^2 \hookrightarrow W = H^1$, the hypotheses of Lemma 7.4 are satisfied and we know that there exists some constant C_4 such that

$$\|M^{-1}\|_{L(H,D(M))} \leqslant C_4. \tag{7.13}$$

Using (2.3), (2.4), (6.1), and (6.2) we also see that

$$\begin{aligned} |\mathcal{M}^{-1}\mathcal{L}(t)u|_W^2 &\leqslant k_m^{-1}|m(\mathcal{M}^{-1}\mathcal{L}(t)u, \mathcal{M}^{-1}\mathcal{L}(t)u)| \\ &= k_m^{-1}|\langle\mathcal{L}(t)u, \mathcal{M}^{-1}\mathcal{L}(t)u\rangle| \\ &= k_m^{-1}|l(t; u, \mathcal{M}^{-1}\mathcal{L}(t)u)| \\ &\leqslant K_l k_m^{-1}|u|_W|\mathcal{M}^{-1}\mathcal{L}(t)u|_W, \end{aligned} \tag{7.14}$$

and we have that, independent of t,

$$\|\mathcal{M}^{-1}\mathcal{L}(t)\|_{L(W,W)} \leqslant K_l k_m^{-1}. \tag{7.15}$$

Then since $\mathcal{M}^{-1}\mathcal{L}(t)\colon W \to W$, $M^{-1}L(t)\colon D(L) \subset H^2 \to D(M) \subset H^2$ and $H^2 \hookrightarrow W = H^1$, we can again apply Lemma 7.4 for each $t \in I_T$ to obtain that $\|M^{-1}L(t)\|_{L(D(L),D(M))}$ is bounded above with the bounds depending upon t. At this point we need the following additional assumption:

(V) The mapping $t \to M^{-1}L(t)$ is continuous in the uniform operator topology on $L(D(L), D(M))$.

Then since $\|M^{-1}L(t)\|_{L(D(L),D(M))}$ is bounded above for each $t \in I_T$, which is a compact set, we see that there exists a constant C_5 such that, uniformly in t,

$$\|M^{-1}L(t)\|_{L(D(L),D(M))} \leqslant C_5. \tag{7.16}$$

Using (7.16) and defining the linear propagator as in [4, 10] we see that

$$\|G(t, s)\|_{L(D(L),D(M))} \leqslant \exp\left|\int_s^t C_5\, d\xi\right| \leqslant \exp C_5 T . \tag{7.17}$$

We now use the integral representation of the solution given by (6.6) to estimate the continuous dependence of w upon p and q from the hyperbolic part of (1.1). Let $w_i(t)$ be the solution generated by $p_i(0, t)$ and $q_i(0, t)$ in (6.6). Note that since $D(L)$ and $D(M)$ are contained in H^2 by (2.9), we are using the H^2 norm on them. Using (6.5), (7.14), (7.16), (7.17) and assumption (III), we obtain

$$\begin{aligned}
|w_1(t) - w_2(t)|_{H^2} &\leqslant \Big|\int_0^t G(t, s)\, M^{-1}[f(s, p_1(0, s), q_1(0, s), w_1(s)) - \\
&\qquad - f(s, p_2(0, s), q_2(0, s), w_2(s))]\, ds\Big|_{H^2} \\
&\leqslant \|G(t, s)\|_{L(D(L),D(M))} \|M^{-1}\|_{L(H,D(M))} \cdot \\
&\qquad \cdot \int_0^t |f(s, p_1(0, s), q_1(0, s), w_1(s)) - f(s, p_2(0, s), q_2(0, s), w_2(s))|_H\, ds \\
&\leqslant (\exp C_5 T) C_4 \int_0^t \{K_4 |p_1(0, s) - p_2(0, s)| + K_5 |q_1(0, s) - q_2(0, s)| + \\
&\qquad + K_3 |w_1(s) - w_2(s)|_{H^2}\}\, ds \\
&\leqslant (\exp C_5 T) C_4 \Big[\max\{K_4, K_5\}\, t \{\|p_1 - p_2\|_t + \|q_1 - q_2\|_t\} + \\
&\qquad + K_3 \int_0^t |w_1(s) - w_2(s)|_{H^2} ds\Big].
\end{aligned} \tag{7.18}$$

Finally from Gronwall's Lemma we have that

$$\begin{aligned}
|w_1(t) - w_2(t)|_{H^2} &\leqslant C_4 \max\{K_4, K_5\} \exp\{C_5 T\} \exp\{K_3 C_4 \exp(C_5 T) T\} t \cdot \\
&\qquad \cdot \{\|p_1 - p_2\|_t + \|q_1 - q_2\|_t\} \\
&\equiv C_6\, t \{\|p_1 - p_2\|_t + \|q_1 - q_2\|_t\} .
\end{aligned} \tag{7.19}$$

We can now apply (6.7) to see that for each $t \in I_T$, we have

$$|w_1(t) - w_2(t)|_{L^\infty(\Omega)} \leqslant K_4 C_6\, t \{\|p_1 - p_2\|_t + \|q_1 - q_2\|_t\} . \tag{7.20}$$

Now due to the definition of the norms $|||\cdot|||_t$ and $\|\cdot\|_t$ in (7.5) and (7.6) and the fact that $K_4 C_6$ is independent of t, we have the following result.

LEMMA 7.5. – Fot $t \in I_T$ there are constants $K_4 C_6$ such that

$$\|\|w_1(0,\cdot) - w_2(0,\cdot)\|\|_t \leq K_4 C_6 t\{\|p_1 - p_2\|_t + \|q_1 - q_2\|_t\}. \tag{7.21}$$

Then from (7.11) we obtain the result

LEMMA 7.6. – For $t \in I_T$

$$\|\|w_1(0,\cdot) - w_2(0,\cdot)\|\|_t \leq 2CK_4 C_6 t \|\|v_1 - v_2\|\|_t. \tag{7.22}$$

Now the restriction (7.10) guarantees that the hyperbolic part of (1.1) is uniquely solvable for a given v. Theorem 6.1 guarantees that the Sobolev part of (1.1) is uniquely solvable. Thus the mapping $\mathcal{T}$ is well-defined for T restricted by (7.10). The continuity of $\mathcal{T}$ follows from Lemma 7.6. Finally by restricting T to satisfy

$$T < (2CK_4 C_6)^{-1}, \tag{7.23}$$

we see by (7.22) that the mapping is a contraction. An application of the contraction mapping theorem guarantees a unique fixed point of the mapping which is a weak solution of (1.1) as defined in section 3.

We thus state the major result of this paper while avoiding a catalog of assumptions upon the data.

THEOREM. – If T satisfies all of the restrictions (7.3), (7.10), and (7.23), a unique weak solution of (1.1) exists. It is composed of a weak solution of the hyperbolic part (H) of (1.1) when $w(0, t)$ in ζ in (3.6) is replaced by a Lipschitz continuous function $v = v(t)$ and a strong solution of the Sobolev part (S) when Lipschitz continuous $p(0, t)$ and $q(0, t)$ are substituted into f.

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