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THE APPROXIMATION OF CERTAIN PARABOLIC EQUATIONS BACKWARD IN TIME BY SOBOLEV EQUATIONS*

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Abstract. For any nonnegative, self-adjoint operator A, which does not depend on time, the backward solution to the parabolic equation, $u'(t) = -Au(t)$, $t \geqq 0$, in a cylinder can be approximated by the solution to the Sobolev equation, $u'(t) = -(I + \beta A)^{-1}Au(t)$. The solution to the backward Sobolev equation can be more readily computed than the solution to the adjoint of the parabolic equation. In a Hilbert space setting, if the norm of the solution is assumed to be bounded by a positive constant E at the base $t = 0$ of the cylinder and the data error at $t = T$ is less than a prescribed $\varepsilon > 0$, then the norm of the difference in the solutions is $O([-\log(\varepsilon/E)]^{-1})$. This logarithmic continuity is essentially the best that can be obtained for this approximation.

The above result can be generalized to operators A which are sectorial with semiangle $\pi/4$ and such that $-A$ generates a contraction semigroup of operators. Simple numerical results for the heat equation in a rectangle illustrate the approximation results.

1. Introduction. Consider the region of the plane given by $0 \leqq x \leqq \pi$ and $0 \leqq t \leqq 1$. Suppose the solution $u(x, t)$ to the heat equation, $u_{xx} = u_t$, in the above region is known approximately for all x when $t = 1$. The object of this paper is to discuss in a Hilbert space setting the numerical approximation and continuous dependence on data of solutions for $t < 1$ to a fairly general class of equations containing the heat equation.

The problem

$$u_{xx} = u_t \quad \text{for } 0 < x < \pi, \quad 0 < t < 1, \tag{1.1a}$$

$$0 = u(0, t) = u(\pi, t) \quad \text{for } 0 < t < 1, \tag{1.1b}$$

$$u(x, 1) = \chi(x) \quad \text{for } 0 < x < \pi, \tag{1.1c}$$

is unstable and not well-posed in the sense of Hadamard [10]. However, continuous dependence of the solution on the data can often be brought about by the additional requirement of a prescribed global bound upon the class of solutions considered [11]. Therefore, we add the restriction

$$|u(x, 0)| < E \quad \text{for } 0 < x < \pi, \tag{1.2}$$

where E is some known positive constant.

Since the heat operator cannot be time-inverted to obtain a well-posed problem (irreversibility), we would like to find an operator "near" the heat operator in some sense for which the backward problem is well-posed. We then compare the solution of the backward problem for the perturbed operator with the desired solution of the original problem (1.1)–(1.2).

Many people have considered this type of problem. Among these are Cannon, Douglas, John, Lattès and Lions, Lavrentiev, Miller, Payne, Pucci, Showalter,

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Buzbee and Carasso, and others [2], [3], [4], [5], [6], [7], [11], [13], [14], [15], [16], [17], [18], [19], [20], [22].

We now consider the differential equation on a Hilbert space,

$$u'(t) = -Au(t), \qquad t \geqq 0, \tag{1.3}$$

where A is a self-adjoint operator that is not dependent on t and is nonnegative, which means the numerical range of A is contained in the right half of the complex plane. The method of *quasi-réversibilité* introduced by Lattés and Lions [13] replaces A in (1.3) by a function of the operator, $f(A) = A - \delta A^2$, with spectrum bounded above and then solves the backward problem for the new operator. Using the final value for this new backward problem as initial data for the original operator, they obtained an approximation which converged to the data in their control theory problem.

Using the quasi-reversibilité idea, Miller [17] employs the requirement of Hölder continuity to determine constraints on $f(A)$. He then shows that an $f(A)$ satisfying these constraints can be found which results in a Hölder degree of approximation. This method leads to rational functions of the operator for which the numerical computations require complex arithmetic and several complicated inversions of the operator at each time step. The purpose of this paper is to consider a perturbation of the operator A which allows much easier numerical computations and still retains logarithmic continuity.

Consider the "pseudoparabolic" [21] or Sobolev equation

$$v'(t) + \beta A v'(t) = -Av(t) \tag{1.4}$$

with $\beta > 0$. Since A is nonnegative and $\beta > 0$, we see that $I + \beta A$ is invertible and we obtain the equation

$$v'(t) = -(I + \beta A)^{-1} A v(t). \tag{1.5}$$

Thus, in the quasi-réversibilité setting we are choosing

$$f(A) = (I + \beta A)^{-1} A. \tag{1.6}$$

The idea of approximating (1.3) by (1.5) is due to Yosida. He uses this idea in his proof of the generation theorem for semigroups of operators [23]. We see that the Sobolev equation (1.4) satisfies the requirement of a bounded spectrum. Also, numerical techniques do not require complex arithmetic. For some numerical methods see [8] and Part II of the author's Ph.D. thesis [7a].

We now state the problem considered in this paper.

Problem. Suppose $u(t)$ is an unknown solution of

$$u'(t) = -Au(t), \qquad t \geqq 0, \tag{1.7a}$$

$$\|u(1) - \chi\| < \varepsilon, \tag{1.7b}$$

$$\|u(0)\| < E, \tag{1.7c}$$

where χ is a given "data" vector in a Hilbert space H, $\varepsilon > 0$ is a known small number, E is a known positive constant, and A is any nonnegative, self-adjoint operator which does not depend on t. H incorporates the side boundary conditions and has norm $\|\cdot\|$. We want to approximate $u(t)$ with $v(t)$, a solution of the approximate problem

$$v'(t) = -(I + \beta A)^{-1}Av(t), \qquad t \geqq 0, \tag{1.8a}$$

$$v(1) = \chi. \tag{1.8b}$$

We shall show that for each $t > 0$, we can choose a β in (1.8) such that

$$\|u(t) - v(t)\| = O([-\log(\varepsilon/E)]^{-1}). \tag{1.9}$$

In § 3, we consider generalizations of the results obtained in § 2 using different techniques. We describe the notion of an operator being sectorial. Then, for any operator A which is sectorial with semiangle $\pi/4$ and such that $-A$ generates a contraction semigroup of operators, we obtain the same type of logarithmic continuity as in (1.9) for the problem related to (1.7). Finally, in § 4, we describe some simple numerical results for the heat equation in the problem (1.1).

2. Continuous dependence on data. It is well known [9], [13], [17] that solutions to (1.7a) have the representation

$$u(t) = e^{-tA}u_0, \qquad t \geqq 0, \quad u_0 \in H, \tag{2.1}$$

where e^{-tA}, the strongly continuous contraction semigroup generated by $-A$, is easily defined in terms of the spectral representation of A. Now we recall a well-known result which stabilizes the problem (1.7).

THEOREM 2.1 (Stability estimate) [1]. *If $u(t)$ is a solution of equation* (1.7), *then* $\log\|u(t)\|$ *is a convex function of t. Consequently, if*

$$\|u(1)\| \leqq \varepsilon, \tag{2.2a}$$

$$\|u(0)\| \leqq E, \tag{2.2b}$$

then

$$\|u(t)\| \leqq \varepsilon^t E^{1-t} \quad \textit{for } 0 \leqq t \leqq 1. \tag{2.3}$$

This stability estimate clearly gives a backward uniqueness result for (1.7). This uniqueness result implies that for $t = 1$, e^{-1A} is a 1–1 operator. Thus the kernel of e^{-A} consists only of the zero vector. An easy computation shows that the kernel of e^{-A} is the orthogonal complement of the range of the adjoint, $(e^{-A})^*$. However, in our discussion, A is self-adjoint and since $(e^{-A})^* = e^{-(A^*)}$ [9], we have that $(e^{-A})^* = e^{-A}$. Thus since the zero vector is the orthogonal complement of the range of e^{-A}, we have the range of e^{-A} is dense in H. Therefore, given any data vector χ in H and $\varepsilon > 0$, we can write for some u_0 in the domain of A,

$$\chi = e^{-A}u_0 + \psi, \tag{2.4a}$$

with

$$\|\psi\| < \varepsilon. \tag{2.4b}$$

Thus we can write any data vector χ in the form (2.4) and be compatible with (1.7b).

Since χ is the exact data for (1.8a), the exact solution for $0 \leqq t \leqq 1$ of (1.8) is given by

$$\begin{aligned} v(t) &= e^{(1-t)(I+\beta A)^{-1}A}\chi \\ &= e^{(1-t)(I+\beta A)^{-1}A}(e^{-A}u_0 + \psi) \\ &= e^{(1-t)(I+\beta A)^{-1}A-A}u_0 + e^{(1-t)(I+\beta A)^{-1}A}\psi, \end{aligned} \tag{2.5}$$

where the strongly continuous semigroup $e^{(1-t)(I+\beta A)^{-1}A}$ is defined in terms of its spectral representation.

THEOREM 2.2. *Let u be a solution of* (1.7) *and let v be given by* (2.5). *If we choose $\beta = 1/\log(E/\varepsilon)$, we obtain for each $t > 0$,*

$$\begin{aligned} \|u(t) - v(t)\| &\leqq \frac{4(1-t)E}{t^2 \log(E/\varepsilon)} + E^{(1-t)}\varepsilon^t \\ &= O([-\log(\varepsilon/E)]^{-1}). \end{aligned} \tag{2.6}$$

Proof. We compare $u(t)$ and $v(t)$ in the norm. From (2.1), (2.4), and (2.5), we have

$$\begin{aligned} \|u(t) - v(t)\| &= \|e^{-tA}u_0 - e^{(1-t)(I+\beta A)^{-1}A-A}u_0 + e^{(1-t)(I+\beta A)^{-1}A}\psi\| \\ &\leqq \|e^{-tA} - e^{(1-t)(I+\beta A)^{-1}A-A}\|\,\|u_0\| + \|e^{(1-t)(I+\beta A)^{-1}A}\|\,\|\psi\|. \end{aligned} \tag{2.7}$$

Thus defining

$$B(t) = e^{-tA} - e^{(1-t)(I+\beta A)^{-1}A-A} \tag{2.8a}$$

and

$$C(t) = e^{(1-t)(I+\beta A)^{-1}A}, \tag{2.8b}$$

it follows from (1.7) and (2.7) that

$$\|u(t) - v(t)\| \leqq \|B(t)\|E + \|C(t)\|\varepsilon. \tag{2.9}$$

Now we consider the roles that $B(t)$ and $C(t)$ play in our problem. We note that $\|B(t)\|$ just measures the amount by which the Sobolev operator differs from the parabolic operator. It is clear that as $\beta \to 0$, $\|B(t)\| \to 0$ in some sense. As in the author's thesis, one can show that $\|B(t)\|$ is at most $O(\beta)$ with the bound

$$\|B(t)\| \leqq \frac{4(1-t)}{t^2}\beta. \tag{2.10}$$

$\|C(t)\|\,\|\psi\|$ measures the effect of the backward Sobolev equation on the error term ψ in the data. We can easily obtain the bound

$$\|C(t)\| < e^{(1-t)/\beta}. \tag{2.11}$$

As $\beta \to 0$, the bound $e^{(1-t)/\beta}$ grows very rapidly. Thus we must balance the two terms against each other to obtain a best estimate. If we could solve for the bound for $\|B(t)\|$ in closed form in terms of β as we did for $\|C(t)\|$, we could obtain the best β in closed form. At present, we can only approximate β.

The best choice of β for the second bound is given by

$$\beta = 1/\log(E/\varepsilon). \tag{2.12}$$

With this choice of β, it follows from (2.9), (2.10), and (2.11) that for any $t > 0$,

$$\|u(t) - v(t)\| \leqq \frac{4(1-t)E}{t^2 \log(E/\varepsilon)} + E^{(1-t)}\varepsilon^t = O([-\log(\varepsilon/E)]^{-1}). \tag{2.13}$$

Remark. The choice (2.12) is not the best possible choice of β since it gives the second term significantly better continuity properties than the first. In regard to this problem, we conducted a simple numerical experiment on the computer for the heat equation (1.1). If we take

$$\chi = e^{-1}\sin x, \tag{2.14}$$

we know a priori the exact solution to the backward heat equation. Using these data and then perturbing it we considered the differences in the Fourier series representations of $u(t)$ and $v(t)$ as we varied ε. The literature tells us we cannot expect usable results all the way back to the time $t = 0$. For $t = 0.5$, we obtained numerically

$$\|u(.5) - v(.5)\| \leqq (.130)[\log(1/\varepsilon)]^{-1.113}. \tag{2.15}$$

Thus even with this very simple problem, where an essentially best possible β was used, we don't get significantly better than logarithmic continuity.

3. Generalizations. We recall that all the results in § 2 hold for any non-negative self-adjoint operator A which does not depend on t. The techniques that were used depended heavily on the self-adjointness of A. We now extend these results to more general operators.

First we assume that $-A$ generates a strongly continuous contraction semigroup of operators on the complex Hilbert space H. We shall add another restriction later and shall need the following theorem.

THEOREM 3.1 [23]. *The operator $-A$ is the generator of a contraction semigroup if and only if $-A$ is closed, densely defined, each $\lambda > 0$ is in the resolvent set of $-A$, and*

$$\|(\lambda + A)^{-1}\| \leqq 1/\lambda \quad \textit{for all } \lambda > 0. \tag{3.1}$$

COROLLARY 3.2 [23]. *If the operator $-A$ is the generator of a contraction semigroup, then for every $\beta > 0$, the operator $J_\beta = (I + \beta A)^{-1}$ is a contraction, or*

$$\|J_\beta\| \leqq 1 \quad \textit{for every } \beta > 0. \tag{3.2}$$

From the identity

$$J_\beta A = (1/\beta)(1 - J_\beta) = AJ_\beta, \tag{3.3}$$

we clearly see that A commutes with J_β. Then from (3.2) and (3.3) we see that $J_\beta A$ is a bounded linear operator:

$$\begin{aligned} \|J_\beta A\| &= \|(1/\beta)(I - J_\beta)\| \\ &\leqq (1/\beta)(1 + \|J_\beta\|) \\ &\leqq 2/\beta. \end{aligned} \tag{3.4}$$

Due to (3.4) we can define the group of linear operators by

$$T_\beta(t) \equiv \exp(-tJ_\beta A), \tag{3.5}$$

where we use the power series to define the exponential function. Yosida showed [23] that for each $\beta > 0$, $t \geqq 0$, $T_\beta(t)$ is a contraction and that the strong limit

$$T(t)x = \operatorname*{s-lim}_{\beta \to 0} T_\beta(t)x, \qquad x \in H, \tag{3.6}$$

exists and is the semigroup generated by $-A$. Yosida also showed that for $x \in D(A)$, the domain of A, then $x \in D((d/dt)T(t))$, $x \in D((d/dt)T_\beta(t))$,

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax, \qquad t \geqq 0, \tag{3.7a}$$

and

$$\frac{d}{dt}T_\beta(t)x = J_\beta AT_\beta(t)x = T_\beta(t)J_\beta Ax, \qquad t \geqq 0. \tag{3.7b}$$

From (3.7b) we see that for any $\alpha, \beta > 0$,

$$T_\alpha(s)T_\beta(t) = T_\beta(t)T_\alpha(s), \qquad s, t > 0. \tag{3.8}$$

Also from (3.7b) if $x \in D(A)$, then $T_\beta(t)x \in D(A)$ and we see that from (3.6) and (3.8), we have for $s, t > 0$,

$$T(s)T_\beta(t)x = \operatorname*{s-lim}_{\alpha \to 0} T_\alpha(s)T_\beta(t)x = \operatorname*{s-lim}_{\alpha \to 0} T_\beta(t)T_\alpha(s)x. \tag{3.9}$$

Now from (3.6),

$$\operatorname*{s-lim}_{\alpha \to 0} T_\alpha(s)x = T(s)x. \tag{3.10}$$

Then since $T_\beta(t)$ is continuous and thus closed,

$$\operatorname*{s-lim}_{\alpha \to 0} T_\beta(t)T_\alpha(s)x = T_\beta(t) \quad \operatorname*{s-lim}_{\alpha \to 0} T_\alpha(s)x = T_\beta(t)T(s)x. \tag{3.11}$$

Thus combining (3.9) and (3.11) we see that for $x \in D(A)$, $s, t > 0$,

$$T(s)T_\beta(t)x = T_\beta(t)T(s)x. \tag{3.12}$$

Now we need to consider some additional terminology. An unbounded operator A on H is called *sectorial* with semiangle θ if the numerical range of A, $\{(Ax, x) : x \in D(A)\}$, is contained in the sector $\{z : |\arg(z)| \leqq \theta\}$. We recall the following theorem.

THEOREM 3.3 [12]. *If $-A$ generates a contraction semigroup T and A is sectorial with semiangle θ, where $0 \leqq \theta < \pi/2$, then T is a holomorphic semigroup. For each $t > 0$ and $x \in H$, $T(t)x \in D(A)$ and $AT(t)$ is a bounded linear operator on H with $\|AT(t)\| \leqq M_1/t$, where M_1 is a positive constant. The identity $T(t) = T(t/m)^m$ holds for $t > 0$ and $m \geqq t$, and we also have*

$$\|A^mT(t)\| \leqq M_m/t^m, \tag{3.13}$$

where M_m are a sequence of positive constants.

Note. The last inequality in Theorem 3.1 is crucial in obtaining the last results above.

Showalter [22] introduces a collection of semigroups which he calls Q-R semigroups generated by $-(A - J_\beta A)$ and denoted by

$$E_\beta(t) \equiv \exp(-t(A - J_\beta A)), \tag{3.14}$$

and proves the following theorem.

THEOREM 3.4 [22]. *Let $-A$ generate a contraction semigroup and define the Q-R semigroups, E_β, by (3.14). Then the Q-R semigroups are each contractions if and only if A is sectorial with semiangle $\pi/4$. In this case we have $\lim_{\beta\to 0} E_\beta(t)x = x$ for each $x \in H$, uniformly on bounded intervals, and the following estimates hold for any $t \geqq 0$:*

$$\|E_\beta(t)x - x\| \leqq t\|Ax - J_\beta Ax\|, \qquad x \in D(A), \tag{3.15a}$$

$$\|E_\beta(t)x - x\| \leqq t\beta\|A^2x\|, \qquad x \in D(A^2). \tag{3.15b}$$

We can now state the main theorem of this section.

THEOREM 3.5. *Let $-A$ generate a contraction semigroup and let A be sectorial with semiangle $\pi/4$. Let u be an unknown solution to*

$$u'(t) = -Au(t), \qquad t \geqq 0, \tag{3.16a}$$

$$\|u(1) - \chi\| < \varepsilon, \tag{3.16b}$$

$$\|u(0)\| < E, \tag{3.16c}$$

where $\chi = e^{-A}u_0 + \psi$ and $\|\psi\| < \varepsilon$. Let

$$v(t) = e^{(1-t)J_\beta A}\chi \tag{3.17}$$

be a solution to

$$v'(t) = -J_\beta Av(t), \tag{3.18a}$$

$$v(1) = \chi, \tag{3.18b}$$

where $J_\beta = (I + \beta A)^{-1}$. For the choice

$$\beta = 2/\log(E/\varepsilon), \tag{3.19}$$

$$\|u(t) - v(t)\| = O([-\log(\varepsilon/E)]^{-1}) \tag{3.20}$$

holds for each $t > 0$, where the constant depends on t and is displayed below in (3.30).

Proof. In defining his Q-R semigroups, instead of the notation in (3.9), Showalter [22] defines

$$E_\beta(t) \equiv T(t)T_\beta(-t), \qquad t \geqq 0, \quad \beta > 0, \tag{3.21}$$

where $T(t)$ and $T_\beta(t)$ are defined in (3.6) and (3.5). In this form, (3.15b) becomes, for $t = 1$,

$$\lim_{\beta\to 0} T(1)T_\beta(-1)x = x, \qquad x \in D(A). \tag{3.22}$$

Since by Theorem 3.1, $-A$ is densely defined, the range of $T(1) = e^{-A}$ is dense in

H and the requirement that

$$\chi = e^{-A}u_0 + \psi, \tag{3.23a}$$

where

$$\|\psi\| < \varepsilon \tag{3.23b}$$

is not a restriction.

Consider $u(t)$ and $v(t)$. From (3.5), (3.6), (3.16), (3.17), and (3.18) we have

$$\begin{aligned}\|u(t) - v(t)\| &= \|e^{-tA}u_0 - e^{(1-t)J_\beta A}(e^{-A}u_0 + \psi)\| \\ &\leqq \|T(t)u_0 - e^{-(A-J_\beta A)}T_\beta(t)u_0\| + \|e^{(1-t)J_\beta A}\psi\|.\end{aligned} \tag{3.24}$$

Using the fact that

$$e^{-(A-J_\beta A)} = E_\beta(1) = T(1)T_\beta(-1) \tag{3.25}$$

from (3.14) and (3.21), we have from (3.8),

$$\begin{aligned}\|u(t) - v(t)\| &\leqq \|T(t)u_0 - T(1)T_\beta(-1)T_\beta(t)u_0\| + \|e^{(1-t)J_\beta A}\psi\| \\ &= \|T(t)u_0 - T(1-t)T_\beta(-1+t)T(t)u_0\| + \|e^{(1-t)J_\beta A}\psi\| \\ &= \|T(t)u_0 - E_\beta(1-t)T(t)u_0\| + \|e^{(1-t)J_\beta A}\psi\|.\end{aligned} \tag{3.26}$$

Theorem 3.3 implies that $-A$ generates a holomorphic semigroup. Hence, $u_0 \in D(A)$ implies $u_0 \in D(A^2)$. Then from (3.4), (3.7), and (3.15b), we have for $t > 0$,

$$\|u(t) - v(t)\| \leqq (1-t)\beta\|A^2T(t)u_0\| + e^{2(1-t)/\beta}\varepsilon. \tag{3.27}$$

Now from (3.13) of Theorem 3.3, we have for $t > 0$,

$$\|u(t) - v(t)\| \leqq \frac{(1-t)}{t^2}M_2E\beta + e^{2(1-t)/\beta}\varepsilon. \tag{3.28}$$

Setting

$$\beta = 2/\log(E/\varepsilon) \tag{3.29}$$

in (3.28), we obtain for each $t > 0$,

$$\|u(t) - v(t)\| \leqq \frac{2(1-t)M_2E}{t^2}\frac{1}{\log(E/\varepsilon)} + E^{(1-t)}\varepsilon^t = O([-\log(\varepsilon/E)]^{-1}). \tag{3.30}$$

4. Numerical results. In this section we present some results of numerical comparison of the Fourier sine series representations of the solutions to the heat equation in a rectangle (1.1) and the corresponding Sobolev equation approximation. We also considered the Crank–Nicolson method and got very comparable numerical results, but these results are not presented here.

First we describe the numerical method for choosing the parameter β in (1.4). We know from the literature that it would be overly optimistic to expect very good numerical results all the way back to $t = 0$. Thus we built in the requirement that the choice of β would be best for results at $t = .5$, half the way back. We noted that for $t = .5$ we can obtain the bound

$$\|u(t) - v(t)\| < (1 - e^{-8\beta})E + e^{1/(2\beta)}\varepsilon. \tag{4.1}$$

To minimize this bound, we differentiated, set the result equal to zero, and used an interval halving method on the computer to obtain an approximation for the required β. Table 1 gives an idea of the optimal choice of β for different data errors with $E = 1$. Using the choices of β found in Table 1 we obtained the result in (2.15).

TABLE 1

Data error	β
10^{-2}	0.196
10^{-3}	0.113
10^{-4}	0.080
10^{-5}	0.061
10^{-6}	0.049
10^{-7}	0.041

Now we describe a few experiments consisting of various perturbations of the fundamental mode of sine curves for which the exact solution of the heat equation is known a priori. We first perturbed the fundamental mode with .01 sin $2x$ and then .01 sin $3x$ in Figs. 1 and 2 for $t = 0.5$. Then in Fig. 3 we perturbed the sine curve

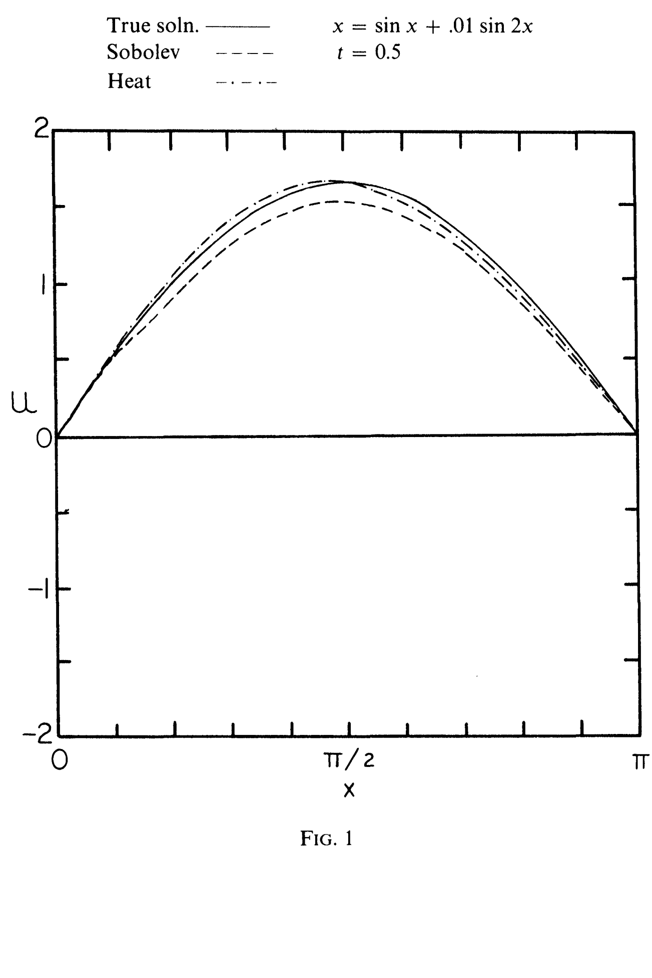

FIG. 1

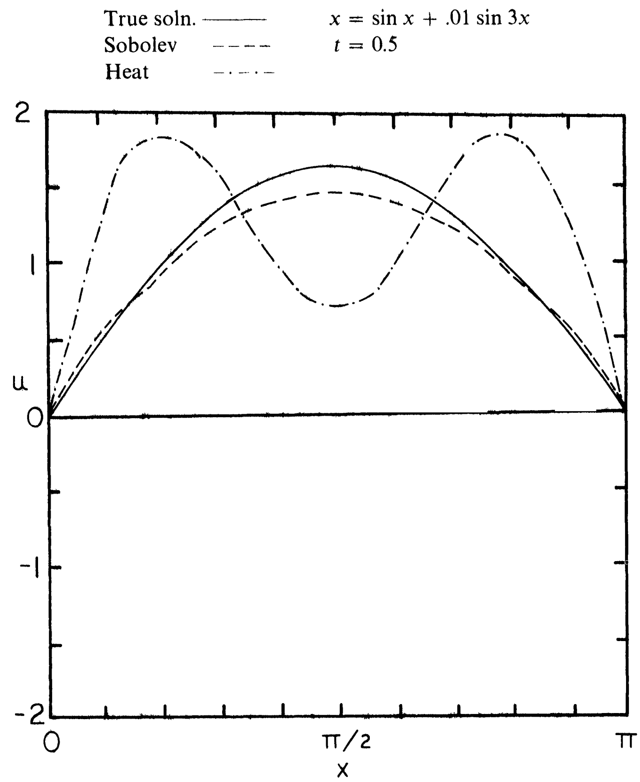

FIG. 2

with a uniform error of .01 (expanded in the first ten terms of this Fourier series) and considered the results at $t = .5$. In each case 100 intervals were used in the x-direction and 10 intervals in the y-direction. Simpson's rule was used for the numerical integration. Truncation in the Fourier series always occurred after 20 terms. In each of the figures all solutions have zero boundary data on $x = 0$ and $x = \pi$. "True solution" means the exact solution u of the heat equation with $u(x, 1) = \sin x$, "Sobolev" refers to the solution w of the Sobolev equation $w_{xx} = w_t - \beta w_{xxt}$ with $w(x, 1) = \sin x$ + perturbation, and "Heat" refers to the solution z of the heat equation (which was obtained by setting $\beta = 0$ in the above equation) with $z(x, 1) = \sin x$ + perturbation. The severe problems with the backward numerical computations on the heat equation are apparent.

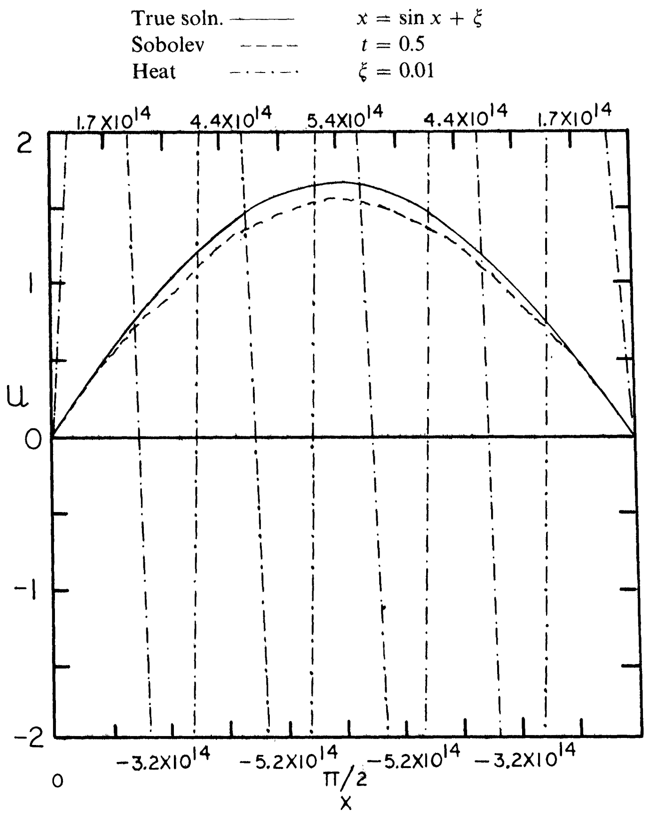

FIG. 3

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