Backward Euler Mixed FEM and Regularity of Parabolic Integro-Differential Equations with Non-smooth Data

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Abstract. The nonFickian flow of fluid in porous media is complicated by the history effect which characterizes various mixing length growth of the flow, and can be modeled by an integro-differential equation. This paper studies a backward Euler scheme for the mixed finite element approximate solution of such problems with non-smooth initial data. A new regularity result is derived for the model problem, which can be used to design high order numerical schemes in time.

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1 Introduction

The transport of contaminants in the subsurface is currently a very active research area in porous media flow modeling. The evolution of a reactive chemical within a velocity field exhibiting excitations on many scales typically represented using classical Fickian dispersion theory. The evolution in such velocity field, when modeled with Fickian type constitutive laws, leads to a dispersion tensor depending upon the timescales of observation. In order to avoid this difficulty, non-local Fickian models have proposed recently. This type of models can represent some important history or memory effects which deserves much attention both in theoretical analysis and numerical approximations.

Cushman and his colleagues [4] have developed a non-local theory and some applications for the flow of fluid in porous media. Furtado, Glimm, Lindquist and Pereira [16] and Glimm, Lindquist, Pereir, and Zhang [17] Neuman and Zhang [21], and Ewing [10] [11] [12] also studied the history effect of various mixing length growth for flow in heterogeneous porous media. In a recent laboratory experimental investigation of contaminant transport in heterogeneous porous media [24], some nonlocal behavior of dispersion tensors have been observed.
Let us consider a simple example where a conservative tracer is transported by convection and dispersion under a steady, saturated, incompressible groundwater flow in a nondeformable porous medium of constant porosity \[5\] [6]. The Darcy’s scale transport equation is thus
\[
\frac{\partial C}{\partial t} + \nabla \cdot (V C) - \nabla \cdot d \nabla C = 0,
\] (1.1)
where \(C\) is the phase concentration of a conservative tracer, \(V = (v_1, v_2, v_3)^T\) is the Darcy scale velocity, and \(d\) is the local scale dispersion tensor assumed to be constant. Because of uncertainty in the data, it is assumed that \(C\) and \(V\) are random. We decompose them into
\[
C = \bar{C} + c, \quad V = \bar{V} + v,
\] (1.2)
where, for example, \(\bar{C}\) is the mean and \(c\) is the fluctuation. The key point here is that there is no “smallness” assumption on any of these fluctuations.

With the decomposition (1.2) and the assumption that the mean of \(V\) is in the \(x_1\) direction, the following equation for \(\bar{C}\) has been derived \[5\]:
\[
\frac{\partial \bar{C}}{\partial t} + V \frac{\partial \bar{C}}{\partial x_1} - \nabla \cdot d \nabla \bar{C} = \int_0^t \nabla \cdot D'(t-s) \nabla \bar{C} ds,
\] (1.3)
where \(D' = (d'_{ij})\) and \(d'_{ij}(t) = d_{ij} \frac{v_i v_j}{\bar{v}_j}(x) B(x, t)\), and \(B\) is a function in space and time related to \(d\). We note that the correlation between \(v_i v_j\) is one of the main sources generating the nonlocal effects from microscale to macroscale level. Readers are referred to \[4\] [5] [6] [16] [10] [11] [12] and the references therein for the mathematical modeling and other related problems in detail.

In this paper, we are concerned with approximate solutions by mixed finite element methods and some regularity of the solution with non-smooth initial data of the following model problem: Find \(u = u(x, t)\) satisfying
\[
\begin{align*}
\frac{\partial u}{\partial t} &= -\nabla \cdot \sigma - cu + f \quad \text{in } \Omega \times J, \\
\sigma &= -A(t)\nabla u - \int_0^t B(t, s) \nabla u(s) ds \quad \text{in } \Omega \times J, \\
u &= -g \quad \text{on } \partial \Omega \times J, \\
u &= u_0(x), \quad x \in \Omega, \quad t = 0,
\end{align*}
\] (1.4)
where \(\Omega \subset \mathbb{R}^d (d = 2, 3)\) is an open bounded domain with smooth boundary \(\partial \Omega\), \(J = (0, T)\) with \(T > 0\), \(A(t) = A(x, t)\) and \(B(t, s) = B(x, t, s)\) are two \(2 \times 2\) or \(3 \times 3\) matrices, and \(A^{-1}\) exists and is bounded, \(c \geq 0\), \(f\), \(g\) and \(u_0\) are known smooth functions. Clearly (1.3) is a special case of (1.4).

For numerical approximations, several finite difference methods were studied for the approximate solution of (1.4). The method of backward Euler and Crank-Nicolson combined with a certain numerical quadrature rule is employed by Thomée and Sloan \[23\] to deal with the time direction which aims at reducing the computational cost and storage spaces due to the memory effect. In the finite element method, there is extensive literature from
the last ten years [2] [19], in which optimal and superconvergence can be found for the corresponding finite element approximations in various norms. In particular, the method of using a Ritz-Volterra projection, discovered by Cannon and Lin [2], proved to be a powerful technique behind the analysis. The mathematical difficulty associated with the analysis of numerical approximations to the solution of (1.4) lies on the integral term added to the standard parabolic equation. Our main contribution of this paper is the establishment of a new regularity result for the problem with non-smooth data. This result can be employed to yield high order numerical schemes in time.

The paper is organized as follows. In Section 2, we propose a backward Euler time-stepping numerical scheme for the general parabolic integro-differential equation (1.4) based on the variables \( u \) and \( \sigma \). In Section 3, we derive an optimal order error estimate for the fully-discretized mixed finite element approximations in the \( L^2 \) norm. Finally in Section 4, we examine the regularity of the solution with non-smooth data.

## 2 Discretization by Mixed Finite Elements

In this section, we discretize the parabolic integro-differential equation (1.4) by using backward Euler in time and mixed finite elements in space variable. For simplicity, the method will be presented on plane domains.

Let \( W = L^2(\Omega) \) be the standard \( L^2 \) space on the domain \( \Omega \) with norm \( \| \cdot \| \). Denote by

\[
V = H(div, \Omega) = \{ \sigma \in (L^2(\Omega))^2 \mid \nabla \cdot \sigma \in L^2(\Omega) \}
\]

the Hilbert space equipped with the following norm:

\[
\| \sigma \|_V = \left( \| \sigma \|^2 + \| \nabla \cdot \sigma \|^2 \right)^{1/2}.
\]

There are several ways to discretize the problem (1.4) based on the variables \( \sigma \) and \( u \); each of the methods corresponds to a particular variational form of (1.4) [14].

Let \( T_h \) be a finite element partition of \( \Omega \) into triangles or quadrilaterals which is quasi-regular. Let \( V_h \times W_h \) denote a pair of finite element spaces satisfying the Brezzi-Babuška condition. For example, the elements of Raviart and Thomas [22] would be a good choice for \( V_h \) and \( W_h \). Our result in this paper is based on the use of Raviart-Thomas element of order \( k \geq 1 \). Extensions to other stable elements can be discussed without any difficulty.

Let us recall from [14] that the weak mixed formulation for (1.4) seeks \((u, \sigma) \in W \times V\) such that

\[
(u_t, w) + (\nabla \cdot \sigma, w) + (cu, w) = (f, w), \quad \forall w \in W,
\]

\[
(\alpha \sigma, v) + \int_0^t (M(t, s)\sigma(s), v)ds - (\nabla \cdot v, u) = (g, v \cdot n), \quad \forall v \in V,
\]

\[
u(0, x) = u_0(x) \text{ in } L^2(\Omega),
\]

\[
(2.1)
\]
where $\alpha = A^{-1}(t)$, $M(t, s) = R(t, s)A^{-1}(s)$ and $R(t, s)$ is the resolvent of the matrix $A^{-1}(t)B(t, s)$ and is given by

$$R(t, s) = -A^{-1}(t)B(t, s) - \int_s^t A^{-1}(t)B(t, \tau) R(\tau, s)d\tau, \quad t > s \geq 0. \quad (2.2)$$

Here $\langle \cdot, \cdot \rangle$ indicates the $L^2$-inner product on $\partial \Omega$.

The corresponding discrete version seeks a pair $(u_h, \sigma_h) \in W_h \times V_h$ such that

$$(u_{h,t}, w_h) + (\nabla \cdot \sigma_h, w_h) + (cu_h, w_h) = (f, w_h), \quad \forall w_h \in W_h,$$

$$(\alpha \sigma_h, v_h) + \int_0^t (M(t, s)\sigma_h(s), v_h)ds - (u_h, \nabla \cdot v_h) = \langle g, n \cdot v_h \rangle, \quad \forall v_h \in V_h. \quad (2.3)$$

The discrete initial condition $u_h(0, x) = u_{0,h}$, where $u_{0,h} \in W_h$ is some appropriately chosen approximation of the initial data $u_0(x)$, should be added to $(2.3)$ for starting. The pair $(u_h, \sigma_h)$ is a semi-discrete approximation of the true solution of $(1.4)$ in the finite element space $W_h \times V_h$ [1] [22] [14], where $\sigma_h(0)$ is chosen to satisfy the equation $(2.3)$ with $t = 0$; namely, it is related to $u_{0,h}$ as follows:

$$(\alpha \sigma_h(0), v_h) - (u_{0,h}, \nabla \cdot v_h) = \langle g_0, n \cdot v_h \rangle, \quad (2.4)$$

where $-g_0 = -g(0, x)$ is the boundary data of the initial value.

Some optimal order error estimates have been established for the semi-discrete approximation $(u_h, \sigma_h)$ by the authors [14]. Our objective in this paper is to study a backward Euler scheme of $(2.3)$. A corresponding error estimate of optimal order shall be derived for the fully-discretized problem.

To this end, let $\Delta t > 0$ be a time-step size and $t_n = n\Delta t$, $n = 0, 1, 2, \ldots$, and $\partial g^n = (g^n - g^{n-1})/\Delta t$. The backward Euler scheme seeks $(u^n_h, \sigma^n_h)$ satisfying

$$(\alpha \sigma^n_h + \sum_{j=1}^{n-1} \Delta t M_{n, j}\sigma^n_h, v_h) - (u^n_h, \nabla \cdot v_h) = \langle g^n, v_h \cdot n \rangle, \quad \forall v_h \in V_h, \quad (2.5)$$

$$(\partial u^n_h, w_h) + (\nabla \cdot \sigma^n_h, w_h) + (cu_n, w_h) = (f^n, w_h), \quad \forall w_h \in W_h,$$

where $M_{n, j} = M(t_n, t_j)$, $u^n_0$ and $\sigma^n_0$ are some appropriate approximations of $u_0$ and $-\Delta u_0$ in $W_h$ and $V_h$, respectively. The quadrature error is given by

$$q_n(g) = \int_0^{t_n} M(t_n, s)g(s)ds - \sum_{j=1}^{n-1} \Delta t M_{n, j}g(t_j),$$

which satisfies

$$|q_n(g)| \leq C\Delta t \int_0^{t_n} (|g| + |g'|)dt.$$
3 Error Estimates for Backward Euler Scheme

Here we derive an optimal order error estimate for the backward Euler approximation (2.5). The result is similar to those of the mixed method applied to parabolic problems without memory effects [7] [25].

In the finite element analysis for parabolic problems, it is often convenient to consider projections of the true solution by using the elliptic part of the differential operator. For example, the standard Ritz projection is a good candidate in the Galerkin method for parabolic equations [29] [25]. For integro-differential equations, the Ritz-Volterra projection [2] played a useful role in the error analysis. The use of Ritz-Volterra projection often leads to results with less requirement on the regularity of the true solution than pure Ritz projection, though the Ritz projection has a simpler analysis in the error estimate.

For simplicity, we shall use the standard Ritz projection in our analysis. But a modified mixed Volterra projection can be defined and used if one prefers to have a better error estimate. This fact will be outlined at the end of this section.

Let \((\bar{u}, \bar{\sigma})\) be the solution of (1.4). Consider a pair \((\bar{u}_h, \bar{\sigma}_h) : [0, T] \to W_h \times V_h\) which is defined as the solution of

\[
\begin{align*}
(\alpha (\sigma - \sigma_h) - (\nabla \cdot v_h, u - \bar{u}_h) &= 0, \quad \forall v_h \in V_h, \quad (3.1) \\
(\varepsilon (u - \bar{u}_h), w_h) + (\nabla \cdot (\sigma - \sigma_h), w_h) &= 0, \quad \forall w_h \in W_h. \quad (3.2)
\end{align*}
\]

The pair \((\bar{u}_h, \bar{\sigma}_h)\) is known as the mixed Ritz finite element projection of \((u, \sigma)\) onto \(W_h \times V_h\). The error between \((\bar{u}_h, \bar{\sigma}_h)\) and \((u, \sigma)\) has been well-studied in many literatures such as [1] [22] [8]. We recall from [8] [22] the following results.

**Lemma 3.1** Assume that \((u, \sigma)\) is the solution of (2.1) and the Raviart-Thomas element is used in the discrete problem (3.1) and (3.2). Then, there exists a unique pair \((\bar{u}_h, \bar{\sigma}_h)\) satisfying (3.1)-(3.2). Furthermore, there exists a positive constant \(C > 0\) independent of \(h > 0\) small such that, for \(m \geq 0\),

\[
\begin{align*}
||D^m_t(u - \bar{u}_h)|| &\leq C h^r ||u(t)||_{r,m}, \quad \text{if } k \geq 1, \quad \text{and } 2 \leq r \leq k + 1, (3.3) \\
||D^m_t(\sigma - \sigma_h)|| &\leq C h^{r+1} ||u(t)||_{r+1,m}, \quad \text{if } 1 \leq r \leq k + 1, \quad (3.4) \\
||D^m_t(\nabla \cdot (\sigma - \sigma_h))|| &\leq C h^r ||u(t)||_{r+1,m}, \quad \text{if } 0 \leq r \leq k + 1, \quad (3.5)
\end{align*}
\]

where ||u(t)||_{r,m} = \sum_{j=0}^{m} ||D^j_t(u(t))||_r, \quad r \in R, \quad t \geq 0, \text{and } ||\cdot||_r \text{ stands for the norm in the Sobolev space } H^r(\Omega).

We are now in a position to present the main result of this section.

**Theorem 3.1** Assume that \((u, \sigma)\) and \((u^h_0, \sigma^h_0)\) are the solutions of (2.1) and (2.5), respectively, and \(||P_h u_0 - \bar{u}_h^0(0)|| \leq C h^r ||u_0||_r\) and \(||\sigma(0) - \bar{\sigma}_h^0(0)|| \leq
\[ Ch^r \|u_0\|_{r+1}. \text{ Then we have for } k \geq 1 \]
\[
\|u(t_n) - u_h^n\|^2 + \|\sigma(t_n) - \sigma_h^n\|^2 \\
\leq Ch^{2r} \left( \|u_0\|_{r+1}^2 + \int_0^{t_n} (\|u(s)\|^2_{r+1} + \|u_t(s)\|^2_{r+1}) \, ds \right) \\
+ C(\Delta t)^2 \int_0^{t_n} (\|u_t\|^2 + \|u_{tt}\|^2) \, ds. \tag{3.6}
\]

**Proof:** For simplicity, we shall assume \( c = 0 \) in the discrete problem (2.5).

Let
\[
u_h(t_n) - u_h^n = (u - \bar{u}_h)(t_n) + (\bar{u}_h(t_n) - u_h^n) = \rho^n + \rho_h^n,
\]
\[
\sigma(t_n) - \sigma_h^n = (\sigma - \bar{\sigma}_h)(t_n) + (\bar{\sigma}_h(t_n) - \sigma_h^n) = \theta^n + \theta_h^n,
\]
where \((\bar{u}_h, \bar{\sigma}_h)\) is the mixed Ritz projection defined by (3.1)-(3.2). It follows from Lemma 3.1 that it is sufficient to bound \(\rho^n_h\) and \(\theta^n_h\).

We find easily from (2.5) and (3.1)-(3.2) that \((\rho^n_h, \theta^n_h)\) satisfies
\[
(\alpha \theta^n_h + \sum_{j=1}^{n-1} \Delta t M_{n,j} \theta_j^n, v_h) - (\rho^n_h, \nabla \cdot v_h) = (\mu^n, v_h), \quad \forall v_h \in V_h, \tag{3.7}
\]
\[
(\partial \rho^n_h, w_h) + (\nabla \cdot \theta^n_h, w_h) = (-\partial \rho^n + \tau^n, w_h), \quad \forall w_h \in W_h, \tag{3.8}
\]
where
\[
\tau^n = \partial u(t_n) - u_t(t_n), \quad \mu^n = \int_0^t M(t_n, s) \sigma(s) \, ds - \Delta t \sum_{j=1}^{n-1} M_{n,j} \theta_j^n.
\]

Let \(v_h = \theta^n_h\) in (3.7) and \(w_h = \rho^n_h\) in (3.8). It follows that for \(\epsilon > 0\), small
\[
\frac{1}{2} \Delta t (\|\rho^n_h\|^2 - \|\rho^n_{h-1}\|^2 - \|\rho^n_h - \rho^n_{h-1}\|^2) + (\alpha \theta^n_h, \theta^n_h) \\
= (\partial \rho^n + \tau^n, \rho^n_h) + (\mu^n, \theta^n_h) - \sum_{j=1}^{n-1} \Delta t M_{n,j} \theta_j^n, \theta^n_h) \\
\leq \epsilon \|\theta^n_h\|^2 + C \sum_{j=1}^{n-1} \Delta t (\|\theta^n_j\|^2 + \|\partial \rho^n - \tau^n\|^2) + \|\mu^n\|^2.
\]

Multiplying the above by \(2\Delta t\), summing on \(n\) and then applying Gronwall’s inequality, we obtain
\[
\|\rho^n_h\|^2 + \sum_{j=1}^{n-1} \Delta t (\|\theta^n_j\|^2 \leq C \left\{ \|\rho^0_h\|^2 + \sum_{j=1}^{n-1} \Delta t (\|\partial \rho^n - \tau^n\|^2 + \|\mu^n\|^2) \right\}.
\]

We find from an elementary calculation and Lemma 3.1 that
\[
\sum_{j=1}^{n-1} \Delta t (\|\partial \rho^n - \tau^n\|^2 \leq Ch^{2r} \left( \|u(t_n)\|_{r+1}^2 + \int_0^{t_n} \|u_t\|_{r+1}^2 \, ds \right) \\
+ C(\Delta t)^2 \int_0^{t_n} \|u_{tt}\|^2 \, ds,
\]
\[ \sum_{j=1}^{n-1} \Delta t ||\mu^n||^2 \leq C(\Delta t)^2 \sum_{j=1}^{n} \Delta t \int_0^{t_j} (||\sigma||^2 + ||D_t\sigma||^2)ds + Ch^2r \sum_{j=0}^{n-1} \Delta t ||\sigma(t_j)||_r \]

Thus, we obtain by the triangle inequality

\[ ||u(t_n) - u^n_h||^2 \leq Ch^{2r} \left( \int_0^{t_n} (||u(t)||^2 + ||u(t)\partial_t u||^2 + ||u_{tt}||^2)dt \right)^{1/2}. \]

In order to bound \( \theta^n_h \), we apply \( \partial_t \) to (3.7) to obtain

\[ \begin{align*}
(\alpha \partial_t \theta^n_h, v_h) - (\partial \rho^n_h, \nabla \cdot v_h) &= (\partial \mu^n(\sigma_h), v_h) \\
&\quad + \left( M_{n,n-1}\theta^n_h + \sum_{j=1}^{n-2} \Delta t \partial M_{n,j} \theta^n_j, v_h \right).
\end{align*} \]

By letting \( v_h = \theta^n_h \) in the above identity and \( u_h = \partial \rho^n_h \) in (3.8), we obtain

\[ \frac{1}{2\Delta t} (||\theta^n_h||_{A^{-1}}^2 - ||\theta^{n-1}_h||_{A^{-1}}^2) + ||\partial \rho^n_h||^2 \leq \epsilon ||\rho^n_h||^2 + C(\epsilon) \Delta t \sum_{j=1}^{n-1} \Delta t ||\partial \rho^n - \tau^n||^2 \]

\[ + C(||\theta^n_h||^2 + ||\theta^{n-1}_h||^2) + C(\Delta t) \sum_{j=1}^{n-1} ||\theta_j^n||^2 + C(\Delta t) \sum_{j=1}^{n-1} ||\mu^n||^2, \]

which, together with an use of Gronwall’s inequality, implies the estimate (3.6).

We remark that similar storage saving schemes by combining higher and lower order numerical quadratures proposed in [23] can be formulated and error estimates can be obtained by using the same techniques developed in this paper.

Let \((u, \sigma)\) be the solution of (1.4). Consider a pair \((\bar{u}_h, \sigma_h) : [0, T] \to W_h \times V_h\) which is defined as the solution of

\[ \begin{align*}
(\alpha(\sigma - \bar{\sigma}_h) + \int_0^t M(t,s)(\sigma - \bar{\sigma}_h)(s)ds, v_h) - (\nabla \cdot v_h, u - \bar{u}_h) &= 0, \\
(\epsilon(u - \bar{u}_h), w_h) + (\nabla \cdot (\sigma - \bar{\sigma}_h), w_h) &= 0.
\end{align*} \]
for any $v_h \in V_h$ and $w_h \in W_h$. The pair $(\bar{u}_h, \bar{\sigma}_h)$ is called the mixed Ritz-Volterra finite element projection of $(u, \sigma)$ onto $W_h \times V_h$. If this projection [15] is used in the place of the standard mixed Ritz projection in the proof of Theorem 3.1, the following error estimate can be obtained:

$$
||u(t_n) - u_h^n||^2 \leq C h^2r \left( ||u_0||^{2+1}_r + \int_0^{t_n} ||u(s)||_r^2 + ||u_t(s)||_r^2 ds \right)
+ C(\Delta t)^2 \int_0^{t_n} (||u||^2 + ||u_t||^2 + ||u_{tt}||^2) ds.
$$

(3.11)

The above is an improved estimate over Theorem 3.1 since the regularity requirement is one order lower than that in (3.6).

## 4 Solution Regularity with Non-Smooth Initial Data

The parabolic problem without memory term is known to have a smoothing property for non-smooth initial data. This smoothing property no longer holds true if a memory term is included in the parabolic equation. For example, Thomee and Zhang [27] have derived the following sharp regularity estimate for solutions of parabolic integro-differential equations:

$$
||u(t)|| \leq C t^{-l/2}||u_0||, \quad 0 \leq l \leq 2, \quad t > 0,
$$

(4.1)

where the initial data $u_0$ belongs to $L^2(\Omega)$. The above estimate was established with homogeneous data $f = g = 0$.

Our objective in this section is to improve the regularity estimate (4.1) by showing that, for any $m \geq 0$,

$$
||D^m_t u(t)|| \leq C t^{-m-l/2}||u_0||, \quad 0 \leq l \leq 2, \quad t > 0.
$$

(4.2)

The above estimate indicates that the memory term has impact only on the space variable.

For simplicity, assume that the matrix $A(t)$ is positive definite. It follows that $||\sigma||^2 = (\sigma, \sigma)$ and $||\sigma||^2_\alpha = (\alpha \sigma, \sigma)$ are equivalent.

**Lemma 4.1** Assume that $f = g = 0$. Then, the solution pair $(u, \sigma)$ satisfies, for some $C_k > 0$ and any $t > 0$,

$$
I^{2k} ||D^k_t u(t)||^2 + \int_0^t s^{2k} ||D^k_t \sigma(s)||^2 ds \leq C_k ||u_0||^2, \quad k \geq 0,
$$

(4.3)

$$
\int_0^t s^{2k-1} ||D^k u(s)||^2 ds + I^{2k-1} ||D^{k-1}_t \sigma(t)||^2 \leq C_k ||u_0||^2, \quad k \geq 1.
$$

(4.4)

**Proof:** Without loss of generality, we assume that $A(x, t) = A(x)$ is independent of the time variable $t$ in (2.1). Thus, (2.1) becomes

$$
(u_t, w) + (\nabla \cdot \sigma, w) = 0, \quad \forall w \in W;
$$

(4.5)

$$
(\alpha \sigma + \int_0^t M(t, s) \sigma(s) ds, v) - (u, \nabla \cdot v) = 0, \quad \forall v \in V.
$$

(4.6)
Let $w = u$ and $v = \sigma$ in (4.5) and (4.6), respectively. By Summing up (4.5) and (4.6) we have from the Cauchy inequality that

$$ \frac{1}{2} \frac{d}{dt} ||u||^2 + (\alpha \sigma, \sigma) = -\left( \int_0^t M(t,s)\sigma(s)ds, \sigma \right) \leq \frac{1}{2} ||\sigma||^2_a + C \int_0^t ||\sigma(s)||^2 ds. $$

An application of the norm equivalence and Gronwall’s Lemma implies that

$$ ||u(t)||^2 + \int_0^t ||\sigma(s)||^2 ds \leq C||u(0)||^2, $$

which gives the desired estimate (4.3) for $k = 0$.

Now differentiate the second equation of (4.6) in time to obtain

$$ (\alpha \sigma_t + M(t,t)\sigma(t) + \int_0^t M_t(t,s)\sigma(s)ds , v) - (u_t, \nabla \cdot v) = 0, \quad \forall v \in V, $$

where $M_t(t,s)$ denotes the partial derivative of $M(t,s)$ with respect to $t$. Let $w = u_t$ in (4.5) and $v = \sigma$ in the last identity. It follows that

$$ ||u_t||^2 + \frac{1}{2} \frac{d}{dt} ||\sigma||^2_a = - \left( M(t,t)\sigma, \sigma \right) - \left( \int_0^t M_t(t,s)\sigma(s)ds, \sigma \right) \leq C||\sigma||^2 + C \int_0^t ||\sigma(s)||^2 ds, $$

so that it follows by multiplying the above inequality by $t$ and using (4.3) with $k = 0$ and integrating from 0 to $t$ that

$$ \int_0^t s||u_t(s)||^2 ds + t||\sigma(t)||^2 \leq C||u(0)||^2. $$

which is (4.4) for $k = 1$.

The rest of the proof is an argument of mathematical induction. Assume that (4.3) and (4.4) are valid for some $k \geq 1$ (note that it can be proved easily that (4.3) is valid for $k = 1$). Differentiate (4.5) and (4.6) $k$-times to obtain for $w \in W$, $v \in V$

$$ (D^{k+1}_t u, w) + (\nabla \cdot D^k_t \sigma, w) = 0, \quad (4.7) $$

$$ (\alpha D^k_t \sigma + \sum_{j=0}^{k-1} M_j(t)D^j_t \sigma(t) + \int_0^t D^k_t M(t,s)\sigma(s)ds, v) - (D^k_t u, \nabla \cdot v) = 0, \quad (4.8) $$

where $M_j(t)$ denote some mixed partial derivatives of $M(t,s)$ evaluated at $s = t$. 


Let \( w = D_t^k u \) in (4.7) and \( v = D_t^k \sigma \) in (4.8), respectively. We find that
\[
\frac{1}{2} \frac{d}{dt} \|D_t^k u\|^2 + \|D_t^k \sigma\|^2
\]
\[
= - \left( \sum_{j=0}^{k-1} M_j(t) D_t^j \sigma(t) + \int_0^t D_t^k M(t, s) \sigma(s) ds , D_t^k \sigma \right)
\]
\[
\leq \frac{1}{2} \|D_t^k \sigma\|^2 + C \sum_{j=0}^{k-1} \|D_t^j \sigma(t)\|^2 + C \int_0^t \|\sigma(s)\|^2 ds.
\]
Therefore, we have by multiplying the above inequality by \( t^{2k} \) and integrating from 0 to 0 together with the mathematical induction hypothesis (4.3) that
\[
t^{2k} \|D_t^k u\|^2 + \int_0^t s^{2k} \|D_t^k \sigma(s)\|^2 ds \leq C \int_0^t \left( s^{2k-1} \|D_t^k \sigma(s)\|^2 + \|u(0)\|\right) ds
\]
\[
\leq C_k \|u(0)\|^2, \quad t > 0. \tag{4.9}
\]
Similarly, we have by differentiating (4.8) that
\[
(\alpha D_t^{k+1} \sigma) + \sum_{j=0}^k \hat{M}_j(t) D_t^j \sigma(t) + \int_0^t D_t^{k+1} M(t, s) \sigma(s) ds , v)
\]
\[-(D_t^{k+1} u, \nabla \cdot v) = 0,
\]
where \( \hat{M}_j \), like \( M_j \), also denotes some mixed partial derivatives of \( M(t, s) \) evaluated at \( s = t \). Thus, by setting \( w = D_t^{k+1} u \) in (4.7) and \( v = D_t^k \sigma \) in the above equality, we find easily from a simple calculation and the mathematical induction hypotheses and (4.9) that
\[
t^{2k+1} \|D_t^{k+1} u\|^2 + \frac{1}{2} \frac{d}{dt} \left( t^{2k+1} \|D_t^k \sigma\|^2 \right)
\]
\[
\leq C t^{2k+1} \left( \sum_{j=0}^k \|D_t^j \sigma(t)\|^2 + \|u(0)\|^2 \right) + C t^{2k} \|D_t^k u(t)\|^2,
\]
which leads to (4.4) by integrating from 0 to \( t \).

**Lemma 4.2** Under the assumptions of Lemma 4.1, there holds for some positive constants \( C_k > 0 \) and \( t > 0 \) that
\[
\|\nabla \cdot D_t^k \sigma(t)\| \leq C_k t^{-(k+1)} \|u(0)\|, \quad k \geq 0, \tag{4.10}
\]
\[
\|\nabla D_t^k u(t)\| \leq C_k t^{-(k+1/2)} \|u(0)\|, \quad k \geq 0. \tag{4.11}
\]

**Proof:** Notice that from (4.5) we have \( u_t = \nabla \cdot \sigma \) in \( L^2(\Omega) \) for \( t > 0 \). Thus, it follows from Lemma 4.1 that \( D_t^{k+1} u_t = \nabla \cdot D_t^{k+1} \sigma \) in \( L^2(\Omega) \). Moreover, we have
\[
\|\nabla \cdot D_t^k \sigma(t)\| = \|D_t^{k+1} u_t(t)\| \leq C t^{-(k+1)} \|u(0)\|, \quad t > 0, \tag{4.12}
\]
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which verifies (4.10).

Next, we observe that from (4.6)

\[-\nabla u = \alpha \sigma + \int_0^t M(t, s)\sigma(s)ds \quad \text{in } L^2(\Omega).\]

Thus, we have

\[\nabla D^k_t u(t) = \alpha D^k_t \sigma(t) + \sum_{j=0}^{k-1} M_j(t)D^j_t \sigma(t) + \int_0^t D^k_t M(t, s)\sigma(s)ds,\]

where \(M_j(t)\) is the same as before. The estimate (4.11) then follows from Lemma 4.1.

**Lemma 4.3** Under the assumptions of Lemma 4.1, there holds for some positive constants \(C_k > 0\) and \(t > 0\) that

\[||D^k_t u(t)||_2 \leq C_k t^{-k-1}||u(0)||, \quad k \geq 0.\]

**Proof:** It is easy to see from the definition of \(\sigma\) that

\[\nabla \cdot \sigma = Au + \int_0^t B(t, s)u(s)ds = Au + B(t, t) \left(\int_0^t u(s)ds\right) + \int_0^t B_s(t, s) \left(\int_0^s u(\tau)d\tau\right)ds\]

in \(L^2(\Omega)\), where \(A = -\nabla \cdot A(x)\nabla\) and \(B = -\nabla \cdot B(x, t, s)\nabla\) are two differential operators of second order in space. In particular, the operator \(A\) is elliptic and self-adjoint. Consequently, we have \(u \in H^2(\Omega)\) for \(t > 0\). By integration we find that

\[u(t) - u(0) = \int_0^t \nabla \cdot \sigma(s)ds = A \left(\int_0^t u(s)ds\right) + \int_0^t B_s(t, s) \left(\int_0^s u(\tau)d\tau\right)ds + \int_0^t \left(\int_0^s B_s(t, \tau) \int_0^\tau u(\eta)d\eta \right)d\tau ds.\]

It follows from the ellipticity of the operator \(A\) that

\[\left\|\int_0^t u(s)ds\right\|_2 \leq C \left\|A\int_0^t u(s)ds\right\| \leq ||u(t) - u(0)|| + C \int_0^t \left\|\int_0^s u(\tau)d\tau\right\| ds \leq C||u(0)|| + C \int_0^t \left\|\int_0^s u(\tau)d\tau\right\| ds,\]

which together with Gronwall’s inequality implies

\[||\int_0^t u(s)ds||_2 \leq C||u(0)||, \quad t \geq 0.\]
Hence, we have easily for $k = 0$ that
\[
\|u(t)\|_2 \leq C\|Au(t)\| \leq C\|\int_0^t u(s)ds\|_2 + \|\nabla \cdot \sigma\| \\
\leq C\|u(0)\| + Ct^{-1}\|u(0)\| \leq Ct^{-1}\|u(0)\|, \quad t > 0.
\]
The case of $k \geq 1$ can be shown similarly for which we omit the detail.

**Theorem 4.1** There holds:
\[
\|u(t)\|_{4-\beta} \leq C_{k,\beta} t^{-k-(4-\beta)/2}\|u(0)\|, \quad k \geq 0,
\]
where $0 \leq \beta \leq 2$ is the order of the operator $B(t, s)$ in space variables.

**Proof:** The proof in fact follows from Lemmas above.

Finally, we remark that the new regularity result proved above can be used to derive high order schemes in time.

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**References**


