Robust Solvers for Symmetric Positive Definite Operators and Weighted Poincaré Inequalities

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Abstract. An abstract setting for robustly preconditioning symmetric positive definite (SPD) operators is presented. The term "robust" refers to the property of the condition numbers of the preconditioned systems being independent of mesh parameters and problem parameters. Important instances of such problem parameters are in particular (highly varying) coefficients. The method belongs to the class of additive Schwarz preconditioners. The core of the method is a robust stable decomposition of functions into several local and one coarse component. The coarse component is contained in a coarse space whose construction is based on the solution of local generalized eigenvalue problems. The paper gives an overview of the results obtained in a recent paper by the authors. It, furthermore, focuses on the importance of weighted Poincaré inequalities (WPIs) for the analysis of stable decompositions. WPIs have recently received attention in the setting of the scalar elliptic case. In our abstract framework we extend the notion of WPIs to general SPD operators. To demonstrate the applicability of the abstract preconditioner the scalar elliptic equation and the stream function formulation of Brinkman's equations in two spatial dimensions are considered. Several numerical examples are presented.

Keywords: domain decomposition, robust additive Schwarz preconditioner, spectral coarse spaces, high contrast, Brinkman's problem, generalized weighted Poincaré inequalities

1 Introduction

The robust preconditioning of symmetric positive definite (SPD) operators has been an important topic in the numerical analysis community. These operators correspond to symmetric coercive bilinear forms appearing in the weak formulation of various partial differential equations (PDEs) and systems modeling e.g. heat conduction or fluid flow in porous media. The condition numbers of the resulting linear systems typically depend on the mesh parameters of the underlying discretizations and variations in physical problem parameters, e.g. (highly) varying thermal conductivities in compound media. Thus, the convergence rates of iterative methods like conjugate gradients (CG) deteriorate as the mesh parameters decrease and the variations in problem parameters increase. One is, therefore, interested in designing preconditioners yielding preconditioned systems whose condition numbers are robust with respect to problem and mesh parameters.

Commonly used approaches include domain decomposition methods (cf. e.g. [12, 17]) and multilevel/multigrid algorithms (cf. e.g. [1, 9, 19]). For certain classes of problems, including the scalar elliptic equation, $-\nabla\cdot(\kappa(\boldsymbol{x})\nabla\phi) = f$, these methods are successful in making the condition number of the preconditioned system independent of the mesh parameter. E.g., it is known that for the scalar elliptic equation a two-level overlapping domain decomposition preconditioner with generous overlap yields a condition number independent of mesh parameters (cf. e.g. [17, Section 3]). However, designing preconditioners that are robust with respect to variations in the physical parameters, e.g., the contrast in the conductivity $\max_{\boldsymbol{x}\in\Omega}\kappa(\boldsymbol{x})/\min_{\boldsymbol{x}\in\Omega}\kappa(\boldsymbol{x})$, where Ω is the domain, is more challenging. Some improvements in standard domain decomposition methods were made in the case of special arrangements of the highly conductive regions with respect to the coarse cells. The construction of preconditioners for these problems has been extensively studied in the last decades (see e.g. [8, 12, 17]). E.g., it was shown that nonoverlapping domain decomposition methods converge independently of the contrast (e.g. [11, 13] and [17, Sections 6.4.4 and 10.2.4]) when conductivity variations within coarse regions are bounded.

Classical arguments to estimate the condition number of a two level overlapping domain decomposition method for the scalar elliptic case use weighted Poincaré inequalities (WPIs) of the form

$$\int_\omega \kappa(\psi - I_0^\omega \psi)^2 \, d\boldsymbol{x} \leq C \int_\omega \kappa |\nabla \psi|^2 \, d\boldsymbol{x}, \tag{1}$$

where $\omega \subset \Omega$ is a local subdomain and $\psi \in H^1(\omega)$. The operator $I_0^\omega \psi$ is a local representation of the function ψ in the coarse space. The constant C in (1) appears in the final bound for the condition number of the operator. Thus, it is desirable to obtain (1) with a constant independent of the contrast. In [15, Section 4.5] this is achieved for the case of quasi-monotonic coefficients. Other approaches consider a more general WPI, given by (1) with κ on the left hand side being replaced by $\kappa(\boldsymbol{x})|\nabla \xi|^2$, where ξ denotes a partition of unity function subordinate to ω (cf. [7]).

For (1) to hold with a constant independent of the contrast in κ the choice of the coarse space is crucial. In particular, two main sets of coarse basis functions have been used in previous works: (1) multiscale finite element functions with various boundary conditions (see e.g. [5, 8, 10]) and (2) energy minimizing or trace minimizing functions (see e.g. [18, 21]). In these cases the coarse spaces have one coarse basis function per coarse node, and the corresponding overlapping domain decomposition methods are robust when the high-conductivity regions are isolated islands.

In [6, 7] the construction of coarse basis functions is based on the solution of local generalized eigenvalue problems. Using a proper projection onto the resulting coarse space, WPIs with contrast independent constants are obtained. This result holds true for general configurations of κ. Using multiscale partition of unity functions accounts for isolated local features within coarse blocks yielding a reduced dimension of the coarse space (see [7]). The idea of using local and global eigenvectors to construct coarse spaces within two-level and multi-level techniques has been used before (e.g. [2, 16]). However, these authors did not study the convergence with respect to physical parameters and did not use generalized eigenvalue problems to achieve small dimensional coarse spaces.

In [4] the construction of coarse spaces based on generalized eigenfunctions has been generalized to abstract SPD bilinear forms. It is shown that the resulting coarse spaces yield robust additive Schwarz preconditioners, such that the resulting condition numbers are controlled by the maximal number of overlaps of subdomains and a predefined threshold determining which generalized eigenfunctions enter the coarse space construction. Also, to reduce the dimension of the coarse space multiscale partition of unity functions are considered, which as in [7] are shown to capture local features. The general framework of [4] was shown to be applicable to scalar elliptic equations and the stream function formulation of the corresponding mixed forms, Stokes' and Brinkman's equations. The latter models fluid flow in highly porous media and can be viewed as a generalization of Stokes' and Darcy's equations (see [20] and the references therein).

The robustness properties of the methods in [6, 7] and [4] are similar. Nevertheless, the generalized eigenvalue problems of the general framework in [4] applied to the scalar elliptic case differ from those studied in [6, 7]. In the paper at hand we investigate the relation between the two approaches. In particular we show that in the scalar elliptic case the validity of (1), with a coarse space as in [6], is equivalent to an estimate in [4] which only involves the bilinear form of the problem. Since the latter estimate can be formulated in an abstract setting, it can be considered a generalization of the concept of WPIs.

The paper is organized as follows. In Section 2 we introduce the problem setting and outline the construction of abstract robust preconditioners as discussed in [4]. Section 3 briefly addresses the application of this abstract framework to the scalar elliptic equation and Brinkman's problem. In Section 4, we discuss the relation between the abstract framework in [4] and WPIs. Section 5 is devoted to some numerical results showing the robustness of the abstract preconditioner when applied to the scalar elliptic and Brinkman's equations.

2 Constructing Coarse Spaces in Robust Stable Decompositions

Let $\Omega \subset \mathbb{R}^n$ be a bounded polyhedral domain, and let $\mathcal{T}_H$ be a quasiuniform quadrilateral ($n = 2$) or hexahedral ($n = 3$) triangulation of Ω with mesh-parameter H. Let $\mathcal{X} = \{\boldsymbol{x}_j\}_{j=1}^{n_{\boldsymbol{x}}}$ be the set of nodes of $\mathcal{T}_H$, and for each $\boldsymbol{x}_j \in \mathcal{X}$ we set $\Omega_j := interior(\bigcup\{\overline{T} \,|\, T \in \mathcal{T}_H,\ \boldsymbol{x}_j \in \overline{T}\})$, i.e., Ω_j is the union of all cells surrounding $\boldsymbol{x}_j$. We define $I_j := \{i = 1, \ldots, n_{\boldsymbol{x}} \,|\, \Omega_i \cap \Omega_j \neq \emptyset\}$ and set $n_I := \max_{j=1,\ldots,n_{\boldsymbol{x}}} \#I_j$. Thus, n_I denotes the maximal number of overlaps of the Ω_j's.

For a separable Hilbert space $\mathscr{V}_0 = \mathscr{V}_0(\Omega)$ of functions defined on Ω and for any subdomain $\omega \subset \Omega$ we set $\mathscr{V}(\omega) := \{\phi|_\omega \,|\, \phi \in \mathscr{V}_0\}$. Using this notation, we make the following assumptions:

(A1) $a_\omega(\cdot, \cdot) : (\mathscr{V}(\omega), \mathscr{V}(\omega)) \to \mathbb{R}$, for subdomains $\omega \subset \Omega$, is a family of symmetric positive semi-definite bounded bilinear forms. For $\omega = \Omega$ we drop the subindex and we assume that $a(\cdot, \cdot)$ is positive definite. For ease of notation we write $a_\omega(\phi, \psi)$ instead of $a_\omega(\phi|_\omega, \psi|_\omega)$ for all $\phi, \psi \in \mathscr{V}_0$.
(A2) For any $\phi \in \mathscr{V}_0$ and any family of pairwisely disjoint subdomains $\{\omega_j\}_{j=1}^{n_\omega}$ with $\cup_{j=1}^{n_\omega} \overline{\omega}_j = \overline{\Omega}$ we have $a(\phi, \phi) = \sum_{j=1}^{n_\omega} a_{\omega_j}(\phi, \phi)$.
(A3) For a suitable subspace $\mathscr{V}_0(\Omega_j)$ of $\mathscr{V}(\Omega_j)$ we have that $a_{\Omega_j}(\cdot, \cdot) : (\mathscr{V}_0(\Omega_j), \mathscr{V}_0(\Omega_j)) \to \mathbb{R}$ is positive definite for all $j = 1, \ldots, n_{\boldsymbol{x}}$.
(A4) $\{\xi_j\}_{j=1}^{n_{\boldsymbol{x}}} : \Omega \to [0, 1]$ is a family of functions such that: (a) $\sum_{j=1}^{n_{\boldsymbol{x}}} \xi_j \equiv 1$ on Ω; (b) $\mathrm{supp}(\xi_j) = \overline{\Omega}_j$ for $j = 1, \ldots, n_{\boldsymbol{x}}$; (c) For $\phi \in \mathscr{V}_0$ we have $\xi_j \phi \in \mathscr{V}_0$ and $(\xi_j \phi)|_{\Omega_j} \in \mathscr{V}_0(\Omega_j)$ for $j = 1, \ldots, n_{\boldsymbol{x}}$.

Now, we would like to construct a "coarse" subspace $\mathscr{V}_H = \mathscr{V}_H(\Omega)$ of $\mathscr{V}_0$ with the following property: For any $\phi \in \mathscr{V}_0$ there is a representation

$$\phi = \sum_{j=0}^{n_{\boldsymbol{x}}} \phi_j \text{ with } \phi_0 \in \mathscr{V}_H,\ \phi_j \in \mathscr{V}_0(\Omega_j),\ j = 1, \ldots, n_{\boldsymbol{x}} \quad \text{such that} \quad \sum_{j=0}^{n_{\boldsymbol{x}}} a(\phi_j, \phi_j) \le C a(\phi, \phi)\,. \tag{2}$$

By abstract domain decomposition theory (see e.g. [17, Section 2.3]) we know that the additive Schwarz preconditioner corresponding to (2) yields a condition number that only depends on the constant C in (2) and n_I. Thus, we would like to "control" this constant and keep the dimension of $\mathscr{V}_H$ "as small as possible".

For the construction of $\mathscr{V}_H$ we define the following symmetric bilinear form for $j = 1, \ldots, n_{\boldsymbol{x}}$.

$$m_{\Omega_j}(\cdot, \cdot) : (\mathscr{V}(\Omega_j), \mathscr{V}(\Omega_j)) \to \mathbb{R}, \quad \text{with} \quad m_{\Omega_j}(\phi, \psi) := \sum_{i \in I_j} a_{\Omega_j}(\xi_j \xi_i \phi, \xi_j \xi_i \psi) \tag{3}$$

To ease the notation, as we did for the bilinear form $a(\cdot, \cdot)$, we write $m_{\Omega_j}(\phi, \psi)$ instead of $m_{\Omega_j}\big(\phi|_{\Omega_j}, \psi|_{\Omega_j}\big)$ for any $\phi, \psi \in \mathscr{V}_0$.

Due to (A4) we have that (3) is well-defined. Also note, that since $\mathrm{supp}(\xi_j) = \overline{\Omega}_j$ we have $\xi_j \phi|_{\Omega_j} \equiv 0 \Leftrightarrow \phi|_{\Omega_j} \equiv 0$, which implies that $m_{\Omega_j}(\cdot, \cdot) : (\mathscr{V}(\Omega_j), \mathscr{V}(\Omega_j)) \to \mathbb{R}$ is positive definite.

Now for $j = 1, \ldots, n_{\boldsymbol{x}}$ we consider the generalized eigenvalue problems: Find $(\lambda_i^j, \varphi_i^j) \in (\mathbb{R}, \mathscr{V}(\Omega_j))$ such that

$$a_{\Omega_j}\Big(\psi, \varphi_i^j\Big) = \lambda_i^j\, m_{\Omega_j}\Big(\psi, \varphi_i^j\Big), \quad \forall \psi \in \mathscr{V}(\Omega_j). \tag{4}$$

Without loss of generality we assume that the eigenvalues are ordered, i.e., $0 \le \lambda_1^j \le \lambda_2^j \le \ldots \le \lambda_i^j \le \lambda_{i+1}^j \le \ldots$. We now state our final assumption.

(A5) For a sufficiently small "threshold" $\tau_\lambda^{-1} > 0$ we may choose $L_j \in \mathbb{N}_0$ such that $\lambda_{L_j+1}^j \ge \tau_\lambda^{-1}$ for all $j = 1, \ldots, n_{\boldsymbol{x}}$. Without loss of generality we may assume $\tau_\lambda^{-1} < 1$.

It is easy to see that any two eigenfunctions corresponding to two distinct eigenvalues are $a_{\Omega_j}(\cdot, \cdot)$- and $m_{\Omega_j}(\cdot, \cdot)$-orthogonal. By orthogonalizing those eigenfunctions corresponding to one and the same eigenvalue, we can thus without loss of generality assume that all computed eigenfunctions are pairwisely $m_{\Omega_j}(\cdot, \cdot)$-orthonormal. Now, every function in $\mathscr{V}(\Omega_j)$ has an expansion with respect to the eigenfunctions of (4).

For $\phi \in \mathscr{V}_0$ let ϕ_0^j be the $m_{\Omega_j}(\cdot, \cdot)$-orthogonal projection of $\phi|_{\Omega_j}$ onto the first L_j eigenfunctions of (4), where $L_j \in \mathbb{N}_0$ is some non-negative integer, i.e.,

$$m_{\Omega_j}\Big(\phi - \phi_0^j, \varphi_i^j\Big) = 0, \quad \forall\, i = 1, \ldots, L_j. \tag{5}$$

If $L_j = 0$, we set $\phi_0^j \equiv 0$. It is easy to see that $\phi|_{\Omega_j} - \phi_0^j = \sum_{i>L_j} m_{\Omega_j}\left(\phi,\, \varphi_i^j\right) \varphi_i^j$. Now, we note that

$$m_{\Omega_j}(\phi,\, \phi) = m_{\Omega_j}\left(\phi,\, \sum_{i\geq 1} m_{\Omega_j}\left(\phi,\, \varphi_i^j\right) \varphi_i^j\right) = \sum_{i\geq 1} m_{\Omega_j}\left(\phi,\, \varphi_i^j\right)^2,$$

and by (4) we see that

$$a_{\Omega_j}(\phi,\, \phi) = \sum_{i\geq 1} m_{\Omega_j}\left(\phi,\, \varphi_i^j\right) a_{\Omega_j}\left(\phi,\, \varphi_i^j\right) = \sum_{i\geq 1} \lambda_i^j m_{\Omega_j}\left(\phi,\, \varphi_i^j\right)^2.$$

Combining these two observations we note that

$$m_{\Omega_j}\left(\phi - \phi_0^j,\, \phi - \phi_0^j\right) \leq \tau_\lambda\, a_{\Omega_j}\left(\phi - \phi_0^j,\, \phi - \phi_0^j\right) \leq \tau_\lambda\, a_{\Omega_j}(\phi,\, \phi)\,. \tag{6}$$

With these preliminaries we are now able to define a decomposition described in (2): First, we specify the coarse space by

$$\mathscr{V}_H := \operatorname{span}\{\xi_j \varphi_i^j \,|\, j = 1, \ldots, n_{\boldsymbol{x}} \text{ and } i = 1, \ldots, L_j\}. \tag{7}$$

Then, for any $\phi \in \mathscr{V}$ let

$$\phi_0 := \sum_{j=1}^{n_{\boldsymbol{x}}} \xi_j \phi_0^j \in \mathscr{V}_H, \quad \text{and for } j = 1, \ldots, n_{\boldsymbol{x}} \quad \phi_j := (\xi_j(\phi - \phi_0))|_{\Omega_j} \in \mathscr{V}_0(\Omega_j), \tag{8}$$

where ϕ_0^j is chosen according to (5). Note that with these definitions $\phi = \sum_{j=0}^{n_{\boldsymbol{x}}} \phi_j$.

Theorem 1. *Using the notation above and assuming that (A1)–(A5) hold, the decomposition defined in* (8) *satisfies* $\sum_{j=0}^{n_{\boldsymbol{x}}} a(\phi_j,\, \phi_j) \leq (2 + C\,\tau_\lambda) a(\phi,\, \phi)$, *where* C *only depends on* n_I.

Proof. See [4, Theorem 3.4].

3 Application to Scalar Elliptic and Brinkman's Equations

Scalar Elliptic Equation

The scalar elliptic equation is given by

$$-\nabla \cdot (\kappa \nabla \phi) = f, \quad \boldsymbol{x} \in \Omega, \text{ and } \quad \phi = 0, \quad \boldsymbol{x} \in \partial\Omega, \tag{9}$$

where $0 < \kappa \in L^\infty(\Omega)$, $\phi \in H_0^1(\Omega)$, and $f \in L^2(\Omega)$. With $\mathscr{V}_0 := H_0^1(\Omega)$, the corresponding variational formulation reads: Find $\phi \in \mathscr{V}_0$ such that for all $\psi \in \mathscr{V}_0$

$$a^{SE}(\phi,\, \psi) := \int_\Omega \kappa(\boldsymbol{x}) \nabla\phi \cdot \nabla\psi \, d\boldsymbol{x} = \int_\Omega f\psi \, d\boldsymbol{x}.$$

It is easy to see that with $\mathscr{V}_0(\Omega_j) := H_0^1(\Omega_j) \subset \mathscr{V}_0|_{\Omega_j}$ and ξ_j the Lagrange finite element function of degree 1 corresponding to x_j, $j = 1, \ldots, n_{\boldsymbol{x}}$, we have that (A1)–(A4) hold. Now consider (4) with $a(\cdot,\, \cdot)$ and $m(\cdot,\, \cdot)$ replaced by $a^{SE}(\cdot,\, \cdot)$ and $m^{SE}(\cdot,\, \cdot)$, where $m^{SE}(\cdot,\, \cdot)$ is given by (3) with $a^{SE}(\cdot,\, \cdot)$ instead of $a(\cdot,\, \cdot)$. It can then be shown (see [4, Section 4.1]), that for a binary medium, i.e., a medium with

$$\kappa(\boldsymbol{x}) = \begin{cases} \kappa_{\min} \text{ in } \Omega^s \\ \kappa_{\max} \text{ in } \Omega^p, \end{cases} \text{ with } \overline{\Omega}^s \cup \overline{\Omega}^p = \overline{\Omega} \text{ and } \kappa_{\max} \gg \kappa_{\min} > 0,$$

the $(L_j + 1)$-st eigenvalue of (4), i.e., λ_{L_j+1} is uniformly (with respect to H and $\kappa_{\max}/\kappa_{\min}$) bounded from below. Here L_j denotes the number of connected components of $\Omega^s \cap \Omega_j$. Thus, also (A5) is established with τ_λ independent of the contrast $\kappa_{\max}/\kappa_{\min}$. Note that by Theorem 1 this implies the robustness of the stable decomposition (8) and the corresponding additive Schwarz preconditioner.

Brinkman's Equations

Brinkman's equations modeling flows in highly porous media are given by

$$-\mu\Delta\boldsymbol{u} + \nabla p + \mu\kappa^{-1}\boldsymbol{u} = \boldsymbol{f} \text{ in } \Omega, \qquad \nabla\cdot\boldsymbol{u} = 0 \text{ in } \Omega, \qquad \text{and} \qquad \boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\Omega, \tag{10}$$

where $p \in L^2(\Omega)/\mathbb{R}$, $\boldsymbol{u} \in (H_0^1(\Omega))^2$, $\boldsymbol{f} \in (L^2(\Omega))^2$, $\mu \in \mathbb{R}^+$, and $\kappa \in L^\infty(\Omega)$, with $\kappa > 0$. Here we assume that $\Omega \subset \mathbb{R}^2$ is simply connected. The variational formulation of the Brinkman problem is: Find $(\boldsymbol{u}, p) \in ((H_0^1(\Omega))^2, L_0^2(\Omega))$ such that for all $(\boldsymbol{v}, q) \in ((H_0^1(\Omega))^2, L_0^2(\Omega))$ we have

$$\int_\Omega \mu\nabla\boldsymbol{u} : \nabla\boldsymbol{v}\, d\boldsymbol{x} + \int_\Omega \mu\kappa^{-1}\boldsymbol{u}\cdot\boldsymbol{v}\, d\boldsymbol{x} - \int_\Omega p\nabla\cdot\boldsymbol{v}\, d\boldsymbol{x} - \int_\Omega q\nabla\cdot\boldsymbol{u}\, d\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\, d\boldsymbol{x}. \tag{11}$$

As described in [4, Section 4] we may adopt the setting of stream functions. For $\mathscr{V}_0 := \left\{\psi \in H^2(\Omega) \cap H_0^1(\Omega) \,|\, \frac{\partial\psi}{\partial\boldsymbol{n}}|_{\partial\Omega} = 0\right\}$ the variational stream function formulation reads: Find $\phi \in \mathscr{V}_0$ such that for all $\psi \in \mathscr{V}_0$ we have

$$a^B(\phi, \psi) := \int_\Omega \mu\left(\nabla(\nabla\times\phi) : \nabla(\nabla\times\psi) + \kappa^{-1}\nabla\times\phi\cdot\nabla\times\psi\right)\, d\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\nabla\times\psi\, d\boldsymbol{x}. \tag{12}$$

Here $\{\xi_j\}_{j=1}^{n_{\boldsymbol{x}}}$ denotes a sufficiently regular partition of unity and $\mathscr{V}_0(\Omega_j) := \left\{\psi \in H^2(\Omega_j) \cap H_0^1(\Omega_j) \,|\, \frac{\partial\psi}{\partial\boldsymbol{n}}|_{\partial\Omega_j} = 0\right\}$. As shown in [4, Section 4.4] we can verify (A1)-(A5), where for the case of a binary medium, τ_λ in (A5) can again be chosen independently of the contrast $\kappa_{\max}/\kappa_{\min}$.

Remark 1. Instead of solving (12) for the stream function ϕ and then recovering $\boldsymbol{u} = \nabla\times\phi$, one may equivalently use the coarse space corresponding to (12) to construct a coarse space corresponding to the original formulation (11) by applying $\nabla\times$ to the coarse stream basis functions. By this, one obtains an equivalent robust additive Schwarz preconditioner for the formulation of Brinkman's problem in the primal variables $\boldsymbol{u}$ and p (for details see [12, Section 10.4.2]).

4 Connection to Weighted Poincaré inequalities

Poincaré type inequalities play a crucial role in the analysis of domain decomposition methods. In the scalar elliptic case, taking $\mathscr{V}_H$ to be the space of Lagrange finite elements of degree 1 corresponding to the coarse mesh $\mathcal{T}_H$, one can show by using the standard Poincaré inequality that the constant in (2) is independent of the mesh parameter H (see [17, Chapter 3] and the references therein).

Recently, for scalar elliptic problems with varying $\kappa(\boldsymbol{x})$, so called "weighted" Poincaré inequalities (WPIs) have received increasing attention (see [6, 7, 14]). According to [6, 7, 14] the validity of an appropriate WPI with a constant independent of the contrast $\kappa_{\max}/\kappa_{\min}$ yields a corresponding stable decomposition such that the constant in (2) is independent of the contrast.

For giving the precise form of the WPI used in [6] we consider the following generalized eigenvalue problem for each Ω_j, $j = 1, \ldots, n_{\boldsymbol{x}}$: Find $(\widehat{\lambda}_i^j, \widehat{\varphi}_i^j) \in (\mathbb{R}, H^1(\Omega_j))$ such that

$$a_{\Omega_j}^{SE}\left(\psi, \widehat{\varphi}_i^j\right) = \widehat{\lambda}_i^j \widehat{m}_{\Omega_j}^{SE}\left(\psi, \widehat{\varphi}_i^j\right), \quad \forall\psi \in H^1(\Omega_j), \tag{13}$$

where $\widehat{m}_{\Omega_j}^{SE}(\phi, \psi) := H^{-2}\displaystyle\int_{\Omega_j} \kappa(\boldsymbol{x})\,\phi\,\psi\, d\boldsymbol{x}$ for $\phi, \psi \in H^1(\Omega_j)$ is the weighted mass matrix scaled by H^{-2}. This scaling is not considered in [6], however, we introduce it here to simplify the exposition.

The WPI derived in [6] then reads as follows:

$$\widehat{m}_{\Omega_j}^{SE}\left(\phi - \widehat{\phi}_0^j, \phi - \widehat{\phi}_0^j\right) \leq \tau_{\widehat{\lambda}}\, a_{\Omega_j}^{SE}(\phi, \phi)\,, \tag{14}$$

where similarly to above $\widehat{\phi}_0^j$ is the $\widehat{m}_{\Omega_j}^{SE}(\cdot, \cdot)$-orthogonal projection of ϕ onto those eigenfunctions of (13) whose eigenvalues are below a suitably chosen threshold $\tau_{\widehat{\lambda}}^{-1}$. Note, that (14) can be obtained by exactly the same reasoning as (6). Due to the scaling by H^{-2} the eigenvalues in (13) are independent of H. In addition, it can be shown by the same reasoning as in [4, Section 4.1], that for a binary medium the (L_j+1)-st eigenvalue of (13), i.e., $\widehat{\lambda}_{L_j+1}$ is bounded from below, uniformly with respect to $\kappa_{\max}/\kappa_{\min}$, where as above, L_j denotes the number of connected components of $\Omega^s \cap \Omega_j$. Note, that this result in particular implies the existence of $\tau_{\widehat{\lambda}}^{-1} > 0$ such that only a finite number of eigenvalues are below this threshold. This is analogous to (A5), and as for τ_{λ}^{-1} we may assume without loss of generality that $\tau_{\widehat{\lambda}}^{-1} < 1$.

Before pointing out a connection between this WPI and our estimate (6), we introduce a triangular (for $n = 2$) or tetrahedral (for $n = 3$) mesh corresponding to $\mathcal{T}_H$. According to e.g. [3] we can obtain a quasiuniform triangular/tetrahedral mesh, denoted by $\widetilde{\mathcal{T}}_H$, by subdividing cells in $\mathcal{T}_H$ without introducing new nodes, i.e., the nodes of $\widetilde{\mathcal{T}}_H$ are given by $\mathcal{X}$.

Proposition 1. *Let $\{\xi_j\}_{j=1}^{n_{\boldsymbol{x}}}$ be the piecewise linear Lagrange finite element functions corresponding to $\widetilde{\mathcal{T}}_H$. Then we have that inequalities* (6) *(with $a(\cdot,\cdot) = a^{SE}(\cdot,\cdot)$ and $m(\cdot,\cdot) = m^{SE}(\cdot,\cdot)$) and* (14) *are up to constants equivalent in the following sense:*

$$\text{For } \widehat{\phi}_0^j \text{ satisfying (14) we have} \qquad m_{\Omega_j}^{SE}\Big(\phi - \widehat{\phi}_0^j,\, \phi - \widehat{\phi}_0^j\Big) \le C\tau_{\widehat{\lambda}}\, a_{\Omega_j}^{SE}(\phi,\, \phi)\,. \tag{15}$$

$$\text{For } \phi_0^j \text{ satisfying (6) we have} \qquad \widehat{m}_{\Omega_j}^{SE}\Big(\phi - \phi_0^j,\, \phi - \phi_0^j\Big) \le C\tau_{\lambda}\, a_{\Omega_j}^{SE}(\phi,\, \phi)\,. \tag{16}$$

I.e., (up to constant C) (6) *holds with $\widehat{\phi}_0^j$ instead of ϕ_0^j and $\tau_{\widehat{\lambda}}$ instead of τ_λ, and vice versa,* (14) *holds with ϕ_0^j instead of $\widehat{\phi}_0^j$ and τ_λ instead of $\tau_{\widehat{\lambda}}$. Here the constants C are independent of H, $\tau_{\widehat{\lambda}}$, and τ_λ but may depend on n_I and the geometry of Ω_j.*

Proof. Since ξ_j is piecewise linear we note that

$$\frac{1}{CH} \le \min\{|\nabla\xi_j(\boldsymbol{x})| \mid \boldsymbol{x} \in \mathring{T},\ \text{for } T \in \widetilde{\mathcal{T}}_H,\ \mathring{T} \subset \Omega_j\} \quad \text{and} \quad \|\nabla\xi_i\|_{L^\infty(\Omega_j)} \le \frac{C}{H} \text{ for } i \in I_j, \tag{17}$$

where $|\cdot|$ denotes some norm on $\mathbb{R}^n$ and C is a constant only depending on the geometry of Ω_j.

For proving (15) we note that by (3) we have

$$\begin{aligned} m_{\Omega_j}^{SE}\Big(\phi - \widehat{\phi}_0^j,\, \phi - \widehat{\phi}_0^j\Big) &= \sum_{i\in I_j} a_{\Omega_j}^{SE}\Big(\xi_j\xi_i(\phi - \widehat{\phi}_0^j),\, \xi_j\xi_i(\phi - \widehat{\phi}_0^j)\Big) = \sum_{i\in I_j}\int_{\Omega_j}\kappa\left|\nabla\left(\xi_j\xi_i(\phi - \widehat{\phi}_0^j)\right)\right|^2 d\boldsymbol{x} \\ &\le 2\sum_{i\in I_j}\int_{\Omega_j}\kappa\left(\left|\nabla(\phi - \widehat{\phi}_0^j)\right|^2 + \left|\nabla(\xi_j\xi_i)(\phi - \widehat{\phi}_0^j)\right|^2\right) d\boldsymbol{x}, \end{aligned}$$

where we have used Schwarz' inequality and the fact that $|\xi_i| < 1$ for $i \in I_j$. Thus by (17),

$$\begin{aligned} m_{\Omega_j}^{SE}\Big(\phi - \widehat{\phi}_0^j,\, \phi - \widehat{\phi}_0^j\Big) &\le C a_{\Omega_j}^{SE}\Big(\phi - \widehat{\phi}_0^j,\, \phi - \widehat{\phi}_0^j\Big) + \frac{C}{H^2}\sum_{i\in I_j}\int_{\Omega_j}\kappa(\phi - \widehat{\phi}_0^j)^2\, d\boldsymbol{x} \\ &\le C\, a_{\Omega_j}^{SE}(\phi,\, \phi) + C\,\widehat{m}_{\Omega_j}^{SE}\Big(\phi - \widehat{\phi}_0^j,\, \phi - \widehat{\phi}_0^j\Big) \le C\tau_{\widehat{\lambda}}\, a_{\Omega_j}^{SE}(\phi,\, \phi)\,, \end{aligned}$$

where we have used that $\widehat{\phi}_0^j$ satisfies (14), $\tau_{\widehat{\lambda}} > 1$, and $a_{\Omega_j}^{SE}\Big(\phi - \widehat{\phi}_0^j,\, \phi - \widehat{\phi}_0^j\Big) \le a_{\Omega_j}^{SE}(\phi,\, \phi)$, which holds by an analogous reasoning as the second inequality in (6). This establishes (15).

For verifying (16) we first note that by the definition of $\widehat{m}_{\Omega_j}^{SE}(\cdot,\, \cdot)$ and (17) we have

$$\begin{aligned} \widehat{m}_{\Omega_j}^{SE}\Big(\phi - \phi_0^j,\, \phi - \phi_0^j\Big) &= H^{-2}\int_{\Omega_j}\kappa(\phi - \phi_0^j)^2\, d\boldsymbol{x} \\ &\le C\int_{\Omega_j}\kappa\,|\nabla\xi_j|^2\,(\phi - \phi_0^j)^2\, d\boldsymbol{x} = C\int_{\Omega_j}\kappa\Big|\sum_{i\in I_j}\nabla(\xi_i\xi_j)(\phi - \phi_0^j)\Big|^2 d\boldsymbol{x}, \end{aligned}$$

where we have used that $\sum_{i\in I_j}\xi_i \equiv 1$ in Ω_j. Since $\nabla(\xi_i\xi_j(\phi-\phi_0^j)) = \nabla(\xi_i\xi_j)(\phi-\phi_0^j)+\xi_i\xi_j\nabla(\phi-\phi_0^j)$, Schwarz' inequality yields

$$\begin{aligned} \frac{1}{C}\,\widehat{m}_{\Omega_j}^{SE}\Big(\phi - \phi_0^j,\, \phi - \phi_0^j\Big) &\le \sum_{i\in I_j}\int_{\Omega_j}\kappa\left|\nabla(\xi_i\xi_j(\phi - \phi_0^j))\right|^2 + \kappa\,\underbrace{(\xi_i\xi_j)^2}_{\le 1}\left|\nabla(\phi - \phi_0^j)\right|^2 d\boldsymbol{x} \\ &\le \sum_{i\in I_j} a_{\Omega_j}^{SE}\Big(\xi_i\xi_j(\phi - \phi_0^j),\, \xi_i\xi_j(\phi - \phi_0^j)\Big) + C a_{\Omega_j}^{SE}\Big(\phi - \phi_0^j,\, \phi - \phi_0^j\Big) \\ &\le m_{\Omega_j}^{SE}\Big(\phi - \phi_0^j,\, \phi - \phi_0^j\Big) + C a_{\Omega_j}^{SE}(\phi,\, \phi)\,, \end{aligned}$$

where we have used the second inequality in (6). Thus, $\widehat{m}^{SE}_{\Omega_j}\Big(\phi - \phi_0^j,\, \phi - \phi_0^j\Big) \le C\,\tau_\lambda\, a^{SE}_{\Omega_j}(\phi,\, \phi)$, by our assumption that ϕ_0^j satisfies (6) and $\tau_\lambda > 1$. This establishes (16).

Remark 2 (Generalized Weighted Poincaré Inequality). Proposition 1 allows the interpretation of (6) as a *generalized* weighted Poincaré inequality (GWPI). For the scalar elliptic case, i.e., for $a(\cdot,\,\cdot) = a^{SE}(\cdot,\,\cdot)$ and $m(\cdot,\,\cdot) = m^{SE}(\cdot,\,\cdot)$, Proposition 1 shows that in terms of the robustness of the stable decomposition it does not matter whether the coarse space is based on eigenmodes corresponding to (4) or (13). For more complicated bilinear forms, such as $a^B(\cdot,\,\cdot)$, it may, however, be rather difficult to formulate a suitable analogue to the weighted Poincaré inequality (14). Using our abstract approach this is straightforward, since the bilinear form $m_{\Omega_j}(\cdot,\,\cdot)$ is entirely based on the partition of unity $\{\xi_i\}_{i=1}^{n_{\boldsymbol{x}}}$ and the bilinear form $a(\cdot,\,\cdot)$ (see (3)).

5 Numerical Results

In this section, we give some numerical examples showing the robustness of the additive Schwarz preconditioner using the spectral coarse space $\mathscr{V}_H$ introduced in [4] and outlined above (see (7)). We consider applications to the equations in Section 3 with varying contrast $\kappa_{\max}/\kappa_{\min}$. For more details and further numerical experiments we refer the reader to [4]. We consider the geometry shown in Figure 1 where the black parts denote regions of high conductivity (for the scalar elliptic case) and low permeability (in the Brinkman case), respectively. Tables 1(a) and 1(b) show the results for the scalar elliptic and Brinkman case, respectively.

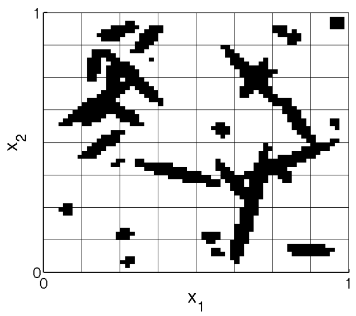

Fig. 1. Sample geometry. The mesh indicates the 8×8 coarse triangulation. The fine triangulation is 64×64.

In the scalar elliptic case we use a conforming finite element discretization with bilinear Lagrange elements yielding a fine space of dimension 4225. The right hand side f in (9) is chosen to compensate for the boundary condition of linear temperature drop in x-direction, i.e., $\phi(\boldsymbol{x}) = 1 - x_1$ on $\partial\Omega$. As partition of unity $\{\xi_j\}_{j=1}^{n_{\boldsymbol{x}}}$, we use bilinear Lagrange finite element functions corresponding to the coarse mesh $\mathcal{T}_H$.

Brinkman's equations are discretized with an $H(div)$-conforming Discontinuous Galerkin discretization (cf. [20] and the references therein) using Raviart-Thomas finite elements of degree 1 (RT1) yielding a fine space of dimension 49408. It is well-known that in two spatial dimensions the stream function space corresponding to the RT1 space is given by Lagrange biquadratic finite elements. The coarse (divergence free) velocity space is constructed as outlined in Remark 1. The right hand side $\boldsymbol{f}$ is chosen to compensate for the boundary condition of unit flow in x-direction, i.e., $\boldsymbol{u} = \boldsymbol{e}_1$ on $\partial\Omega$. As partition of unity we choose piecewise polynomials of degree 3, such that all first derivatives and the lowest mixed derivatives are continuous and $\xi_j(\boldsymbol{x}_i) = \delta_{i,j}$ for $i, j = 1, \ldots, n_{\boldsymbol{x}}$.

In all numerical examples we choose $\tau_\lambda^{-1} = 0.5$ and prescribe a relative reduction of the preconditioned residual by a factor of $1e-6$ as stopping criterion. For comparison reasons we also provide numerical results for additive Schwarz preconditioners using a standard coarse space, denoted by $\mathscr{V}_H^{st}$. For the scalar elliptic case this is given by bilinear Lagrange finite element functions corresponding to the coarse mesh. For the Brinkman problem the standard coarse space is given by the span of the curl of the partition of unity functions corresponding to interior coarse mesh nodes. The numerical results clearly show the robustness of the preconditioners using spectral coarse spaces, whereas the condition numbers corresponding to the preconditioners using standard coarse spaces deteriorate with increasing contrasts.

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(a) Scalar elliptic case.

	Standard coarse space $\mathcal{V}_H^{st}$		Spectral coarse space $\mathcal{V}_H$	
$\kappa_{\max}/\kappa_{\min}$	dim $\mathcal{V}_H^{st}$	cond. #	dim. $\mathcal{V}_H$	cond. #
$1e2$	49	$3.50e1$	70	15.5
$1e3$	49	$3.11e2$	124	26.7
$1e4$	49	$3.06e3$	145	7.92
$1e5$	49	$3.06e4$	148	7.90
$1e6$	49	$2.71e5$	148	7.90

(b) Brinkman case.

	Standard coarse space $\mathcal{V}_H^{st}$		Spectral coarse space $\mathcal{V}_H$	
$\kappa_{\max}/\kappa_{\min}$	dim $\mathcal{V}_H^{st}$	cond. #	dim. $\mathcal{V}_H$	cond. #
$1e2$	49	$1.88e1$	66	12.6
$1e3$	49	$2.68e1$	75	19.2
$1e4$	49	$1.32e2$	99	21.9
$1e5$	49	$1.00e3$	125	17.7
$1e6$	49	$9.31e3$	145	26.0

Table 1. Numerical results for standard and spectral coarse spaces. We report the dimension of the coarse space and the preconditioned condition number for varying contrasts.

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