A domain decomposition preconditioner for multiscale high-contrast problems

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1 Summary

We present a new class of coarse spaces for two-level additive Schwarz preconditioners that yield condition number bound independent of the contrast in the media properties. These coarse spaces are an extension of the spaces discussed in [3]. Second order elliptic equations are considered. We present theoretical and numerical results. Detailed description of the results and numerical studies will be presented elsewhere.

2 Introduction

Many problems in applied sciences occur in the media that contains multiple scales and has high contrast in the properties. For example, it is very common to have several orders of magnitude of variations in the permeability field in natural porous formations. Domain decomposition preconditioners are often used to solve the fine-scale system that arises from the discretization of partial differential equations. The number of iterations required by domain decomposition preconditioners is typically affected by the contrast in the media properties that are within each coarse-grid block. It is known that if high and low conductivity regions can be encompassed within coarse-grid blocks such that the variation of the conductivity within each coarse region is bounded, domain decomposition preconditioners result to a system with the condition number independent of the contrast (e.g., [5, 6]). Because of complex geometry of fine-scale features, it is often impossible to separate low and high conductivity regions into different coarse-grid blocks. Thus, the contrast will adversly affect the number of iterations required by domain decomposition preconditioners.

The design and analysis of preconditioners that converge independent of the contrast is important for many applications, such as porous media flows where flow problems are solved multiple times. In [3], we introduce a coarse

space based on local spectral problems (see also [1]). These spaces are motivated by weighted Poincaré estimates that arise in the proofs of L^2 approximation property of the coarse interpolation. In particular, the spectrum of local eigenvalue problem contains eigenvalues that are small and asymptotically vanish as the contrast increases, and thus, there is a gap in the spectrum. The eigenvectors corresponding to these small (asymptotically vanishing) eigenvalues represent the high-conducting features. The number of these eigenvectors is the same as the number of disconnected high-conductivity inclusions. The coarse space is constructed such that the basis functions span the eigenfunctions corresponding to these small (asymptotically vanishing) eigenvalues as well as some nodal multiscale basis functions. In [3], we prove that if the coarse space includes the basis functions associated to these eigenfunctions, then the condition number of the two level additive method is bounded independent of the contrast of the media.

In many applications where the flow equations are solved multiple times, it is important to choose a coarse space with a minimal dimension. The coarse spaces constructed in [3] represent both high-conductivity channels (high-conductivity inclusions that connect the boundaries of a coarse-grid block) and high-conductivity isolated inclusions. Consequently, these coarse spaces can have a large dimension. In [3], we note that one only needs to represent channels within coarse blocks and present a procedure for removing high-contrast isolated inclusions. In this paper, we present a more general approach that removes the inclusions. In fact, one can consider the proposed construction as an approach that *complements* the coarse spaces constructed using partition of unity functions. In particular, starting with an initial partition of unity functions, e.g., multiscale basis functions, one adds new basis functions by using eigenvectors of weighted eigenvalue problem. In this eigenvalue problem, the weight is computed using the gradient of the initial partition of unity functions, see (10) below. The eigenfunctions corresponding to small (asymptotically vanishing) eigenvalues are chosen and new basis functions that span these eigenfunctions are added to the coarse space. With a correct choice of partition of unity functions, one can remove the inclusions and obtain the coarse space with a small dimension. We present a theoretical result that states that the condition number of the preconditioned system is independent of contrast. Numerical results are presented to demonstrate our theoretical findings.

3 Problem setting and domain decomposition framework

Let $D \subset \mathbb{R}^2$ (or $\mathbb{R}^3$) be a polygonal domain which is the union of a disjoint polygonal subregions $\{D_i\}_{i=1}^N$. We consider the following problem. Find $u^* \in H_0^1(D)$ such that

$$a(u^*, v) = f(v) \quad \text{for all } v \in H_0^1(D). \tag{1}$$

Here the bilinear form a and f are defined by $a(u,v) = \int_D \kappa(x)\nabla u(x)\nabla v(x)dx$, and $f(v) = \int_D f(x)v(x)dx$, for all $u, v \in H_0^1(D)$.

We assume that $\{D_i\}_{i=1}^N$ form a quasiuniform triangulation of D and denote $H = \max_i \operatorname{diam}(D_i)$.

Let $\mathcal{T}^h$ be a fine triangulation which refine $\{D_i\}_{i=1}^N$. We denote by $V^h(D)$ the usual finite element discretization of piecewise linear continuous functions with respect to the fine triangulation $\mathcal{T}^h$. Denote also by $V_0^h(D)$ the subset of $V^h(D)$ with vanishing values on ∂D. Similar notations, $V^h(\Omega)$ and $V_0^h(\Omega)$, are used for subdomains $\Omega \subset D$.

The Galerkin formulation of (1) is to find $u^* \in V_0^h(D)$ with $a(u^*, v) = f(v)$ for all $v \in V_0^h(D)$, or in matrix form

$$Au^* = b, \tag{2}$$

where for all $u, v \in V^h(D)$ we have $u^T Av = \int_D \kappa \nabla u \nabla v$, and $v^T b = \int_D fv$.

It is sufficient to consider the case of piecewise constant coefficient κ. From now on we will assume that κ is piecewise constant coefficient in $\mathcal{T}^h$ with value $\kappa = \kappa_e$ on each fine triangulation element $e \in \mathcal{T}^h$.

We denote by $\{D_i'\}_{i=1}^N$ the overlapping decomposition obtained from the original nonoverlapping decomposition $\{D_i\}_{i=1}^N$ by enlarging each subdomain D_i to $D_i' = D_i \cup \{x \in D, \operatorname{dist}(x, D_i) < \delta_i\}, \quad i = 1, \ldots, N$, where dist is some distance function and let $\delta = \max_{1 \le i \le N} \delta_i$. Let $V_0^i(D_i')$ be the set of finite element functions with support in D_i'. We also denote by $R_i^T : V_0^i(D_i') \to V^h$ the extension by zero operator.

We will use a partition of unity $\{\xi_i\}_{i=1}^N$ subordinated to the covering $\{D_i'\}_{i=1}^N$ such that

$$\sum_{i=1}^N \xi_i = 1, \quad \xi_i \in V^h, \quad \text{and} \quad \operatorname{Supp}(\xi_i) \subset D_i', i = 1, \ldots, N, \tag{3}$$

where $\operatorname{Supp}(\xi_i)$ stands for the support of the function ξ_i. This will be the partition of unity used to truncate global functions to local ones in the proof of the stable decomposition.

Given a coarse triangulation $\mathcal{T}^H$ we introduce N_c coarse basis functions $\{\Phi_i\}_{i=1}^{N_c}$. We define the coarse space by

$$V_0 = \operatorname{span}\{\Phi_i\}_{i=1}^{N_c}, \tag{4}$$

and the coarse matrix $A_0 = R_0 A R_0^T$ where $R_0^T = [\Phi_1, \ldots, \Phi_{N_c}]$.

We use a two level additive preconditioner of the form

$$B^{-1} = R_0^T A_0^{-1} R_0 + \sum_{i=1}^N R_i^T A_i^{-1} R_i, \tag{5}$$

where the local matrices are defined by $vA_i w = a(v, w)$ for all $v, w \in V^i = V_0^h(D_i')$, $i = 1, \ldots, N$. See [5, 6].

We denote by $\{y_i\}_{i=1}^{N_v}$ the vertices of the coarse mesh $\mathcal{T}^H$ and define

$$\omega_i = \bigcup\{K \in \mathcal{T}^H; \quad y_i \in \overline{K}\}, \quad \omega_K = \bigcup\{\omega_j; \quad y_j \in \overline{K}\}. \tag{6}$$

We will use a partition of unity $\{\chi_i\}_{i=1}^{N_v}$ subordinated to the covering $\{\omega_i\}_i^{N_v}$ such that

$$\sum_{i=1}^{N_v} \chi_i = 1, \quad \chi_i \in V^h, \quad \text{and} \quad \text{Supp}(\chi_i) \subset \omega_i, i = 1, \ldots, N_c. \tag{7}$$

4 Coarse-space-completing eigenvalue problem and stability estimates

In this section we define the new local spectral multiscale coarse space using eigenvectors of high contrast eigenvalue problems. Fist we introduce the notation for eigenvalue problems.

For any $\Omega \subset D$ define the matrix A^Ω by

$$v^T A^\Omega w = \int_\Omega \kappa \nabla v \nabla w \quad \text{for all } v, w \in \widetilde{V}^h(\Omega), \tag{8}$$

and the *modified mass matrix* of same dimension M^Ω by

$$v^T M^\Omega w = \int_\Omega \widetilde{\kappa} v w \quad \text{for all } v, w \in \widetilde{V}^h(\Omega), \tag{9}$$

where $\widetilde{V}_h = V_h(\Omega)$ if $\overline{\Omega} \cap \partial D = \emptyset$ and $\widetilde{V}_h = \{v \in V_h(\Omega) : v = 0 \text{ on } \partial\Omega \cap \partial D\}$ otherwise. Here $\widetilde{\kappa}$ in (9) is a weight derived from the high contrast coefficient κ and contains the relevant information we need for the construction of the coarse basis functions. Several possible choices for $\widetilde{\kappa}$ can be considered. We refer to [3] for the case $\widetilde{\kappa} = \kappa$. Here we will consider only the case of the piecewise constant $\widetilde{\kappa}$ given by

$$\widetilde{\kappa} = \max\left\{ \kappa \sum_{i=1}^{N} |\nabla \xi_i|^2, \kappa \sum_{j=1}^{N_v} |\nabla \chi_j|^2 \right\}, \tag{10}$$

where $\{\xi\}_{j=1}^N$ and $\{\chi_i\}_{i=1}^{N_v}$ are the partition of unity introduced in (3) and (7), respectively. From now on, we assume that overlapping decomposition is constructed from the coarse mesh and $\xi_i = \chi_i$ for all $i = 1, \ldots, N = N_v$. We consider the finite dimensional symmetric eigenvalue problem

$$A^\Omega \phi = \widetilde{\lambda} M^\Omega \phi \tag{11}$$

and denote its eigenvalues and eigenvectors by $\{\widetilde{\lambda}_\ell^\Omega\}$ and $\{\psi_\ell^\Omega\}$, respectively. Note that the eigenvectors $\{\psi_\ell^\Omega\}$ form an orthonormal basis of $\widetilde{V}^h(\Omega)$ with

respect to the M^{Ω} inner product. Assume that $\widetilde{\lambda}_1^{\Omega} \leq \widetilde{\lambda}_2^{\Omega} \leq \cdots \leq \widetilde{\lambda}_i^{\Omega} \leq \ldots$, and note that $\widetilde{\lambda}_1^{\Omega} = 0$. In particular, $\psi_{\ell}^{\omega_i}$ denotes the ℓ-th eigenvector of the matrix associated to the neighborhood of y_i, $i = 1, \ldots, N_v$.

In general, when $\tilde{\kappa} = \kappa$ and for the Neumann boundary case, if there are n inclusions and channels, then one can observe n small (asymptotically vanishing) eigenvalues. The eigenvectors corresponding to these eigenvalues will be used to construct the coarse space V_0. In this case, the term $\widetilde{\kappa} = \kappa$ on the right hand side of the eigenvalue problem results to eigenvectors that are nearly constant inside each high conductivity inclusion/channel. When $\widetilde{\kappa}$ is chosen based on (10), then the number of asymptotically small eigenvalues is the same as the number of high-conductivity inclusions in $\widetilde{\kappa}$. In particular, if the partition of unity functions are piecewise linear polynomials then $\widetilde{\kappa}$ and κ have the same high-contrast structure. We are interested in partition of unity functions that can "eliminate" isolated high-conductivity inclusions and thus reduce the size of the coarse space. This can be achieved by minimizing high-conductivity components in $\widetilde{\kappa}$. In particular, by choosing multiscale finite element basis functions or energy minimizing basis functions, we can eliminate all isolated high-conductivity inclusions, while preserving the channels. This can be observed in our numerical experiments. In Figure 1 (below) and Figure 2 (on page 7), we depict κ (middle picture) and $\widetilde{\kappa}$ (right picture) using multiscale basis functions on the coarse grid. The coarse grid is depicted on the left pictures. One can observe that isolated inclusions are removed in $\widetilde{\kappa}$, and consequently, the coarse space contains only long channels that connect boundaries of the coarse grid.

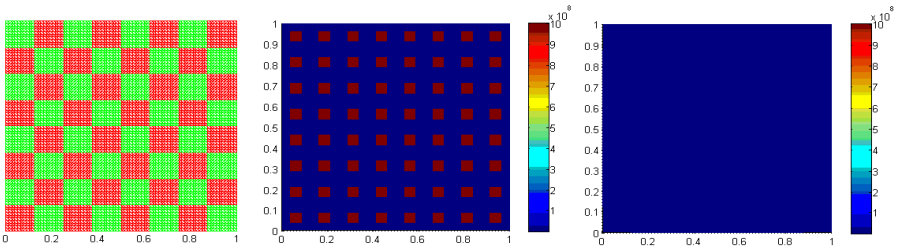

Fig. 1. Left: Coarse mesh. Center: Original coefficient. Here $\eta = 10^9$. Right: Coefficient $\widetilde{\kappa}$ computed as in (10) using (linear) multiscale basis functions.

We note that for the proposed methods, we only need to specify the eigenvectors based on the quantities $\{1/\widetilde{\lambda}_l^{\omega_i}\}$ in each ω_i, $i = 1, \ldots, N_v$. These eigenvectors are used to construct the coarse space.

We assume that the elements of $\mathcal{T}^h$ contained in Ω form a triangulation of Ω. Let $n_h(\Omega)$ denote the number of degrees of freedom in $\overline{\Omega}$. Given an integer L and $v \in V^h(\Omega)$ define

$$I_L^{\Omega} v = \sum_{\ell=1}^{L} \left(\int_{\Omega} \widetilde{\kappa} v \psi_{\ell}^{\Omega} \right) \psi_{\ell}^{\Omega}. \tag{12}$$

Let $\{\chi_i\}_{i=1}^{N_v}$ be a partition of unity (3). Define the coarse basis functions

$$\Phi_{i,\ell} = I^h(\chi_i \psi_{\ell}^{\omega_i}) \quad \text{for } 1 \le i \le N_v \text{ and } 1 \le \ell \le L_i, \tag{13}$$

where I^h is the fine-scale nodal value interpolation and L_i is an integer number for each $i = 1, \ldots, N_v$. Denote by V_0 the *local spectral multiscale* space

$$V_0 = \text{span}\{\Phi_{i,\ell} : 1 \le i \le N_v \text{ and } 1 \le \ell \le L_i\}. \tag{14}$$

Define also the coarse interpolation $I_0 : V^h(D) \to V_0$ by

$$I_0 v = \sum_{i=1}^{N_v} \sum_{\ell=1}^{L_i} \left(\int_{\omega_i} \widetilde{\kappa} v \psi_{\ell}^{\omega_i} \right) I^h(\chi_i \psi_{\ell}^{\omega_i}) = \sum_{i=1}^{N_v} I^h \left((I_{L_i}^{\omega_i} v) \chi_i \right), \tag{15}$$

where I^h is the fine-scale nodal value interpolation and $I_{L_i}^{\omega_i}$ is defined in (12).

We have the following weighted L^2 approximation and weighted H^1 stability properties.

Lemma 1. *For all coarse element K we have*

$$\int_K \widetilde{\kappa}(v - I_0 v)^2 \preceq \frac{1}{\widetilde{\lambda}_{K,L+1}} \int_{\omega_K} \kappa |\nabla v|^2 \tag{16}$$

$$\int_K \kappa |\nabla I_0 v|^2 \preceq \max\{1, \frac{1}{\widetilde{\lambda}_{K,L+1}}\} \int_{\omega_K} \kappa |\nabla v|^2 \tag{17}$$

where $\widetilde{\lambda}_{K,L+1} = \min_{y_i \in K} \widetilde{\lambda}_{L_i+1}^{\omega_i}$ and ω_K is the union of the elements that share common edge with K defined in (6).

The proof of this lemma follows from the results presented in [3] and will be presented elsewhere.

Using Lemma 1, we can estimate the condition number of the preconditioned operator $B^{-1}A$ with B^{-1} defined in (5) using the coarse space V_0 in (14). From the abstract domain decomposition theory we only need to prove the stable decomposition property; see [5, 6]. From this stable decomposition property, one has the following Lemma.

Lemma 2. *The condition number of the preconditioned operator $B^{-1}A$ with B^{-1} defined in (5) is of order*

$$cond(B^{-1}A) \preceq C_0^2 \preceq 1 + \frac{1}{\widetilde{\lambda}_{L+1}}$$

where $\widetilde{\lambda}_{L+1} = \min\limits_{1 \le i \le N_v} \widetilde{\lambda}_{L_i+1}^{\omega_i}$.

It can be easily shown that the eigenvalues of the local problem scale as $O(1)$ assuming $\xi_i = \chi_i$, $i = 1, \ldots, N = N_v$, in (10). The dependency of the condition number of overlapping decomposition (δ) and coarse grid size (H) is controlled by the partition of unity $\{\xi_i\}$ and $\{\chi_i\}$ in (10).

5 Numerical results

In this section, we present representative numerical results for the additive preconditioner (5) with the local spectral multiscale coarse space defined in (14). We take $D = [0,1] \times [0,1]$ that is divided into 8×8 equal square subdomains. Inside each subdomain we use a fine-scale triangulation where triangular elements constructed from 10×10 squares are used.

In our first numerical example, we choose a simple permeability field that only has isolated inclusions, see middle picture of Figure 1. The coarse grid is demonstrated in the left picture. Multiscale finite element basis functions with linear boundary conditions are chosen as a partition of unity functions in (10). The purpose of this example is to demonstrate that $\widetilde{\kappa}$ does not have any high-conductivity components with this choice of partition of unity functions. As a result, we have only one eigenfunction (constant) per coarse grid. Thus, there is no need to complement the space of multiscale basis functions with linear boundary conditions. Note that if we use the eigenvalue problem with the weight function κ, then there will be four basis functions per node that represent inclusions. One can choose any $\widetilde{\kappa}$ that is larger than the one defined by (10). In our simulations, we add a positive constant to $\widetilde{\kappa}$ to avoid a numerical instability. In our numerical results, we observed that the number of iterations with the weight $\widetilde{\kappa} = \kappa$ and the weight $\widetilde{\kappa}$ defined in (10) (which results in the multiscale finite element basis functions) does not change for the contrast $\eta = 10^4, 10^5, 10^6, 10^8, 10^8$. The number of iterations is 22 iterations. Due to space limitation, we do not present detailed numerical results.

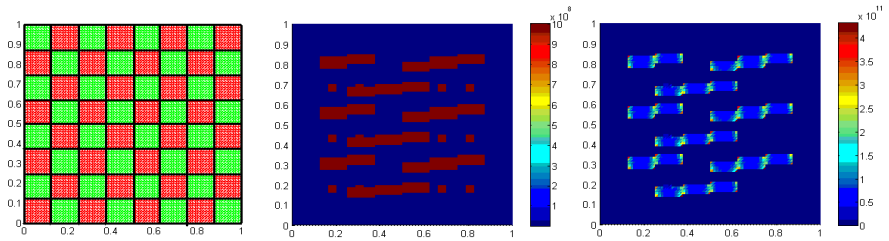

Fig. 2. Left: Coarse mesh. Center: Original coefficient. Right: Coefficient $\widetilde{\kappa}$ computed as in (10) using (linear) multiscale basis functions. The numerical results are presented in Table 1.

In the second example, we test our approach on a more complicated permeability field that contains inclusions and channeles (see middle picture of Figure 2). As before we use multiscale finite element basis functions as the initial partition of unity. From the right picture of Figure 2 we see that the modified weight $\widetilde{\kappa}$ does not contain any isolated inclusions and only contains long channels connecting boundaries of the coarse-grid block. This is achieved automatically from the choice of the partition of unity functions. There are fewer

small (asymptotically vanishing) eigenvalues when local eigenvalue problem is solved with the modified weight $\widetilde{\kappa}$. Thus, with a good choice of partition of unity functions in (10), there are fewer new multiscale basis functions needed to achieve an optimal, in terms of the contrast, convergence. Numerical results are presented in Table 1. We observe that using the proposed coarse spaces, the number of iterations is independent of contrast. In Table 1 we also show the dimension of the coarse spaces. The dimension of the local spectral coarse space is smaller if we use $\widetilde{\kappa}$ in (10) instead of $\widetilde{\kappa} = \kappa$ as in [3].

η	MS	EMF	LSM ($\widetilde{\kappa} = \kappa$)	LSM($\widetilde{\kappa}$ in (10))
10^4	98(2490.75)	62(257.86)	27(6.19)	28(7.34)
10^5	123(24866.24)	62(283.29)	28(6.19)	29(7.35)
10^6	144(248621.33)	62(286.12)	29(6.19)	29(7.35)
10^7	174(2486172.35)	63(286.41)	29(6.19)	30(7.35)
Dim	49	49	102	69

Table 1. Number of iterations until convergence and estimated condition number for the PCG and different values of the contrast η with the coefficient depicted in Figure 2. We set the tolerance to 1e-10. Here $H = 1/8$ with $h = 1/80$. The notation MS stands for the (linear boundary condition) multiscale coarse space, EMF is the energy minimizing coarse space, see e.g., [7], and LSM is the local spectral multiscale coarse space defined in (14).

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