PARABOLIC FINITE VOLUME ELEMENT EQUATIONS IN NONCONVEX POLYGONAL DOMAINS

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ABSTRACT. We study spatially semidiscrete and fully discrete finite volume element approximations of the heat equation with homogeneous Dirichlet boundary conditions in a plane polygonal domain with one reentrant corner. We show that, as a result of the singularity in the solution near the reentrant corner, the convergence rate is reduced from optimal second order, similarly to what was shown for the finite element method in the earlier work [5]. Optimal order convergence may be restored by mesh refinement near the corners of the domain.

1. INTRODUCTION

We shall consider the finite volume method, using continuous, piecewise linear approximating functions, for the model parabolic initial boundary value problem

$$\begin{aligned} &u_t - \Delta u = f(t) \quad \text{in } \Omega, \quad \text{with } u(\cdot,t) = 0 \quad \text{on } \partial\Omega, \quad \text{for } t > 0, \\ &u(\cdot,0) = v \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where Ω is a nonconvex polygonal domain in $\mathbb{R}^2$. We assume for simplicity that exactly one interior angle ω is reentrant, i.e., such that $\omega \in (\pi, 2\pi)$, and set $\beta = \pi/\omega \in (\frac{1}{2}, 1)$.

In [4] we showed an $O(h^2)$ error bound in L_2 in the case of a convex Ω, and in [5] we discussed the error in the nonconvex case for the finite element method. In the latter case the error in L_2 is reduced from $O(h^2)$ to $O(h^{2\beta})$, as a result of the singularity which is present in the solution of (1.1) at the reentrant corner. In this paper we show the corresponding result for a finite volume method. We also discuss error estimations in H^1 and in the maximum–norm. The present work can be considered as a continuation of [4] and [5], and we refer to these papers for references to the literature.

The finite volume method relies on a local conservation property associated with the differential equation. Namely, integrating (1.1) over any region $V \subset \Omega$ and using Green's formula, we obtain

$$\int_V u_t \, dx - \int_{\partial V} \nabla u \cdot n \, ds = \int_V f \, dx, \qquad \text{for } t > 0, \tag{1.2}$$

where n denotes the unit exterior normal vector to ∂V.

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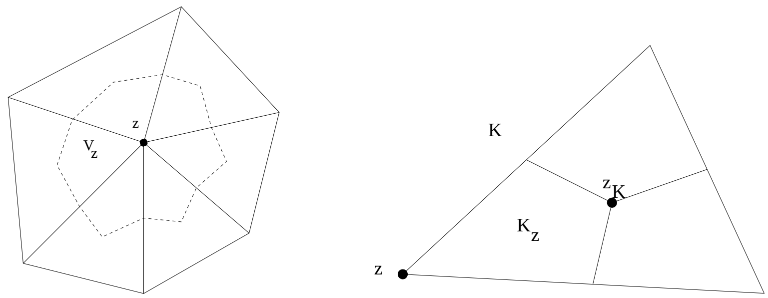

FIGURE 1. *Left:* A union of triangles that have a common vertex z; the dotted line shows the boundary of the corresponding control volume V_z. *Right:* A triangle K partitioned into the three subregions K_z.

There are various approximation strategies in the finite volume (control-volume) method. For comprehensive presentations and references to existing results and various applications we refer to the monographs [7, 9]. Here we shall study spatially semidiscrete approximations of (1.1) by the finite volume element method, which for brevity we will refer to as the finite volume method below. The approximate solution will be sought in the space of piecewise linear finite elements

$$S_h \equiv S_h(\Omega) = \{\chi \in \mathcal{C}(\Omega) : \ \chi|_K \ \text{ linear}, \ \forall K \in \mathcal{T}_h; \ \chi|_{\partial\Omega} = 0\},$$

where $\{\mathcal{T}_h\}_{0<h<1}$ is a family of regular triangulations of Ω, with h denoting the maximum diameter of the triangles of $\mathcal{T}_h$. In the sequel, for simplicity, we shall suppress the index Ω in the notation for functional spaces.

The semidiscrete finite volume approximation $u_h(t) \in S_h$, $t \geq 0$, will satisfy the relation (1.2) for V in a finite collection of subregions of Ω called control volumes, the number of which will be equal to the dimension of the finite element space S_h. These control volumes are constructed as follows. Let z_K be the barycenter of $K \in \mathcal{T}_h$. We connect z_K with line segments to the midpoints of the edges of K, thus partitioning K into three quadrilaterals K_z, $z \in Z_h(K)$, where $Z_h(K)$ are the vertices of K. Then with each vertex $z \in Z_h = \cup_{K\in\mathcal{T}_h} Z_h(K)$ we associate a control volume V_z, which consists of the union of the subregions K_z, sharing the vertex z (see Figure 1). We denote the set of interior vertices of Z_h by Z_h^0. The semidiscrete finite volume method is then to find $u_h(t) \in S_h$ for $t \geq 0$ such that, with $v_h \in S_h$ a given approximation of v,

$$\int_{V_z} u_{h,t}\, dx - \int_{\partial V_z} \nabla u_h \cdot n\, ds = \int_{V_z} f\, dx, \quad \forall z \in Z_h^0, \ t > 0, \quad \text{with } u_h(0) = v_h.$$

This problem may also be expressed in a weak form. For this purpose we introduce the finite dimensional piecewise constant space

$$Y_h = \{\eta \in L_2 : \ \eta|_{V_z} = \text{constant}, \ \forall z \in Z_h^0; \ \eta|_{V_z} = 0, \ \forall z \in \partial\Omega\}.$$

We now multiply the integral relation above by an arbitrary $\eta(z)$, $\eta \in Y_h$, and sum over all $z \in Z_h^0$ to obtain the Petrov–Galerkin formulation

$$(u_{h,t}, \eta) + a_h(u_h, \eta) = (f, \eta), \quad \forall \eta \in Y_h, \ t > 0, \quad \text{with } u_h(0) = v_h, \tag{1.3}$$

where $(v,w) = \int_\Omega vw\,dx$ and the bilinear form $a_h(\cdot,\cdot) : S_h \times Y_h \to \mathbb{R}$ is defined by

$$a_h(v,\eta) = -\sum_{z\in Z_h^0} \eta(z) \int_{\partial V_z} \nabla v \cdot n\,ds, \quad v \in S_h,\ \eta \in Y_h.$$

Obviously, we can extend the definition of $a_h(v,\eta)$ for v in the fractional order Sobolev space H^{1+s}, $s > 1/2$, and using Green's formula we easily see that

$$a_h(v,\eta) = -(\Delta v, \eta), \quad \forall \eta \in Y_h. \quad v \in H^2, \tag{1.4}$$

The stationary elliptic problem corresponding to (1.1) is the Dirichlet problem,

$$-\Delta u = f \quad \text{in } \Omega, \quad \text{with } u = 0 \quad \text{on } \partial\Omega. \tag{1.5}$$

For this problem, the reentrant corner O gives rise to a singularity in the solution with a leading term of the form $c(f) r^\beta \sin(\beta\theta)$, in polar coordinates centered at O, even when f is smooth. This function is not in the space H^{1+s} for any $s \geq \beta$.

The finite volume method approximates the solution of (1.5) by $u_h \in S_h$ from

$$a_h(u_h, \eta) = (f, \eta), \quad \forall \eta \in Y_h,$$

and the error may be shown to satisfy

$$\|u_h - u\| \leq Ch^{2\beta} \|\Delta u\|_{H^{2\beta-1}}, \quad \text{where } \|\cdot\| = \|\cdot\|_{L_2} \text{ and } \tfrac{1}{2} < \beta < 1. \tag{1.6}$$

For the corresponding finite element method,

$$a(\underline{u}_h, \chi) = (f, \chi), \quad \forall \chi \in S_h, \quad \text{where } a(v,w) = \int_\Omega \nabla v \cdot \nabla w\,dx,$$

we have an error bound of the same order, which requires less regularity, namely

$$\|\underline{u}_h - u\| \leq C_s h^{2\beta} \|\Delta u\|_{H^{-1+s}}, \quad \text{for } \beta < s \leq 1. \tag{1.7}$$

As a guide to our analysis of (1.3) we use the corresponding finite element problem,

$$(\underline{u}_{h,t}, \chi) + a(\underline{u}_h, \chi) = (f, \chi), \quad \forall \chi \in S_h,\ t > 0, \quad \text{with } \underline{u}_h(0) = v_h. \tag{1.8}$$

Here, in the error analysis, it is customary to split the error into two terms by

$$\underline{u}_h(t) - u(t) = (\underline{u}_h(t) - R_h u(t)) + (R_h u(t) - u(t)) = \underline{\vartheta}(t) + \underline{\varrho}(t),$$

where $R_h : H_0^1 \to S_h$ denotes the elliptic, or Ritz, projection defined by

$$a(R_h v, \chi) = a(v, \chi), \quad \forall \chi \in S_h. \tag{1.9}$$

As a result of (1.7), we immediately have

$$\|\varrho(t)\| \leq C_s h^{2\beta} \|\Delta u(t)\|_{H^{-1+s}}, \text{ for } \beta < s \leq 1,\ t > 0, \tag{1.10}$$

and we find that $\underline{\vartheta}$ satisfies

$$(\underline{\vartheta}_t, \chi) + a(\underline{\vartheta}, \chi) = -(\varrho_t, \chi), \quad \forall \chi \in S_h,\ t > 0.$$

This leads to an $O(h^{2\beta})$ bound also for $\|\underline{\vartheta}\|$, and thus of the total error.

For the error analysis of the semidiscrete method (1.3) it would seem more natural to split the error using the finite volume elliptic projection $\tilde{R}_h : H^{1+s} \cap H_0^1 \to S_h$, $s > \frac{1}{2}$, defined by

$$a_h(\tilde{R}_h v, \eta) = a_h(v, \eta), \quad \forall \eta \in Y_h, \tag{1.11}$$

and thus write

$$u_h(t) - u(t) = (u_h(t) - \tilde{R}_h u(t)) + (\tilde{R}_h u(t) - u(t)) = \tilde{\vartheta}(t) + \tilde{\varrho}(t). \tag{1.12}$$

The second term, $\tilde\varrho$ then represents the error in an elliptic problem whose exact solution is u, and by (1.6) this term may be bounded by

$$\|\tilde\varrho(t)\| \le Ch^{2\beta}\|\Delta u(t)\|_{H^{2\beta-1}}, \quad \text{for } t>0.$$

For the first term in (1.12), $\tilde\vartheta(t) \in S_h$, we have

$$(\tilde\vartheta_t,\eta) + a_h(\tilde\vartheta,\eta) = -(\tilde\varrho_t,\eta), \quad \forall \eta \in Y_h,\ t>0. \tag{1.13}$$

and it follows easily that $\|\tilde\vartheta(t)\|$, and thus also the total error, are of order $O(h^{2\beta})$. However, as we shall see, these error bounds will require higher regularity assumptions and compatibility conditions on data than for the finite element method.

In an alternative analysis, proposed in [4], we split the error using the finite element Ritz projection R_h, and thus write

$$u_h(t) - u(t) = (u_h(t) - R_h u(t)) + (R_h u(t) - u(t)) = \vartheta(t) + \varrho(t). \tag{1.14}$$

We then have estimate (1.10) for ϱ, whereas ϑ now satisfies the somewhat more complicated equation

$$(\vartheta_t,\eta) + a_h(\vartheta,\eta) = -(\varrho_t,\eta) - a_h(\varrho,\eta), \quad \forall \eta \in Y_h,\ t>0. \tag{1.15}$$

This equation also makes it possible to show an $O(h^{2\beta})$ bound for ϑ and thus for $u_h - u$. It turns out that the regularity requirements using this method, although still slightly higher than for the finite element method, are less stringent than what is needed by using the finite volume elliptic projection $\tilde R_h$.

Using the Ritz projection in the error splitting, i.e., (1.14), we also derive, as for the finite element method in [5], an $O(h^\beta)$ bound for the gradient of the error and an almost $O(h^\beta)$ global error estimate in maximum–norm. In maximum–norm, we also show an $O(h^{2\beta})$ error bound, away from the corners, and finally demonstrate that an almost optimal order $O(h^2)$ error bound holds when the triangulations are appropriately refined near the corners. The regularity requirements for these error bounds, as in the L_2 norm estimate, are higher than those needed for the finite element method.

The following is an outline of the paper. In Section 2 we briefly recall from [5] some definitions of function spaces, regularity results, and error bounds for finite element approximations for elliptic and parabolic problems, that will be useful subsequently. The main section of the paper is Section 3, where error bounds in L_2, and H^1 are shown together with the three maximum–norm error estimates, mentioned above. In Section 4 we derive similar error bounds for a fully discrete scheme by discretizing also in time using the Backward Euler method.

As in [4] and [5], our error bounds will be expressed in terms of norms of data, together with compatibility conditions at $\partial\Omega$ for $t = 0$. This should be interpreted to mean that if these bounds are finite, and the compatibility conditions are satisfied, then the exact solution will have enough regularity to secure the convergence rate stated, uniformly down to $t = 0$. Under weaker regularity assumptions, lower convergence rates have to be expected. In the error bounds, C will denote constants which may depend on Ω and on geometrical properties of $\mathcal{T}_h$, but are independent of h and data. Several of the constants in our error and regularity bounds grow with t, and in order to not to have to account for their precise growth, we will assume throughout that $t \le T$, for some positive T, without indicating the dependence of the constant on T. Also, in our analysis, sometimes norms in the spatial variable of fractional order occur, and, for easier reading, we then often replace such norms with bigger norms of integral order in our statements. Further, for simplicity, we shall choose the discrete initial values v_h as the elliptic projection of the given v. By the stability of (1.3), other natural choices of v_h would give the same error bounds.

2. Review of the error analysis for finite element approximations

In this section we collect some material from [5] that we will need in our subsequent analysis, namely some definitions relating to fractional order Sobolev spaces, regularity results for the Dirichlet problem (1.5) and the parabolic model problem (1.1), and error bounds for the Ritz projection R_h. Also, in order to be able to compare our new results for the finite volume solution of (1.1) with the corresponding error bounds for the finite element solution, we include some of the latter. For more details and references to the literature, we refer to [5].

Letting $H^{-1} = (H_0^1)^\star$ denote the dual space of H_0^1, with respect to the L_2 inner product, we define the variational solution of (1.5) for $f \in H^{-1}$ as the function $u \in H_0^1$ which satisfies

$$(\nabla u, \nabla \varphi) = (f, \varphi), \quad \forall \varphi \in H_0^1, \tag{2.1}$$

thus also defining the operator $\Delta : H_0^1 \to H^{-1}$. It is well–known that this problem has a unique solution, and that

$$\|\nabla u\| \le \|f\|_{H^{-1}}.$$

In order to discuss further regularity results we shall need to use fractional order Sobolev spaces. Let H^m with norm $\|\cdot\|_{H^m}$ denote the standard Sobolev spaces of order m, with m integer. The space H^s, for s non integer, $s = m + \sigma$, $0 < \sigma < 1$, is defined by the real interpolation method, $H^s = [H^m, H^{m+1}]_{\sigma,2}$. Also, let H_0^s, $0 \le s \le 1$, be the fractional order Sobolev space obtained by interpolation between L_2 and H_0^1. Note that $H_0^s = H^s$ for $0 < s < \frac{1}{2}$, which means H_0^s does not require any boundary condition for small s. Further, we denote $H^{-s} = (H_0^s)^*$, for $0 < s < 1$.

For the error analysis of (1.8) and (1.3) we also use the Hilbert spaces $\dot H^s$ defined by the norms

$$\|v\|_{\dot H^s} = \Big(\sum_{j=1}^\infty \lambda_j^s (v, \varphi_j)^2\Big)^{1/2}, \quad \text{for } s \ge -1, \quad v \in H^{-1},$$

where $\{\varphi_j\}_{j=1}^\infty$ are the orthonormal eigenfunctions and $\{\lambda_j\}_{j=1}^\infty$ the corresponding eigenvalues of $-\Delta$ in Ω.

Since both $\dot H^{-s}$ and H^{-s} are the uniquely defined interpolation space between L_2 and H^{-1}, we have $\dot H^{-s} = H^{-s}$ for $0 \le s \le 1$, and for $1 \le s \le 2$, $\dot H^s$ consists of the functions $u \in H_0^1$ such that Δu is in the space H^{s-2}. Further it is obvious that $-\Delta$ gives an isomorphism between $\dot H^{1+s}$ and $\dot H^{-1+s}$. Thus,

$$\|\Delta u\|_{H^{-1+s}} \le C\|\Delta u\|_{\dot H^{-1+s}} = C\|u\|_{\dot H^{1+s}}, \qquad \text{for } 0 \le s \le 1.$$

It is well–known that the nonconvex corner of Ω bounds the regularity of the solution u of (1.5). Thus $u \in H^{1+s}$ for $0 \le s < \beta$ for f smooth enough, and $\|u\|_{H^{1+s}} \le C_s\|f\|_{H^{-1+s}}$, but $u \notin H^{1+\beta}$. A somewhat more sophisticated regularity result makes it possible to show the following error bounds in L_2 and energy norms, for the Ritz finite element projection R_h, defined by (1.9).

Lemma 2.1. *For u the solution of* (1.5) *or* (2.1)*, we have, for $\beta < s \le 1$,*

$$\|R_h u - u\| + h^\beta \|\nabla(R_h u - u)\| \le C_s h^{2\beta} \|\Delta u\|_{H^{-1+s}} \le C_s h^{2\beta} \|u\|_{\dot H^{1+s}}. \tag{2.2}$$

Further,

$$\|R_h u - u\| \le Ch^\beta \|u\|_{H^1}. \tag{2.3}$$

In the maximum–norm $\|v\|_{\mathcal{C}} = \sup_{x\in\Omega} |v(x)|$ the following almost $O(h^\beta)$ error estimate holds.

Lemma 2.2. *Let u be the solution of* (1.5). *If the triangulations $\mathcal{T}_h$ are such that $h_{min} \ge Ch^\gamma$ for some $\gamma > 0$, where $h_{min} = min_{\mathcal{T}_h} diam(\tau)$, then*

$$\|R_h u - u\|_{\mathcal{C}} \le C_{s,s_1} h^s \|u\|_{\dot{H}^{1+s_1}}, \ \textit{for}\ 0 \le s < s_1 < \beta.$$

We recall that for quasiuniform triangulations the logarithmic stability estimate

$$\|R_h v\|_{\mathcal{C}} \le C\ell_h \|v\|_{\mathcal{C}}, \quad \text{where } \ell_h = \max(\log(1/h), 1),$$

may be used to improve the maximum–norm convergence rate to $O(h^\beta \ell_h)$.

Away from the corners of the domain Ω, the convergence in maximum–norm is of the same order $O(h^{2\beta})$ as in the global L_2 error estimate. For this we quote the following lemma, where we denote the norm in $\mathcal{C}^s$ by $\|\cdot\|_{\mathcal{C}^s}$.

Lemma 2.3. *Let u be the solution of* (1.5). *If $\Omega_0 \subset \Omega_1 \subset \Omega$ is such that Ω_1 does not contain any corner of Ω and the distance between $\partial\Omega_1 \cap \Omega$ and $\partial\Omega_0 \cap \Omega$ is positive and if the $\mathcal{T}_h$ are quasiuniform in Ω_1, then we have, for $\beta < s \le 1$,*

$$\|R_h u - u\|_{\mathcal{C}(\Omega_0)} \le C_s h^{2\beta}\big(\|u\|_{\mathcal{C}^{2s}(\Omega_1)} + \|\Delta u\|_{H^{-1+s}}\big) \le C_s h^{2\beta}\big(\|u\|_{\mathcal{C}^{2s}(\Omega_1)} + \|u\|_{\dot{H}^{1+s}}\big).$$

Optimal order $O(h^2)$ and $O(h)$ convergence in L_2 and H^1, respectively, and almost optimal $O(h^2)$ convergence in the maximum–norm, may be obtained by systematically refining the triangulations toward the corners of Ω. In the case of the maximum-norm, to which we shall restrict ourselves here, such triangulations can be described as follows. It is known, cf. [8], that a corner P_i of Ω gives rise to singularities, expressed in terms of polar coordinates centered at P_i, of the form $cr^{m\beta_i}\sin(\beta_i\theta)$, with $\beta_i = \pi/\omega_i \in (0, 2)$, where ω_i is the interior angle, $\frac{1}{2}\pi < \omega_i < 2\pi$, and m is a positive integer. Assuming that the triangulations are quasiuniform away from the corners, the refinement near P_i has the 'local mesh-size' $h(x)$ at $x \in \Omega$ with $h(x) \approx min(hd_i(x)^{1-\beta_i/2+\epsilon}, h^{2/\beta_i})$, with $d_i(x)$ the distance from x to P_i, thus with mesh-sizes smaller near P_i the bigger the ω_i. This may be done in so that $dim\, S_h \le Ch^{-2}$. The condition on $\mathcal{T}_h$ of Lemma 2.2 is satisfied. For details, see [5]; below we shall simply refer to such triangulations $\mathcal{T}_h$ as appropriately refined.

Lemma 2.4. *Let u be the solution of* (1.5), *and let the triangulations $\mathcal{T}_h$ be appropriately refined. Then we have*

$$\|R_h u - u\| + h\|\nabla(R_h u - u)\| \le Ch^2\|\Delta u\| = Ch^2\|f\|. \tag{2.4}$$

Further, for any $s < 2$ and $p < \infty$ sufficiently large, we have with $C = C_{s,p}$,

$$\|R_h u - u\|_{\mathcal{C}} \le Ch^s\|\Delta u\|_{L_p} = Ch^s\|f\|_{L_p}. \tag{2.5}$$

We turn now to the parabolic problem. A basic weak solution of (1.1) is such that $u \in L_2(0, T; H_0^1)$, with $u_t \in L_2(0, T; H^{-1})$, for any $T > 0$, and a unique such solution exists if $v \in L_2$ and $f \in L_2(0, T; H^{-1})$. However, in our search for maximal order convergence, the following stronger regularity results, expressed in terms of the data v and f, will be needed. We will use the notation

$$g_0 = u_t(0) = \Delta v + f(0), \ \text{for } v \in \dot{H}^2, \ f(0) \in L_2. \tag{2.6}$$

Note that $v \in \dot{H}^2$ contains the compatibility condition $v = 0$ on $\partial\Omega$ between initial data and the boundary condition in (1.1).

Lemma 2.5. *Let u be the solution of* (1.1) *and assume $v = 0$ on $\partial\Omega$. Then we have, for $t \le T$,*

$$\int_0^t \|u_t\|_{H^1}^2 d\tau \le C\Big(\|g_0\|^2 + \int_0^t \|f_t\|_{H^{-1}}^2 \, d\tau\Big), \tag{2.7}$$

and, if in addition $g_0 = 0$ on $\partial\Omega$, then

$$\int_0^t (\|\Delta u_t\|^2 + \|u_{tt}\|^2) d\tau \le C\Big(\|g_0\|_{H^1}^2 + \int_0^t \|f_t\|^2 d\tau\Big). \tag{2.8}$$

Further,

$$\int_0^t (\|u_t\|_{\dot{H}^{1+s}} + \|u_{tt}\|_{\dot{H}^{-1+s}}) \, d\tau \le C_s\Big(\|g_0\| + \int_0^t \|f_t\| \, d\tau\Big), \ \text{for } 0 \le s < 1, \tag{2.9}$$

and, if $g_0 \in H^\epsilon$,

$$\int_0^t (\|u_t\|_{\dot{H}^2} + \|u_{tt}\|) d\tau \le C_\epsilon\Big(\|g_0\|_{H^\epsilon} + \int_0^t \|f_t\|_{H^\epsilon} \, d\tau\Big), \quad 0 < \epsilon < \tfrac{1}{2}. \tag{2.10}$$

For comparison with the finite volume results to be shown in Section 3 we state some error estimates for the spatially semidiscrete finite element approximation (1.8) of the solution of (1.1).

Theorem 2.1. *Let $\underline{u}_h$ and u be the solutions of* (1.8) *and* (1.1) *with $v = 0$ on $\partial\Omega$ and let g_0 be defined by* (2.6)*. Then if $v_h = R_h v$, we have, for $t \le T$,*

$$\|\underline{u}_h(t) - u(t)\| \le Ch^{2\beta}\Big(\|\Delta v\| + \|g_0\| + \int_0^t \|f_t\| \, d\tau\Big)$$

and

$$\|\nabla \underline{u}_h(t) - \nabla u(t)\| \le Ch^{\beta}\Big(\|\Delta v\| + \|g_0\| + \int_0^t \|f_t\| \, d\tau + \big(\int_0^t \|f_t\|_{H^{-1}}^2 \, d\tau\big)^{1/2}\Big).$$

Also, we have the following maximum–norm error estimates.

Theorem 2.2. *Under the assumptions of Theorem 2.1, if the triangulations $\mathcal{T}_h$ are such that $h_{min} \ge Ch^\gamma$ for some $\gamma > 0$, we have, for $0 < s < \beta$ and $t \le T$.*

$$\|\underline{u}_h(t) - u(t)\|_{\mathcal{C}} \le C_s h^s\Big(\|v\|_{\mathcal{C}^\beta} + \|f(0)\| + \int_0^t \|f_t\| \, d\tau + \big(\int_0^t \|f_t\|_{H^{-1}}^2 \, d\tau\big)^{1/2}\Big).$$

Further if $\Omega_0 \subset \Omega_1 \subset \Omega$ is such that Ω_1 does not contain any corner of Ω and the distance between $\partial\Omega_1 \cap \Omega$ and $\partial\Omega_0 \cap \Omega$ is positive and if the triangulations $\mathcal{T}_h$ are quasiuniform in Ω_1 and $g_0 = 0$ on $\partial\Omega$, then we have, for $\beta < s < 1$ and $t \le T$,

$$\|\underline{u}_h(t) - u(t)\|_{\mathcal{C}(\Omega_0)} \le C_s h^{2\beta} \ell_h^{1/2}\Big(\|u(t)\|_{\mathcal{C}^{2s}(\Omega_1)} + \|\Delta v\| + \|g_0\|_{H^1} + \big(\int_0^t \|f_t\|^2 d\tau\big)^{1/2}\Big).$$

Note that the first term in the parenthesis is finite provided v and f are smooth in the interior of Ω.

In the presence of the appropriate refinements the convergence is almost $O(h^2)$.

Theorem 2.3. *Under the assumptions of Theorem 2.1, if the triangulations $\mathcal{T}_h$ are appropriately refined, and $g_0 = 0$ on $\partial\Omega$, then we have, for any $s < 2$ and $t \le T$,*

$$\|\underline{u}_h(t) - u(t)\|_{\mathcal{C}} \le C_s h^s\Big(\|g_0\|_{H^1} + \|f(0)\|_{\mathcal{C}} + \int_0^t \|f_t\|_{\mathcal{C}} \, d\tau + \big(\int_0^t \|f_t\|^2 d\tau\big)^{1/2}\Big).$$

3. The semidiscrete finite volume method for the parabolic problem

We begin this section by recalling some basic material concerning the finite volume method, cf. [1, 2, 6, 7, 9], and then proceed with our error bounds.

We shall first rewrite the Petrov-Galerkin method (1.3) as a Galerkin method. For this purpose, we introduce the interpolation operator $J_h : \mathcal{C} \mapsto Y_h$ by

$$J_h u(x) = \sum_{z \in Z_h^0} u(z)\Psi_z(x),$$

where the set $\{\Psi_z : z \in Z_h^0\}$, with Ψ_z the characteristic function of the finite volume V_z, is a basis of Y_h. We recall that the bilinear form $(\chi, J_h\psi)$ is symmetric, positive definite on S_h, thus an inner product, and that the corresponding discrete norm is equivalent to the L_2 norm, uniformly in h, i.e., with $C \geq c > 0$,

$$c\|\chi\| \leq |||\chi||| \leq C\|\chi\|, \quad \forall \chi \in S_h, \quad \text{where } |||\chi||| \equiv (\chi, J_h\chi)^{1/2}. \tag{3.1}$$

It is well–known, cf., e.g., [1], that

$$a(\chi, \psi) = a_h(\chi, J_h\psi), \quad \forall \chi, \psi \in S_h, \tag{3.2}$$

and it follows that there exists $c > 0$, such that

$$a_h(\chi, J_h\chi) \geq c\|\nabla\chi\|^2, \quad \forall \chi \in S_h. \tag{3.3}$$

With this notation, (1.3) may equivalently be written in Galerkin form as

$$(u_{h,t}, J_h\chi) + a_h(u_h, J_h\chi) = (f, J_h\chi), \quad \forall \chi \in S_h,\ t > 0, \quad \text{with } u_h(0) = v_h.$$

In the analysis we shall need the error functional $\varepsilon_h(\cdot,\cdot)$, defined by

$$\varepsilon_h(f, \chi) = (f, J_h\chi) - (f, \chi), \quad \forall f \in H^s, \quad -\tfrac{1}{2} < s \leq 1,\ \chi \in S_h,$$

and recall the following bound, cf. [2, Lemma 5.1]:

Lemma 3.1. *Let $f \in H^s$, with $0 \leq s \leq 1$. Then we have*

$$|\varepsilon_h(f, \chi)| \leq Ch^{i+s}\|f\|_{H^s}\,\|\chi\|_{H^i}, \quad \forall \chi \in S_h,\ i = 0, 1.$$

Our next purpose is to derive an L_2 norm error estimate for the semidiscrete finite volume method (1.3), using the finite volume elliptic projection $\tilde{R}_h$ defined in (1.11). The proof is based on the following error bound. Note that (3.5) requires more regularity than the corresponding result for the Ritz projection in (2.2).

Lemma 3.2. *Let u be the solution of* (1.5). *Then we have*

$$\|\nabla(\tilde{R}_h u - u)\| \leq C_s h^\beta \|\Delta u\|_{H^{-1+s}} \leq C_s h^\beta \|u\|_{\dot{H}^{1+s}}, \quad \text{for } \beta < s \leq 1. \tag{3.4}$$

Further

$$\|\tilde{R}_h u - u\| \leq Ch^{2\beta}\|\Delta u\|_{H^{2\beta-1}}. \tag{3.5}$$

Proof. The estimate (3.4) is shown in [3, Theorem 5.2]. For the proof of (3.5) we employ a duality argument. For $\psi \in H_0^1$ satisfying $-\Delta\psi = \tilde{R}_h u - u$, we have

$$\|\tilde{R}_h u - u\|^2 = a(\tilde{R}_h u - u, \psi - R_h\psi) + a(\tilde{R}_h u - u, R_h\psi) = I + II.$$

For the first term we obtain, using (3.4) and (2.2) for $s = 1$, since $2\beta - 1 > 0$,

$$\begin{aligned} |I| &\leq \|\nabla(\tilde{R}_h u - u)\|\,\|\nabla(R_h\psi - \psi)\| \leq Ch^{2\beta}\|\Delta u\|\,\|\Delta\psi\| \\ &\leq Ch^{2\beta}\|\Delta u\|_{H^{2\beta-1}}\|\tilde{R}_h u - u\|. \end{aligned}$$

To bound now the second term we note that by (3.2), (1.11) and (1.4), we get

$$a(\tilde{R}_h u, R_h\psi) = a_h(\tilde{R}_h u, J_h R_h\psi) = a_h(u, J_h R_h\psi) = -(\Delta u, J_h R_h\psi),$$

so that $II = -\varepsilon_h(\Delta u, R_h\psi)$, and hence, by Lemma 3.1,

$$\begin{aligned}|II| &\le Ch^{2\beta}\|\Delta u\|_{H^{2\beta-1}}\|\nabla R_h\psi\| \le Ch^{2\beta}\|\Delta u\|_{H^{2\beta-1}}\|\nabla\psi\|\\ &\le Ch^{2\beta}\|\Delta u\|_{H^{2\beta-1}}\|\tilde{R}_h u - u\|.\end{aligned}$$

Together these estimates show

$$\|\tilde{R}_h u - u\|^2 \le Ch^{2\beta}\|\Delta u\|_{H^{2\beta-1}}\|\tilde{R}_h u - u\|,$$

which completes the proof. □

We are now ready for our L_2 norm error estimate for (1.3). Here and below we denote

$$g_1 = u_{tt}(0) = \Delta g_0 + f_t(0) \text{ for } g_0 \in \dot{H}^2,\ f_t(0) \in L_2. \tag{3.6}$$

Theorem 3.1. *Let u_h and u be the solutions of* (1.3) *and* (1.1), *respectively, and assume $v = g_0 = 0$ on $\partial\Omega$. Then, if $v_h = \tilde{R}_h v$, we have, for $t \le T$,*

$$\|u_h(t) - u(t)\| \le Ch^{2\beta}\Big(\|\Delta v\|_{H^1} + \|g_1\| + \int_0^t (\|f_{tt}\| + \|f_t\|_{H^1})d\tau\Big).$$

Proof. Writing $u_h - u = \tilde{\vartheta} + \tilde{\varrho}$ as in (1.12), we find by (3.5)

$$\begin{aligned}\|\tilde{\varrho}(t)\| &\le \|\tilde{\varrho}(0)\| + \int_0^t \|\tilde{\varrho}_t\|\,d\tau \le Ch^{2\beta}\Big(\|\Delta v\|_{H^{2\beta-1}} + \int_0^t \|\Delta u_t\|_{H^{2\beta-1}}\,d\tau\Big)\\ &\le Ch^{2\beta}\Big(\|\Delta v\|_{H^1} + \int_0^t (\|u_{tt}\|_{H^1} + \|f_t\|_{H^1})d\tau\Big),\end{aligned} \tag{3.7}$$

where in the last step we have used the fact that $2\beta - 1 \le 1$ and $\Delta u_t = u_{tt} - f_t$. Since u_t satisfies (1.1), with f and v replaced by f_t and $u_t(0) = g_0$, respectively, the regularity estimate (2.9) with $s = 0$ shows

$$\int_0^t \|u_{tt}\|_{H^1}\,d\tau \le C\Big(\|g_1\| + \int_0^t \|f_{tt}\|\,d\tau\Big), \tag{3.8}$$

which applied to (3.7) bounds $\tilde{\varrho}$ as desired. We now turn to $\tilde{\vartheta}$, which satisfies (1.13). Choosing $\eta = J_h\tilde{\vartheta}$ we find

$$(\tilde{\vartheta}_t, J_h\tilde{\vartheta}) + a_h(\tilde{\vartheta}, J_h\tilde{\vartheta}) = -(\tilde{\varrho}_t, J_h\tilde{\vartheta}),$$

and hence by standard energy arguments we obtain

$$\|\tilde{\vartheta}(t)\| \le C\int_0^t \|\tilde{\varrho}_t\|\,d\tau.$$

In view of (3.7) and (3.8), this completes the proof. □

We now show an L_2 norm error estimate for (1.3), using instead the finite element Ritz projection R_h in the analysis. Note that in this case the regularity requirements on data are weaker than in Theorem 3.1.

Theorem 3.2. *Let u_h and u be the solutions of* (1.3) *and* (1.1), *respectively, and assume $v = 0$ on $\partial\Omega$. Then, if $v_h = R_h v$, we have, for $t \le T$,*

$$\|u_h(t) - u(t)\| \le Ch^{2\beta}\Big(\|\Delta v\| + \|g_0\| + \Big(\int_0^t (\|f_t\|^2 + \|f\|_{H^1}^2)\,d\tau\Big)^{1/2}\Big).$$

Proof. We write $u_h - u = \vartheta + \varrho$, as in (1.14). Then (2.2) with $\beta < s < 1$ and (2.9) give

$$\begin{aligned}\|\varrho(t)\| &\le \|\varrho(0)\| + \int_0^t \|\varrho_t\|\,d\tau \le Ch^{2\beta}\Big(\|v\|_{\dot H^{1+s}} + \int_0^t \|u_t\|_{\dot H^{1+s}}\,d\tau\Big)\\ &\le Ch^{2\beta}\Big(\|\Delta v\| + \|g_0\| + \int_0^t \|f_t\|\,d\tau\Big),\end{aligned} \tag{3.9}$$

which yields the desired estimate for ϱ.

We turn to the estimation of ϑ, which satisfies the equation (1.15). In view of (3.2), (1.4) and (1.9), we have

$$a_h(\varrho, J_h\chi) = a(R_h u, \chi) + (\Delta u, J_h\chi) = \varepsilon_h(\Delta u, \chi), \quad \forall \chi \in S_h. \tag{3.10}$$

Using this, (1.15) with $\eta = J_h\vartheta$ is transformed into

$$(\vartheta_t, J_h\vartheta) + a_h(\vartheta, J_h\vartheta) = -(\varrho_t, J_h\vartheta) - \varepsilon_h(\Delta u, \vartheta).$$

By the symmetry of $(\chi, J_h\psi)$ on S_h, this shows, in view of Lemma 3.1,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\,|||\vartheta|||^2 + a_h(\vartheta, J_h\vartheta) &\le C\|\varrho_t\|\|\vartheta\| + Ch^{2\beta}\|\Delta u\|_{H^{2\beta-1}}\|\vartheta\|_{H^1}\\ &\le C\|\varrho_t\|\|\vartheta\| + Ch^{2\beta}\Big(\|u_t\|_{H^1} + \|f\|_{H^1}\Big)\|\nabla\vartheta\|.\end{aligned}$$

Using (3.3) to kick back $\|\nabla\vartheta\|$, and integrating, we obtain, since $\vartheta(0) = 0$, and in view of (3.1) and (2.7), that

$$\begin{aligned}\|\vartheta(t)\|^2 &\le C\int_0^t \|\varrho_t\|\|\vartheta\|\,d\tau + Ch^{4\beta}\int_0^t (\|u_t\|_{H^1}^2 + \|f\|_{H^1}^2)\,d\tau\\ &\le C\int_0^t \|\varrho_t\|\|\vartheta\|\,d\tau + Ch^{4\beta}\Big(\|g_0\|^2 + \int_0^t (\|f_t\|_{H^{-1}}^2 + \|f\|_{H^1}^2)\,d\tau\Big).\end{aligned}$$

Setting $\Theta(t) \equiv \sup_{0<s\le t}\|\vartheta(s)\|$, this shows

$$\|\vartheta(t)\|^2 \le \Theta(t)^2 \le C\big(\int_0^t \|\varrho_t\|\,d\tau\big)\Theta(t) + Ch^{4\beta}\Big(\|g_0\|^2 + \int_0^t (\|f_t\|_{H^{-1}}^2 + \|f\|_{H^1}^2)\,d\tau\Big),$$

which, together with (3.9), gives the desired bound for ϑ. □

Next, we show an $O(h^\beta)$ estimate for the gradient of the error.

Theorem 3.3. *Under the assumptions of Theorem 3.2, we have*

$$\|\nabla(u_h(t) - u(t))\| \le Ch^\beta\Big(\|\Delta v\| + \|g_0\| + \big(\int_0^t \|f_t\|^2\,d\tau\big)^{1/2}\Big), \ \textit{for } t_n \le T.$$

Proof. In view of (2.2) we have, with $\beta < s < 1$,

$$\|\nabla\varrho(t)\| \le \|\nabla\varrho(0)\| + \int_0^t \|\nabla\varrho_t\|\,d\tau \le Ch^\beta\Big(\|v\|_{\dot H^{1+s}} + \int_0^t \|u_t\|_{\dot H^{1+s}}\,d\tau\Big), \tag{3.11}$$

and (2.9) then gives the desired bound for $\nabla\varrho$. To bound $\nabla\vartheta$ we choose $\eta = J_h\vartheta_t$ in (1.15), and using (3.10) with the fact that $a_h(\vartheta, J_h\vartheta_t) = \frac{1}{2}\frac{d}{dt}\,a(\vartheta, \vartheta)$, we get

$$|||\vartheta_t|||^2 + \frac{1}{2}\frac{d}{dt}\,a(\vartheta, \vartheta) = -(\varrho_t, J_h\vartheta_t) - \varepsilon_h(\Delta u, \vartheta_t). \tag{3.12}$$

Substituting $-\Delta u = f - u_t$ yields

$$|||\vartheta_t|||^2 + \frac{1}{2}\frac{d}{dt}\,a(\vartheta, \vartheta) = -(\varrho_t, J_h\vartheta_t) - \varepsilon_h(u_t, \vartheta_t) + \frac{d}{dt}\,\varepsilon_h(f, \vartheta) - \varepsilon_h(f_t, \vartheta).$$

Integrating, using $\vartheta(0)=0$, together with (3.1) and Lemma 3.1, we find

$$\int_0^t \|\vartheta_t\|^2 d\tau + \frac{1}{2}\|\nabla\vartheta(t)\|^2 \le Ch\|f(t)\|\,\|\nabla\vartheta(t)\| + \int_0^t \big(\|\varrho_t\|\,\|\vartheta_t\| + Ch(\|u_t\|_{H^1}\|\vartheta_t\| + \|f_t\|\,\|\nabla\vartheta\|)\big)\,d\tau.$$

After kicking back $\int_0^t \|\vartheta_t\|^2 d\tau$, this together with (2.3) implies, since $\beta<1$, that

$$\|\nabla\vartheta(t)\|^2 \le Ch^{2\beta}\Big(\|f(t)\|^2 + \int_0^t (\|u_t\|_{H^1}^2 + \|f_t\|^2)d\tau\Big) + C\int_0^t \|\nabla\vartheta\|^2 d\tau.$$

Using Gronwall's lemma, the estimate

$$\|f(t)\|^2 \le C\Big(\|\Delta v\|^2 + \|g_0\|^2 + \int_0^t \|f_t\|^2\,d\tau\Big),$$

and (2.7), we finally find

$$\|\nabla\vartheta(t)\|^2 \le Ch^{2\beta}\Big(\|\Delta v\|^2 + \|g_0\|^2 + \int_0^t \|f_t\|^2\,d\tau\Big), \tag{3.13}$$

which shows the desired bound for ϑ. □

By a slight modification of the above analysis of $\nabla\vartheta$, we can show the following "super"–closeness of the gradients of u_h and $R_h u$.

Lemma 3.3. *Under the assumptions of Theorem 3.2, let $g_0=0$ on $\partial\Omega$ and g_1 be defined by* (3.6)*. Then we have, for $t\le T$,*

$$\|\nabla\vartheta(t)\| \le Ch^{2\beta}\Big(\|\Delta v\|_{H^1} + \|g_1\| + \big(\int_0^t (\|f_t\|_{H^1}^2 + \|f_{tt}\|_{H^{-1}}^2)\,d\tau\big)^{1/2}\Big).$$

Proof. Rewriting the right hand side of equation (3.12) in the form

$$|||\vartheta_t|||^2 + \frac{1}{2}\frac{d}{dt}\,a(\vartheta,\vartheta) = -(\varrho_t, J_h\vartheta_t) - \frac{d}{dt}\,\varepsilon_h(\Delta u,\vartheta) + \varepsilon_h(\Delta u_t,\vartheta), \tag{3.14}$$

integrating, using $\vartheta(0)=0$, together with Lemma 3.1, and (3.1), we find

$$\int_0^t |||\vartheta_t|||^2 d\tau + \frac{1}{2}\|\nabla\vartheta(t)\|^2 \le Ch^{2\beta}\|\Delta u(t)\|_{H^{2\beta-1}}\|\nabla\vartheta(t)\| + \int_0^t \big(\|\varrho_t\|\,|||\vartheta_t||| + Ch^{2\beta}\|\Delta u_t\|_{H^{2\beta-1}}\|\nabla\vartheta\|\big)\,d\tau.$$

Together with (2.2) for $s=1$ and the fact that $2\beta-1\le 1$, this gives

$$\|\nabla\vartheta(t)\|^2 \le Ch^{4\beta}\Big(\|\Delta v\|_{H^1}^2 + \int_0^t (\|u_{tt}\|_{H^1}^2 + \|f_t\|_{H^1}^2)d\tau\Big) + C\int_0^t \|\nabla\vartheta\|^2\,d\tau.$$

Since u_t satisfies (1.1), with f and v replaced by f_t and $u_t(0)=g_0$, respectively, the regularity estimate (2.7) shows

$$\int_0^t \|u_{tt}\|_{H^1}^2 \le C\Big(\|g_1\|^2 + \int_0^t \|f_{tt}\|_{H^{-1}}^2\,d\tau\Big). \tag{3.15}$$

Using this together with Gronwall's lemma, we obtain the desired estimate. □

We now turn to error estimates in maximum–norm, and begin with global such estimate of order almost $O(h^\beta)$, cf. Theorem 2.2 for the corresponding result for the finite element method, under almost the same regularity assumptions.

Theorem 3.4. *Under the assumptions of Theorem 3.2, if the triangulations $\mathcal{T}_h$ are such that $h_{min} \geq Ch^\gamma$ for some $\gamma > 0$, then we have, for $s \in (0, \beta)$ and $t \leq T$,*

$$\|u_h(t) - u(t)\|_{\mathcal{C}} \leq C_s h^s \Big(\|\Delta v\| + \|g_0\| + \big(\int_0^t \|f_t\|^2 d\tau\big)^{1/2}\Big).$$

Proof. We have by Lemma 2.2, with $s_1 \in (s, \beta)$,

$$\|\varrho(t)\|_{\mathcal{C}} \leq C_{s,s_1} h^s \|u(t)\|_{\dot{H}^{1+s_1}} \leq C_{s,s_1} h^s \Big(\|v\|_{\dot{H}^{1+s_1}} + \int_0^t \|u_t\|_{\dot{H}^{1+s_1}}\, d\tau\Big),$$

which is bounded as desired by (2.9).

In [5] we showed that the following discrete Sobolev type inequality, is valid for triangulations satisfying the condition assumed in this theorem

$$\|\chi\|_{\mathcal{C}} \leq C\ell_h^{1/2} \|\nabla\chi\|, \quad \forall \chi \in S_h. \tag{3.16}$$

Hence, in view of (3.13) we get

$$\|\vartheta(t)\|_{\mathcal{C}} \leq Ch^\beta \ell_h^{1/2} \Big(\|\Delta v\| + \|g_0\| + \big(\int_0^t \|f_t\|^2\, d\tau\big)^{1/2}\Big),$$

which implies the desired estimate for ϑ. □

We note that under the stronger assumption $\gamma = 1$, i.e., when the $\mathcal{T}_h$ are globally quasiuniform, one can show an $O(h^\beta \ell_h)$ maximum–norm estimate, under slightly weaker regularity assumptions on data, cf. the comment after Lemma 2.2.

Next, we derive an almost $O(h^{2\beta})$ error estimate away from the corners of Ω, cf. Theorem 2.2 for the finite element method.

Theorem 3.5. *Under the assumptions of Theorem 3.4, let $g_0 = 0$ on $\partial\Omega$ and g_1 be defined by (3.6). If $\Omega_0 \subset \Omega_1 \subset \Omega$ is such that Ω_1 does not contain any corner of Ω and the distance between $\partial\Omega_1 \cap \Omega$ and $\partial\Omega_0 \cap \Omega$ is positive, and if the triangulations $\mathcal{T}_h$ are quasiuniform in Ω_1, then we have, for $t \leq T$,*

$$\|u_h(t) - u(t)\|_{\mathcal{C}(\Omega_0)} \leq Ch^{2\beta} \ell_h^{1/2} \big(\|\Delta v\|_{H^1} + \|g_1\| + \big(\int_0^t (\|f_t\|_{H^1}^2 + \|f_{tt}\|^2)\, d\tau\big)^{1/2}\big).$$

Proof. By Lemma 2.3 we have, with $\beta < s < 1$,

$$\|\varrho(t)\|_{\mathcal{C}(\Omega_0)} \leq Ch^{2\beta} \Big(\|u(t)\|_{\mathcal{C}^{2s}(\Omega_1)} + \|\Delta u(t)\|\Big) \leq Ch^{2\beta} \|\Delta u(t)\|_{H^1}. \tag{3.17}$$

Here we have used that by Sobolev's inequality, an interior regularity estimate, and $\|u\| \leq C\|\Delta u\| \leq C\|\Delta u\|_{H^1}$, we have

$$\|u\|_{\mathcal{C}^{2s}(\Omega_1)} \leq C\|u\|_{H^3(\Omega_1)} \leq C(\|\Delta u\|_{H^1} + \|u\|) \leq C\|\Delta u\|_{H^1}.$$

We now bound the last term in (3.17), using $\Delta u_t = u_{tt} - f_t$ and (3.8), as

$$\begin{aligned} \|\Delta u(t)\|_{H^1} &\leq \|\Delta v\|_{H^1} + \int_0^t (\|u_{tt}\|_{H^1} + \|f_t\|_{H^1})\, d\tau \\ &\leq C\Big(\|\Delta v\|_{H^1} + \|g_1\| + \int_0^t (\|f_{tt}\| + \|f_t\|_{H^1})\, d\tau\Big), \end{aligned} \tag{3.18}$$

so that ϱ is estimated as stated. Using the supercloseness result of Lemma 3.3 together with (3.16), ϑ as bounded as desired, which completes the proof. □

We finally show the following almost $O(h^2)$ convergence result in the presence of the appropriate refinements, cf. Theorem 2.3.

Theorem 3.6. *Under the assumptions of Lemma 3.3, with the triangulations $\mathcal{T}_h$ appropriately refined, we have, for $s<2$ and $t \le T$,*

$$\|u_h(t)-u(t)\|_{\mathcal{C}} \le Ch^s\Big(\|\Delta v\|_{H^1}+\|g_1\|+\big(\int_0^t(\|f_t\|_{H^1}^2+\|f_{tt}\|^2)d\tau\big)^{1/2}\Big).$$

Proof. To bound ϱ we use (2.5) for p sufficiently large, and a standard Sobolev inequality, to obtain, with $C=C_{s,p}$,

$$\|\varrho(t)\|_{\mathcal{C}} \le Ch^s\|\Delta u(t)\|_{L_p} \le Ch^s\|\Delta u(t)\|_{H^1}$$

and hence (3.18) gives the desired bound for ϱ.

We next derive a superconvergent $O(h^2)$ order estimate for $\|\nabla\vartheta\|$ based on the L_2 norm error bound of (2.4). For this we follow the proof of Lemma 3.3, and obtain this time, after the application of Gronwall's lemma,

$$\|\nabla\vartheta(t)\|^2 \le Ch^4\Big(\|\Delta v\|_{H^1}^2+\int_0^t(\|u_{tt}\|_{H^1}^2+\|f_t\|_{H^1}^2)d\tau\Big),$$

which, in view of the regularity estimate (3.15) and (3.16) completes the proof. $\square$

We note that under somewhat weaker assumptions on the refinements of the $\mathcal{T}_h$ than those introduced before Lemma 2.4, one can show optimal order $O(h^2)$ and $O(h)$ convergence, in L_2 and H^1 norm, respectively, for the error u_h-u, cf. [5] for a corresponding result for the finite element approximation $\underline{u}_h$.

4. The backward euler fully discrete scheme

In [5], in addition to the semidiscrete finite element problem (1.8), also fully discrete methods were considered. These were obtained by discretizing (1.8) in time by the backward Euler and Crank-Nicolson methods. The time discretization resulted in slightly higher regularity requirements on data than those summarized in Section 2 above, and we refer to [5] for details.

In this section, by application of our analysis of the semidiscrete finite volume problem (1.3) to a fully discrete scheme, we will show some error estimates for the discretization in time by the *Backward Euler* method. Letting k denote the time step, $t=t_n=nk$, U^n the approximation in S_h of u^n, where $\varphi^n=\varphi(t_n)$ for φ defined in $[0,T]$, and $\bar\partial U^n=(U^n-U^{n-1})/k$, we consider the fully discrete scheme

$$(\bar\partial U^n,\eta)+a_h(U^n,\eta)=(f^n,\eta),\quad \forall\eta\in Y_h,\quad \text{with } U^0=v_h=R_hv. \tag{4.1}$$

We first show the following error estimate in L_2, with g_0 defined in (2.6).

Theorem 4.1. *Let U^n and $u(t_n)$ be the solutions of (4.1) and (1.1), respectively, and assume $v=0$ on $\partial\Omega$. Then we have, for $t_n\le T$ and $\epsilon\in(0,\frac12)$, with $C=C_\epsilon$,*

$$\|U^n-u(t_n)\| \le C(h^{2\beta}+k)\Big(\|\Delta v\|+\|g_0\|_{H^\epsilon}+\big(\int_0^{t_n}(\|f\|_{H^1}^2+\|f_t\|_{H^1}^2)d\tau\big)^{1/2}\Big).$$

Proof. Analogously to (1.14) we write

$$U^n-u(t_n)=(U^n-R_hu(t_n))+(R_hu(t_n)-u(t_n))=\vartheta^n+\varrho^n.$$

Here ϱ^n is bounded as desired by (3.9). To bound ϑ^n we note that

$$\begin{aligned}(\bar\partial\vartheta^n,\eta)+a_h(\vartheta^n,\eta)&=-(\bar\partial\varrho^n,\eta)+(u_t^n-\bar\partial u^n,\eta)-a_h(\varrho^n,\eta)\\&=-(\omega^n,\eta)-a_h(\varrho^n,\eta),\quad\forall\eta\in Y_h,\end{aligned} \tag{4.2}$$

where

$$\omega^n = \omega_1^n + \omega_2^n = (R_h - I)\bar\partial u(t_n) + (\bar\partial u(t_n) - u_t(t_n)). \tag{4.3}$$

Choosing $\eta = J_h\vartheta^n$ in (4.2) we obtain, in view of (3.10),

$$\begin{aligned}\frac{1}{2k}(|||\vartheta^n|||^2 - |||\vartheta^{n-1}|||^2) + \frac{1}{2k}|||\vartheta^n - \vartheta^{n-1}|||^2 + a_h(\vartheta^n, J_h\vartheta^n)&\\ = -(\omega^n, J_h\vartheta^n) - \varepsilon_h(u_t^n - f^n, \vartheta^n)&\\ = -(\omega_1^n, J_h\vartheta^n) - (\omega_2^n, \vartheta^n) - \varepsilon_h(\bar\partial u^n - f^n, \vartheta^n).&\end{aligned}$$

Multiplying this by $2k$, employing Lemma 3.1 and (3.1), and kicking back $\|\nabla\vartheta^n\|$, we get

$$|||\vartheta^n|||^2 \le |||\vartheta^{n-1}|||^2 + Ck(\|\omega_1^n\| + \|\omega_2^n\|)|||\vartheta^n||| + Ckh^4(\|\bar\partial u^n\|_{H^1}^2 + \|f^n\|_{H^1}^2).$$

Let now $\Theta^n = \max_{0\le j\le n}\|\vartheta^j\|$. Then, since $\vartheta^0 = 0$ and using again (3.1),

$$\|\vartheta^n\|^2 \le (\Theta^n)^2 \le Ck\sum_{j=1}^n(\|\omega_1^j\| + \|\omega_2^j\|)\Theta^n + Ckh^4\sum_{j=1}^n(\|\bar\partial u^j\|_{H^1}^2 + \|f^j\|_{H^1}^2). \tag{4.4}$$

To bound the first term on the right, we employ the case $p = 1$ of the inequality

$$k\sum_{j=1}^n|\bar\partial g^j|^p \le Ck\sum_{j=1}^n\big(k^{-1}\int_{t_{j-1}}^{t_j}|g_t|d\tau\big)^p \le C\int_0^{t_n}|g_t|^p d\tau, \quad 1 \le p < \infty, \tag{4.5}$$

where $|\cdot|$ is a norm in a linear space, and (2.2) with $s = 1$, to find

$$\begin{aligned}Ck\sum_{j=1}^n(\|\omega_1^j\| + \|\omega_2^j\|) &\le Ck\sum_{j=1}^n\big(h^{2\beta}\|\bar\partial u^j\|_{\dot H^2} + \int_{t_{j-1}}^{t_j}\|u_{tt}\|d\tau\big)\\ &\le Ch^{2\beta}\int_0^{t_n}\|u_t\|_{\dot H^2}d\tau + Ck\int_0^{t_n}\|u_{tt}\|d\tau.\end{aligned} \tag{4.6}$$

For the last term in (4.4), we use the inequality

$$k\sum_{j=1}^n|g^j|^2 \le t_n\max_{\tau\le t_n}|g(\tau)|^2 \le C\int_0^{t_n}(|g|^2 + |g_t|^2)d\tau$$

and (4.5) with $p = 2$ to obtain

$$k\sum_{j=1}^n(\|\bar\partial u^j\|_{H^1}^2 + \|f^j\|_{H^1}^2) \le C\int_0^{t_n}(\|u_t\|_{H^1}^2 + \|f\|_{H^1}^2 + \|f_t\|_{H^1}^2)d\tau. \tag{4.7}$$

Altogether, (4.4), (4.6) and (4.7) give

$$\begin{aligned}\|\vartheta^n\|^2 \le Ch^{4\beta}\Big(\big(\int_0^{t_n}\|u_t\|_{\dot H^2}d\tau\big)^2 + \int_0^{t_n}(\|u_t\|_{H^1}^2 + \|f\|_{H^1}^2 + \|f_t\|_{H^1}^2)d\tau\Big)&\\ + Ck^2\Big(\int_0^{t_n}\|u_{tt}\|d\tau\Big)^2.&\end{aligned}$$

In view of (2.7) and (2.10) this completes the proof. □

Next, we will show the following error estimate for the gradient.

Theorem 4.2. *Under the assumptions of Theorem 4.1, let $g_0 = 0$ on $\partial\Omega$. Then we have, for $t_n \le T$,*

$$\|\nabla(U^n - u(t_n))\| \le C(h^\beta + k)\Big(\|\Delta v\| + \|g_0\|_{H^1} + \big(\int_0^{t_n} \|f_t\|^2\,d\tau\big)^{1/2}\Big).$$

Proof. Here $\nabla\varrho^n$ is bounded as desired by (3.11) and (2.9). To estimate $\nabla\vartheta^n$, we choose $\eta = J_h\bar\partial\vartheta^n$ in (4.2). Using (3.2) together with the identity $2a(\vartheta^n, \bar\partial\vartheta^n) = \bar\partial\|\nabla\vartheta^n\|^2 + k\|\nabla\bar\partial\vartheta^n\|^2$, and (3.10), we then obtain

$$|||\bar\partial\vartheta^n|||^2 + \frac{1}{2}(\bar\partial\|\nabla\vartheta^n\|^2 + k\|\nabla\bar\partial\vartheta^n\|^2) = -(\omega^n, J_h\bar\partial\vartheta^n) - \varepsilon_h(\Delta u^n, \bar\partial\vartheta^n).$$

Multiplying by $2k$, using (3.1), eliminating $|||\bar\partial\vartheta^n|||^2$, and summing in time, we find

$$\|\nabla\vartheta^n\|^2 \le Ck\sum_{j=1}^n \|\omega^j\|^2 - k\sum_{j=1}^n \varepsilon_h(\Delta u^j, \bar\partial\vartheta^j). \tag{4.8}$$

Since $\vartheta^0 = 0$, the last term can be rewritten as

$$-k\sum_{j=1}^n \varepsilon_h(\Delta u^j, \bar\partial\vartheta^j) = -\varepsilon_h(\Delta u^n, \vartheta^n) + k\sum_{j=1}^n \varepsilon_h(\bar\partial\Delta u^j, \vartheta^{j-1}).$$

Thus, employing this identity and Lemma 3.1 (with $i = 1$, $s = 0$) in (4.8), we get

$$\begin{aligned}\|\nabla\vartheta^n\|^2 \le{}& Ck\sum_{j=1}^n (\|\omega_1^j\|^2 + \|\omega_2^j\|^2)\\ &+ Ch^2\big(\|\Delta u^n\|^2 + k\sum_{j=1}^n \|\bar\partial\Delta u^j\|^2\big) + Ck\sum_{j=0}^{n-1}\|\nabla\vartheta^j\|^2.\end{aligned} \tag{4.9}$$

The last term is eliminated by using the discrete version of Gronwall's lemma, and using (4.5) with $p = 2$ we easily find

$$\|\Delta u^n\|^2 + k\sum_{j=1}^n \|\bar\partial\Delta u^j\|^2 \le C\big(\|\Delta v\|^2 + \int_0^{t_n} \|\Delta u_t\|^2 d\tau\big).$$

Hence by (2.8), the second to last term in (4.9) is bounded as desired. Using (2.3) and again (4.5) with $p = 2$, we obtain

$$\begin{aligned}k\sum_{j=1}^n (\|\omega_1^j\|^2 + \|\omega_2^j\|^2) &\le Ck\sum_{j=1}^n \Big(h^{2\beta}\|\bar\partial u^j\|_{H^1}^2 + \big(\int_{t_{j-1}}^{t_j} \|u_{tt}\|d\tau\big)^2\Big)\\ &\le Ch^{2\beta}\int_0^{t_n} \|u_t\|_{H^1}^2\,d\tau + Ck^2\int_0^{t_n} \|u_{tt}\|^2 d\tau,\end{aligned}$$

which is bounded as desired by (2.7) and (2.8). This completes the proof. □

We finally demonstrate the following time discrete version of the maximum–norm error estimate of Theorem 3.4.

Theorem 4.3. *Under the assumptions of Theorem 4.2, if the triangulations $\mathcal{T}_h$ are such that $h_{min} \ge Ch^\gamma$ for some $\gamma > 0$, then we have, for $0 \le s < \beta$ and $t_n \le T$,*

$$\|U^n - u(t_n)\|_{\mathcal{C}} \le C_s(h^s + k\ell_h^{1/2})\Big(\|\Delta v\| + \|g_0\|_{H^1} + \big(\int_0^{t_n} \|f_t\|^2 d\tau\big)^{1/2}\Big).$$

Proof. The term ϱ^n is bounded as stated in the proof of Theorem 3.4. For ϑ^n we have, using (3.16) and the bound for $\nabla\vartheta^n$ of the proof of Theorem 4.2,

$$\|\vartheta^n\|_{\mathcal{C}} \le \ell_h^{1/2}\|\nabla\vartheta^n\| \le C(h^\beta + k)\ell_h^{1/2}\Big(\|\Delta v\| + \|g_0\|_{H^1} + \big(\int_0^{t_n} \|f_t\|^2 d\tau\big)^{1/2}\Big),$$

which completes the proof. □

We refrain from stating and proving the straight–forward analogues of Theorems 3.5 and 3.6.

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