Preconditioning of symmetric interior penalty discontinuous Galerkin FEM for elliptic problems*

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Summary. This is a further development of [10] regarding multilevel preconditioning for symmetric interior penalty discontinuous Galerkin finite element approximations of second order elliptic problems. We assume that the mesh on the finest level is a results of a geometrically refined fixed coarse mesh. The preconditioner is a multilevel method that uses a sequence of finite element spaces of either continuous or piece-wise constant functions. The spaces are nested, but due to the penalty term in the DG method the corresponding forms are not inherited. For the continuous finite element spaces we show that the variable V-cycle provides an optimal preconditioner for the DG system. The piece-wise constant functions do not have approximation property so in order to control the energy growth of the inter-level transfer operator we apply W–cycle MG. Finally, we present a number of numerical experiments that support the theoretical findings.

1 Introduction

Consider the following model second order elliptic problem on a bounded domain with a polygonal boundary $\Omega \subset R^d$, $d = 2, 3$:

$$-\nabla \cdot (a(x)\nabla u) = f(x) \quad \text{in } \Omega, \quad u(x) = g \quad \text{on } \partial\Omega. \tag{1}$$

Here a is a uniformly positive in Ω and piece-wise $W^1_\infty(\Omega)$-function that may have jumps along some interfaces. The theoretical results can be easily extended to a coefficient matrix a and more general boundary conditions.

Our goal is to study iterative methods for a symmetric interior penalty discontinuous Galerkin finite element approximations of (1) over a partition

* The second author has been supported in parts by NSF DEIA-0540136 and by CASC, Lawrence Livermore National Laboratory; the third author was supported in part by NSF DMS-058110, DMS-0619587, and by Lawrence Livermore National Laboratory under subcontract number B551021.

$\mathcal{T}$ of Ω into finite elements denoted by T. We assume that the partition is quasi uniform and regular. For a finite element T we denote by h_T its size and $h = \max_{T\in\mathcal{T}} h_T$. Further, we use the following notations concerning $\mathcal{T}$: $\mathcal{E}^0$ is the set of all interior edges/faces, $\mathcal{E}^b$ is the set of the edges/faces on the boundary $\partial\Omega$ and $\mathcal{E} = \mathcal{E}^0 \cup \mathcal{E}^b$. In fact, $\mathcal{T}$, $\mathcal{E}$, etc are sets depending on the mesh-size h. However, in order to avoid proliferation of indices and since we are dealing exclusively with algebraic problems we shall not explicitly denote this dependence on the mesh-size. We also use a hierarchy of meshes $\mathcal{T}_1 \subset \cdots \subset \mathcal{T}_J$ which are obtained by geometric refinement of a coarse mesh $\mathcal{T}_1$. Thus, $\mathcal{T} = \mathcal{T}_J$ and $\mathcal{T}_k$ is the mesh generated after $k-1$ levels of refinement of $\mathcal{T}_1$. When the index k, showing the dependence on the refinement level, is suppressed this means that the quantities are defined on the finest level.

We introduce the spaces

$$H^s(\mathcal{T}) = \left\{ v \in L^2(\Omega) : v|_T \in H^s(T),\ \forall T \in \mathcal{T} \right\},\ \text{for } s \geq 0 \tag{2}$$

and for $r \geq 0$ integer we define the finite element space

$$\mathcal{V} := \mathcal{V}(\mathcal{T}) := \{ v \in L^2(\Omega) : v|_T \in P_r(T),\ T \in \mathcal{T} \}, \tag{3}$$

where P_r is the set of polynomials of total degree at most r restricted to T. On $\mathcal{V}$ we define the bilinear forms

$$(a\nabla u, \nabla v)_{\mathcal{T}} := \sum_{T\in\mathcal{T}} \int_T a\nabla u, \nabla v\ dx, \quad \langle p, q\rangle_{\mathcal{E}} := \sum_{F\in\mathcal{E}} \int_F pq\ ds.$$

On $F = \bar{T}_1 \cap \bar{T}_2 \in \mathcal{E}$ we define the jump of a scalar function $v \in \mathcal{V}$ by

$$[\![v]\!]_F := \begin{cases} v|_{T_1}\mathbf{n}_{T_1} + v|_{T_2}\mathbf{n}_{T_2}, & F = \bar{T}_1 \cap \bar{T}_2,\ \text{i.e. } F \in \mathcal{E}^0, \\ v|_T\mathbf{n}_T, & F = \bar{T} \cap \partial\Omega,\ \text{i.e. } F \in \mathcal{E}^b \end{cases}$$

and the average value of the traces of $a\nabla v$ for $v \in \mathcal{V}$:

$$\{\{a\nabla v\}\}|_F := \begin{cases} \frac{1}{2}\{a\nabla v|_{T_1} + a\nabla v|_{T_2}\}, & F = \bar{T}_1 \cap \bar{T}_2,\ \text{i.e. } F \in \mathcal{E}^0, \\ a\nabla v|_T, & F = \bar{T} \cap \partial\Omega,\ \text{i.e. } F \in \mathcal{E}^b. \end{cases}$$

Here $\mathbf{n}_T$ is the external unit vector normal to the boundary ∂T of $T \in \mathcal{T}$.

Next, we define the piecewise constant function $h_{\mathcal{E}}$ on $\mathcal{E}$ as

$$h_{\mathcal{E}} = h_{\mathcal{E}}(x) = |F|^{\frac{1}{d-1}}, \quad \text{for } x \in F,\ \ F \in \mathcal{E}, d = 2, 3. \tag{4}$$

And finally, we introduce the following mesh-dependent norm on $\mathcal{V}$:

$$|||v|||^2 = (a\nabla v, \nabla v)_{\mathcal{T}} + \left\langle h_{\mathcal{E}}^{-1}\kappa_{\mathcal{E}} [\![v]\!], [\![v]\!] \right\rangle_{\mathcal{E}}. \tag{5}$$

The stabilization factor $\kappa_{\mathcal{E}}$ is weighted by the coefficient a, namely, $\kappa_{\mathcal{E}} = \kappa\{\{a\}\}$, where $\{\{a\}\}$ is the average value of a from both sides of $F \in \mathcal{E}$. This

choice of the penalty gives rise to a DG bilinear form (7) that is equivalent to the norm (5) with constants independent of the jumps of a.

We consider the following symmetric interior penalty discontinuous Galerkin (SIPG) finite element approximation of (1) (see, e.g. [1, 2]):

$$\text{find} \quad u_h \in \mathcal{V} \quad \text{such that} \quad \mathcal{A}(u_h, v) = \mathcal{L}(v) \quad \forall v \in \mathcal{V}, \tag{6}$$

where $\mathcal{V}$ is the finite element space and $\mathcal{A}(\cdot,\cdot)$, $\mathcal{L}(\cdot)$ are bilinear and linear forms on $\mathcal{V}$ defined by

$$\begin{aligned}\mathcal{A}(u_h, v) \equiv (a\nabla u_h, \nabla v)_{\mathcal{T}} - \langle \{\{a\nabla u_h\}\}, [\![v]\!] \rangle_{\mathcal{E}} - \langle [\![u_h]\!], \{\{a\nabla v\}\} \rangle_{\mathcal{E}} \\ + \left\langle h_{\mathcal{E}}^{-1} \kappa_{\mathcal{E}} [\![u_h]\!], [\![v]\!] \right\rangle_{\mathcal{E}}\end{aligned} \tag{7}$$

and

$$\mathcal{L}(v) = \int_\Omega f v dx + \int_{\partial\Omega} (h_{\mathcal{E}}^{-1} \kappa_{\mathcal{E}} v - a\nabla v \cdot \mathbf{n})\, g\, ds. \tag{8}$$

It is known (see, e.g. [2]) that SIPG (6) – (8) is stable for sufficiently large $\kappa > 0$ and has optimal convergence in H^1-like norm (5). This is just one example of a large number of DG FEM approximations of second order elliptic problems that have been introduced and studied in the last several years (see, e.g. [2, 10]).

The aim of this paper is to introduce and study multilevel iterative methods for the corresponding algebraic problems. Note that the condition number of the DG FE system grows like $O(h^{-2})$ on a quasiuniform mesh with mesh-size h. Therefore construction of optimal solution methods, i.e. with arithmetic work proportional to the numbers of unknowns, that is robust with respect to large variations of the coefficient a is an important problem from both theoretical and practical points of view.

The work of Gopalakrishnan and Kanschat [11], the first one we are aware of, studied the variable V-cycle multigrid operator as a preconditioner of the symmetric DG system. Under certain weak regularity assumptions on geometrically nested meshes it was shown in [11] that the condition number of the preconditioned system is $O(1)$, i.e. bounded independently of h. The analysis of the preconditioner is based on the abstract multigrid theory [5] for non-inherited bilinear forms and the estimates for interior penalty finite element method. Further, Brenner and Zhao [8] studied V-cycle, W-cycle, and F-cycle algorithms for the symmetric DG FE schemes on rectangular meshes and showed that they produce uniform preconditioners for sufficiently many pre- and post smoothing steps. Their analysis is based on certain mesh dependent norms and a relationship of the discontinuous FE spaces to some higher order continuous finite element spaces. Our approach is slightly different, it could be seen as the classical two-level method applied to the DG linear systems. We explore two different possibilities for a choice of the second level, namely, continuous piece-wise polynomial functions and piece-wise constant functions.

2 MG preconditioner using spaces of continuous functions

We assume that we have a sequence of nested globally quasi-uniform triangulations $\mathcal{T}_k$, $k = 1, \dots, J$, of the domain Ω with $\mathcal{T}_1$ being the coarsest triangulation. According to the convention from the introduction the set of all edges/faces of elements in $\mathcal{T}_k$ is denoted by $\mathcal{E}_k$, the sets of the interior and boundary edges/faces are denoted by $\mathcal{E}_k^0$ and $\mathcal{E}_k^b$, respectively, and h_k is the diameter of a typical element in $\mathcal{T}_k$ and $h_{\mathcal{E}_k}$ is defined by (4) on $\mathcal{E}_k$. Then $H^s(\mathcal{T}_k)$ and $\mathcal{V}_k$ are the spaces (2) and (3), respectively, defined on $\mathcal{T}_k$. The corresponding continuous discrete spaces are defined as $\mathcal{V}_k^c = \mathcal{V}_k \cap C(\overline{\Omega})$.

For functions u and v in $H^s(\mathcal{T}_k)$, $s > \frac{3}{2}$, we define the interior penalty (SIPG) bilinear and linear forms according to (7) for the mesh $\mathcal{T}_k$:

$$\begin{aligned}\mathcal{A}_k(u,v) = (a\nabla u, \nabla v)_{\mathcal{T}_k} + \left\langle h_{\mathcal{E}_k}^{-1}\kappa_{\mathcal{E}} [\![u]\!], [\![v]\!]\right\rangle_{\mathcal{E}_k} \\ - \langle \{\{a\nabla u\}\}, [\![v]\!]\rangle_{\mathcal{E}_k} - \langle \{\{a\nabla v\}\}, [\![u]\!]\rangle_{\mathcal{E}_k},\end{aligned}$$

$$\mathcal{L}_k(v) = \int_\Omega fv + \left\langle h_{\mathcal{E}_k}^{-1}\kappa_{\mathcal{E}} g, v\right\rangle_{\mathcal{E}_k^b} - \langle a\nabla v\cdot \mathbf{n}, g\rangle_{\mathcal{E}_k^b}.$$

With these definitions, the interior penalty discontinuous Galerkin method for the elliptic problem (1) reads: find $u_h \in \mathcal{V}_J$ such that

$$\mathcal{A}_J(u_h, v) = \mathcal{L}_J(v), \quad \forall v \in \mathcal{V}_J. \tag{9}$$

Let $|||\cdot|||_k$ be the norm (5) defined on the mesh $\mathcal{T}_k$. It is well known that there exists κ_0 such that for $\kappa > \kappa_0$ the following norm equivalence on $\mathcal{V}_k$ holds $\mathcal{A}_k(v,v) \simeq |||v|||_k^2$, $\forall v \in \mathcal{V}_k$, with constants in the norm equivalence independent of h_k, i.e. $\mathcal{A}_k(v,v)^{\frac{1}{2}}$ is a norm on $\mathcal{V}_k$.

Lemma 1. *Consider the case of homogeneous boundary condition, $g = 0$, and assume that the solution u of (1) belongs to $H^{1+\alpha}(\Omega)$ for some $\frac{1}{2} < \alpha \le 1$. Let $u_k \in \mathcal{V}_k$ (or $\mathcal{V}_k^c$) be the solution of $\mathcal{A}_k(u_k, v) = \mathcal{L}_k(v)$, $\forall v \in \mathcal{V}_k$ ($\mathcal{V}_k^c$). Then the following error estimate holds*

$$|||u - u_k|||_k \le C h_k^\alpha \|u\|_{1+\alpha}$$

with a constant C independent of h_k.

Sketch of the proof. To prove this estimate one can use the Galerkin orthogonality, the boundedness of $\mathcal{A}_k(\cdot,\cdot)$ in the norm $|||u|||_{\alpha,k} = |||u|||_k^2 + \sum_{T\in\mathcal{T}_k} h_k^{2\alpha}|u|_{1+\alpha,T}^2$ for $u \in H^{1+\alpha}(\mathcal{T}_k)$ and the approximation properties of the space $\mathcal{V}_k$. Note that in contrast to the work [11] instead of using the quantity $\mathcal{A}_k(\cdot,\cdot)^{\frac{1}{2}}$, which in general is not a norm on $H^{1+\alpha}(\mathcal{T}_k)$, we work directly in the norm $|||u|||_{\alpha,k}$.

Now we define the variable V-cycle MG preconditioner.

3 Variable V-cycle Multigrid Preconditioner

In this Section we shall follow the general theory of multigrid methods as presented by Bramble and Zhang in [5, Chapter II, Section 7]. We will use the following sequence of nested spaces: $M_{J+1} = \mathcal{V}$, i.e. this is the space where the SIPG method is defined; for $k = 1, \ldots, J$ we take $M_k = \mathcal{V}_k^c$ the continuous finite element space. The corresponding bilinear forms $\mathcal{A}_k(\cdot, \cdot)$ are defined above for $k = 1, \ldots, J$; for $k = J + 1$ we let $\mathcal{A}_{J+1}(u, v) = \mathcal{A}(u, v)$. Define the operators $A_k : M_k \to M_k$, $Q_k : L^2(\Omega) \to M_k$, and $P_k : M_{k+1} \to M_k$ by

$$\begin{aligned}
(A_k u, v) &= \mathcal{A}_k(u, v), && \forall v \in M_k, \quad k = 1, \ldots, J + 1, \\
(Q_k u, v) &= (u, v), && \forall v \in M_k, \quad k = 1, \ldots, J + 1, \\
\mathcal{A}_k(P_k u, v) &= \mathcal{A}_{k+1}(u, v), && \forall v \in M_k, \quad k = 1, \ldots, J,
\end{aligned}$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$. Note that because of the penalty term the forms $\mathcal{A}_k(u, v)$ defined on the spaces $\mathcal{V}_k$ vary. Assume we are given the smoothing operators $R_k : M_k \to M_k$ that satisfy appropriate smoothing property (see, [5, Chapter II, Section 7, p. 260]). One can show that scaled Jacobi and Gauss-Seidel iterations satisfy this requirement.

Let B_k be the operator of the MG method based on the sequence of spaces $M_1 \subset \cdots \subset M_J \subset M_{J+1}$, with m_k pre- and post-smoothing steps with the smoother R_k. Note that to retain the symmetry of certain operators on odd steps we apply R_k, while on even steps we apply R_k^t, where the transposition is with respect to the $(\cdot, \cdot)$-inner product.

The following assumption will be used in the study of the MG method.

Assumption A.1: For any $f \in H^{-1+\rho}(\Omega)$ with $\frac{1}{2} < \rho \leq 1$ and $g = 0$ the problem (1) has a unique solution $u \in H^{1+\rho}(\Omega)$ and $\|u\|_{H^{1+\rho}} \leq C_\Omega \|f\|_{H^{-1+\rho}}$ with a constant C_Ω.

For this setting, we prove the following main result (see, e.g. [5]):

Theorem 1. *Let the Assumption A.1 hold. Assume also that for some* $1 < \beta_0 \leq \beta_1$ *we have* $\beta_0 m_k \leq m_{k-1} \leq \beta_1 m_k$. *Then there is a constant* M *independent of* k *such that*

$$\eta_k^{-1} \mathcal{A}_k(v, v) \leq \mathcal{A}_k(B_k A_k v, v) \leq \eta_k \mathcal{A}_k(v, v), \quad \forall v \in M_k$$

with $\eta_k = \frac{M + m_k^\alpha}{m_k^\alpha}$ *and* α *as in Lemma 1.*

Sketch of the proof. The proof essentially checks the conditions (of "smoothing and approximation") from [5] under which this theorem is proved. The first condition essentially requires that R_k is a smoother. It is well known that Gauss-Seidel or scaled Jacobi satisfy this condition.

Now we outline the main steps in the proof of the second condition which is: for some $\alpha \in (0, 1]$ there is a constant C_P independent of k such that

$$|\mathcal{A}_k((I - P_{k-1})v, v)| \leq C_P \left(\frac{\|A_k v\|^2}{\lambda_k} \right)^\alpha [\mathcal{A}_k(v, v)]^{1-\alpha}, \tag{10}$$

where λ_k is the largest eigenvalue of the operator A_k. This is established in several steps.

First, we show that under the Assumption A.1 for all $u \in M_k$, $k = 2, \ldots, J+1$ we have $|||u - P_{k-1}u|||_k \leq Ch_k^\rho \|A_k u\|_{-1+\rho}$, where $|||\cdot|||_{J+1} = |||\cdot|||_J$. Next, we show that

$$\|A_k u\|_{-1} \leq C \, |||u|||_k \,, \quad \forall u \in M_k \tag{11}$$

and

$$|||u - P_{k-1}u|||_k \leq Ch_k^\rho \|A_k u\|_{-1+\rho} \leq Ch_k^\rho \|A_k u\|_{-1}^{1-\rho} \|A_k u\|^\rho. \tag{12}$$

Finally, using the estimates (12) and (11) and the fact that $H^{-1+\rho}(\Omega)$ is an intermediate space between $H^{-1}(\Omega)$ and $L^2(\Omega)$ we obtain

$$\begin{aligned} |\mathcal{A}_k(u - P_{k-1}u, u)| &\leq Ch_k^\rho \|A_k u\|_{-1}^{1-\rho} \|A_k u\|^\rho \, |||u|||_k \\ &\leq C \frac{\|A_k u\|^\rho}{\lambda_k^{\rho/2}} \, |||u|||_k^{2-\rho} = C \left(\frac{\|A_k u\|^2}{\lambda_k} \right)^{\frac{\rho}{2}} \mathcal{A}_k(u,u)^{1-\frac{\rho}{2}} \end{aligned}$$

which is exactly the required result with $\alpha = \rho/2$.

Remark 1. This results is quite similar to the results of [11] and [8] in the sense that it proves the convergence of the variable V-cycle MG and ensures better convergence for smoother solutions. The difference is the choice of the hierarchy of finite element spaces used on the consecutive levels and the proof of the fundamental estimate (10). After closer inspection of the proof one can see easily that one can take $M_k = \mathcal{V}_k$, for all $k \geq k_0 \geq 1$. In fact, making this choice with $k_0 = 1$ will lead to the result of [11] (with a slightly different proof).

4 Multigrid *W*-cycle for piece wise-constant spaces

In this section we consider a method for the solution of the coarse problem, when a two level method with coarse space, denoted here with M_J, of piecewise constant functions. We will also take a standard multilevel hierarchy of this space, given by the subspaces M_k, of piecewise constant functions on grids with size h_k. Let us note that such two level algorithm is attractive, because of its simplicity and low number of degrees of freedom. However, it is well known that using the hierarchy given by M_k and applying standard V-cycle on M_J does not lead to an optimal algorithm.

In this section we briefly describe how a general ν-fold cycle can be applied to solve the coarse grid problem when piece-wise constant functions are used to define this problem. Note that on general meshes the piecewise constant functions do not provide approximation and one cannot apply the theory of MG methods in a manner used in [6] for cell-centered schemes on regular rectangular meshes. To introduce the ν-fold MG cycle algorithm, we consider

the recursive definition of a general multilevel method as in [5]. Assuming that we know the action of B_{k-1} on M_{k-1}, for a given $f \in M_k$ we define the action $B_k f$ as follows.

Recursive definition of a multilevel algorithm:

1. $x = R_k g$.
2. $y = x + Z_k B_{k-1} Q_{k-1}(f - A_k x)$.
3. $B_k f = y + R_k^t(g - A_k y)$.

Now, for a fixed $e \in M_k$, we consider $E_k e = (I - B_k A_k)e$. It is easy to derive the following error equation:

$$E_k e = (I - R_k^t A_k)(I - Z_k B_{k-1} A_{k-1} P_{k-1})(I - R_k A_k)e.$$

In the case, when $\{M_k\}_{k=1}^J$ are the spaces of discontinuous piece-wise constant functions we shall define Z_k, using the techniques from [13, 3, 4], namely we shall choose Z_k to be a polynomial in $(B_{k-1}A_{k-1})$. Indeed, in such case the second term in the product form of the error equation is as follows.

$$X_k = I - Z_k B_{k-1} A_{k-1} P_{k-1} = I - (I - p_\nu(B_{k-1}A_{k-1}))P_{k-1}.$$

Usually, $p_\nu(t)$ is of degree less than or equal to ν, $p_\nu(t)$ is non-negative for $t \in [0, 1]$, and $p_\nu(0) = 1$. Taking $p_\nu(t) = (1-t)^\nu$ gives the ν-fold MG cycle. For $\nu = 1$ this is the V-cycle and for $\nu = 2$, this is the W-cycle. Note also that, for $p_\nu(t) = (1-t)^\nu$, we have $X_k = I - P_{k-1} + E_{k-1}^\nu P_{k-1}$. Hence, if the degree of the polynomial is sufficiently large and E_{k-1} is a contraction on M_{k-1}, then the corresponding ν-fold cycle can be made as close as we please to a two-level iteration. As it is well known, the two level iteration, is uniformly convergent [10].

We would like to point out that an adaptive choice of the polynomials p_ν is possible, and we refer to [13, 3, 4] for strategies how to make such choices and also for many theoretical results for these methods.

A crucial property of the coarser spaces, that determines the convergence of such multilevel process, in general, is the stability of projections on coarser spaces. A basic assumption in the analysis is the existence of constants $q \geq 1$ and C (both independent of k and l) and such that

$$\|Q_l v\|^2 \leq C q^{k-l} \|v\|_{A_k}, \quad \forall v \in M_k, \quad k > l. \tag{13}$$

Clearly, if $q = 1$, then the resulting V-cycle algorithm has convergence rate depending only logarithmically on the mesh size, without any regularity assumptions on the underlying elliptic equation (see [7]). The ν-fold cycle, however, works even in cases, when $q > 1$, by increasing the polynomial degree ν when needed. Since the goal is to construct an optimal algorithm, the overall computational complexity gives a restriction on ν. Practical values are $\nu = 2$ or $\nu = 3$. In case $\nu = 2$ (W-cycle), which we have used in most of our numerical experiments in the next section, a uniform convergence result can be

proved in a fashion similar to the case of variable V-cycle. In such analysis, an essential ingredient are bounds on q from (13) and such estimates for piecewise constant spaces on uniformly refined hexahedral, quadrilateral as well as simplicial grids are given in [12, 10].

5 Numerical Experiments

We present three test problems of elliptic equation with homogeneous Dirichlet boundary conditions:
Test Problem 1: The equation $-\Delta u = 1$ in the cube $\Omega = (0,1)^3$;
Test Problem 2: The equation $-\nabla \cdot (a\nabla u) = 1$ in $\Omega = (0,1)^3 \setminus [0.5,1)^3$ where the coefficient a has jumps (a 3-D chess-board pattern) as follows: $a = 1$, in $(I_1 \times I_1 \times I_1) \cup (I_2 \times I_2 \times I_1) \cup (I_1 \times I_2 \times I_2) \cup (I_2 \times I_1 \times I_2)$ and $a = \epsilon$, in the other parts of Ω, where $I_1 = (0, 0.5]$ and $I_2 = (0.5, 1]$, and we vary the value of ϵ according to the data in the Tables;
Test Problem 3: The equation $-\Delta u = 1$ in the domain shown on Figure 1.

The second test problem is designed to check the robustness of the methods with respect to jumps of the coefficient a. The mesh of test problem 3 has a number of finite elements of high aspect ration and the aim was to see how the iteration methods perform on such grids.

For all test examples we have used a coarse tetrahedral mesh which is uniformly refined to form a sequence of nested meshes. In SIPG we use linear and quadratic finite elements. The value of the penalty term was experimentally chosen to be $\kappa = 15$ for linear, and $\kappa = 30$ for quadratic finite elements (cf. (5), (7)).

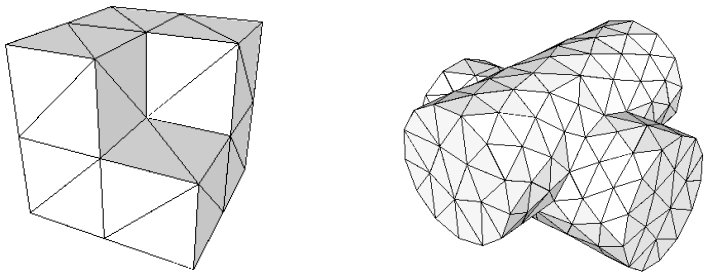

Fig. 1. Coarse meshes for the second (left) and third (right) test problems.

We test the following multilevel preconditioners for the SIPG method:

1. the V-cycle preconditioner based on continuous elements with one pre- and one post-smoothing Gauss-Seidel iteration.
2. W-cycle preconditioner based on piecewise constant coarse spaces using one pre- and post-smoothing steps of symmetric Gauss-Seidel smoother.

3. variable V-cycle preconditioner based on continuous elements described in Section 3 with one pre- and post-smoothing Gauss-Seidel iteration on the finest level and double the pre- and post-smoothing iteration on each consecutive coarser level.

The numerical results are summarised below. In each table we give the number of iterations in the PCG algorithm and the corresponding average reduction factor for each test run. In addition we include the number of degrees of freedom (DOF) in the DG space, $\mathcal{V}$, and the DOF for the first coarse space (defined on the finest mesh) of either continuous piecewise polynomial functions or piecewise constants.

Table 1. Numerical results for SIPG with **linear** FE: V-cycle based on continuous linear FE and W-cycle based on piece-wise constant functions with one pre- and one post-smoothing Gauss-Seidel iteration.

Test Problem 1	Level 2	Level 3	Level 4	Level 5	Level 6
DOF SIPG	3 072	24 576	196 608	1 572 864	12 582 912
preconditioner DOF	189	1 241	9 009	68 705	536 769
continuous FE	14/0.2556	14/0.2614	14/0.2572	14/0.2487	13/0.2344
preconditioner DOF	768	6 144	49 152	393 216	3 145 728
piecewise constant	24/0.4493	29/0.5238	30/0.5374	30/0.5342	29/0.5276

Table 2. Numerical results for SIPG with **quadratic** FE: V-cycle preconditioner based on continuous FE and W-cycle preconditioner based on piecewise constant functions each with one pre- and one post-smoothing Gauss-Seidel iteration.

Test Problem 1	Level 1	Level 2	Level 3	Level 4	Level 5
DOF SIPG	960	7 680	61 440	491 520	3 932 160
preconditioner DOF	189	1 241	9 009	68 705	536 769
continuous FE	10/0.1414	11/0.1717	11/0.1747	11/0.1657	10/0.1514
preconditioner DOF	96	768	6 144	49 152	393 216
piecewise constant	22/0.4315	35/0.5810	42/0.6442	43/0.6509	43/0.6496

In Tables 1 and 2 we present the computational results for test problem 1. These results show that both preconditioners, the V-cycle, that uses continuous finite elements, and the W-cycle, that uses piece-wise constant function on all coarser levels are optimal with respect to the number of iterations. The W-cycle preconditioner, based on piecewise constant functions, performs according to the W-cycle theory. However, it needs two times more iterations compared with the V-cycle, based on continuous functions. While the former has a matrix of size about 6 times larger than size of the matrix of the latter (for linear FE), one should have in mind that in the case of piece-wise constant functions the corresponding matrix has only five nonzero entries per row, i.e.

it is about five times sparser than the matrix produced by continuous linear elements. Unfortunately, we do not have a theory for the V-cycle.

Table 3. V-cycle and variable V-cycle based on continuous coarse spaces for the SIPG with **linear** elements and stabilization factor $\kappa_{\mathcal{E}}$ that does not depend on the jumps of a.

Test Problem 2	Level 1	Level 2	Level 3	Level 4
DOF of SIPG	1 344	10 752	86 016	688 128
precond. DOF - continuous linear	117	665	4 401	31 841
$\epsilon = 1$, V-cycle	15/0.2750	16/0.2946	15/0.2908	15/0.2838
$\epsilon = 0.1$, V-cycle	17/0.3322	19/0.3645	19/0.3717	19/0.3675
$\epsilon = 0.01$, V-cycle	17/0.3219	19/0.3632	19/0.3746	19/0.3713
$\epsilon = 0.001$, V-cycle	15/0.2929	17/0.3377	18/0.3527	18/0.3488
$\epsilon = 1$, variable V-cycle	15/0.2738	15/0.2900	15/0.2850	15/0.2759
$\epsilon = 0.1$, variable V-cycle	17/0.3310	18/0.3593	19/0.3658	18/0.3566
$\epsilon = 0.01$, variable V-cycle	17/0.3211	18/0.3568	19/0.3684	18/0.3582
$\epsilon = 0.001$, variable V-cycle	15/0.2919	17/0.3333	18/0.3457	17/0.3337

It is known that the choice of the stabilization factor $\kappa_{\mathcal{E}}$ could affect the properties of the method. To test sensitivity of the preconditioners with respect to the jumps of the coefficient a we considered two different choices, $\kappa_{\mathcal{E}} = \kappa \{\{a\}\}$, as defined in the SIPG method, and $\kappa_{\mathcal{E}} = \kappa \|a\|_{L^\infty} = 15$, which obviously is independent of the jumps. As shown in Table 3 the variable V-cycle preconditioner, covered by our theory, gives the same number of iterations as the V-cycle. Both preconditioners are not sensitive to the choice of $\kappa_{\mathcal{E}}$. From Table 3 one can see that the preconditioners based on continuous coarse spaces are robust in this case with respect to the jumps in a. However, this is not the case for the preconditioners based on piece-wise constant coarse spaces. We observe this in Table 4 where the performance of the W-cycle is given. From these experiments we see that a proper weighting of the jumps is essential for the performance of the W-cycle iteration based on piece-wise constant functions. In Table 5 we present results for test problem 2 with properly

Table 4. W-cycle based on piece-wise constant coarse spaces for the SIPG with **linear** elements and stabilization factor $\kappa_{\mathcal{E}}$ that does not depend on the jumps of a

Test Problem 2	Level 1	Level 2	Level 3	Level 4
DOF of SIPG	1 344	10 752	86 016	688 128
precond. DOF - piecewise constant	336	2 688	21 504	172 032
$\epsilon = 1$, W-cycle	22/0.4151	27/0.4940	29/0.5224	29/0.5297
$\epsilon = 0.1$, W-cycle	38/0.6106	72/0.7706	85/0.8027	91/0.8160
$\epsilon = 0.01$, W-cycle	48/0.6804	157/0.8869	210/0.9156	238/0.9255

scaled stabilization parameter: $\kappa_{\mathcal{E}} = \kappa \{\{a\}\}$. We tested the following preconditioners: V-cycle and variable V-cycle based on continuous coarse spaces and W-cycle based on piece-wise constant coarse spaces. Once again one can see that V-cycle and variable V-cycle based on continuous coarse spaces perform almost identically. Note that the iteration counts are slightly larger than those of the case $\kappa_{\mathcal{E}} = \kappa \|a\|_{L^\infty}$ (cf. Table 3) but they are insensitive to large jumps. In the case of piece-wise constant coarse spaces (W-cycle) the advantage of the weighted stabilization is evident – the numerical experiments show that the number of PCG iterations is essentially independent of the jumps.

Table 5. Numerical results for Test Problem 2: SIPG with **linear** elements and stabilization parameter $\kappa_{\mathcal{E}} = \kappa \{\{a\}\}$.

Test Problem 2	Level 1	Level 2	Level 3	Level 4	Level 5
DOF of SIPG	1 344	10 752	86 016	688 128	5 505 024
precond. DOF - continuous	117	665	4 401	31 841	241 857
$\epsilon = 1$, V-cycle	15/0.2750	16/0.2946	15/0.2908	15/0.2838	15/0.2766
$\epsilon = 0.1$, V-cycle	16/0.3161	20/0.3812	21/0.4105	22/0.4187	22/0.4196
$\epsilon = 0.01$, V-cycle	20/0.3800	24/0.4539	29/0.5228	31/0.5518	33/0.5687
$\epsilon = 0.001$, V-cycle	19/0.3782	24/0.4603	30/0.5377	33/0.5674	36/0.5957
$\epsilon = 10^{-4}$, V-cycle	18/0.3546	24/0.4535	30/0.5312	32/0.5622	34/0.5753
$\epsilon = 10^{-5}$, V-cycle	18/0.3411	23/0.4488	28/0.5100	30/0.5405	32/0.5622
$\epsilon = 10^{-6}$, V-cycle	17/0.3279	23/0.4416	26/0.4911	29/0.5298	30/0.5375
$\epsilon = 1$, var. V-cycle	15/0.2738	15/0.2900	15/0.2850	15/0.2759	14/0.2628
$\epsilon = 0.1$, var. V-cycle	16/0.3157	20/0.3782	21/0.4038	21/0.4107	21/0.4056
$\epsilon = 0.01$, var. V-cycle	20/0.3796	24/0.4508	29/0.5170	31/0.5448	32/0.5559
$\epsilon = 0.001$, var. V-cycle	19/0.3779	24/0.4574	30/0.5329	33/0.5612	35/0.5886
precond. DOF - p.w. constant	336	2 688	21 504	172 032	1 376 256
$\epsilon = 1$, W-cycle	22/0.4151	27/0.4940	29/0.5224	29/0.5297	29/0.5251
$\epsilon = 0.1$, W-cycle	23/0.4400	28/0.5057	29/0.5284	30/0.5357	30/0.5343
$\epsilon = 0.01$, W-cycle	22/0.4300	28/0.5012	30/0.5321	30/0.5385	31/0.5420
$\epsilon = 0.001$, W-cycle	23/0.4410	28/0.5001	30/0.5332	30/0.5403	31/0.5438
$\epsilon = 10^{-4}$, W-cycle	22/0.4302	27/0.4980	30/0.5333	30/0.5405	31/0.5442
$\epsilon = 10^{-5}$, W-cycle	22/0.4209	26/0.4880	30/0.5333	30/0.5405	31/0.5442
$\epsilon = 10^{-6}$, W-cycle	21/0.4112	25/0.4730	30/0.5333	30/0.5405	31/0.5442

Finally, in Table 6 we present the results iteration for V-cycle and W-cycle preconditioners for test Problem 3. The mesh of this example has a number of finite elements with high aspect ratio. The computations show that the preconditioner based on piecewise constant functions is slightly more sensitive with respect to the aspect ratio.

Acknowledgments. The authors thanks the referees for their helpful remarks and useful comments.

Table 6. Numerical results for Test Problem 3 for V-cycle and W-cycle for the SIPG with **linear** elements.

Test Problem 3	Level 1	Level 2	Level 3	Level 4
precond. DOF of SIPG	24 032	192 256	1 538 048	12 304 384
precond. DOF of cont. FE	1 445	9 693	70 633	538 513
V-cycle	18/0.3530	18/0.3559	18/0.3529	19/0.3785
precond. DOF p.w. constants	6 008	48 064	384 512	3 076 096
W-cycle	35/0.5907	40/0.6307	45/0.6578	48/0.6788

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