Derivation of a Darcy's law for composite porous solids using a homogenization procedure

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Abstract

The objective of this article is to describe the macroscopic fluid flow within a moving porous medium composed of a solid matrix, an ice matrix and unfrozen water, with the additional assumption that there is no direct contact between the solid and ice matrices. The derived fluid flow equations for this type of saturated porous material are obtained using two-space homogenization techniques for periodic structures. The pore size is assumed to be small compared to the macroscopic scale under consideration. At the microscopic scale the two *weakly coupled* solids are described by the linear elastic equations, and the fluid by the linearized Navier-Stokes equations, with appropriate boundary conditions at the solid-fluid interfaces. After performing the homogenization procedure, a generalized Darcy's law for the macroscopic fluid velocity is obtained. Also, a formal relation with a previous macroscopic fluid flow equation derived using a phenomenological approach is given.

Keywords: composite porous solids, homogenization, Darcy's law.

1 Introduction

The study of fluid flow in porous saturated media is a subject of interest in many fields such as geophysics, rock physics and materials science.

The fundamental concepts about the stress-strain relations and the dynamics of deformable porous single-phase solids fully saturated by a fluid were established in the works of M. Biot [3, 4, 5]. This formulation assumes that the quantities measured at the macroscopic scale can be described using the concepts of continuum mechanics.

When the porous matrix is composed by two (or more) different solid phases, more complicated models are required. Based on Biot's theory, Leclaire et al. [11] developed a phenomenological model to describe wave propagation in a porous solid matrix where the pore space is filled with ice and water, assuming no interaction between the solid and ice particles.

This model has been recently generalized to the case of variable porosity [17]. As a consequence of the models in [11] and [17] a generalized Darcy's law for this type of materials is obtained.

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The macroscopic description of porous media can also be obtained by means of the *homogenization method*, which consists of passing from the microscopic description at the pore and grain scales to the macroscopic scale. Important contributions to the solution of this problem were given by Sanchez-Palencia [15] and Bensoussan et al.[2], who developed the so called two-space homogenization technique. This method provides a systematic procedure for deriving macroscopic dynamical equations starting from the governing equations for the medium valid at the microscale. It was successfully applied by different authors to obtain theoretical justifications of Darcy's law and Biot's equations for single phase porous media [1, 7, 12, 16]. The procedure was recently applied to derive the equations of motion for saturated composite porous media for the special case when only one of the solid phases is in contact with the fluid phase [18].

Following these ideas, the aim of this paper is to apply the homogenization procedure to obtain a description of the macroscopic fluid flow within a saturated porous medium composed of a solid matrix, an ice matrix and unfrozen water, where as in [11] it is assumed that there is no contact between the solid matrix and the ice. This assumption is valid for example in finely dispersed frozen media, for which there exists a layer of unfrozen water around the solid particles isolating them from the ice, as explained in [11].

The analysis is restricted to the range of small deformations and for Newtonian fluids, under the assumption of spatial periodicity. As a result, a generalized Darcy's law for the material is obtained, in which the macroscopic fluid flow represents the contributions from the moving boundaries of the two solid phases as well as the gradient of the fluid pressure. The argument employs the concept of *very weak* solutions of the local Stokes problems in order to obtain an explicit forms of permeability tensors in terms of nonhomogeneous boundary data. The derived Darcy's law is formally in agreement with those derived in [11] and [17] using phenomenological arguments.

The organization of the paper is as follows. In §2 we state the local equations and apply the homogenization procedure to obtain our form of Darcy's law containing three permeability tensors, whose properties are analyzed in §3. Also §3 contains a formal relation of our Darcy's law with previously derived forms using phenomenological arguments. Finally in §4 we prove existence and uniqueness results for the local Stokes problems with nonhomogeneous boundary data using the concept of *very weak* solutions.

2 The homogenization procedure and Darcy's law

2.1 The local description and formal expansion

Let us consider a composite porous medium consisting of a porous solid matrix, an ice matrix and unfrozen water, i.e, two solid phases and one single-phase fluid. It will be assumed that there is no contact between the solid matrix and the ice, or equivalently, there exists a layer of unfrozen water around the solid particles isolating them from the ice. The solid matrix and the ice will be referred to by the subscripts or superscripts 1 and 3, while the fluid phase will be indicated by the subscript or superscript f. The porous medium will be considered to be periodic and composed of a large number of periods, with l and L denoting the length of the period and the macroscopic length, respectively, so that $\epsilon = \dfrac{l}{L} << 1$. The microscopic and macroscopic behaviors will be described by the two dependent spatial variables $\mathbf{x}$ and $\mathbf{y} = \dfrac{\mathbf{x}}{\epsilon}$.

Let Ω denote a periodic porous medium consisting of the solid and the ice matrices, Ω_1 and

Ω_3, and the fluid phase Ω_f. Also let Y denote one period in Ω so that

$$Y = Y_1 \cup Y_3 \cup Y_f,\ Y_j = Y \cap \Omega_j, j = 1, 3, f.$$

Also, let

$$\Gamma_{jf} = \partial Y_j \cap \partial Y_f, \quad \Gamma_{je} = \partial Y_j \cap \partial Y, \quad j = 1, 3,$$

$$\Gamma_{sf} = \Gamma_{1f} \cup \Gamma_{3f}, \quad \Gamma_{fe} = \partial Y_f \cap \partial Y,$$

so that

$$\partial Y_f = \Gamma_{1f} \cup \Gamma_{3f} \cup \Gamma_{fe}, \quad \partial Y_1 = \Gamma_{1f} \cup \Gamma_{1e}, \quad \partial Y_3 = \Gamma_{3f} \cup \Gamma_{3e}.$$

We assume that all phases are connected and that at the local level the two solid phases are linear elastic and the fluid is viscous Newtonian. We further assume that the transient Reynolds number is $O(1)$ at the local level so that the fluid viscosities η and κ are scaled by ϵ^2. Let $\mathbf{u}_j = \mathbf{u}_j(\omega)$ and $\boldsymbol{\sigma}_j = \boldsymbol{\sigma}_j(\omega), j = 1, 3, f$ denote the time Fourier transforms at the angular frequency ω of the displacement vectors and stress tensors of the three phases, respectively, let $p_f = p_f(\omega)$ be the fluid pressure and set $\mathbf{v}_j = i\omega\mathbf{u}_j$. The local variables are defined in their domain of definition and taken to be zero elsewhere. In what follows, to avoid cumbersome notation the explicit dependence on the frequency ω of the field variables will be omitted except when it is desired to emphasize this dependence. The local equations are given by

$$\begin{aligned}
\text{solid 1:}\quad \nabla\cdot\boldsymbol{\sigma}_1 &= -\omega^2\rho_1\mathbf{u}_1, \quad Y_1, && \text{(2.1a)}\\
\boldsymbol{\sigma}_1 &= \mathbf{a}_1 : \mathbf{e}(\mathbf{u}_1), \quad Y_1, && \text{(2.1b)}\\
\text{solid 3:}\quad \nabla\cdot\boldsymbol{\sigma}_3 &= -\omega^2\rho_3\mathbf{u}_3, \quad Y_3, && \text{(2.1c)}\\
\boldsymbol{\sigma}_3 &= \mathbf{a}_3 : \mathbf{e}(\mathbf{u}_3), \quad Y_3, && \text{(2.1d)}\\
\text{fluid:}\quad \nabla\cdot\boldsymbol{\sigma}_f &= i\omega\rho_f\mathbf{v}_f, \quad Y_f, && \text{(2.1e)}\\
\boldsymbol{\sigma}_f &= -p_f I + \boldsymbol{\tau}_f, \quad Y_f, && \text{(2.1f)}\\
\boldsymbol{\tau}_f &= 2\eta\epsilon^2\mathbf{e}(\mathbf{v}_f) + \epsilon^2\left(\kappa - \frac{2}{3}\eta\right)\nabla\cdot\mathbf{v}_f, \quad Y_f, && \text{(2.1g)}\\
i\omega p_f &= B_f\nabla\cdot\mathbf{v}_f, \quad Y_f. && \text{(2.1h)}
\end{aligned}$$

Here ρ_1, ρ_3, $\mathbf{a}_1$ and $\mathbf{a}_3$ are respectively the mass densities and fourth-order positive definite elastic tensors associated with the two solid phases, depending on the space variable and Y-periodic. Also, $\mathbf{e}$ denotes the linear strain tensor, i.e.,

$$e_{lm}(\mathbf{v}_f) = \frac{1}{2}\left(\frac{\partial v_{f,l}}{\partial x_m} + \frac{\partial v_{f,m}}{\partial x_l}\right).$$

Here, and in what follows, if $\mathbf{a}$ and $\mathbf{b}$ are, respectively, fourth and second order tensors, then $\mathbf{a} : \mathbf{b}$ denotes the index contraction operation $a_{klst}b_{st}$, with the usual Einstein's convention of summing on repeated indices.

Also, with $\boldsymbol{\nu}_{jk}, j,k = 1,3,f, j \neq k$, denoting the unit outer normal at the interface Γ_{jk}, the boundary conditions among the different solid and fluid phases are

$$\boldsymbol{\nu}_{1f} \cdot \boldsymbol{\sigma}_1 = \boldsymbol{\nu}_{1f} \cdot \boldsymbol{\sigma}_f, \quad \Gamma_{1f}, \tag{2.2a}$$
$$\boldsymbol{\nu}_{3f} \cdot \boldsymbol{\sigma}_3 = \boldsymbol{\nu}_{3f} \cdot \boldsymbol{\sigma}_f, \quad \Gamma_{3f}, \tag{2.2b}$$
$$\mathbf{v}_1 = \mathbf{v}_f, \quad \Gamma_{1f}, \tag{2.2c}$$
$$\mathbf{v}_3 = \mathbf{v}_f, \quad \Gamma_{3f}. \tag{2.2d}$$

Next, following Sanchez-Palencia [15, 16] and Auriault [1], we expand the unknowns $\mathbf{u}_1, \mathbf{u}_3, \mathbf{u}_f$ in the form

$$\mathbf{u}_j^\epsilon = \mathbf{u}_j(x,y) = \mathbf{u}_j^{(0)}(x,y) + \epsilon \mathbf{u}_j^{(1)}(x,y) + \epsilon^2 \mathbf{u}_j^{(2)}(x,y) + \cdots, \quad j = 1,3,f, \tag{2.3}$$

where the functions $\mathbf{u}_j^{(n)}(x,y), n = 0,1,\cdots$, are Y-periodic. Then we substitute the expansions (2.3) into Equations (2.1)–(2.2) describing the local behavior recalling that for the spatial derivatives we have that $\frac{d}{dx}$ becomes

$$\frac{\partial}{\partial x} + \epsilon^{-1}\frac{\partial}{\partial y}.$$

Similarly,

$$\mathbf{e} = \mathbf{e}_x + \epsilon^{-1}\mathbf{e}_y, \quad \nabla = \nabla_x + \epsilon^{-1}\nabla_y, \quad \text{etc.}$$

2.2 Solution of the local equations for the solid phases

Let us consider the local equations for the solid phases at the lowest order. First, from (2.1b) and (2.1d),

$$\begin{aligned} \boldsymbol{\sigma}_j &= \mathbf{a}_j : (\mathbf{e}_x + \epsilon^{-1}\mathbf{e}_y)(\mathbf{u}_j^{(0)} + \epsilon \mathbf{u}_j^{(1)} + \cdots) \\ &= \epsilon^{-1}\mathbf{a}_j : \mathbf{e}_y(\mathbf{u}_j^{(0)}) + \mathbf{a}_j : \left(\mathbf{e}_x(\mathbf{u}_j^{(0)}) + \mathbf{e}_y(\mathbf{u}_j^{(1)})\right) + \epsilon\, \mathbf{a}_j : \left(\mathbf{e}_x(\mathbf{u}_j^{(1)}) + \mathbf{e}_y(\mathbf{u}_j^{(2)})\right) + \cdots \\ &= \epsilon^{-1}\boldsymbol{\sigma}_j^{(-1)} + \boldsymbol{\sigma}_j^{(0)} + \epsilon\, \boldsymbol{\sigma}_j^{(1)} + \cdots, \quad Y_j,\ j = 1,3. \end{aligned} \tag{2.4}$$

Next, from (2.1a) and (2.4)

$$\begin{aligned} &\epsilon^{-2}\nabla_y \cdot \boldsymbol{\sigma}_j^{(-1)} + \epsilon^{-1}\left(\nabla_x \cdot \boldsymbol{\sigma}_j^{(-1)} + \nabla_y \cdot \boldsymbol{\sigma}_j^{(0)}\right) + \epsilon^0 \left(\nabla_x \cdot \boldsymbol{\sigma}_j^{(0)} + \nabla_y \cdot \boldsymbol{\sigma}_j^{(1)}\right) + \cdots \\ &\quad = -\rho_j \omega^2 \left(\mathbf{u}_j^{(0)} + \epsilon \mathbf{u}_j^{(1)} + \cdots\right), \quad Y_j,\ j = 1,3. \end{aligned} \tag{2.5}$$

Also, from (2.1f)-(2.1g),

$$\begin{aligned} \boldsymbol{\sigma}_f^{(0)} + \epsilon \boldsymbol{\sigma}_f^{(1)} + \cdots &= -p_f^{(0)} I + \epsilon\left(-p_f^{(1)} I + 2\eta \mathbf{e}(\mathbf{v}_f^{(0)})\right) + \cdots \\ &= -p_f^{(0)} I + \epsilon\left(-p_f^{(1)} I + \boldsymbol{\tau}_f^{(1)}\right) + \cdots, \quad Y_f. \end{aligned} \tag{2.6}$$

Next we use (2.2a)–(2.2d) to obtain the boundary conditions for the local problems. First, from (2.2a), (2.2b) and (2.6),

$$\begin{aligned}&\boldsymbol{\nu}_{jf}\cdot\left(\epsilon^{-1}\boldsymbol{\sigma}_j^{(-1)}+\boldsymbol{\sigma}_j^{(0)}+\epsilon\boldsymbol{\sigma}_j^{(1)}+\cdots\right)\\&=\boldsymbol{\nu}_{jf}\cdot\left(-p_f^{(0)}I+\epsilon\left(-p_f^{(1)}I+2\eta\mathbf{e}(\mathbf{v}_f^{(0)})\right)+\cdots\right),\ \Gamma_{jf},\ j=1,3.\end{aligned}\tag{2.7}$$

From (2.5) at ϵ^{-2} and (2.4) and (2.7) at ϵ^{-1} we obtain the following elliptic system for $\mathbf{u}_1^{(0)}$:

$$\nabla_y\cdot\boldsymbol{\sigma}_1^{(-1)} = 0, \quad Y_1, \tag{2.8a}$$
$$\boldsymbol{\sigma}_1^{(-1)} = \mathbf{a}_1:\mathbf{e}_y(\mathbf{u}_1^{(0)}), \quad Y_1, \tag{2.8b}$$
$$\boldsymbol{\nu}_{1f}\cdot\boldsymbol{\sigma}_1^{(-1)} = 0, \quad \Gamma_{1f}, \tag{2.8c}$$
$$\mathbf{u}_1^{(0)} \text{ is } \quad Y-\text{periodic}. \tag{2.8d}$$

Let us formulate (2.8) in variational form. Set

$$\mathcal{W}_{Y_j}=\left\{\boldsymbol{\varphi}\in[H^1(Y_j)]^3:\ \boldsymbol{\varphi}\text{ is }Y-\text{periodic},\ \int_{Y_j}\boldsymbol{\varphi}\,dy=0,\ \int_{Y_j}\nabla\times\boldsymbol{\varphi}\,dy=0\right\},j=1,3.$$

Also, for an open set $S\subset\mathbb{R}^3$ and a two-dimensional manifold γ, let $(\cdot,\cdot)_S$ and $\langle\cdot,\cdot\rangle_\gamma$ denote the $L^2(S)$ and $L^2(\gamma)$ complex inner products. Then a weak form of (2.8) is given to find $\mathbf{u}_1^{(0)}\in\mathcal{W}_{Y_1}$ such that

$$\left(\mathbf{a}_1:\mathbf{e}_y(\mathbf{u}_1^{(0)}),\mathbf{e}_y(\boldsymbol{\varphi})\right)_{Y_1}=0,\quad \boldsymbol{\varphi}\in\mathcal{W}_Y^1. \tag{2.9}$$

Note that thanks to Korn's second inequality [9, 14] and the fact that $\mathbf{a}_1$ is positive definite the sesquilinear form $(\mathbf{a}_1:\mathbf{e}_y(\mathbf{u}),\mathbf{e}_y(\mathbf{v}))_{Y_1}$ defines an inner product in the Hilbert space $\mathcal{W}_{Y_1}$ equivalent to the H^1-inner product [6]. Thus the Lax-Milgram lemma implies that $\mathbf{u}_1^{(0)}=0\in\mathcal{W}_{Y_1}$ is the unique solution of (2.9), or equivalently, the solution of (2.8a)–(2.8c) is independent of the $y-$variable, so that

$$\mathbf{u}_1^{(0)}(x,y) = \mathbf{u}_1^{(0)}(x), \quad Y_1, \tag{2.10a}$$
$$\boldsymbol{\sigma}_1^{(-1)} = 0, \quad Y_1, \tag{2.10b}$$

where (2.10b) follows from (2.8b). With an identical argument, for the solid phase 3 we get

$$\mathbf{u}_3^{(0)}(x,y) = \mathbf{u}_3^{(0)}(x), \quad Y_3, \tag{2.11a}$$
$$\boldsymbol{\sigma}_3^{(-1)} = 0, \quad Y_3. \tag{2.11b}$$

2.3 Solution of the local equations for the fluid phase: A generalized Darcy's law

We now consider the local equations for the fluid at the lowest order. First, from the fluid equations (2.1e)–(2.1h) it follows that

$$\begin{aligned}
&\eta\epsilon^2 \left(\Delta_x + \epsilon^{-1}\left(\nabla_x \cdot \nabla_y + \nabla_y \cdot \nabla_x\right) + \epsilon^{-2}\Delta_y\right)\left(\mathbf{v}_f^{(0)} + \epsilon\mathbf{v}_f^{(1)} + \epsilon^2\mathbf{v}_f^{(2)} + \cdots\right) \\
&+\epsilon^2\left(\kappa - \frac{1}{2}\eta\right)\left[\nabla_x\nabla_x + \epsilon^{-1}\left(\nabla_x\nabla_y + \nabla_y\nabla_x\right) + \epsilon^{-2}\nabla_y\nabla_y\right]\cdot\left(\mathbf{v}_f^{(0)} + \epsilon\mathbf{v}_f^{(1)} + \epsilon^2\mathbf{v}_f^{(2)} + \cdots\right) \\
&= \left(\nabla_x + \epsilon^{-1}\nabla_y\right)\left(p_f^{(0)} + \epsilon p_f^{(1)} + \epsilon^2 p_f^{(2)} + \cdots\right) + i\omega\rho_f\left(\mathbf{v}_f^{(0)} + \epsilon\mathbf{v}_f^{(1)} + \cdots\right),
\end{aligned} \tag{2.12}$$

and

$$i\omega\left(p_f^{(0)} + \epsilon p_f^{(1)} + \cdots\right) = B_f\left(\nabla_x + \epsilon^{-1}\nabla_y\right)\cdot\left(\mathbf{v}_f^{(0)} + \epsilon\mathbf{v}_f^{(1)} + \epsilon^2\mathbf{v}_f^{(2)} + \right). \tag{2.13}$$

Thus from (2.12) at ϵ^{-1} we get

$$p_f^{(0)}(x,y) = p_f^{(0)}(x), \tag{2.14}$$

and then it follows from (2.13) at ϵ^{-1} that

$$\nabla_y \cdot \mathbf{v}_f^{(0)} = 0, \quad Y_f. \tag{2.15}$$

Hence, from (2.15) and (2.12) at ϵ^0 we get

$$\eta\Delta_y\mathbf{v}_f^{(0)} = \nabla_y \cdot \boldsymbol{\tau}_f^{(1)}(\mathbf{v}_f^{(0)}) = \nabla_y p_f^{(1)} + \nabla_x p_f^{(0)} + i\omega\rho_f\mathbf{v}_f^{(0)}, \quad Y_f. \tag{2.16}$$

Also, it follows from (2.2c)–(2.2d) that

$$\mathbf{v}_f^{(0)} = i\omega\mathbf{u}_1^{(0)}(x), \quad \Gamma_{1f}, \tag{2.17a}$$

$$\mathbf{v}_f^{(0)} = i\omega\mathbf{u}_3^{(0)}(x), \quad \Gamma_{3f}. \tag{2.17b}$$

Let us split Problem (2.15), (2.16), and (2.17a)-(2.17b) into two subproblems as follows. First, let $\mathbf{v}_f^{(0),I}$ and $p_f^{(1),I}$ be Y-periodic such that

$$i\omega\rho_f\mathbf{v}_f^{(0),I} - \eta\Delta_y\mathbf{v}_f^{(0),I} + \nabla_y p_f^{(1),I} = -\nabla_x p_f^{(0)}, \quad Y_f, \tag{2.18a}$$

$$\nabla_y \cdot \mathbf{v}_f^{(0),I} = 0, \quad Y_f, \tag{2.18b}$$

$$\mathbf{v}_f^{(0),I} = 0, \quad \Gamma_{sf}. \tag{2.18c}$$

Second, let $\mathbf{v}_f^{(0),B}$ and $p_f^{(1),B}$ be the Y-periodic solution of

$$i\omega\rho_f\mathbf{v}_f^{(0),B} - \eta\Delta_y\mathbf{v}_f^{(0),B} + \nabla_y p_f^{(1),B} = 0, \quad Y_f, \tag{2.19a}$$

$$\nabla_y \cdot \mathbf{v}_f^{(0),B} = 0, \quad Y_f, \tag{2.19b}$$

$$\mathbf{v}_f^{(0),B} = i\omega\mathbf{u}_1^{(0)}(x), \quad \Gamma_{1f}, \tag{2.19c}$$

$$\mathbf{v}_f^{(0),B} = i\omega\mathbf{u}_3^{(0)}(x), \quad \Gamma_{3f}. \tag{2.19d}$$

Let us solve the cell problem (2.18)- (2.18c) for $\mathbf{v}_f^{(0),I}$. Let

$$\mathcal{V}^1_{Y_f} = \left\{ \boldsymbol{\varphi} \in \left[H^1(Y_f)\right]^3 : \quad \nabla_y \cdot \boldsymbol{\varphi} = 0 \text{ in } Y_f, \quad \boldsymbol{\varphi} = 0 \text{ on } \Gamma_{sf}, \quad \boldsymbol{\varphi} \text{ is } Y-\text{periodic} \right\},$$

provided with the natural (complex) inner product in $\left[H^1(Y_f)\right]^3$. Then a variational formulation of (2.18), (2.18c) can be stated as follows: Find $\mathbf{v}_f^{(0),I} \in \mathcal{V}^1_{Y_f}$ such that

$$i\omega \left(\rho_f \mathbf{v}_f^{(0),I}, \boldsymbol{\varphi}\right)_{Y_f} + \left(\eta \nabla_y \mathbf{v}_f^{(0),I}, \nabla_y \boldsymbol{\varphi}\right)_{Y_f} = -\nabla_x p_f^{(0)}(x) \cdot \int_{Y_f} \boldsymbol{\varphi} dy, \quad \boldsymbol{\varphi} \in \mathcal{V}^1_{Y_f}. \tag{2.20}$$

It is known that (2.20) has a unique solution, which can be found as usual by solving the following set of problems [15]. For $s = 1, 2, 3$ let $\mathbf{V}^s = (V_t^s)_{1 \le t \le 3} \in \mathcal{V}^1_{Y_f}$ be the solution of

$$i\omega \left(\rho_f \mathbf{V}^s, \boldsymbol{\varphi}\right)_{Y_f} + \left(\eta \nabla_y \mathbf{V}^s, \nabla_y \boldsymbol{\varphi}\right)_{Y_f} = \mathbf{e}^s \cdot \int_{Y_f} \boldsymbol{\varphi} dy, \quad \boldsymbol{\varphi} \in \mathcal{V}^1_{Y_f} \tag{2.21}$$

where $\mathbf{e}^s$ denotes the standard basis in $\mathbf{R}^3$ and set

$$\mathbf{K}(x, y, \omega) = \left(\mathbf{K}(x, y, \omega)\right)_{ts} = V_t^s(x, y, \omega). \tag{2.22}$$

Then,

$$\mathbf{v}_f^{(0),I}(x, y, \omega) = -\mathbf{K}(x, y, \omega) \nabla p_f^{(0)}(x, \omega). \tag{2.23}$$

We turn to analyze the second subproblem (2.19a)–(2.19d). First, notice that it follows from (2.10a) and (2.11a) that

$$\begin{aligned} 0 &= \int_{Y_j} \nabla_y \cdot \mathbf{u}_j^{(0)}(x) dy = \mathbf{u}_j^{(0)}(x) \cdot \int_{\partial Y_j} \boldsymbol{\nu}_j dy \\ &= \mathbf{u}_j^{(0)}(x) \cdot \int_{\Gamma_{jf}} \boldsymbol{\nu}_j dy + \mathbf{u}_j^{(0)}(x) \cdot \int_{\Gamma_{je}} \boldsymbol{\nu}_j dy = \mathbf{u}_j^{(0)}(x) \cdot \int_{\Gamma_{jf}} \boldsymbol{\nu}_j dy, \quad j = 1, 3, \end{aligned} \tag{2.24}$$

since $\int_{\Gamma_{je}} \boldsymbol{\nu}_j dy = 0$ due to the periodicity of the boundary Γ_{je}. Thus the boundary data function defined by

$$\mathbf{g}(x, y, \omega) = \begin{cases} i\omega \mathbf{u}_1^{(0)}(x, \omega), & \Gamma_{1f}, \\ i\omega \mathbf{u}_3^{(0)}(x, \omega), & \Gamma_{3f}, \\ \text{periodic in} & \Gamma_{fe}, \end{cases} \tag{2.25}$$

satisfies the consistency condition

$$\int_{\partial Y_f} \mathbf{g} \cdot \boldsymbol{\nu}_f dy = 0. \tag{2.26}$$

Remark 2.1. *Notice that our boundary data take constant values on each boundary component of Y_f, which implies that the solution of Problem (2.19) is smooth. Therefore a classical abstract theory of existence of solutions of Stokes problems (e.g. [19]) can be applied to our case. However, it is our intention to derive a form of the permeability tensor depending explicitly on the boundary data, and consequently we analyze the problem in the very weak sense [8, 13].*

Set

$$\mathcal{V}^2_{Y_f} = \left\{\boldsymbol{\varphi} \in \left[H^2(Y_f)\right]^3 : \quad \nabla_y \cdot \boldsymbol{\varphi} = 0 \text{ in } Y_f, \quad \boldsymbol{\varphi} = 0 \text{ on } \Gamma_{sf}, \quad \boldsymbol{\varphi} \text{ is } Y-\text{periodic}\right\},$$

$$\mathcal{W} = \left\{q \in H^1(Y_f) : \quad \int_{Y_f} q\,dy = 0, \quad q \text{ is } Y-\text{periodic}\right\}.$$

Now we state the existence and uniqueness results on the solution of Problem (2.19).

Theorem 2.1. *There exists a unique Y-periodic very weak solution $\mathbf{v}_f^{(0),B} \in \left[L^2(Y_f)\right]^3$ and $\mathbf{v}_f^{(0),B}|_{\Gamma_{sf}} \in \left[L^2(\Gamma_{sf})\right]^3$ of Problem (2.19) in the following sense:*

$$i\omega\left(\rho_f \mathbf{v}_f^{(0),B}, \boldsymbol{\varphi}\right)_{Y_f} - \left(\eta \mathbf{v}_f^{(0),B}, \Delta_y \boldsymbol{\varphi}\right)_{Y_f} = -\left\langle \eta \mathbf{g}, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{\nu}}\right\rangle_{\Gamma_{sf}}, \quad \boldsymbol{\varphi} \in \mathcal{V}^2_{Y_f}, \tag{2.27a}$$

$$\left(\mathbf{v}_f^{(0),B}, \nabla q\right)_{Y_f} = \langle \mathbf{g}\cdot\boldsymbol{\nu}, q\rangle_{\Gamma_{sf}}, \quad q \in \mathcal{W}. \tag{2.27b}$$

The proof of Theorem 2.1 is given in §4.

Set

$$\boldsymbol{\gamma}_j^s(x,y) = \chi_{\Gamma_{jf}}(x,y)\mathbf{e}^s, \quad j = 1,3, \quad s = 1,2,3, \tag{2.28}$$

where $\chi_{\Gamma_{jf}}(x,y)$ denotes the characteristic function of Γ_{jf}. Also on Γ_{sf} write the boundary data vector $\mathbf{g}$ in (2.25) in the form

$$\mathbf{g} = \sum_s \left(g_{1,s}\boldsymbol{\gamma}_1^s + g_{3,s}\boldsymbol{\gamma}_3^s\right), \quad y \in \Gamma_{sf}, \tag{2.29}$$

where $g_{j,s}$ denotes the s-component of $i\omega\mathbf{u}_j^{(0)}$ on Γ_{jf}. Let $\mathbf{Z}^{j,s} = (Z_t^{j,s})_{1\le t\le 3}$ be the solution of (2.27) with $\mathbf{g}$ replaced by $\boldsymbol{\gamma}_j^s, j = 1,3, s = 1,2,3$. Set

$$\mathbf{K}^j(x,y,\omega) = \left(\mathbf{K}^j(x,y,\omega)\right)_{ts} = Z_t^{j,s}(x,y,\omega), \quad j = 1,3. \tag{2.30}$$

Then, by linearity the solution of (2.27) is given by

$$\mathbf{v}_f^{(0),B}(x,y,\omega) = \mathbf{K}^1(x,y,\omega)\left[i\omega\mathbf{u}_1^{(0)}(x,\omega)\right] + \mathbf{K}^3(x,y,\omega)\left[i\omega\mathbf{u}_3^{(0)}(x,\omega)\right]. \tag{2.31}$$

Combining (2.23) and (2.31), we conclude that

$$\begin{aligned}\mathbf{v}_f^{(0)}(x,y,\omega) &= \mathbf{v}_f^{(0),I}(x,y,\omega) + \mathbf{v}_f^{(0),B}(x,y,\omega) \\ &= -\mathbf{K}(x,y,\omega)\nabla p_f^{(0)}(x,\omega) + \mathbf{K}^1(x,y,\omega)\left[i\omega\mathbf{u}_1^{(0)}(x,\omega)\right] + \mathbf{K}^3(x,y,\omega)\left[i\omega\mathbf{u}_3^{(0)}(x,\omega)\right].\end{aligned} \tag{2.32}$$

Let

$$\langle\langle\theta\rangle\rangle = \frac{1}{|Y|}\int_Y \theta(y)dy,$$

denote the average of $\theta(y)$ over Y, where θ is defined to be zero outside its domain of definition. Then, averaging (2.32) over Y yields

$$\begin{aligned}\left\langle\left\langle\mathbf{v}_f^{(0)}\right\rangle\right\rangle(x,\omega) &= -\langle\langle\mathbf{K}\rangle\rangle(x,\omega)\nabla_x p_f^{(0)}(x,\omega) \\ &\quad + \langle\langle\mathbf{K}^1\rangle\rangle(x,\omega)\left[i\omega\mathbf{u}_1^{(0)}(x,\omega)\right] + \langle\langle\mathbf{K}^3\rangle\rangle(x,\omega)\left[i\omega\mathbf{u}_3^{(0)}(x,\omega)\right].\end{aligned} \tag{2.33}$$

which is a generalized Darcy's law for our composite system.

3 Properties of the permeability tensors

In this section we analyze some properties of the permeability tensors $\mathbf{K}$, $\mathbf{K}^1$, and $\mathbf{K}^3$ which appear in the Darcy's law (2.33).

3.1 Properties of K

Note that defining on $\mathcal{V}^1_{Y_f}$ the sesquilinear form

$$B(\mathbf{u},\mathbf{v}) = i\omega\,(\rho_f\mathbf{u},\mathbf{v})_{Y_f} + (\eta\nabla\mathbf{u},\nabla\mathbf{v})_{Y_f}\,, \quad \mathbf{u},\mathbf{v}\in\mathcal{V}^1_{Y_f}, \tag{3.1}$$

and the continuous linear functional

$$L_{\mathbf{f}}(\boldsymbol{\varphi}) = \mathbf{f}\cdot\int_{Y_{\mathbf{f}}}\boldsymbol{\varphi}dy, \tag{3.2}$$

with $\mathbf{f} = \mathbf{f}(x,\omega) = -\nabla p_f^{(0)}(x,\omega)$, Problem (2.20) can be stated in the form: find $\mathbf{u}\in\mathcal{V}^1_{Y_f}$ such that

$$B(\mathbf{u},\mathbf{v}) = L_{\mathbf{f}}(\mathbf{v}), \quad \mathbf{v}\in\mathcal{V}^1_{Y_f}. \tag{3.3}$$

Note that $B(\mathbf{u},\mathbf{v})$ is continuous and coercive in $\mathcal{V}^1_{Y_f}$ since

$$|B(\mathbf{u},\mathbf{u})| \geq \frac{1}{2}\left(|\mathrm{Re}(B(\mathbf{u},\mathbf{u}))| + |\mathrm{Im}(B(\mathbf{u},\mathbf{u}))|\right) = \frac{1}{2}\left[(\eta\nabla\mathbf{u},\nabla\mathbf{u}) + \omega\,(\rho_f\mathbf{u},\mathbf{u})\right] \geq C(\omega)\|\mathbf{u}\|_1^2. \tag{3.4}$$

Thus, by Lax-Milgram Lemma, Problem (3.3) has a unique solution, which implies that the solution operator

$$\mathbf{f}\to\mathbf{u} = T_{\mathbf{f}}$$

where $\mathbf{u}$ solves (3.3) is injective. Thus if $\{\mathbf{e}^s : s = 1,2,3\}$ denotes the standard basis in $\mathbf{R}^3$, $\{\mathbf{V}^s = T_{\mathbf{f}}^{-1}(\mathbf{e}^s), s = 1,2,3\}$ forms a linearly independent set in $\mathcal{V}^1_{Y_f}$. Thus the tensor $\mathbf{K}$ is invertible. Set $\mathbf{V}^s = \mathrm{Re}(\mathbf{V}^s) + i\,\mathrm{Im}(\mathbf{V}^s) = \mathbf{V}^s_R + i\mathbf{V}^s_I$ and recall that $\langle\langle\mathrm{Re}\,(\mathbf{K})_{st}\rangle\rangle = \frac{1}{|Y|}\left(\mathbf{e}^s,\mathbf{V}^t_R\right)_{Y_f}$

Assuming $\boldsymbol{\varphi}$ to be real, take the real and imaginary parts in (2.21) to obtain

$$-\,\omega\,(\rho_f\mathbf{V}^s_I,\boldsymbol{\varphi})_{Y_f} + (\eta\nabla_y\mathbf{V}^s_R,\nabla_y\boldsymbol{\varphi})_{Y_f} = \mathbf{e}^s\cdot\int_{Y_f}\boldsymbol{\varphi}dy, \quad \boldsymbol{\varphi}\in\mathcal{V}^1_{Y_f}, \tag{3.5a}$$

$$\omega\,(\rho_f\mathbf{V}^s_R,\boldsymbol{\varphi})_{Y_f} + (\eta\nabla_y\mathbf{V}^s_I,\nabla_y\boldsymbol{\varphi})_{Y_f} = 0, \quad \boldsymbol{\varphi}\in\mathcal{V}^1_{Y_f}, \quad s = 1,2,3. \tag{3.5b}$$

Choose $\boldsymbol{\varphi} = \mathbf{V}^t_R$ in (3.5a) and $\boldsymbol{\varphi} = \mathbf{V}^s_I$ in (3.5b) with s replaced by t and add the resulting equations to get

$$\langle\langle\mathrm{Re}\,(\mathbf{K})_{st}\rangle\rangle = \frac{1}{|Y|}\left[\left(\eta\ \nabla_y\mathbf{V}^s_R,\nabla_y\mathbf{V}^t_R\right)_{Y_f} + \left(\eta\nabla_y\mathbf{V}^t_I,\nabla_y\mathbf{V}^s_I\right)_{Y_f}\right],$$

which shows that $\langle\langle\mathrm{Re}\,(\mathbf{K})\rangle\rangle$ is symmetric. Observe that for any $\boldsymbol{\xi}\in\mathbb{R}^3$,

$$\boldsymbol{\xi}^T\,\langle\langle\mathrm{Re}\,(\mathbf{K})\rangle\rangle\,\boldsymbol{\xi} = \frac{1}{|Y|}\left[\|\eta^{\frac{1}{2}}\nabla_y\mathbf{V}^s_R\xi_s\|^2_{0,Y_f} + \|\eta^{\frac{1}{2}}\nabla_y\mathbf{V}^s_I\xi_s\|^2_{0,Y_f}\right] \geq 0,$$

where the equality holds if and only if $\nabla_y \mathbf{V}_R^s \xi_s = 0$ and $\nabla_y \mathbf{V}_I^s \xi_s = 0$. Since $\mathbf{V}^s = 0$ on Γ_{sf}, by Poincaré inequality, $\mathbf{V}_R^s \xi_s = \mathbf{V}_I^s \xi_s = 0$, and therefore, $\langle\langle \mathrm{Re}\,(\mathbf{K})\rangle\rangle$ is positive-definite.

Next, recalling that $\langle\langle \mathrm{Im}\,(\mathbf{K})_{st}\rangle\rangle = \frac{1}{|Y|}\left(\mathbf{e}^s, \mathbf{V}_I^t\right)_{Y_f}$ we analyze the properties of $\langle\langle \mathrm{Im}\,(\mathbf{K})\rangle\rangle$. For this, we choose $\boldsymbol{\varphi} = \mathbf{V}_I^t$ in (3.5a) and $\boldsymbol{\varphi} = \mathbf{V}_R^s$ in (3.5b) with s replaced by t. Then

$$\langle\langle \mathrm{Im}\,(\mathbf{K})_{st}\rangle\rangle = -\frac{\omega}{|Y|}\left[\left(\rho_f^{\frac{1}{2}}\mathbf{V}_R^t, \mathbf{V}_R^s\right)_{Y_f} + \left(\rho_f^{\frac{1}{2}}\mathbf{V}_I^s, \mathbf{V}_I^t\right)_{Y_f}\right],$$

which implies that $\langle\langle \mathrm{Im}\,(\mathbf{K})\rangle\rangle$ is symmetric and negative-definite. This in turn implies that both the real and imaginary parts of $(\langle\langle \mathbf{K}\rangle\rangle)^{-1}$ are symmetric and positive-definite.

3.2 Properties of $\mathbf{K}^1$ and $\mathbf{K}^3$

Le us turn to analyze the $\mathbf{K}^j$-tensors, $j = 1, 3$, having the contribution from the boundaries $\Gamma_{jf}, j = 1, 3$. For this purpose, it is convenient to analyze the properties of the solution $\mathbf{Z}^{j,s,(m)}$ of (4.5) in §4, with right hand side $\mathbf{g} = \mathbf{e}^s$, $s = 1, 2, 3$. Thus, set

$$H^1(\mathrm{div}\,0; Y_f) = \left\{\boldsymbol{\varphi} \in \left[H^1(Y_f)\right]^3 : \quad \nabla_y \cdot \boldsymbol{\varphi} = 0, \quad \boldsymbol{\varphi} \text{ is } Y-\text{periodic}\right\}$$

and let $\mathbf{Z}^{j,s,(m)} \in H^1(\mathrm{div}\,0; Y_f)$ be the solution of

$$\begin{aligned} i\omega\left(\rho_f \mathbf{Z}^{j,s,(m)}, \boldsymbol{\varphi}\right)_{Y_f} + \left(\eta\nabla_y \mathbf{Z}^{j,s,(m)}, \nabla_y\boldsymbol{\varphi}\right)_{Y_f} + m\left\langle \mathbf{Z}^{j,s,(m)}, \boldsymbol{\varphi}\right\rangle_{\Gamma_{jf}} \\ = m\left\langle \mathbf{e}^s, \boldsymbol{\varphi}\right\rangle_{\Gamma_{jf}}, \qquad \boldsymbol{\varphi} \in H^1(\mathrm{div}\,0; Y_f),\ j = 1, 3. \end{aligned} \tag{3.6}$$

Set $\mathbf{Z}^{j,s,(m)} = \mathrm{Re}\left(\mathbf{Z}^{j,s,(m)}\right) + i\,\mathrm{Im}\left(\mathbf{Z}^{j,s,(m)}\right) = \mathbf{Z}_R^{j,s,(m)} + i\mathbf{Z}_I^{j,s,(m)}$. Assuming $\boldsymbol{\varphi}$ to be real, take the real and imaginary parts in (3.6) to obtain

$$\begin{aligned} -\omega\left(\rho_f \mathbf{Z}_I^{j,s,(m)}, \boldsymbol{\varphi}\right)_{Y_f} + \left(\eta\nabla_y \mathbf{Z}_R^{j,s,(m)}, \nabla_y\boldsymbol{\varphi}\right)_{Y_f} + m\left\langle \mathbf{Z}_R^{j,s,(m)}, \boldsymbol{\varphi}\right\rangle_{\Gamma_{jf}} \\ = m\left\langle \mathbf{e}^s, \boldsymbol{\varphi}\right\rangle_{\Gamma_{jf}}, \end{aligned} \tag{3.7a}$$

$$\begin{aligned} \omega\left(\rho_f \mathbf{Z}_R^{j,s,(m)}, \boldsymbol{\varphi}\right)_{Y_f} + \left(\eta\nabla_y \mathbf{Z}_I^{j,s,(m)}, \nabla_y\boldsymbol{\varphi}\right)_{Y_f} + m\left\langle \mathbf{Z}_I^{j,s,(m)}, \boldsymbol{\varphi}\right\rangle_{\Gamma_{jf}} = 0, \\ \boldsymbol{\varphi} \in H^1(\mathrm{div}0 : Y_f). \end{aligned} \tag{3.7b}$$

Next, with the choice $\boldsymbol{\varphi} = \mathbf{Z}_I^{j,t,(m)}$ in (3.7a) and $\boldsymbol{\varphi} = \mathbf{Z}_R^{j,s,(m)}$ in (3.7b) with s replaced by t, take the difference in the resulting equations to obtain

$$-\omega\left[\left(\rho_f \mathbf{Z}_R^{j,t,(m)}, \mathbf{Z}_R^{j,s,(m)}\right)_{Y_f} + \left(\rho_f \mathbf{Z}_I^{j,s,(m)}, \mathbf{Z}_I^{j,t,(m)}\right)_{Y_f}\right] = m\left\langle \mathbf{e}^s, \mathbf{Z}_I^{j,t,(m)}\right\rangle_{\Gamma_{jf}}. \tag{3.8}$$

Also, add (3.7a) with the choice $\boldsymbol{\varphi} = \mathbf{Z}_R^{j,t,(m)}$ with (3.7b) with the choice $\boldsymbol{\varphi} = \mathbf{Z}_I^{j,s,(m)}$ with s replaced by t to get

$$\begin{aligned} \left(\eta\nabla \mathbf{Z}_R^{j,s,(m)}, \nabla \mathbf{Z}_R^{j,t,(m)}\right)_{Y_f} + \left(\eta\nabla \mathbf{Z}_I^{j,t,(m)}, \nabla \mathbf{Z}_I^{j,s,(m)}\right)_{Y_f} \\ + m\left[\left\langle \mathbf{Z}_R^{j,s,(m)}, \mathbf{Z}_R^{j,t,(m)}\right\rangle_{\Gamma_{jf}} + \left\langle \mathbf{Z}_I^{j,t,(m)}, \mathbf{Z}_I^{j,s,(m)}\right\rangle_{\Gamma_{jf}}\right] = m\left\langle \mathbf{e}^s, \mathbf{Z}_R^{j,t,(m)}\right\rangle_{\Gamma_{jf}}. \end{aligned} \tag{3.9}$$

Similarly, take $\boldsymbol{\varphi} = \mathbf{e}^t$ in (3.7a) and $\boldsymbol{\varphi} = \mathbf{e}^s$ in (3.7b) with s replaced by t to have

$$-\omega\left(\rho_f Z_{I,t}^{j,s,(m)},1\right)_{Y_f} + m\left\langle Z_{R,t}^{j,s,(m)},1\right\rangle_{\Gamma_{jf}} = m\delta_{st}\left|\Gamma_{jf}\right|, \tag{3.10a}$$

$$\omega\left(\rho_f Z_{R,s}^{j,t,(m)},1\right)_{Y_f} + m\left\langle Z_{I,s}^{j,t,(m)},1\right\rangle_{\Gamma_{jf}} = 0. \tag{3.10b}$$

Using (3.10b) in (3.8) and (3.10a) (interchanging s and t) in (3.9) we have

$$\left(Z_{R,s}^{j,t,(m)},1\right)_{Y_f} = \left(\mathbf{Z}_R^{j,t,(m)},\mathbf{Z}_R^{j,s,(m)}\right)_{Y_f} + \left(\mathbf{Z}_I^{j,s,(m)},\mathbf{Z}_I^{j,t,(m)}\right)_{Y_f} \tag{3.11}$$

and

$$\begin{aligned}\omega\rho_f\left(Z_{I,s}^{j,t,(m)},1\right)_{Y_f} &= \eta\left[\left(\nabla\mathbf{Z}_R^{j,s,(m)},\nabla\mathbf{Z}_R^{j,t,(m)}\right)_{Y_f} + \left(\nabla\mathbf{Z}_I^{j,t,(m)},\nabla\mathbf{Z}_I^{j,s,(m)}\right)_{Y_f}\right] \\ &\quad + m\left[\left\langle\mathbf{Z}_R^{j,s,(m)},\mathbf{Z}_R^{j,t,(m)}\right\rangle_{\Gamma_{jf}} + \left\langle\mathbf{Z}_I^{j,t,(m)},\mathbf{Z}_I^{j,s,(m)}\right\rangle_{\Gamma_{sf}} - \delta_{st}\left|\Gamma_{jf}\right|\right].\end{aligned} \tag{3.12}$$

Let

$$\mathbf{K}^{j,(m)} = \left(\mathbf{K}^{j,(m)}\right)_{st} = Z_s^{j,t,(m)},$$

so that

$$\left\langle\left\langle \mathrm{Re}\left(\mathbf{K}^{j,(m)}\right)_{st}\right\rangle\right\rangle = \frac{1}{|Y|}\left(Z_{R,s}^{j,t,(m)},1\right)_{Y_f}, \quad \left\langle\left\langle \mathrm{Im}\left(\mathbf{K}^{j,(m)}\right)_{st}\right\rangle\right\rangle = \frac{1}{|Y|}\left(Z_{I,s}^{j,t,(m)},1\right)_{Y_f}.$$

It follows from (3.11) and (3.12) that, for each m, $\langle\langle \mathrm{Re}\left(\mathbf{K}^{j,(m)}\right)\rangle\rangle$ is symmetric positive-definite and $\langle\langle \mathrm{Im}\left(\mathbf{K}^{j,(m)}\right)\rangle\rangle$ is symmetric. Due to the weak convergence of $\mathbf{Z}^{j,t,(m)}$ to $\mathbf{Z}^{j,t}$ in $[L^2(Y_f)]^3$, we have (c.f. (4.18)),

$$\lim_{m\to\infty}(Z_{R,s}^{j,t,(m)},1)_{Y_f} = (Z_{R,s}^{j,t},1)_{Y_f}, \quad \lim_{m\to\infty}(Z_{I,s}^{j,t,(m)},1)_{Y_f} = (Z_{I,s}^{j,t},1)_{Y_f},\ j=1,3. \tag{3.13}$$

Consequently, $\langle\langle \mathrm{Re}\left(\mathbf{K}^j\right)\rangle\rangle$ is symmetric, positive semi-definite and $\langle\langle \mathrm{Im}\left(\mathbf{K}^j\right)\rangle\rangle$ is symmetric. Also, thanks to (3.13) the first term in the left hand side in (3.10b) is bounded. Thus taking the limit in (3.8) as m tends to ∞, it follows that

$$\left\langle Z_{I,s}^{j,t},1\right\rangle_{\Gamma_{jf}} = 0. \tag{3.14}$$

Next note that from (3.10a)

$$\left\langle Z_{R,t}^{j,s,(m)},1\right\rangle_{\Gamma_{jf}} - \delta_{st}\left|\Gamma_{jf}\right| = \frac{1}{m}\omega\rho_f\left(Z_{I,t}^{j,s,(m)},1\right)_{Y_f}.$$

In the above equation, the first term in left hand side converges due to the weak convergence of $\mathbf{Z}^{j,s,(m)}$ to $\mathbf{Z}^{j,s}$ in $[L^2(\Gamma_{sf}]^3$ and the right hand side tends to zero as m goes to infinity thanks to (3.13). Thus,

$$\left\langle Z_{R,t}^{j,s},1\right\rangle_{\Gamma_{jf}} = \delta_{st}\left|\Gamma_{jf}\right|. \tag{3.15}$$

3.3 A formal relation of Darcy's law (2.33) with a previous phenomenological derivation

Consider Equation (13) in [11] for the fluid part in the steady-state case (*i.e.* the velocities are time-independent). In terms of our notation it can be stated as follows:

$$-\phi_w \nabla p_f^{(0)}(x,t) = b_{12}\left[\mathbf{v}_f^{(0)}(x,t) - \mathbf{v}_1^{(0)}(x,t)\right] + b_{23}\left[\mathbf{v}_f^{(0)}(x,t) - \mathbf{v}_3^{(0)}(x,t)\right], \tag{3.16}$$

where ϕ_w, b_{12}, and b_{23} are positive coefficients independent of time as defined in [11]. For the sake of convenience, the above equation is written in the form:

$$\mathbf{v}_f^{(0)}(x,t) = -\frac{\phi_w}{b_{12}+b_{23}}\nabla p_f^{(0)}(x,t) + \frac{b_{12}}{b_{12}+b_{23}}\mathbf{v}_1^{(0)}(x,t) + \frac{b_{23}}{b_{12}+b_{23}}\mathbf{v}_3^{(0)}(x,t). \tag{3.17}$$

Next, note that the inverse Fourier transform of the solutions $\mathbf{V}^s$ and $\mathbf{Z}^s$ of the local Stokes problems (2.21) and (2.27) vanish for negative times (*i.e* they are causal functions of time t). Consequently, the inverse Fourier transforms of the permeability tensors $\langle\langle \mathbf{K}\rangle\rangle, \langle\langle \mathbf{K}^j\rangle\rangle, j = 1,3$ are causal, and Darcy's Law (2.33) can be restated in the space-time domain using convolutions as follows. Let

$$\langle\langle \mathbf{S}\rangle\rangle(x,\omega) = \frac{\langle\langle \mathbf{K}\rangle\rangle(x,\omega)}{i\omega}, \quad \langle\langle \mathbf{S}^j\rangle\rangle(x,\omega) = \frac{\langle\langle \mathbf{K}^j\rangle\rangle(x,\omega)}{i\omega}, j = 1,3, \tag{3.18}$$

and for any function $f(\omega)$ let $\widehat{f}(t)$ denote its inverse Fourier transform. Then, (2.33) in the space-time domain becomes

$$\begin{aligned}
\left\langle\left\langle \mathbf{v}_f^{(0)}\right\rangle\right\rangle(x,t) &= -\int_0^t \widehat{\langle\langle \mathbf{S}\rangle\rangle}(x,t-\tau)\frac{\partial}{\partial\tau}\nabla_x \widehat{p_f^{(0)}}(x,\tau)d\tau \\
&\quad - \sum_{j=1,3}\int_0^t \widehat{\langle\langle \mathbf{S}^j\rangle\rangle}(x,t-\tau)\frac{\partial}{\partial\tau}\widehat{\mathbf{v}_j^{(0)}}(x,\tau)d\tau \\
&= -\widehat{\langle\langle \mathbf{S}\rangle\rangle}(x,0+)\nabla_x\widehat{p_f^{(0)}}(x,t) + \int_0^t \widehat{\langle\langle \mathbf{K}\rangle\rangle}(x,t-\tau)\nabla_x\widehat{p_f^{(0)}}(x,\tau)d\tau \\
&\quad - \sum_{j=1,3}\left[\widehat{\langle\langle \mathbf{S}^j\rangle\rangle}(x,0+)\widehat{\mathbf{v}_j^{(0)}}(x,t) + \int_0^t \widehat{\langle\langle \mathbf{K}^j\rangle\rangle}(x,t-\tau)\widehat{\mathbf{v}_j^{(0)}}(x,\tau)d\tau\right].
\end{aligned} \tag{3.19}$$

Thus in the isotropic case we can identify the coefficients in (3.17) with the $\widehat{\langle\langle \mathbf{S}\rangle\rangle}(x,0+)$ and $\widehat{\langle\langle \mathbf{S}^j\rangle\rangle}(x,0+)$ terms in (3.19).

4 Proof of Theorem 2.1

In this section we give a proof of Theorem 2.1. To avoid cumbersome notations we restate Problem (2.19) in the following form: find Y-periodic functions $\mathbf{v}$ and p such that

$$i\omega\rho_f\mathbf{v} - \eta\Delta_y\mathbf{v} + \nabla_y p = 0, \quad Y_f, \tag{4.1a}$$

$$\nabla_y\cdot\mathbf{v} = 0, \quad Y_f, \tag{4.1b}$$

$$\mathbf{v} = \mathbf{g}, \quad \Gamma_{sf}, \tag{4.1c}$$

where $\mathbf{g} \in \left[L^2(\Gamma_{sf})\right]^3$ is Y-periodic and satisfies the consistency condition

$$\int_{\Gamma_{sf}} \mathbf{g} \cdot \boldsymbol{\nu}_f dy = 0. \tag{4.2}$$

For the proof of Theorem 2.1, we prove first three auxiliary lemmas, and then employ a compactness argument to show the existence and uniqueness of Problem (4.1).

As in [13], for $m \in \mathbf{Z}^+$, consider the sequence of penalized problems: find Y-periodic functions $\mathbf{v}^m$ and p^m such that

$$\begin{aligned} i\omega\rho_f \mathbf{v}^m - \eta\Delta_y \mathbf{v}^m + \nabla_y p^m &= 0, \quad Y_f, &(4.3a)\\ \nabla_y \cdot \mathbf{v}^m &= 0, \quad Y_f, &(4.3b)\\ \mathbf{v}^m + \frac{1}{m}\left(\eta \frac{\partial \mathbf{v}^m}{\partial \boldsymbol{\nu}_f} - p^m \boldsymbol{\nu}_f\right) &= \mathbf{g}, \quad \Gamma_{sf}. &(4.3c)\end{aligned}$$

Define the sesquilinear form $A_{\omega,m} : H^1(\operatorname{div} 0; Y_f) \times H^1(\operatorname{div} 0; Y_f) \to \mathbb{C}$ by the rule

$$A_{\omega,m}(\mathbf{v}, \boldsymbol{\varphi}) = i\omega\left(\rho_f \mathbf{v}, \boldsymbol{\varphi}\right)_{Y_f} + \left(\eta\nabla_y \mathbf{v}, \nabla_y \boldsymbol{\varphi}\right)_{Y_f} + m\left\langle \mathbf{v}, \boldsymbol{\varphi}\right\rangle_{\Gamma_{sf}}, \ \mathbf{v}, \boldsymbol{\varphi} \in H^1(\operatorname{div} 0; Y_f). \tag{4.4}$$

Then, testing (4.3a) against $\boldsymbol{\varphi} \in H^1(\operatorname{div} 0; Y_f)$, we obtain the following variational formulation of Problem (4.3): find $\mathbf{v}^m \in H^1(\operatorname{div} 0; Y_f)$ such that

$$A_{\omega,m}(\mathbf{v}^m, \boldsymbol{\varphi}) = m\left\langle \mathbf{g}, \boldsymbol{\varphi}\right\rangle_{\Gamma_{sf}}, \quad \boldsymbol{\varphi} \in H^1(\operatorname{div} 0; Y_f). \tag{4.5}$$

Lemma 4.1. *For each positive integer m there exists a unique solution $\mathbf{v}^m \in H^1(\operatorname{div} 0; Y_f)$ of (4.5) such that*

$$\|\mathbf{v}^m\|_{L^2(\Gamma_{sf})} \leq \|\mathbf{g}\|_{L^2(\Gamma_{sf})}. \tag{4.6}$$

Proof. First note that

$$\begin{aligned} |A_{\omega,m}(\mathbf{v}^m, \mathbf{v}^m)| &\geq \frac{1}{2}\left(\left(\omega\rho_f \mathbf{v}^m, \mathbf{v}^m\right)_{Y_f} + \left(\eta\nabla_y \mathbf{v}^m, \nabla_y \mathbf{v}^m\right)_{Y_f} + m\left\langle \mathbf{v}^m, \mathbf{v}^m\right\rangle_{\Gamma_{sf}}\right)\\ &\geq C(\omega)\|\mathbf{v}^m\|^2_{1,Y_f} + \frac{m}{2}\|\mathbf{v}^m\|^2_{L^2(\Gamma_{sf})}, \qquad \mathbf{v} \in H^1(\operatorname{div} 0; Y_f),\end{aligned}$$

where $C(\omega) = \frac{1}{2}\min(\rho_f\omega, \eta)$. Thus the Lax-Milgram lemma implies the existence and uniqueness of the solution of (4.5). Next, taking the real part in the equation

$$A_{\omega,m}(\mathbf{v}^m, \mathbf{v}^m) = m\left\langle \mathbf{g}, \mathbf{v}^m\right\rangle_{\Gamma_{sf}}$$

leads to

$$\begin{aligned} \eta\|\nabla_y \mathbf{v}^m\|^2_{1,Y_f} + m\|\mathbf{v}^m\|^2_{L^2(\Gamma_{sf})} &\leq |A_{\omega,m}(\mathbf{v}^m, \mathbf{v}^m)| = m\left|\left\langle \mathbf{g}, \mathbf{v}^m\right\rangle_{\Gamma_{sf}}\right|\\ &\leq m\|\mathbf{g}\|_{L^2(\Gamma_{sf})}\|\mathbf{v}^m\|_{L^2(\Gamma_{sf})}.\end{aligned}$$

Thus, (4.6) follows.This completes the proof. □

Next we proceed to get an estimate for $\|\mathbf{v}^m\|_{L^2(Y_f)}$. First we prove an auxiliary result which is an extension of a regularity estimate given in Galdi [10].

Lemma 4.2. *Let* $\mathbf{f} \in \left[L^2(Y_f)\right]^3$ *be* Y*-periodic. Then there exist a* Y*-periodic unique solution* $(\mathbf{u}, \pi) \in H^2(Y_f) \times H^1(Y_f)$ *such that*

$$\begin{aligned} i\omega\rho_f \mathbf{u} - \eta\Delta_y \mathbf{u} + \nabla_y \pi &= \mathbf{f}, \quad Y_f, && (4.7a)\\ \nabla_y \cdot \mathbf{u} &= 0, \quad Y_f, && (4.7b)\\ \mathbf{u} &= 0, \quad \Gamma_{sf}, && (4.7c) \end{aligned}$$

satisfying

$$\|\mathbf{u}\|_{H^2(Y_f)} + \|\pi\|_{H^1(Y_f)} \leq C_2 \|\mathbf{f}\|_{L^2(Y_f)}, \tag{4.8}$$

where C_2 *is independent of* ω.

Proof. Let $\mathbf{F} \in \left[L^2(Y_f)\right]^3$ be Y-periodic and $(\mathbf{U}, P)$ be the Y-periodic solution of

$$\begin{aligned} -\eta\Delta_y \mathbf{U} + \nabla_y P &= \mathbf{F}, \quad Y_f, && (4.9a)\\ \nabla_y \cdot \mathbf{U} &= 0, \quad Y_f, && (4.9b)\\ \mathbf{U} &= 0, \quad \Gamma_{sf}. && (4.9c) \end{aligned}$$

According to [10] (Theorem 6.1, pp.225), the following regularity estimate holds:

$$\|\mathbf{U}\|_{H^2(Y_f)} + \|P\|_{H^1(Y_f)} \leq \widetilde{C}_1 \|\mathbf{F}\|_{L^2(Y_f)}. \tag{4.10}$$

Consequently, applying (4.10) in (4.7) we get

$$\|\mathbf{u}\|_{H^2(Y_f)} + \|\pi\|_{H^1(Y_f)} \leq \widetilde{C}_1 \left(\|\mathbf{f}\|_{L^2(Y_f)} + \omega \|\mathbf{u}\|_{L^2(Y_f)} \right). \tag{4.11}$$

By testing (4.7a) against $\mathbf{u}$ and taking the imaginary part in the resulting equation, it follows that

$$\omega \|\mathbf{u}\|_{L^2(Y_f)} \leq \frac{1}{\rho_f} \|\mathbf{f}\|_{L^2(Y_f)}, \tag{4.12}$$

which combined with (4.11) proves the validity of (4.8). □

Using the result in Lemma 4.2, we now obtain an estimate for $\|\mathbf{v}^m\|_{L^2(Y_f)}$.

Lemma 4.3. *The solution* $\mathbf{v}^m$ *of* (4.5) *satisfies the estimate*

$$\|\mathbf{v}^m\|_{L^2(Y_f)} \leq C_3 \|\mathbf{g}\|_{L^2(\Gamma_{sf})}, \tag{4.13}$$

where $C_3 > 0$ *is a constant independent of* m *and* ω.

Proof. Consider the auxiliary Y-periodic problem to find $\mathbf{u}$ and π satisfying

$$\begin{aligned} -i\omega\rho_f \mathbf{u} - \eta\Delta_y \mathbf{u} + \nabla_y \pi &= \mathbf{v}^m, \quad Y_f, && (4.14a)\\ \nabla_y \cdot \mathbf{u} &= 0, \quad Y_f, && (4.14b)\\ \mathbf{u} &= 0, \quad \Gamma_{sf}. && (4.14c) \end{aligned}$$

Take $\boldsymbol{\varphi} = \mathbf{u}$ in (4.5), use integration by parts in the η-term and apply (4.14) in the resulting equation. After integrating by parts the term $(\mathbf{v}^m, \nabla\pi)$, owing to the fact that $\mathbf{v}^m$ is divergence free, we obtain

$$(\mathbf{v}^m, \mathbf{v}^m)_{Y_f} + \left\langle \mathbf{v}^m, \left(\eta \frac{\partial \mathbf{u}}{\partial \boldsymbol{\nu}} - \boldsymbol{\nu}\pi\right)\right\rangle_{\Gamma_{sf}} = 0. \tag{4.15}$$

Next, recall the continuity of the trace operators

$$\begin{array}{ccc} H^2(Y_f) & \longrightarrow & L^2(\Gamma_{sf}) \\ \mathbf{u} & \longmapsto & \dfrac{\partial \mathbf{u}}{\partial \boldsymbol{\nu}} \end{array} \quad \text{and} \quad \begin{array}{ccc} H^1(Y_f) & \longrightarrow & L^2(\Gamma_{sf}) \\ \mathbf{u} & \longmapsto & \mathbf{u}_{|\Gamma_{sf}} \end{array}. \tag{4.16}$$

Then from (4.6) and (4.8) it follows that

$$\begin{aligned} \left|\left\langle \mathbf{v}^m, \left(\eta \frac{\partial \mathbf{u}}{\partial \boldsymbol{\nu}} - \boldsymbol{\nu}\pi\right)\right\rangle_{\Gamma_{sf}}\right| &\leq \|\mathbf{v}^m\|_{L^2(\Gamma_{sf})} \left\|\left(\eta \frac{\partial \mathbf{u}}{\partial \boldsymbol{\nu}} - \boldsymbol{\nu}\pi\right)\right\|_{L^2(\Gamma_{sf})} \\ &\leq C\|\mathbf{g}\|_{L^2(\Gamma_{sf})} \left(\|\mathbf{u}\|_{H^2(Y_f)} + \|\pi\|_{H^1(Y_f)}\right) \\ &\leq C_3\|\mathbf{g}\|_{L^2(\Gamma_{sf})}\|\mathbf{v}^m\|_{L^2(Y_f)}. \end{aligned} \tag{4.17}$$

Next, using (4.17) in (4.15) we get

$$\|\mathbf{v}^m\|^2_{L^2(Y_f)} \leq C_3\|\mathbf{g}\|_{L^2(\Gamma_{sf})}\|\mathbf{v}^m\|_{L^2(Y_f)},$$

which shows the validity of (4.13). This completes the proof. □

Now we proceed to derive the desired existence and uniqueness result on the solution of Problem (4.1). Note that the bounds in Lemmas 4.1 and 4.3 imply that there exists a subsequence of $\mathbf{v}^m$, that we denote again $\mathbf{v}^m$, such that

$$\mathbf{v}^m \rightharpoonup \mathbf{v}^0 \quad \text{weakly in} \quad \left[L^2(Y_f)\right]^3, \tag{4.18a}$$

$$\mathbf{v}^m \rightharpoonup \mathbf{z}^0 \quad \text{weakly in} \quad \left[L^2(\Gamma_{sf})\right]^3. \tag{4.18b}$$

We wish to show that $\mathbf{z}^0 = \mathbf{g}$ on Γ_{sf}. First, since $\nabla\cdot\mathbf{v}^m = 0$ in Y_f, we notice that $\displaystyle\int_{\Gamma_{sf}} \mathbf{z}^0\cdot\boldsymbol{\nu} dS = 0$. Then, take a Y-periodic function $\boldsymbol{\varphi} \in \left[C^2(\overline{Y}_f)\right]^3$ with $\nabla\cdot\boldsymbol{\varphi} = 0$ as a test function in (4.5) and use (4.6) and (4.13) to obtain

$$\begin{aligned} \left|m\left\langle \mathbf{v}^m - \mathbf{g}, \boldsymbol{\varphi}\right\rangle_{\Gamma_{sf}}\right| &= \left|i\omega\left(\rho_f\mathbf{v}^m, \boldsymbol{\varphi}\right) - (\eta\mathbf{v}^m, \Delta\boldsymbol{\varphi}) + \left\langle \eta\mathbf{v}^m, \frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\nu}}\right\rangle_{\Gamma_{sf}}\right| \\ &\leq C\left(\omega\|\mathbf{v}^m\|_{L^2(Y_f)}\|\boldsymbol{\varphi}\|_{L^2(Y_f)} + \|\mathbf{v}^m\|_{L^2(Y_f)}\|\Delta\boldsymbol{\varphi}\|_{L^2(Y_f)}\right. \\ &\qquad \left. + \|\mathbf{v}^m\|_{L^2(\Gamma_{sf})}\left\|\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\nu}}\right\|_{L^2(\Gamma_{sf})}\right) \\ &\leq C(\boldsymbol{\varphi})\max\{1,\omega\}\|\mathbf{g}\|_{L^2(\Gamma_{sf})} \end{aligned} \tag{4.19}$$

Taking limit as $m \to \infty$ in (4.19), we get that

$$\langle \mathbf{z}^0 - \mathbf{g}, \boldsymbol{\varphi} \rangle_{\Gamma_{sf}} = 0,$$

for all of such test functions $\boldsymbol{\varphi}$. Since the traces of functions $\boldsymbol{\varphi} \in \left[C^2(\overline{Y}_f)\right]^3$ are dense in $L^2(\Gamma_{sf})$, we conclude that

$$\mathbf{z}^0 = \mathbf{g}, \quad \text{a.e. on } \Gamma_{sf}.$$

Again, take $\boldsymbol{\varphi} \in \mathcal{V}^2_{Y_f}$ in (4.5) and use integration by parts in the η-term to obtain

$$i\omega \left(\rho_f \mathbf{v}^m, \boldsymbol{\varphi}\right)_{Y_f} - \left(\eta \mathbf{v}^m, \Delta_y \boldsymbol{\varphi}\right)_{Y_f} + \left\langle \eta \mathbf{v}^m, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{\nu}} \right\rangle_{\Gamma_{sf}} = 0. \tag{4.20}$$

Next, using (4.18), take limit when $m \to \infty$ in (4.20) to obtain

$$i\omega \left(\rho_f \mathbf{v}^0, \boldsymbol{\varphi}\right)_{Y_f} - \left(\eta \mathbf{v}^0, \Delta_y \boldsymbol{\varphi}\right)_{Y_f} = -\left\langle \eta \mathbf{g}, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{\nu}} \right\rangle_{\Gamma_{sf}}, \quad \boldsymbol{\varphi} \in \mathcal{V}^2_{Y_f}.$$

Also, note that for $\boldsymbol{\psi} \in \mathcal{W}$, since $\nabla \cdot \mathbf{v}^m = 0$,

$$\left(\mathbf{v}^m, \nabla \boldsymbol{\psi}\right)_{Y_f} = \langle \mathbf{v}^m \cdot \boldsymbol{\nu}, \boldsymbol{\psi} \rangle_{\Gamma_{sf}}. \tag{4.21}$$

Due to $\nabla \boldsymbol{\psi} \in \left[L^2(Y_f)\right]^3$ and (4.18a), by taking limit in (4.21) as $m \to \infty$, we get

$$\left(\mathbf{v}^0, \nabla \boldsymbol{\psi}\right)_{Y_f} = \langle \mathbf{g} \cdot \boldsymbol{\nu}, \boldsymbol{\psi} \rangle_{\Gamma_{sf}}, \quad \boldsymbol{\psi} \in \mathcal{W}.$$

Thus $\mathbf{v}^0$ is a *very-weak* solution of (4.1) in the sense defined in (2.27). This completes the proof of Theorem 2.1.

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