Finite Element Methods for the Simulation of Waves in Composite Saturated Poroviscoelastic Media

Juan E. Santos* Dongwoo Sheen†

Abstract

This work presents and analyzes a collection of finite element procedures for the simulation of wave propagation in a porous medium composed of two weakly coupled solids saturated by a single-phase fluid. The equations of motion, formulated in the space-frequency domain, include dissipation due to viscous interaction between the fluid and solid phases and intrinsic anelasticity of the solids modeled using linear viscoelasticity. This formulation leads to the solution of a Helmholtz-type boundary value problem for each temporal frequency. For the spatial discretization, nonconforming finite element spaces are employed for the solid phases, while for the fluid phase the vector part of the Raviart-Thomas-Nedelec mixed finite element space is used. Optimal a priori error estimates for *global* standard and *hybridized* Galerkin finite element procedures are derived. An iterative nonoverlapping domain decomposition procedure is also presented and convergence results are derived. Numerical experiments showing the application of the numerical procedures to simulate wave propagation in partially frozen porous media are presented.

Keywords: poroviscoelasticity, finite element method, error estimate, domain decomposition

1 Introduction

Wave propagation in composite porous materials has applications in many branches of science and technology, such as seismic methods in the presence of shaley sandstones [8], frozen or partially frozen sandstones [31, 10, 11], gas-hydrates in ocean-bottom sediments [12] and evaluation of the freezing conditions of foods by ultrasonic techniques [27].

A theory to describe wave propagation in frozen porous media was first presented by Leclaire *et al.* [25]. This model, valid for uniform porosity, predicts the existence of three compressional and two shear waves; the verification that additional (slow) waves can be observed in laboratory experiments was published by Leclaire *et al.* [26]. Later, Carcione and Tinivella [12] generalized this theory to include the interaction between the solid and ice particles and grain cementation with decreasing temperature. Also, Carcione *et al.* [8] applied this theory to study the acoustic properties of shaley sandstones, assuming that sand and clay are *non-welded* and form a continuous and inter-penetrating porous composite skeleton. Both frozen porous media and

*CONICET, Departamento de Geofísica Aplicada, Fac. Ciencias Astronómicas y Geofísicas, UNLP, Paseo del Bosque S/N, La Plata, 1900, Argentina, and Purdue University, West Lafayette, IN 47907, USA; E-mail: santos@math.purdue.edu

†Department of Mathematics, Seoul National University, Seoul 151–747, Korea; E-mail: sheen@snu.ac.kr; Supported in part by the Korea Research Foundation Grant (KRF-2004-C00007) and Korea Science and Engineering Foundation Grant (KOSEF R14-2003-019-01000-0).

shaley sandstones are two examples of porous materials where the two solid phases are *weakly-coupled* or *non-welded*, i.e, both solids form a continuous and interacting composite structure, interchanging mechanical energy. Similar *weakly-coupled* formulations have previously been proposed. For instance, McCoy [30] has proposed a mixture theory appropriate for the combination of two *acoustic phases*.

This article presents a differential and numerical model to describe wave propagation in a heterogeneous poroviscolastic frame consisting of two weakly-coupled solid phases saturated by a single phase fluid. The equations of motion, stated in the space-frequency domain, generalizes that presented in [39] and [9] by the inclusion of solid matrix dissipation using a linear viscoelastic model and frequency dependent mass and viscous coupling coefficients. It also generalizes the models of Leclaire *et al.* [25] and Carcione *et al.* [12] for the case of uniform porosity, and consequently is the appropriate model to perform numerical simulation in heterogeneous materials.

The numerical procedures presented employ the nonconforming rectangular element defined in [17] to approximate the displacement vector in the solid phases. The dispersion analysis presented in [40] shows that employing this nonconforming element allows for a reduction in the number of points per wavelength necessary to reach a desired accuracy. On the other hand, the displacement in the fluid phase is approximated by using the vector part of the Raviart-Thomas-Nedelec mixed finite element space of zero order, which is a conforming space [36, 32].

The error analysis yields optimal *a priori* error estimates for the *global* standard and *hybridized* Galerkin methods.

Numerical simulation of waves in porous media is computationally expensive due to a large number of degrees of freedom needed to calculate wave fields accurately; the use of a domain decomposition iteration is a convenient approach to overcome this difficulty. Here we define a nonoverlapping domain decomposition iterative scheme and derive convergence results similar to those presented in [14] for solving second-order elliptic problems.

This iterative procedure was used for the simulation of waves in a sample of water saturated partially frozen Berea sandstone [9, 12], perturbed by a point source at seismic frequencies. The sample has an interior plane interface defined by a change in ice content in the pores, and the snapshots of the generated wave fields show clearly the events associated with the different types of waves.

2 The differential model

In this section we review and generalize a model recently presented by one of the authors and some of his colleagues [39] to describe the propagation of waves in a poroviscoelastic domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, in which the matrix consists of two different solids indicated by the super-indices (1) and (3), saturated by a single phase fluid indicated by the super-index (2). Thus, for any reference element E of bulk material we have

$$E = E^{(1)} \cup E^{(2)} \cup E^{(3)}.$$

Let $V^{(i)}$ denote the volumetric measure of the phase $E^{(i)}$ and by $V^{(b)}$ and $V^{(sm)}$ the volumetric measures of E and the solid matrix $E^{(sm)} = E^{(1)} \cup E^{(3)}$, respectively, so that

$$V^{(sm)} = V^{(1)} + V^{(3)}, \qquad V^{(b)} = V^{(1)} + V^{(2)} + V^{(3)}.$$

We introduce the bulk volumetric fractions of the different components in the form:

$$\phi = \frac{V^{(2)}}{V^{(b)}}, \quad \phi^{(1)} = \frac{V^{(1)}}{V^{(b)}}, \quad \phi^{(3)} = \frac{V^{(3)}}{V^{(b)}},$$

and the solid fractions of the composite matrix

$$S^{(1)} = \frac{V^{(1)}}{V^{(sm)}}, \quad S^{(3)} = \frac{V^{(3)}}{V^{(sm)}}, \quad \text{with} \quad S^{(1)} + S^{(3)} = 1.$$

For some practical applications it is convenient to define the *absolute* or *effective porosity* $\phi^{(a)}$ of the medium, defined as the ratio of the volume of the interconnected pores $V^{(p)}$ and the total volume of the sample, *i.e.*,

$$\phi^{(a)} = \frac{V^{(p)}}{V^{(b)}}.$$

These set of fractions can have different meanings depending on the physical model considered. For example, in the case of a sandstone or soil at very low temperature, it is reasonable to consider that a part of the fluid which saturates the pore space is at a liquid state and the rest is frozen. If $E^{(1)}$ represents the mineral grains and $E^{(3)}$ the ice, for a given porosity $\phi^{(a)}$ and *bulk water content* ϕ, the following relations hold:

$$\phi^{(1)} = 1 - \phi^{(a)}, \quad \phi^{(3)} = \phi^{(a)} - \phi, \quad S^{(3)} = \frac{\phi^{(3)}}{1 - \phi}. \tag{2.1}$$

It is useful to introduce an additional fraction $S^{(3)'}$ to account for the *ice content in the pores*, given by

$$S^{(3)'} = \frac{V^{(3)}}{V^{(p)}} = \frac{\phi^{(3)}}{1 - \phi^{(1)}}.$$

A different application of this model would be the case of a shaley sandstone, that is, a porous rock mainly composed of quartz grains and clay particles, saturated by a fluid (such as water, brine, gas or oil). In this case we assume that the fluid completely saturates the pore space of the composite rock so that $V^{(2)} \equiv V^{(p)}$. Then, if $E^{(1)}$ represents the grains of the rock and $E^{(3)}$ the clay part, for a given *matrix clay content* $S^{(3)}$ and water content ϕ, instead of (2.1) the following hold

$$\phi = \phi^{(a)}, \quad \phi^{(1)} = S^{(1)}(1 - \phi), \quad \phi^{(3)} = S^{(3)}(1 - \phi).$$

Let us now consider a unit cube $\Omega = \Omega^{(1)} \cup \Omega^{(2)} \cup \Omega^{(3)} \subset \mathbb{R}^d$ of our fluid-saturated poroviscoelastic material with boundary $\Gamma = \partial\Omega$. Since by hypothesis the two solids are non-welded (or weakly coupled), we assume that they can move independently and consequently we can distinguish three different particle displacement fields for this model. Let $u^{(m)} \equiv u^{(m)}(x,\omega) = \left(u_1^{(m)}(x,\omega), \cdots, u_d^{(m)}(x,\omega)\right)^t$, $m = 1, 3$, be the averaged solid displacements over the bulk material Ω at the angular frequency ω and let $\widetilde{u}^{(2)} \equiv \widetilde{u}^{(2)}(x,\omega) = \left(\widetilde{u}_1^{(2)}(x,\omega), \cdots, \widetilde{u}_d^{(2)}(x,\omega)\right)^t$ denote the absolute fluid displacement. Also, let the relative flow of the fluid phase with respect to the composite solid matrix be defined by

$$u^{(2)} = \phi(\widetilde{u}^{(2)} - S^{(1)}u^{(1)} - S^{(3)}u^{(3)})$$

and set $u = \left(u^{(1)}, u^{(2)}, u^{(3)}\right)^t$. As explained in [39], the variable

$$\zeta = -\nabla \cdot u^{(2)}$$

represents the change in fluid content. Next we introduce the local stress tensors $\sigma_{jk}^{(1,s)}$ and $\sigma_{jk}^{(3,s)}$ in the solid parts $\Omega^{(1)}$ and $\Omega^{(3)}$, averaged over the bulk material and the fluid pressure p_f. Following [39], we define the second order tensors

$$\sigma_{jk}^{(1)} = \sigma_{jk}^{(1,s)} - S^{(1)}\phi p_f \delta_{jk}, \qquad \sigma_{jk}^{(3)} = \sigma_{jk}^{(3,s)} - S^{(3)}\phi p_f \delta_{jk},$$

associated with the total stresses in $\Omega^{(1)}$ and $\Omega^{(3)}$, respectively. Then the constitutive equations, stated in the space-frequency domain are follows [39]:

$$\sigma_{jk}^{(1)}(u) = \left[K_G^{(1)} e^{(1)} - B^{(1)}\zeta + B^{(3)} e^{(3)}\right]\delta_{jk} + 2\mu^{(1)} d_{jk}^{(1)} + \mu^{(13)} d_{jk}^{(3)}, \tag{2.2a}$$

$$\sigma_{jk}^{(3)}(u) = \left[K_G^{(3)} e^{(3)} - B^{(2)}\zeta + B^{(3)} e^{(1)}\right]\delta_{jk} + 2\mu^{(3)} d_{jk}^{(3)} + \mu^{(13)} d_{jk}^{(1)}, \tag{2.2b}$$

$$p_f(u) = -B^{(1)} e^{(1)} - B^{(2)} e^{(3)} + K_{av}\zeta, \tag{2.2c}$$

where the elastic coefficients $K_G^{(m)}, B^{(m)}, K_{av}, \mu^{(m)}, \mu^{(13)}$ are given in Appendix A.1, and

$$d_{jk}^{(m)} = \epsilon_{jk}(u^{(m)}) - \frac{1}{d}\, e^{(m)}\delta_{jk}, \;\; m = 1, 3, \text{ in } \mathbb{R}^d,$$

denotes the deviatoric tensor in $\Omega^{(m)}$, with $\epsilon_{jk}(u^{(m)})$ being the strain tensor with linear invariant $e^{(m)}$. In [39] the constitutive relations (2.2) were stated in the space-time domain with real coefficients that were determined in terms of the properties of the individual solid and fluid phases. In this work the coefficients in (2.2) will be assumed to be complex and frequency dependent in order to include viscoelasticity in the solid composite matrix, using a linear viscoelastic model as explained in Appendix A.

Next, by writing

$$\lambda^{(m)} = K_G^{(m)} - \frac{2}{d}\mu^{(m)}, \quad D^{(3)} = B^{(3)} - \frac{1}{d}\mu^{(13)} \text{ in } \mathbb{R}^d,$$

the constitutive relations (2.2) are then stated in the following equivalent form, which will be used in the analysis that follows:

$$\sigma_{jk}^{(1)}(u) = \left[\lambda^{(1)} e^{(1)} - B^{(1)}\zeta + D^{(3)} e^{(3)}\right]\delta_{jk} + 2\mu^{(1)}\epsilon_{jk}(u^{(1)}) + \mu^{(13)}\epsilon_{jk}(u^{(3)}), \tag{2.3a}$$

$$\sigma_{jk}^{(3)}(u) = \left[\lambda^{(3)} e^{(3)} - B^{(2)}\zeta + D^{(3)} e^{(1)}\right]\delta_{jk} + 2\mu^{(3)}\epsilon_{jk}(u^{(3)}) + \mu^{(13)}\epsilon_{jk}(u^{(1)}), \tag{2.3b}$$

$$p_f(u) = -B^{(1)} e^{(1)} - B^{(2)} e^{(3)} + K_{av}\zeta. \tag{2.3c}$$

Let the positive definite mass matrix $\mathcal{P} = \mathcal{P}(\omega)$ and the nonnegative dissipation matrix $\mathcal{B} = \mathcal{B}(\omega)$ be defined by

$$\mathcal{P} = \begin{bmatrix} p_{11}I & p_{12}I & p_{13}I \\ p_{12}I & p_{22}I & p_{23}I \\ p_{13}I & p_{23}I & p_{33}I \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} b_{11}I & -b_{12}I & -b_{11}I \\ -b_{12}I & b_{22}I & b_{12}I \\ -b_{11}I & b_{12}I & b_{11}I \end{bmatrix},$$

where I denotes the identity matrix in $\mathbb{R}^{d\times d}$. The nonnegative coefficients $p_{jk} = p_{jk}(\omega), b_{jk} = b_{jk}(\omega)$ in the definition of the matrices $\mathcal{P}$ and $\mathcal{B}$ can be computed as explained in Appendix A. The coefficients $b_{11},\ b_{12}$ and b_{22} satisfy the condition

$$b_{11}b_{22} - b_{12}^2 > 0, \quad \omega > 0, \tag{2.4}$$

which is needed in order that the dissipation function be positive in the variables $u^{(2)}$ and $u^{(1)} - u^{(3)}$. Next, let $\mathcal{L}(u)$ be the second order differential operator defined by

$$\mathcal{L}(u) = \left(\nabla\cdot\sigma^{(1)}(u), -\nabla p_f(u), \nabla\cdot\sigma^{(3)}(u)\right)^t.$$

Then the equations of motion in Ω, stated in the space-frequency domain, are given as follows [39]:

$$-\omega^2\mathcal{P}u(x,\omega) + i\omega\mathcal{B}u(x,\omega) - \mathcal{L}(u(x,\omega)) = F(x,\omega), \quad (x,\omega)\in\Omega\times(0,\omega^*), \tag{2.5}$$

where $F(x,\omega) = \left(F^{(1)}(x,\omega), F^{(2)}(x,\omega), F^{(3)}(x,\omega)\right)^t$ denotes the external source and ω^* is an upper temporal frequency of interest.

A plane wave analysis shows that three different compressional waves (P1, P2 and P3) and two shear waves (S1, S2) can propagate [25, 39]. The P1 and S1 waves correspond to the classical fast P and S waves propagating in elastic or viscoelastic isotropic solids. The additional slow waves are related to motions out of phase of the different phases. The experimental observation of the additional (slow) waves was reported by Leclaire *et al.* [26].

Let us denote by ν the unit outer normal on Γ. In the 2D case let χ be a unit tangent on Γ so that $\{\nu,\chi\}$ is an orthonormal system on Γ. In the 3D case let χ^1 and χ^2 be two unit tangents on Γ so that $\{\nu,\chi^1,\chi^2\}$ is an orthonormal system on Γ.

Then, in the 2D case set

$$\mathcal{G}_\Gamma(u) = \left(\sigma^{(1)}(u)\nu\cdot\nu, \sigma^{(1)}(u)\nu\cdot\chi, p_f(u), \sigma^{(3)}(u)\nu\cdot\nu, \sigma^{(3)}(u)\nu\cdot\chi\right)^t, \tag{2.6a}$$

$$S_\Gamma(u) = \left(u^{(1)}\cdot\nu, u^{(1)}\cdot\chi, u^{(2)}\cdot\nu, u^{(3)}\cdot\nu, u^{(3)}\cdot\chi\right)^t, \tag{2.6b}$$

and in the 3D case set

$$\mathcal{G}_\Gamma(u) = \Big(\sigma^{(1)}(u)\nu\cdot\nu, \sigma^{(1)}(u)\nu\cdot\chi^1, \sigma^{(1)}(u)\nu\cdot\chi^2, p_f(u), \tag{2.7a}$$

$$\sigma^{(3)}(u)\nu\cdot\nu, \sigma^{(3)}(u)\nu\cdot\chi^1, \sigma^{(3)}(u)\nu\cdot\chi^2\Big)^t,$$

$$S_\Gamma(u) = \left(u^{(1)}\cdot\nu, u^{(1)}\cdot\chi^1, u^{(1)}\cdot\chi^2, u^{(2)}\cdot\nu, u^{(3)}\cdot\nu, u^{(3)}\cdot\chi^1, u^{(3)}\cdot\chi^2\right)^t. \tag{2.7b}$$

Let us consider the solution of (2.3) with the following absorbing boundary condition, which is derived in Appendix B:

$$-\mathcal{G}_\Gamma(u(x,\omega)) = i\omega\mathcal{D}S_\Gamma(u(x,\omega)), \qquad (x,\omega)\in\Gamma\times(0,\omega^*). \tag{2.8}$$

The matrix $\mathcal{D}$ in (2.8) is positive definite.

3 A weak formulation

For $X \subset \mathbb{R}^d$ with boundary ∂X, let $(\cdot,\cdot)_X$ and $\langle \cdot,\cdot \rangle_{\partial X}$ denote the complex $L^2(X)$ and $L^2(\partial X)$ inner products for scalar, vector, or matrix valued functions. Also, for $s \in \mathbb{R}$, $\|\cdot\|_{s,X}$ and $|\cdot|_{s,X}$ will denote the usual norm and seminorm for the Sobolev space $H^s(X)$. In addition, if $X = \Omega$ or $X = \Gamma$, the subscript X may be omitted such that $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$ or $\langle \cdot,\cdot \rangle = \langle \cdot,\cdot \rangle_\Gamma$. Also, set

$$H(\mathrm{div};\Omega) = \{v \in [L^2(\Omega)]^d : \nabla\cdot v \in L^2(\Omega)\}, \quad H^1(\mathrm{div};\Omega) = \{v \in [H^1(\Omega)]^d : \nabla\cdot v \in H^1(\Omega)\},$$

with the norms

$$\|v\|_{H(\mathrm{div};\Omega)} = \left[\|v\|_0^2 + \|\nabla\cdot v\|_0^2\right]^{1/2}; \quad \|v\|_{H^1(\mathrm{div};\Omega)} = \left[\|v\|_1^2 + \|\nabla\cdot v\|_1^2\right]^{1/2}.$$

We will assume that the solution of (2.5) with the boundary condition (2.8) exists and satisfies the regularity assumption

$$\|u^{(1)}\|_2 + \|u^{(3)}\|_2 + \|u^{(2)}\|_1 + \|\nabla\cdot u^{(2)}\|_1 \le C(w)\|F\|_0. \tag{3.1}$$

Let us introduce the space $\mathcal{V} = \left[H^1(\Omega)\right]^d \times H(\mathrm{div};\Omega) \times \left[H^1(\Omega)\right]^d$. Then multiply Equation (2.3) by $v \in \mathcal{V}$, use integration by parts in the $(\mathcal{L}(u),v)$-term, and apply the boundary condition (2.8) to see that the solution u of (2.5) and (2.8) satisfies *the weak form*:

$$-\omega^2 (\mathcal{P}u,v) + i\omega(\mathcal{B}u,v) + \mathcal{A}(u,v) + i\omega\langle \mathcal{D}\, S_\Gamma(u), S_\Gamma(v)\rangle = (F,v), \tag{3.2}$$
$$v = \left(v^{(1)}, v^{(2)}, v^{(3)}\right)^t \in \mathcal{V},$$

where $\mathcal{A}(u,v)$ is the bilinear form defined as follows:

$$\mathcal{A}(u,v) = \left(\sigma_{jk}^{(1)}(u), \varepsilon_{jk}(v^{(1)})\right) + \left(\sigma_{jk}^{(3)}(u), \varepsilon_{jk}(v^{(3)})\right) - \left(p_f(u), \nabla\cdot v^{(2)}\right), \ u,v \in \mathcal{V}. \tag{3.3}$$

In (3.3), and the rest of the paper, Einstein's convention of sum on repeated indices is used. Note that the bilinear form $\mathcal{A}(u,v)$ can be written in the form

$$\mathcal{A}(u,v) = (\mathbf{E}\,\widetilde{\epsilon}(u), \widetilde{\epsilon}(v)) = (\mathbf{E}_r\,\widetilde{\epsilon}(u), \widetilde{\epsilon}(v)) + i\,(\mathbf{E}_i\,\widetilde{\epsilon}(u), \widetilde{\epsilon}(v)), \ u,v \in \mathcal{V},$$

where $\mathbf{E} = \mathbf{E}_r + i\mathbf{E}_i$ is a complex matrix. Furthermore, we assume that the real part $\mathbf{E}_r$ is positive definite since in the elastic limit it is associated with the strain energy density. On the other hand, the imaginary part $\mathbf{E}_i$ is assumed to be positive definite because of the restriction imposed on our system by the First and Second Laws of Thermodynamics. A similar assumption was used in [35] to obtain restrictions on the imaginary parts of the coefficients in the constitutive relations for the case of a poroviscoelastic matrix saturated by a two-phase fluid. In the 2D case the matrix $\mathbf{E}$ is defined as follows, with the obvious extension to the 3D case:

$$\mathbf{E} = \begin{pmatrix} \widehat{\mathbf{E}} & 0 \\ 0 & \widehat{\mathbf{S}} \end{pmatrix}, \quad \widehat{\mathbf{S}} = \begin{pmatrix} 2\mu^{(1)} & \mu^{(13)} \\ \mu^{(13)} & 2\mu^{(3)} \end{pmatrix},$$

$$\widehat{\mathbf{E}} = \begin{pmatrix} \lambda^{(1)}+2\mu^{(1)} & \lambda^{(1)} & D^{(3)}+\mu^{(13)} & D^{(3)} & B^{(1)} \\ \lambda^{(1)} & \lambda^{(1)}+2\mu^{(1)} & D^{(3)} & D^{(3)}+\mu^{(13)} & B^{(1)} \\ D^{(3)}+\mu^{(13)} & D^{(3)} & \lambda^{(3)}+2\mu^{(3)} & \lambda^{(3)} & B^{(2)} \\ D^{(3)} & D^{(3)}+\mu^{(13)} & \lambda^{(3)} & \lambda^{(3)}+2\mu^{(3)} & B^{(2)} \\ B^{(1)} & B^{(1)} & B^{(2)} & B^{(2)} & K_{av} \end{pmatrix}, \quad \widetilde{\varepsilon}(u) = \begin{pmatrix} \varepsilon_{11}(u^{(1)}) \\ \varepsilon_{22}(u^{(1)}) \\ \varepsilon_{11}(u^{(3)}) \\ \varepsilon_{22}(u^{(3)}) \\ \nabla\cdot u^{(2)} \\ \varepsilon_{12}(u^{(1)}) \\ \varepsilon_{12}(u^{(3)}) \end{pmatrix}.$$

Let us analyze the uniqueness of the solution of our differential model for the case of a unit square $\Omega = (0,1)^2$ in the (x_1, x_2)-plane to shorten the argument; the 3D case follows with the same argument. Then, set $F = 0$ and choose $v = u$ in (3.2). Taking the imaginary part in the resulting equation, we obtain

$$\omega\,(\mathcal{B}u, u) + (\mathbf{E}_i\,\widetilde{\epsilon}(u), \widetilde{\epsilon}(u)) + \omega\,\langle \mathcal{D}\,S_\Gamma(u), S_\Gamma(u)\rangle = 0.$$

Using (2.4) and that $\mathbf{E}_i$ and $\mathcal{D}$ are positive definite and $\mathcal{B}$ is nonnegative, we conclude that

$$u^{(2)} = 0, \quad u^{(1)} - u^{(3)} = 0, \qquad \Omega, \tag{3.4a}$$

$$u^{(1)} = 0, \quad u^{(3)} = 0, \qquad \Gamma, \tag{3.4b}$$

$$u^{(2)} \cdot \nu = 0, \qquad \Gamma. \tag{3.4c}$$

Consider the part Γ_1 of the boundary Γ defined by $\Gamma_1 = \{x = (x_1, x_2) \in \Gamma : x_1 = 1, 0 < x_2 < 1\}$. Notice that (3.4b) and (3.4c) imply that

$$\frac{\partial u_1^{(1)}}{\partial x_2} = \frac{\partial u_2^{(1)}}{\partial x_2} = \frac{\partial u_1^{(3)}}{\partial x_2} = \frac{\partial u_2^{(3)}}{\partial x_2} = 0, \quad \Gamma. \tag{3.5}$$

Owing to (2.8) $\mathcal{G}_\Gamma(u) = 0$ leads to the following relations on Γ_1

$$\sigma_{11}^{(1)}(u) = \left(\lambda^{(1)} + 2\mu^{(1)}\right)\frac{\partial u_1^{(1)}}{\partial x_1} + \left(D^{(3)} + \mu^{(13)}\right)\frac{\partial u_1^{(3)}}{\partial x_1} + B^{(1)}\nabla\cdot u^{(2)} = 0, \tag{3.6a}$$

$$\sigma_{11}^{(3)}(u) = \left(\lambda^{(3)} + 2\mu^{(3)}\right)\frac{\partial u_1^{(3)}}{\partial x_1} + \left(D^{(3)} + \mu^{(13)}\right)\frac{\partial u_1^{(1)}}{\partial x_1} + B^{(2)}\nabla\cdot u^{(2)} = 0, \tag{3.6b}$$

$$\sigma_{12}^{(1)}(u) = \mu^{(1)}\frac{\partial u_2^{(1)}}{\partial x_1} + \frac{1}{2}\mu^{(13)}\frac{\partial u_2^{(3)}}{\partial x_1} = 0, \tag{3.6c}$$

$$\sigma_{12}^{(3)}(u) = \mu^{(3)}\frac{\partial u_2^{(3)}}{\partial x_1} + \frac{1}{2}\mu^{(13)}\frac{\partial u_2^{(1)}}{\partial x_1} = 0, \tag{3.6d}$$

$$-p_f(u) = B^{(1)}\frac{\partial u_1^{(1)}}{\partial x_1} + B^{(2)}\frac{\partial u_1^{(3)}}{\partial x_1} + K_{av}\nabla\cdot u^{(2)} = 0. \tag{3.6e}$$

Next we observe that (3.6c) and (3.6d) form a homogeneous 2×2 linear system of equations with coefficient matrix $2\,\widehat{\mathbf{S}}$, while (3.6a), (3.6b) and (3.6e) is another homogeneous linear system of equations with matrix coefficients

$$\mathcal{E}^{(\mathbf{p})} = \begin{bmatrix} \lambda^{(1)} + 2\mu^{(1)} & D^{(3)} + \mu^{(13)} & B^{(1)} \\ D^{(3)} + \mu^{(13)} & \lambda^{(3)} + 2\mu^{(3)} & B^{(2)} \\ B^{(1)} & B^{(2)} & K_{av} \end{bmatrix}.$$

We make the assumption (valid in any physically meaningful situation) that the coefficients in the matrix $\mathbf{E}_i$ fulfill

$$\mathrm{Im}\left(det\left(\mathcal{E}^{(\mathbf{p})}\right)\right) > 0, \tag{3.7a}$$

$$\mathrm{Im}\left(det\left(\widehat{\mathbf{S}}\right)\right) > 0. \tag{3.7b}$$

For example, a calculation shows that (3.7) is satisfied if the coefficients $\mu^{(13)}, B^{(1)}, B^{(2)}$ and $D^{(3)}$ are real, the coefficients $\lambda^{(m)},\ \mu^{(m)},\ m=1,3,$ and K_{av} are complex and $\mathrm{Im}(K_{av})$ is chosen sufficiently small but different from zero in order that $\mathbf{E}_i$ be positive definite. Thus, under the condition (3.7), from (3.6) we conclude that

$$\frac{\partial u_2^{(1)}}{\partial x_1} = \frac{\partial u_2^{(3)}}{\partial x_1} = 0, \quad \Gamma_1, \tag{3.8a}$$

$$\frac{\partial u_1^{(1)}}{\partial x_1} = \frac{\partial u_1^{(3)}}{\partial x_1} = \nabla \cdot u^{(2)} = 0, \quad \Gamma_1. \tag{3.8b}$$

The same argument applies for the validity of (3.5) and (3.8) in the rest of the boundary. Thus by the Cauchy-Kowalevsky theorem $u^{(1)} = 0,\ u^{(3)} = 0$ in a neighborhood of any point on Γ where the coefficients are analytic and with the possible exception at the corners. Then the unique continuation principle [33] implies

$$u^{(1)} = u^{(3)} = 0, \quad \Omega. \tag{3.9}$$

Now from (3.4a) and (3.9) we have uniqueness. The 3D case follows with the identical argument.

We summarize the result in the following theorem.

Theorem 3.1. *Under the assumption made in the above argument concerning the validity of* (3.7)*, Problem (2.5) with (2.8) has a unique solution for any* $\omega \neq 0$.

For the analysis that follows a similar result can be demonstrated for the adjoint problem to (2.5) and (2.8). Thus, the solution $\psi = \left(\psi^{(1)}, \psi^{(2)}, \psi^{(3)}\right)^t$ of the problem

$$-\omega^2 \mathcal{P}\psi - i\omega\mathcal{B}\psi - \mathcal{L}^*(\psi) \;=\; F, \quad \Omega \times (0, \omega^*), \tag{3.10a}$$

$$\mathcal{G}_\Gamma^*(\psi) - i\omega\mathcal{D}S_\Gamma(\psi) \;=\; 0, \quad \Gamma \times (0, \omega^*), \tag{3.10b}$$

is unique and satisfies the regularity assumption

$$\|\psi^{(1)}\|_2 + \|\psi^{(3)}\|_2 + \|\psi^{(2)}\|_1 + \|\nabla \cdot \psi^{(2)}\|_1 \le C(w)\|F\|_0. \tag{3.11}$$

In (3.10a),

$$\mathcal{L}^*(\psi) = \left(\nabla \cdot \sigma^{(1,*)}(\psi), -\nabla p_f^*(\psi), \nabla \cdot \sigma^{(3,*)}(\psi)\right)^t,$$

where $\sigma^{(m,*)}(\psi), m = 1,3,$ and $p_f^*(\psi)$ are defined as in (2.3) but using the complex conjugates of the coefficients. Similarly, $\mathcal{G}_\Gamma^*(\psi)$ is defined as in (2.6) but using $\sigma^{(m,*)}(\psi), m = 1,3,$ and $p_f^*(\psi)$ in those definitions. As before, existence for (3.10) will be assumed.

4 The global finite element procedure

The numerical procedures will be defined and analyzed in detail in two dimensions and for rectangular elements. The changes for triangular elements and the three dimensional case will be described in Section 9.

Let $\mathcal{T}^h(\Omega)$ be a nonoverlapping partition of Ω into rectangles Q_j of diameter bounded by h such that $\overline{\Omega} = \cup_{j=1}^J \overline{Q}_j$. Denote by ξ_j and ξ_{jk} the midpoints of $\partial Q_j \cap \Gamma$ and $\partial Q_j \cap \partial Q_k$, respectively. Let $\langle\langle\cdot,\cdot\rangle\rangle_{\Gamma_{jk}}$ denote the approximation to the (complex) inner product $\langle\cdot,\cdot\rangle_{\Gamma_{jk}}$ in $L^2(\Gamma_{jk})$ computed using the mid-point quadrature rule; more precisely,

$$\langle\langle u,v\rangle\rangle_{\Gamma_{jk}} = (u\overline{v})(\xi_{jk})|\Gamma_{jk}|,$$

where $|\Gamma_{jk}|$ denotes the measure of Γ_{jk},

Let us denote by ν_{jk} the unit outer normal on $\partial Q_j \cap \partial Q_k$ from Q_j to Q_k and by ν_j the unit outer normal to ∂Q_j. Let χ_j and χ_{jk} be unit tangents on $\partial Q_j \cap \Gamma$ and $\partial Q_j \cap \partial Q_k$ so that $\{\nu_j, \chi_j\}$ and $\{\nu_{jk}, \chi_{jk}\}$ are orthonormal systems on $\partial Q_j \cap \Gamma$ and $\partial Q_j \cap \partial Q_k$, respectively.

To approximate each component of the solid displacement vector we employ the nonconforming finite element space as in [17], while to approximate the fluid displacement vector we choose the vector part of the Raviart-Thomas-Nedelec space [36, 32] of zero order. More specifically, set

$$\widehat{R} = [-1,1]^2, \qquad \widehat{\mathcal{NC}}(\widehat{R}) = \mathrm{Span}\{1, \widehat{x}_1, \widehat{x}_2, \alpha(\widehat{x}_1) - \alpha(\widehat{x}_2)\}, \quad \alpha(\widehat{x}_1) = \widehat{x}_1^2 - \frac{5}{3}\widehat{x}_1^4.$$

with the degrees of freedom being the values at the midpoint of each edge of $\widehat{R}$. Also, if $\psi^L(\widehat{x}_1) = \frac{-1+\widehat{x}_1}{2}$, $\psi^R(\widehat{x}_1) = \frac{1+\widehat{x}_1}{2}$, $\psi^B(\widehat{x}_2) = \frac{-1+\widehat{x}_2}{2}$, $\psi^T(\widehat{x}_2) = \frac{1+\widehat{x}_2}{2}$, we have that

$$\widehat{\mathcal{W}}(\widehat{R}) = \mathrm{Span}\big\{(\psi^L(\widehat{x}_1),0)^t, (\psi^R(\widehat{x}_1),0)^t, (0,\psi^B(\widehat{x}_2))^t, (0,\psi^T(\widehat{x}_2))^t\big\}.$$

For each Q_j, let $F_{Q_j} : \widehat{R} \to Q_j$ be an invertible affine mapping such that $F_{Q_j}(\widehat{R}) = Q_j$, and define

$$\begin{aligned}
\mathcal{NC}_j^h &= \{v = (v_1,v_2)^t : v_i = \widehat{v}_i \circ F_{Q_j}^{-1}, \ \widehat{v}_i \in \widehat{\mathcal{NC}}(\widehat{R}), \ i = 1,2\}, \\
\mathcal{W}_j^h &= \{w : w = \widehat{w} \circ F_{Q_j}^{-1}, \ \widehat{w} \in \widehat{\mathcal{W}}(\widehat{R})\}.
\end{aligned}$$

Setting

$$\begin{aligned}
&\mathcal{NC}^h = \{v : v_j = v|_{Q_j} \in \mathcal{NC}_j^h, \ v_j(\xi_{jk}) = v_k(\xi_{jk}) \ \forall (j,k)\}, \\
&\mathcal{W}^h = \{w \in H(\mathrm{div};\Omega) : w_j = w|_{Q_j} \in \mathcal{W}_j^h\},
\end{aligned}$$

the global finite element space to approximate the solution u of (3.2) is defined by

$$\mathcal{V}^h = \mathcal{NC}^h \times \mathcal{W}^h \times \mathcal{NC}^h.$$

In order to state the approximation properties of $\mathcal{V}^h$ let us introduce the space

$$\widetilde{\Lambda}_s^h = \left\{\widetilde{\lambda}_s^h : \widetilde{\lambda}_s^h|_{\partial Q_j \cap \partial Q_k} = \widetilde{\lambda}_{s,jk}^h \in [P_0(\partial Q_j \cap \partial Q_k)]^2 \equiv \widetilde{\Lambda}_{s,jk}^h, \quad \widetilde{\lambda}_{s,jk}^h + \widetilde{\lambda}_{s,kj}^h = 0\right\},$$

where $P_0(\partial Q_j \cap \partial Q_k)$ denotes the constant functions defined on $\partial Q_j \cap \partial Q_k$. Also, define the projections $\Pi_h : [H^2(\Omega)]^2 \to \mathcal{NC}^h$ and $P_h^{(m)} : [H^2(\Omega)]^2 \times H^1(\mathrm{div};\Omega) \times [H^2(\Omega)]^2 \to \widetilde{\Lambda}_s^h, m = 1,3,$ associated with the two solid phases by

$$\begin{aligned}
&(\varphi^{(m)} - \Pi_h \varphi^{(m)})(\xi) = 0, \ \xi = \xi_{jk} \text{ or } \xi_j, \\
&\left\langle \sigma^{(m)}(\psi_j)\nu - P_h^{(m)}(\psi_j), 1\right\rangle_B = 0, \quad B = \partial Q_j \cap \partial Q_k \ \text{ or } \ \partial Q_j \cap \Gamma,
\end{aligned}$$

for all $\varphi \in [H^2(\Omega)]^2$ and $\psi \in [H^2(\Omega)]^2 \times H^1(\mathrm{div};\Omega) \times [H^2(\Omega)]^2$. Then, standard approximation theory implies that, for all $\varphi = \left(\varphi^{(1)}, \varphi^{(2)}, \varphi^{(3)}\right)^t \in [H^2(\Omega)]^2 \times H^1(\mathrm{div};\Omega) \times [H^2(\Omega)]^2$,

$$\sum_{m=1,3} \Bigg[\|\varphi^{(m)} - \Pi_h \varphi^{(m)}\|_0 + h \Big(\sum_j \|\varphi^{(m)} - \Pi_h \varphi^{(m)}\|_{1,Q_j}^2 \Big)^{\frac{1}{2}} + h^2 \Big(\sum_j \|\varphi^{(m)} - \Pi_h \varphi^{(m)}\|_{2,Q_j}^2 \Big)^{\frac{1}{2}}$$
$$+ h^{\frac{1}{2}} \Big(\sum_j |\varphi^{(m)} - \Pi_h \varphi^{(m)}|_{0,\partial Q_j}^2 \Big)^{\frac{1}{2}} + h^{\frac{3}{2}} \Big(\sum_j |\sigma^{(m)}(\varphi_j)\nu_j - P_h^{(m)} \varphi_j|_{0,\partial Q_j}^2 \Big)^{1/2} \Bigg]$$
$$\leq Ch^2 \left(\|\varphi^{(1)}\|_2 + \|\varphi^{(3)}\|_2 + \|\nabla \cdot \varphi^{(2)}\|_1 \right). \tag{4.1}$$

We also notice the orthogonality to constants of the difference $\varphi_j^{(m)} - \varphi_k^{(m)}$ on the interfaces $\partial Q_j \cap \partial Q_k$ of Q_j and Q_k; that is,

$$\left\langle \varphi_j^{(m)} - \varphi_k^{(m)}, 1 \right\rangle_{\partial Q_j \cap \partial Q_k} = 0 \text{ for all interfaces } \partial Q_j \cap \partial Q_k, \quad \varphi^{(m)} \in \mathcal{NC}^h, \ m = 1, 3.$$

Next, let us define the projection $\mathbf{Q}_h$ associated with the displacement vector of the fluid phase as follows:

$$\mathbf{Q}_h : [H^1(\Omega)]^2 \to \mathcal{W}^h : \left\langle \left(\mathbf{Q}_h \varphi^{(2)} - \varphi^{(2)} \right) \cdot \nu, 1 \right\rangle_B = 0,$$
$$B = \partial Q_j \cap \partial Q_k \text{ or } B = \partial Q_j \cap \Gamma.$$

Then, it follows from [32, 36] that

$$\|\varphi^{(2)} - \mathbf{Q}_h \varphi^{(2)}\|_0 \leq Ch \|\varphi^{(2)}\|_1, \tag{4.2a}$$
$$\|\varphi^{(2)} - \mathbf{Q}_h \varphi^{(2)}\|_{H(\mathrm{div};\Omega)} \leq Ch \left(\|\varphi^{(2)}\|_1 + \|\nabla \cdot \varphi^{(2)}\|_1 \right). \tag{4.2b}$$

Set

$$\mathcal{A}_h(u, v) = \sum_j \left[\left(\sigma_{jk}^{(1)}(u), \varepsilon_{jk}(v^{(1)}) \right)_{Q_j} + \left(\sigma_{jk}^{(3)}(u), \varepsilon_{jk}(v^{(3)}) \right)_{Q_j} - \left(p_f(u), \nabla \cdot v^{(2)} \right)_{Q_j} \right] \tag{4.3}$$

and

$$\Theta_h(u, v) = -\omega^2 \left(\mathcal{P} u, v \right) + i\omega \left(\mathcal{B} u, v \right) + \mathcal{A}_h(u, v) + i\omega \left\langle \mathcal{D}\ S_\Gamma(u), S_\Gamma(v) \right\rangle.$$

Then the *global* finite element procedure is defined as follows: find $u^h = \left(u^{(1,h)}, u^{(2,h)}, u^{(3,h)} \right)^t \in \mathcal{V}^h$ such that

$$\Theta_h(u^h, v) = (F, v), \quad v = \left(v^{(1)}, v^{(2)}, v^{(3)} \right)^t \in \mathcal{V}^h. \tag{4.4}$$

Let us denote by $u_j^{(m,h)}$, $j = 1, 2,$, the components of the vector $u^{(m,h)}$, $m = 1, 2, 3$. The following theorem analyzes the uniqueness of the solution of (4.4).

Theorem 4.1. *Problem (4.4) has a unique solution for any $\omega \neq 0$.*

Proof. Set $F = 0$, choose $v = u^h$ in (4.4) and take the imaginary part in the resulting equation to obtain

$$\omega\left(\mathcal{B}u^h, u^h\right) + \sum_{Q_j}\left(\mathbf{E}_i\,\widetilde{\epsilon}(u^h), \widetilde{\epsilon}(u^h)\right)_{Q_j} + \omega\left\langle \mathcal{D}\,S_\Gamma(u^h), S_\Gamma(u^h)\right\rangle = 0. \tag{4.5}$$

Since each term in the left-hand side of (4.5) is nonnegative, in particular we have that $\left(\mathcal{B}u^h, u^h\right) = 0$, and the argument in the proof of Theorem 3.1 can be repeated to show that

$$u^{(2,h)} = 0, \quad u^{(1,h)} = u^{(3,h)}, \quad \Omega. \tag{4.6}$$

To show that $u^{(1,h)} = u^{(3,h)} = 0$, take an element, say Q_1, among the four elements which intersect Γ at the vertices of Ω; two faces of Q_1 are contained in Γ. After a proper transformation, without loss of generality we can assume that $Q_1 = (-1,1)^2$ with the faces $\Gamma^R = \{(x_1,x_2) \in \Gamma : x_1 = 1\}$ and $\Gamma^T = \{(x_1,x_2) \in \Gamma : x_2 = 1\}$ contained in Γ. Set

$$\begin{aligned} u_1^{(1,h)} &= a_1 + b_1 x_1 + c_1 x_2 + d_1(\alpha(x_1) - \alpha(x_2)),\\ u_2^{(1,h)} &= a_2 + b_2 x_1 + c_2 x_2 + d_2(\alpha(x_1) - \alpha(x_2)). \end{aligned}$$

Since the boundary term in (4.5) must vanish and the matrix $\mathcal{D}$ is positive definite, we conclude that $S_\Gamma(u^h) = 0$ and consequently $u^{(m,h)}(x_1,x_2), m = 1,3$, must vanish on $\Gamma^R \cup \Gamma^T$. In particular at the mid point of $\Gamma^R \cup \Gamma^T$ we have

$$\begin{aligned} u_1^{(1,h)}(1,0) &= a_1 + b_1 - \frac{2}{3}d_1 = 0, \quad u_1^{(1,h)}(0,1) = a_1 + c_1 + \frac{2}{3}d_1 = 0,\\ u_2^{(1,h)}(1,0) &= a_2 + b_2 - \frac{2}{3}d_2 = 0, \quad u_2^{(1,h)}(0,1) = a_2 + c_2 + \frac{2}{3}d_2 = 0. \end{aligned} \tag{4.7}$$

Next, since the second term in the left-hand side of (4.5) is nonnegative and the matrix $\mathbf{E}_i$ is positive definite, for $(x_1,x_2) \in Q_1$ we must have

$$\varepsilon_{11}(u^{(1,h)}) = b_1 + 2d_1\left(x_1 - \frac{10}{3}x_1^3\right) = 0, \tag{4.8a}$$

$$\varepsilon_{22}(u^{(1,h)}) = c_2 - 2d_2\left(x_2 - \frac{10}{3}x_2^3\right) = 0, \tag{4.8b}$$

$$\varepsilon_{12}(u^{(1,h)}) = \frac{1}{2}\left[c_1 + b_2 - 2d_1\left(x_1 - \frac{10}{3}x_1^3\right) + 2d_2\left(x_2 - \frac{10}{3}x_2^3\right)\right] = 0. \tag{4.8c}$$

From (4.7) and (4.8) it follows that $u_1^{(1,h)}|_{Q_1} = u_2^{(1,h)}|_{Q_1} = 0$. and using (4.6) we also have $u_1^{(3,h)}|_{Q_1} = u_2^{(3,h)}|_{Q_1} = 0$. Let us take an element Q_2 adjacent to Q_1 that intersects Γ and has a common face Γ_{12} with Q_1. Then $u_1^{(1,h)}$ and $u_2^{(1,h)}$ vanish at the mid points of Γ_2 and Γ_{12} and $\varepsilon_{11}(u^{(1,h)}), \varepsilon_{22}(u^{(1,h)})$ and $\varepsilon_{22}(u^{(1,h)})$ vanish identically on Q_2, so that repeating the above argument we verify that

$$u_1^{(m,h)}|_{Q_2} = u_2^{(m,h)}|_{Q_2} = 0, \; m = 1,3. \tag{4.9}$$

Repeating the argument, one can show that (4.9) holds for all elements with a face contained in Γ. Next stripping out such boundary elements, take a boundary element with two faces common with the corner of stripped out domain and repeat the argument to show the validity of (4.9) for those elements. Then continue the process until the domain is exhausted. This completes the proof. □

5 A priori error estimates for the global procedure

In this section, we derive an error estimate between the solutions u and u^h defined by (3.2) and (4.4), respectively. The argument in this section is close to that given in [22] which uses a boot-strapping argument similar to [15] for nonconforming finite element methods for Helmholtz-type problems. Also, see [16] for such a boot-strapping argument for conforming finite element methods for the Helmholtz equation.

Set

$$\mathbf{Z_h} = \left(\Pi_h u^{(1)}, \mathbf{Q}_h u^{(2)}, \Pi_h u^{(3)}\right)^t, \quad \delta = u - u^h, \quad \gamma = \mathbf{Z_h} u - u^h.$$

Our first goal is to derive an estimate for $\|\gamma\|_0$, and for that purpose we will solve the adjoint problem (3.10) to (2.5) and (2.8) with γ as a source term. It is convenient to define the following broken norms and seminorms:

$$\|v\|_{s,h}^2 = \sum_j \|v\|_{s,Q_j}^2, \quad |v|_{s,h}^2 = \sum_j |v|_{s,Q_j}^2, \quad |v|_{s,h,\Gamma}^2 = \sum_j |v|_{s,\partial Q_j \cap \Gamma}^2.$$

First note that for $v = (v^{(1)}, v^{(2)}, v^{(3)})^t \in \left[L^2(\Omega)\right]^6$ such that $v^{(1)}, v^{(3)} \in [H^1(Q_j)]^2, v^{(2)} \in H(\mathrm{div}; Q_j)$. Using integration by parts on each Q_j, we obtain

$$\begin{aligned}\Theta_h(u,v) \quad &= \quad \sum_j \left(-\omega^2 \mathcal{P} u + i\omega \mathcal{B} u - \mathcal{L}(u), v\right)_{Q_j} \\ &\quad + \sum_j \left\langle \left(\sigma^{(1)}(u)\nu, -p_f(u)\nu, \sigma^{(3)}(u)\nu\right)^t, (v^{(1)}, v^{(2)}, v^{(3)})^t \right\rangle_{\partial Q_j \setminus \Gamma}.\end{aligned} \tag{5.1}$$

Thus from (4.4) and (5.1) we see that for $v \in \mathcal{V}^h$

$$\Theta_h(\delta, v) = \sum_j \left[\sum_{m=1,3} \left\langle \sigma^{(m)}(u)\nu, v^{(m)} \right\rangle_{\partial Q_j \setminus \Gamma} - \left\langle p_f(u), v^{(2)} \cdot \nu \right\rangle_{\partial Q_j \setminus \Gamma} \right]. \tag{5.2}$$

Notice that the regularity assumption (3.1) implies that $p_f(u) \in H^{1/2}(\partial Q_j \cap \partial Q_k)$, which together with the fact that $v_j^{(2)} \cdot \nu_{jk} + v_k^{(2)} \cdot \nu_{kj} = 0$ in the sense of $H^{-1/2}(\partial Q_j \cap \partial Q_k)$, leads to

$$\sum_j \left\langle p_f(u), v^{(2)} \cdot \nu \right\rangle_{\partial Q_j \setminus \Gamma} = 0. \tag{5.3}$$

Hence, thanks to (5.3) and that $v^{(1)}$ and $v^{(3)}$ are orthogonal to constants, (5.2) can be rewritten in the form

$$\Theta_h(\delta, v) = \sum_j \sum_{m=1,3} \left\langle \sigma^{(m)}(u)\nu - P_h^{(m)}(u), v^{(m)} \right\rangle_{\partial Q_j \setminus \Gamma}, \quad v \in \mathcal{V}^h. \tag{5.4}$$

Let $\psi = \left(\psi^{(1)}, \psi^{(2)}, \psi^{(3)}\right)^t$ be the solution of the adjoint problem to (2.5) and (2.8):

$$-\omega^2 \mathcal{P}\psi - i\omega \mathcal{B}\psi - \mathcal{L}^*(\psi) \quad = \quad \gamma, \quad \Omega \times (0, \omega^*), \tag{5.5a}$$
$$\mathcal{G}_\Gamma^*(\psi) - i\omega \mathcal{D} S_\Gamma(\psi) \quad = \quad 0, \quad \Gamma \times (0, \omega^*). \tag{5.5b}$$

According to (3.11), ψ satisfies the regularity assumption

$$\|\psi^{(1)}\|_2 + \|\psi^{(3)}\|_2 + \|\psi^{(2)}\|_1 + \|\nabla \cdot \psi^{(2)}\|_1 \le C(w)\|\gamma\|_0. \tag{5.6}$$

Using integration by parts on each Q_j and applying the boundary condition (5.5b), we get

$$\begin{aligned} -\left(\gamma, \mathcal{L}^*(\psi)\right) &= \mathcal{A}_h(\gamma,\psi) + i\omega \left\langle \mathcal{D}\ S_\Gamma(\gamma), S_\Gamma(\psi)\right\rangle - \sum_j \Big[\sum_{m=1,3} \left\langle \gamma^{(m)}, \sigma^{(m,*)}(\psi)\nu \right\rangle_{\partial Q_j \setminus \Gamma} \\ &\quad - \left\langle \gamma^{(2)} \cdot \nu, p_f^*(\psi) \right\rangle_{\partial Q_j \setminus \Gamma} \Big] \end{aligned} \tag{5.7}$$

Next, the argument used to show the validity of (5.3) can be applied to see that the last term in the right-hand side of (5.7) vanishes. Thus (5.7) implies that

$$\begin{aligned} \|\gamma\|_0^2 &= \left(\gamma, -\omega^2 \mathcal{P}\psi - i\omega\mathcal{B}\psi - \mathcal{L}^*(\psi)\right) \\ &= \Theta_h(\gamma,\psi) - \sum_j \sum_{m=1,3} \left\langle \gamma^{(m)}, \sigma^{(m,*)}(\psi)\nu \right\rangle_{\partial Q_j \setminus \Gamma}. \end{aligned} \tag{5.8}$$

Next, since $\sigma^{(m,*)}(\psi)\nu - P_h^{(m,*)}(\psi)$ has average value zero on $\partial Q_j \setminus \Gamma$, we have that for any $q^{(m)} \in [P_0(Q_j)]^2, m = 1,3,$

$$\left\langle q^{(m)}, \sigma^{(m,*)}(\psi)\nu - P_h^{(m,*)}(\psi) \right\rangle_{\partial Q_j \setminus \Gamma} = 0, \quad m = 1,3,$$

so that (5.8) can be stated in the form

$$\|\gamma\|_0^2 = \Theta_h(\gamma,\psi) - \sum_j \sum_{m=1,3} \left\langle \gamma^{(m)} - q^{(m)}, \sigma^{(m,*)}(\psi)\nu - P_h^{(m,*)}(\psi) \right\rangle_{\partial Q_j \setminus \Gamma}. \tag{5.9}$$

Next use (5.4) to see that for $v \in \mathcal{V}^h$,

$$\begin{aligned} \Theta_h(\gamma, v) &= \Theta_h(\delta, v) - \Theta_h(u - \mathbf{Z_h}u, v) = \sum_j \sum_{m=1,3} \left\langle \sigma^{(m)}(u)\nu - P_h^{(m)}(u), v^{(m)} \right\rangle_{\partial Q_j \setminus \Gamma} \\ &\quad - \Theta_h(u - \mathbf{Z_h}u, v). \end{aligned} \tag{5.10}$$

Then use (5.10) in (5.9) to obtain

$$\begin{aligned} \|\gamma\|_0^2 &= \Theta_h(\gamma, \psi - v) - \Theta_h(u - \mathbf{Z_h}u, v) \\ &\quad + \sum_j \sum_{m=1,3} \left\langle \sigma^{(m)}(u)\nu - P_h^{(m)}(u), v^{(m)} \right\rangle_{\partial Q_j \setminus \Gamma} \\ &\quad - \sum_j \sum_{m=1,3} \left\langle \gamma^{(m)} - q^{(m)}, \sigma^{(m,*)}(\psi)\nu - P_h^{(m,*)}(\psi) \right\rangle_{\partial Q_j \setminus \Gamma}. \end{aligned} \tag{5.11}$$

Next, since $\psi^{(m)} \in [H^2(\Omega)]^2, m = 1,3,$ (5.11) can be put in the equivalent form

$$\begin{aligned} \|\gamma\|_0^2 &= \Theta_h(\gamma, \psi - v) - \Theta_h(u - \mathbf{Z_h}u, v) \\ &\quad + \sum_j \sum_{m=1,3} \left\langle \sigma^{(m)}(u)\nu - P_h^{(m)}(u), v^{(m)} - \psi^{(m)} \right\rangle_{\partial Q_j \setminus \Gamma} \\ &\quad - \sum_j \Big[\sum_{m=1,3} \left\langle \gamma^{(m)} - q^{(m)}, \sigma^{(m,*)}(\psi)\nu - P_h^{(m,*)}(\psi) \right\rangle_{\partial Q_j \setminus \Gamma}. \end{aligned} \tag{5.12}$$

Let us bound each term in the right-hand side of (5.12). First, choose $v = \left(v^{(1)}, v^{(2)}, v^{(3)}\right)^t = \mathbf{Z_h}\psi \in \mathcal{V}^h$ such that

$$\sum_{m=1,3} \|\psi^{(m)} - v^{(m)}\|_0 + h\|\psi^{(m)} - v^{(m)}\|_{1,h} + h^2\|v^{(m)}\|_{2,h}$$
$$\leq Ch^2\left(\|\psi^{(1)}\|_2 + \|\psi^{(3)}\|_2\right) \leq Ch^2\|\gamma\|_0, \tag{5.13a}$$
$$\|\psi^{(2)} - v^{(2)}\|_0 \leq Ch\|\psi^{(2)}\|_1 \leq Ch\|\gamma\|_0, \tag{5.13b}$$
$$\left\|\nabla\cdot\left(\psi^{(2)} - v^{(2)}\right)\right\|_0 + h\left\|\nabla\cdot\left(\psi^{(2)} - v^{(2)}\right)\right\|_{1,h} \leq Ch\|\nabla\cdot\psi^{(2)}\|_1 \leq Ch\|\gamma\|_0. \tag{5.13c}$$

For the first term in the right-hand side of (5.12), using (5.13) we see that

$$|\Theta_h(\gamma, \psi - v)| \leq C(\omega)\Big[\|\gamma\|_0\|\psi - v\|_0 + \sum_{m=1,3}\|\gamma^{(m)}\|_{1,h}\|\psi^{(m)} - v^{(m)}\|_{1,h}$$
$$+\|\nabla\cdot\gamma\|_0\|\nabla\cdot(\psi - v)\|_0 + |\langle S_\Gamma(\gamma), S_\Gamma(\psi - v)\rangle|\Big]$$
$$\leq C(\omega)h\|\gamma\|_0\left[\|\gamma^{(1)}\|_{1,h} + \|\gamma^{(3)}\|_{1,h} + \|\nabla\cdot\gamma^{(2)}\|_0 + |\langle S_\Gamma(\gamma), S_\Gamma(\psi - v)\rangle|\right]. \tag{5.14}$$

The boundary integral in the right-hand side of (5.14) can be bounded using (5.6) and the trace inequality as follows:

$$|\langle S_\Gamma(\gamma), S_\Gamma(\psi - v)\rangle| \leq C\|\gamma\|_0 h^{3/2}\left[\|\gamma^{(1)}\|_{1,h} + |\gamma^{(3)}\|_{1,h}\right], \tag{5.15}$$

where we have used that

$$\sum_j \left\langle\left(\psi^{(2)} - \mathbf{Q}_h\psi^{(2)}\right)\cdot\nu, \gamma^{(2)}\cdot\nu\right\rangle_{\partial Q_j\backslash\Gamma} = 0.$$

Hence, using (5.15) in (5.14), we get

$$|\Theta_h(\gamma, \psi - v)| \leq C(\omega)h\|\gamma\|_0\left[\|\gamma^{(1)}\|_{1,h} + \|\gamma^{(3)}\|_{1,h} + \|\nabla\cdot\gamma^{(2)}\|_0\right]. \tag{5.16}$$

By choosing $q_j^{(m)} = q^m|_{Q_j}$, $m = 1, 3$, to be the average value of $\gamma^{(m)}$ on Q_j and using the trace inequality, (4.1) and (5.6), the last term in (5.12) is bounded as follows:

$$\left|\sum_{m=1,3}\sum_j \left\langle\gamma^{(m)} - q^{(m)}, \sigma^{(m,*)}(\psi)\nu - P_h^{(m,*)}(\psi)\right\rangle_{\partial Q_j\backslash\Gamma}\right|$$
$$\leq \sum_{m=1,3}\left(\sum_j |\gamma^{(m)} - q^{(m)}|^2_{0,\partial Q_j\backslash\Gamma}\right)^{1/2}\left(\sum_j |\sigma^{(m,*)}(\psi)\nu - P_h^{(m,*)}(\psi)|^2_{0,\partial Q_j\backslash\Gamma}\right)^{1/2}$$
$$\leq \sum_{m=1,3}\left(\sum_j h\|\gamma^{(m)}\|^2_{1,Q_j}\right)^{1/2} h^{1/2}\left(\|\psi^{(1)}\|_2 + \|\psi^{(3)}\|_2 + \|\nabla\cdot\psi^{(2)}\|_1\right)$$
$$\leq Ch\|\gamma\|_0\left(\|\gamma^{(1)}\|_{1,h} + \|\gamma^{(3)}\|_{1,h}\right). \tag{5.17}$$

Next, using integration by parts in the $\mathcal{A}_h(u-\mathbf{Z_h}u,v)$-term and the boundary condition (5.5b), the second term in the right-hand side of (5.12) can be written in the form

$$\begin{aligned}
\Theta_h(u-\mathbf{Z_h}u,v) = & \sum_j \left(u-\mathbf{Z_h}u, -\omega^2\mathcal{P}v - i\omega\mathcal{B}v - \mathcal{L}^*(v)\right)_{Q_j} \\
& + \sum_j \left\langle S_\Gamma\left(u-\mathbf{Z_h}u\right), \mathcal{G}^*_\Gamma(v)\right\rangle_{\partial Q_j\setminus\Gamma} + \sum_j \left\langle S_\Gamma\left(u-\mathbf{Z_h}u\right), \mathcal{G}^*_\Gamma(v)\right\rangle_{\partial Q_j\cap\Gamma} \\
& + i\omega\left\langle \mathcal{D}S_\Gamma\left(u-\mathbf{Z_h}u\right), S_\Gamma(v)\right\rangle \\
= \quad & \sum_j \left(u-\mathbf{Z_h}u, -\omega^2\mathcal{P}v - i\omega\mathcal{B}v - \mathcal{L}^*(v)\right)_{Q_j} \\
& + \sum_j \left\langle S_\Gamma\left(u-\mathbf{Z_h}u\right), \mathcal{G}^*_\Gamma(v) - \mathcal{G}^*_\Gamma(\psi)\right\rangle_{\partial Q_j\cap\Gamma} + \sum_j \left\langle S_\Gamma\left(u-\mathbf{Z_h}u\right), \mathcal{G}^*_\Gamma(v)\right\rangle_{\partial Q_j\setminus\Gamma} \\
& + i\omega\left\langle \mathcal{D}S_\Gamma\left(u-\mathbf{Z_h}u\right), S_\Gamma(v-\psi)\right\rangle \\
\equiv\ & T_1 + T_2 + T_3 + T_4.
\end{aligned} \tag{5.18}$$

Let us bound each term in the right-hand side of (5.18). First, using (4.1), (4.2), and (5.13) we see that

$$|T_1| \le Ch\|\gamma\|_0\left(\|u^{(1)}\|_2 + \|u^{(3)}\|_2 + \|u^{(2)}\|_1 + \|\nabla\cdot u^{(2)}\|_1\right).$$

For the T_2 term, applying the trace inequality, (4.1), (4.2), (3.11), and (5.13), one has

$$\begin{aligned}
|T_2| \le & \sum_{m=1,3}\sum_j |u^{(m)} - \Pi_h u^{(m)}|_{0,\partial Q_j\cap\Gamma} \left|\left(\sigma^{(m,*)}(v) - \sigma^{(m,*)}(\psi)\right)\cdot\nu\right|_{0,\partial Q_j\cap\Gamma} \\
& + \sum_j \left|\left(u^{(2)} - \mathbf{Q}_h u^{(2)}\right)\cdot\nu\right|_{-1/2,\partial Q_j\cap\Gamma} \left|p^*_f(v) - p^*_f(\psi)\right|_{1/2,\partial Q_j\cap\Gamma} \\
\le\ & C\|\gamma\|_0\left[h^2\left(\|u^{(1)}\|_2 + \|u^{(3)}\|_2\right) + h\left(\|u^{(2)}\|_1 + \|\nabla\cdot u^{(2)}\|_1\right)\right].
\end{aligned} \tag{5.19}$$

Next, we decompose T_3 as follows:

$$\begin{aligned}
T_3 = & \sum_j \left\langle S_\Gamma\left(u-\mathbf{Z_h}u\right), \mathcal{G}^*_\Gamma(v) - \mathcal{G}^*_\Gamma(\psi)\right\rangle_{\partial Q_j\setminus\Gamma} + \sum_j \left\langle S_\Gamma\left(u-\mathbf{Z_h}u\right), \mathcal{G}^*_\Gamma(\psi)\right\rangle_{\partial Q_j\setminus\Gamma} \\
\equiv\ & T_{3,1} + T_{3,2}.
\end{aligned} \tag{5.20}$$

Then, as in (5.19), the first term is bounded as follows:

$$|T_{3,1}| \le C\|\gamma\|_0\left[h^2\left(\|u^{(1)}\|_2 + \|u^{(3)}\|_2\right) + h\left(\|u^{(2)}\|_1 + \|\nabla\cdot u^{(2)}\|_1\right)\right].$$

The other term in (5.20) can be bounded by using again the fact that $\Pi_h u^{(m)}_j - \Pi_h u^{(m)}_k$ is orthogonal to constants

$$\begin{aligned}
|T_{3,2}| \le & \left|\sum_{m=1,3}\sum_j \left\langle\left(u^{(m)} - \Pi_h u^{(m)}\right)\cdot\nu, \sigma^{(m,*)}(\psi)\nu\cdot\nu\right\rangle_{\partial Q_j\setminus\Gamma}\right. \\
& + \left\langle\left(u^{(m)} - \Pi_h u^{(m)}\right)\cdot\chi, \sigma^{(m,*)}(\psi)\nu\cdot\chi\right\rangle_{\partial Q_j\setminus\Gamma} \\
& \left. - \sum_j \left\langle\left(u^{(2)} - \mathbf{Q}_h u^{(2)}\right)\cdot\nu, p^*_f(\psi)\right\rangle_{\partial Q_j\setminus\Gamma}\right| \\
\le\ & C\|\gamma\|_0\left[h^2\left(\|u^{(1)}\|_2 + \|u^{(3)}\|_2\right)\right],
\end{aligned}$$

where we have used again the argument in (5.3) to cancel out the terms involving $u^{(2)}$ in the inequality above. Finally, in order to bound T_4, applying the trace inequality, (4.1), (4.2), and (5.13), we obtain

$$\begin{aligned}
|T_4| &\leq C\Bigg[\sum_{m=1,3}\sum_j \left|u^{(m)}-\Pi_h u^{(m)}\right|_{0,\partial Q_j\cap\Gamma}\left|v^{(m)}-\psi^{(m)}\right|_{0,\partial Q_j\cap\Gamma} \\
&\quad +\sum_j\left|\left(u^{(2)}-\mathbf{Q}_h u^{(2)}\right)\cdot\nu\right|_{0,\partial Q_j\cap\Gamma}\left|\left(v^{(2)}-\psi^{(2)}\right)\cdot\nu\right|_{0,\partial Q_j\cap\Gamma}\Bigg] \\
&\leq \sum_{m=1,3}\sum_j \|u^{(m)}-\Pi_h u^{(m)}\|_{0,Q_j}^{\frac12}\|u^{(m)}-\Pi_h u^{(m)}\|_{1,Q_j}^{\frac12}\|\psi^{(m)}-v^{(m)}\|_{0,Q_j}^{\frac12}\|\psi^{(m)}-v^{(m)}\|_{1,Q_j}^{\frac12} \\
&\quad +\sum_j h^{\frac12}|u^{(2)}\cdot\nu|_{\frac12,\partial Q_j\cap\Gamma}h^{\frac12}|\psi^{(2)}\cdot\nu|_{\frac12,\partial Q_j\cap\Gamma} \\
&\leq C\|\gamma\|_0\left[h^3\left(\|u^{(1)}\|_2+\|u^{(3)}\|_2\right)\right]+Ch\|u^{(2)}\|_1\|\psi^{(2)}\|_1 \\
&\leq C\|\gamma\|_0\left[h^3\left(\|u^{(1)}\|_2+\|u^{(3)}\|_2\right)+Ch\|u^{(2)}\|_1\right].
\end{aligned}$$

Collecting the estimates for T_1, T_2, T_3, and T_4, we conclude that

$$|\Theta_h(u-\mathbf{Z_h}u,v)|\leq C\|\gamma\|_0\left[h^2\left(\|u^{(1)}\|_2+\|u^{(3)}\|_2\right)+h\left(\|u^{(2)}\|_1+\|\nabla\cdot u^{(2)}\|_1\right)\right]. \tag{5.21}$$

Next, use the trace inequality, (4.1), and (5.16) to bound the third term in the right-hand side of (5.12) as follows:

$$\begin{aligned}
&\left|\sum_j\Bigg[\sum_{m=1,3}\left\langle\sigma^{(m)}(u)\nu-P_h^m(u),v^{(m)}-\psi^{(m)}\right\rangle_{\partial Q_j\setminus\Gamma}\right| \\
&\leq \sum_{m=1,3}\left(\sum_j|\sigma^{(m)}(u)\nu-P_h^{(m)}(u)|^2_{0,\partial Q_j\setminus\Gamma}\right)^{1/2}\left(\sum_j|v^{(m)}-\psi^{(m)}|^2_{0,\partial Q_j\setminus\Gamma}\right)^{1/2} \\
&\leq Ch^{1/2}\left(\|u^{(1)}\|_2+\|u^{(3)}\|_2+\|\nabla\cdot u^{(2)}\|_1\right)h^{3/2}\left(\|\psi^{(1)}\|_2+\|\psi^{(3)}\|_2\right) \\
&\leq Ch^2\|\gamma\|_0\left(\|u^{(1)}\|_2+\|u^{(3)}\|_2+\|\nabla\cdot u^{(2)}\|_1\right).
\end{aligned} \tag{5.22}$$

Thus collecting the bounds in (5.16), (5.17), (5.21), and (5.22), we obtain

$$\begin{aligned}
\|\gamma\|_0\leq C(\omega)\Big[&h\left(\|\gamma^{(1)}\|_{1,h}+\|\gamma^{(3)}\|_{1,h}+\|\nabla\cdot\gamma^{(2)}\|_0\right) \\
&+h^2\left(\|u^{(1)}\|_2+\|u^{(3)}\|_2\right)+h\left(\|u^{(2)}\|_1+\|\nabla\cdot u^{(2)}\|_1\right)\Big].
\end{aligned} \tag{5.23}$$

Using the triangle inequality, the last estimate (5.23), and the approximation properties of

Π_h and $\mathbf{Q}_h$ in (4.1) and (4.2), we get

$$\begin{aligned}\|\delta\|_0 &\le \|\gamma\|_0 + \|\mathbf{Z}_h u - u\|_0 \le C(\omega)\Big[h\left(\|\delta^{(1)}\|_{1,h} + \|\delta^{(3)}\|_{1,h} + \|\nabla\cdot\delta^{(2)}\|_0\right)\\ &\quad + h\left(\|u^{(1)} - \Pi_h u^{(1)}\|_{1,h} + \|u^{(3)} - \Pi_h u^{(3)}\|_{1,h} + \|\nabla\cdot(u^{(2)} - \mathbf{Q}_h u^{(2)})\|_0\right)\\ &\quad + h^2\left(\|u^{(1)}\|_2 + \|u^{(3)}\|_2\right) + h\left(\|u^{(2)}\|_1 + \|\nabla\cdot u^{(2)}\|_1\right)\Big]\\ &\le C(\omega)\Big[h\left(\|\delta^{(1)}\|_{1,h} + \|\delta^{(3)}\|_{1,h} + \|\nabla\cdot\delta^{(2)}\|_0\right)\\ &\quad + h^2\left(\|u^{(1)}\|_2 + \|u^{(3)}\|_2\right) + h\left(\|u^{(2)}\|_1 + \|\nabla\cdot u^{(2)}\|_1\right)\Big].\end{aligned} \tag{5.24}$$

We next use a Gärding-type inequality to bound the δ-terms in (5.24) in terms of the u-terms in that inequality. First note that using Korn's second inequality [18, 34] and that $\mathbf{E}_i$ is positive definite, we get

$$\begin{aligned}|\mathrm{Im}\,(\Theta_h(\delta,\delta))| &= \omega\,(\mathcal{B}\delta,\delta) + \sum_j (\mathbf{E}_i\;\widetilde{\epsilon}(\delta),\widetilde{\epsilon}(\delta))_{Q_j} + \omega\,\langle\mathcal{D}\;S_\Gamma(\delta),S_\Gamma(\delta)\rangle\\ &\ge C_1(\omega)\left[\|\delta^{(1)}\|_{1,h}^2 + \|\delta^{(3)}\|_{1,h}^2 + \|\nabla\cdot\delta^{(2)}\|_0^2 + \langle S_\Gamma(\delta),S_\Gamma(\delta)\rangle\right]\\ &\quad - C_2(\omega)\|\delta\|_0^2.\end{aligned}$$

Hence,

$$\begin{aligned}&\|\delta^{(1)}\|_{1,h}^2 + \|\delta^{(3)}\|_{1,h}^2 + \|\nabla\cdot\delta^{(2)}\|_0^2 + \langle S_\Gamma(\delta),S_\Gamma(\delta)\rangle\\ &\quad\le C_3(\omega)\,|\Theta_h(\delta,\delta)| + C_2(\omega)\|\delta\|_0^2\\ &\quad\le C_3(\omega)\left[\|\delta\|_0^2 + |\Theta_h(\delta,u-\mathbf{Z}_h u)| + |\Theta_h(\delta,\gamma)|\right].\end{aligned} \tag{5.25}$$

Since $\gamma\in\mathcal{V}^h$, the expression for $\Theta_h(\delta,\gamma)$ given in (5.4) can be replaced by using (5.25) so that

$$\begin{aligned}&\|\delta^{(1)}\|_{1,h}^2 + \|\delta^{(3)}\|_{1,h}^2 + \|\nabla\cdot\delta^{(2)}\|_0^2 + \langle S_\Gamma(\delta),S_\Gamma(\delta)\rangle\\ &\le C_3(\omega)\Bigg[\|\delta\|_0^2 - \omega^2\,(\mathcal{P}\delta,u-\mathbf{Z}_h u) + i\omega\,(\mathcal{B}\delta,u-\mathbf{Z}_h u) + \mathcal{A}_h(\delta,u-\mathbf{Z}_h u)\\ &\quad + i\omega\,\langle\mathcal{D}\;S_\Gamma(\delta),S_\Gamma(u-\mathbf{Z}_h u)\rangle + \sum_j\sum_{m=1,3}\left\langle\sigma^{(m)}(u)\nu - P_h^{(m)}(u),\gamma^{(m)}\right\rangle_{\partial Q_j\setminus\Gamma}\Bigg].\end{aligned} \tag{5.26}$$

Let us bound the last five terms in the right-hand side of (5.26). First, thanks to the approximation properties of Π_h and $\mathbf{Q}_h$ given in (4.1) and (4.2), it follows that

$$\begin{aligned}&\left|-\omega^2\,(\mathcal{P}\delta,u-\mathbf{Z}_h u) + i\omega\,(\mathcal{B}\delta,u-\mathbf{Z}_h u)\right|\\ &\quad\le C(\omega)\left[\|\delta\|_0^2 + h^4\left(\|u^{(1)}\|_2^2 + \|u^{(3)}\|_2^2\right) + h^2\|u^{(2)}\|_1^2\right].\end{aligned} \tag{5.27}$$

Again, due to (4.1) and (4.2),

$$
\begin{aligned}
|\mathcal{A}_h(\delta, u - \mathbf{Z}_h u)| &\leq C(\omega)\Big[\sum_{m=1,3} \left(\|\delta^{(m)}\|_{1,h} \|u^{(m)} - \Pi_h u^{(m)}\|_{1,h} \right) \\
&\qquad + \|\nabla\cdot\delta^{(2)}\|_0 \|\nabla\cdot(u^{(2)} - \mathbf{Q}_h u^{(2)})\|_0 \Big] \\
&\leq \epsilon \left(\|\delta^{(1)}\|_{1,h}^2 + \|\delta^{(3)}\|_{1,h}^2 + \|\nabla\cdot\delta^{(2)}\|_0^2 \right) \\
&\qquad + C(\omega)\left[h^2 \left(\|u^{(1)}\|_2^2 + \|u^{(3)}\|_2^2 \right) + h^2 \|\nabla\cdot u^{(2)}\|_1^2 \right].
\end{aligned}
\tag{5.28}
$$

Next, using the trace inequality and approximation properties (4.1) and (4.2) again, we have

$$
\begin{aligned}
&|\omega \langle \mathcal{D}\, S_\Gamma(\delta), S_\Gamma(u - \mathbf{Z}_h u)\rangle| \\
&\quad \leq \epsilon \langle \mathcal{D} S_\Gamma(\delta), S_\Gamma(\delta)\rangle + C(\omega) \langle \mathcal{D} S_\Gamma(u - \mathbf{Z}_h u), S_\Gamma(u - \mathbf{Z}_h u)\rangle \\
&\quad \leq \epsilon \langle \mathcal{D} S_\Gamma(\delta), S_\Gamma(\delta)\rangle \\
&\qquad + C(\omega) \left[\sum_{m=1,3} \sum_j |u^{(m)} - \Pi_h u^{(m)}|_{0,\partial Q_j\cap\Gamma}^2 + \sum_j |(u^{(2)} - \mathbf{Q}_h u^{(2)})\cdot\nu|_{0,\partial Q_j\cap\Gamma}^2 \right] \\
&\quad \leq \epsilon \langle \mathcal{D} S_\Gamma(\delta), S_\Gamma(\delta)\rangle + C(\omega) \left[h^3 \left(\|u^{(1)}\|_2^2 + \|u^{(3)}\|_2^2 \right) + \sum_j h^2 |u^{(2)}\cdot\nu|_{1,\partial Q_j\cap\Gamma}^2 \right] \\
&\quad \leq \epsilon \langle \mathcal{D} S_\Gamma(\delta), S_\Gamma(\delta)\rangle + C(\omega) \left[h^3 \left(\|u^{(1)}\|_2^2 + \|u^{(3)}\|_2^2 \right) + \sum_j h^2 \|u^{(2)}\|_{\frac{3}{2},Q_j}^2 \right] \\
&\quad \leq \epsilon \langle \mathcal{D} S_\Gamma(\delta), S_\Gamma(\delta)\rangle + C(\omega) \left[h^3 \left(\|u^{(1)}\|_2^2 + \|u^{(3)}\|_2^2 \right) + h^2 \|u^{(2)}\|_{\frac{3}{2}}^2 \right].
\end{aligned}
\tag{5.29}
$$

Finally, owing to the orthogonality property of $\gamma^{(m)}$ to constants on $\partial Q_j \setminus \Gamma$, the trace inequality, and (4.1), it follows that

$$
\begin{aligned}
&\left| \sum_j \sum_{m=1,3} \left\langle \sigma^{(m)}(u)\nu - P_h^{(m)}(u), \gamma^{(m)} \right\rangle_{\partial Q_j \setminus \Gamma} \right| \\
&\quad = \left| \sum_j \sum_{m=1,3} \left\langle \sigma^{(m)}(u)\nu - P_h^{(m)}(u), \gamma^{(m)} - q^{(m)} \right\rangle_{\partial Q_j \setminus \Gamma} \right| \\
&\quad \le C \sum_{m=1,3} \left(\sum_j |\sigma^{(m)}(u)\nu - P_h^{(m)}(u)|^2_{0,\partial Q_j \setminus \Gamma} \right)^{1/2} \left(\sum_j |\gamma^{(m)} - q^{(m)}|^2_{0,\partial Q_j \setminus \Gamma} \right)^{1/2} \\
&\quad \le C h^{1/2} \left(\|u^{(1)}\|_2 + \|u^{(3)}\|_2 + \|\nabla \cdot u^{(2)}\|_1 \right) \sum_{m=1,3} \left(\sum_j h \|\gamma^{(m)}\|_{1,Q_j} \right)^{1/2} \\
&\quad \le C h \sum_{m=1,3} \|\gamma^{(m)}\|_{1,h} \left(\|u^{(1)}\|_2 + \|u^{(3)}\|_2 + \|\nabla \cdot u^{(2)}\|_1 \right) \\
&\quad \le C h \left(\sum_{m=1,3} \|\delta^{(m)}\|_{1,h} + \|u^{(m)} - \Pi_h u^{(m)}\|_{1,h} \right) \left(\|u^{(1)}\|_2 + \|u^{(3)}\|_2 + \|\nabla \cdot u^{(2)}\|_1 \right) \\
&\quad \le \epsilon \left(\|\delta^{(1)}\|^2_{1,h} + \|\delta^{(3)}\|^2_{1,h} \right) + C h^2 \left(\|u^{(1)}\|^2_2 + \|u^{(3)}\|^2_2 + \|\nabla \cdot u^{(2)}\|^2_1 \right).
\end{aligned}
\tag{5.30}
$$

Hence using (5.27), (5.28), (5.29), and (5.30) in (5.26), we have the following estimate:

$$
\begin{aligned}
&\|\delta^{(1)}\|_{1,h} + \|\delta^{(3)}\|_{1,h} + \|\nabla \cdot \delta^{(2)}\|_0 + \langle S_\Gamma(\delta), S_\Gamma(\delta) \rangle^{\frac{1}{2}} \\
&\qquad \le C(\omega) \left[\|\delta\|_0 + h \left(\|u^{(1)}\|_2 + \|u^{(3)}\|_2 + \|u^{(2)}\|_{\frac{3}{2}} + \|\nabla \cdot u^{(2)}\|_1 \right) \right].
\end{aligned}
\tag{5.31}
$$

Next, use (5.31) in (5.24) to obtain

$$
\begin{aligned}
\|\delta\|_0 \quad \le \quad C(\omega) \Big[& h\|\delta\|_0 + h^2 \left(\|u^{(1)}\|_2 + \|u^{(3)}\|_2 + \|u^{(2)}\|_{\frac{3}{2}} \right) \\
& + h \left(\|u^{(2)}\|_1 + \|\nabla \cdot u^{(2)}\|_1 \right) \Big].
\end{aligned}
\tag{5.32}
$$

Then, for sufficiently small $h > 0$ such that $0 < C(\omega)h < 1$, the term $\|\delta\|_0$ in the right hand side of (5.32) is absorbed in the left hand side, and therefore,

$$
\|\delta\|_0 \le C(\omega) \left[h^2 \left(\|u^{(1)}\|_2 + \|u^{(3)}\|_2 + \|u^{(2)}\|_{\frac{3}{2}} \right) + h \left(\|u^{(2)}\|_1 + \|\nabla \cdot u^{(2)}\|_1 \right) \right].
\tag{5.33}
$$

Finally using (5.33) in (5.31), we arrive at the following error estimate.

$$
\begin{aligned}
&\|\delta^{(1)}\|_{1,h} + \|\delta^{(3)}\|_{1,h} + \|\nabla \cdot \delta^{(2)}\|_0 + \langle S_\Gamma(\delta), S_\Gamma(\delta) \rangle \\
&\qquad \le C(\omega) h \left[\|u^{(1)}\|_2 + \|u^{(3)}\|_2 + \|u^{(2)}\|_{\frac{3}{2}} + \|\nabla \cdot u^{(2)}\|_1 \right].
\end{aligned}
$$

We summarize the above in the following theorem:

Theorem 5.1. *Let $u \in \mathcal{V}$ and $u^h \in \mathcal{V}^h$ be the solutions of (3.2) and (4.4), respectively. We then have the following energy-norm error estimate: for sufficiently small $h > 0$,*

$$\sum_{m=1,3} \|u^{(m)} - u^{(m,h)}\|_{1,h} + \|\nabla\cdot(u^{(2)} - u^{(2,h)})\|_0 + \sum_{m=1,3} |u^{(m)} - u^{(m,h)}|_{0,\Gamma} + |(u^{(2)} - u^{(2,h)})\cdot\nu|_{0,\Gamma} \le C(\omega) h \left[\|u^{(1)}\|_2 + \|u^{(3)}\|_2 + \|u^{(2)}\|_{\frac{3}{2}} + \|\nabla\cdot u^{(2)}\|_1\right].$$

Also, we have the $[L^2(\Omega)]^6$-error estimate as follows: for sufficiently small $h > 0$,

$$\|u - u^h\|_0 \le C(\omega)\left[h^2\left(\|u^{(1)}\|_2 + \|u^{(3)}\|_2 + \|u^{(2)}\|_{\frac{3}{2}}\right) + h\left(\|u^{(2)}\|_1 + \|\nabla\cdot u^{(2)}\|_1\right)\right].$$

6 A global hybridized nonconforming finite element procedure

Let us decompose $\Omega \in \mathbb{R}^2$ into nonoverlapping subdomains $\Omega_1, \cdots, \Omega_N$ such that each Ω_j is composed of the union of disjoint rectangles $Q \in \mathcal{T}^h(\Omega)$, with the interfaces $\Gamma_{jk} = \partial\Omega_j \cap \partial\Omega_k$. Also, let $\Gamma_j = \partial\Omega_j \cap \Gamma$. Set

$$\begin{aligned}
\mathcal{T}^h(\Omega_j) &= \{Q \in \mathcal{T}^h(\Omega) : Q \in \Omega_j\},\\
\mathcal{NC}^h(\Omega_j) &= \{v_j : \Omega_j \to \mathbb{C}^2,\ v_j|_Q \in \mathcal{NC}_j^h\ \forall Q \in \mathcal{T}^h(\Omega_j), v_j|_{Q_k}(\xi_{kl}) = v_j|_{Q_l}(\xi_{kl})\ \forall(k,l)\},\\
\mathcal{W}^h(\Omega_j) &= \{w \in H(\mathrm{div};\Omega_j) : w_k = w|_{Q_k} \in \mathcal{W}_k^h\},\\
\mathcal{V}^h(\Omega_j) &= \mathcal{NC}^h(\Omega_j) \times \mathcal{W}^h(\Omega_j) \times \mathcal{NC}^h(\Omega_j).
\end{aligned}$$

Our global hybridized finite element space is then defined by

$$\mathcal{V}_{-1}^h = \{v \in [L^2(\Omega)]^6 : v|_{\Omega_j} \in \mathcal{V}^h(\Omega_j)\}.$$

In order to define a hybridized procedure, we follow the ideas in [1, 19, 20, 14] to impose the continuity constraints across interior interfaces using Lagrange multipliers. Thus we introduce the space $\widetilde{\Lambda}_{-1,j}^h$, with $\widetilde{\lambda}_{jk}^h \in \widetilde{\Lambda}_{-1,j}^h$ associated with $\mathcal{G}_{\Gamma_{jk}}(u_j)$ on Γ_{jk}:

$$\widetilde{\Lambda}_{-1,j}^h = \{\widetilde{\lambda}_j^h : \widetilde{\lambda}_{jk}^h = \widetilde{\lambda}_j^h|_{\partial Q\cap\Gamma_{jk}} \in \widetilde{\Lambda}_{-1,jk}^h\ \forall Q \in \Omega_j \text{ such that } \overline{Q}\cap\Gamma_{jk} \ne \emptyset\},$$

where

$$\widetilde{\Lambda}_{-1,jk}^h = \{\widetilde{\lambda}_{jk}^h : \widetilde{\lambda}_{jk}^h \in [P_0(\partial Q\cap\Gamma_{jk})]^5\ \forall Q \in \Omega_j \text{ such that } \overline{Q}\cap\Gamma_{jk} \ne \emptyset, \widetilde{\lambda}_{jk}^h = \widetilde{\lambda}_{kj}^h\} \text{ for all } j,k.$$

Set

$$\widetilde{\Lambda}_{-1}^h = \cup_j \widetilde{\Lambda}_{-1,j}^h.$$

The global hybridized nonconforming procedure is defined in the following fashion: find $(\widetilde{u}^h, \widetilde{\lambda}^h) \in \mathcal{V}_{-1}^h \times \widetilde{\Lambda}_{-1}^h$ such that

$$\sum_j \sum_{Q\in\mathcal{T}^h(\Omega_j)} \left[-\omega^2(\mathcal{P}\widetilde{u}_j^h, v)_Q + i\omega(\mathcal{B}\widetilde{u}_j^h, v)_Q + \mathcal{A}_{h,Q}(\widetilde{u}_j^h, v)\right] - \sum_{j,k}\left\langle\left\langle\widetilde{\lambda}_{jk}^h, S_{\Gamma_{jk}}(v)\right\rangle\right\rangle_{\Gamma_{jk}} + i\omega\sum_j\left\langle\mathcal{D}S_{\Gamma_j}(\widetilde{u}_j^h), S_{\Gamma_j}(v)\right\rangle_{\Gamma_j} = (F, v), \quad v \in \mathcal{V}_{-1}^h, \tag{6.1a}$$

$$\sum_{j,k}\left\langle\left\langle\theta, S_{\Gamma_{jk}}(\widetilde{u}_j^h)\right\rangle\right\rangle_{\Gamma_{jk}} = 0, \quad \theta \in \widetilde{\Lambda}_{-1}^h, \tag{6.1b}$$

where $\mathcal{A}_{h,Q}$ indicates the restriction to Q of the bilinear form $\mathcal{A}_h$ defined in (4.3) and $S_{\Gamma_{jk}}, S_{\Gamma_j}$ are defined as in (2.6)–(2.7). The following theorem gives an existence and uniqueness result for the procedure (6.1).

Theorem 6.1. *Problem* (6.1) *has a unique solution.*

Proof. It is enough to show uniqueness due to finite dimensionality. For this, set $F = 0$ and add (6.1a) with the choice of $v = \widetilde{u}^h$ and (6.1b) with the choice $\theta = \widetilde{\lambda}^h$. Then the imaginary part in the resulting equation reduces to

$$\sum_j \left(\sum_{Q \in \mathcal{T}^h(\Omega_j)} \left[\omega(\mathcal{B}\widetilde{u}_j^h, \widetilde{u}_j^h)_Q + (\mathbf{E}_i \widetilde{\varepsilon}(\widetilde{u}_j^h), \widetilde{\varepsilon}(\widetilde{u}_j^h))_Q \right] + \omega \left\langle \mathcal{D} S_{\Gamma_j}(\widetilde{u}_j^h), S_{\Gamma_j}(\widetilde{u}_j^h) \right\rangle_{\Gamma_j} \right) = 0. \qquad (6.2)$$

Now an argument similar to that given in the proof of Theorem 4.1 shows that

$$\widetilde{u}^h = 0 \quad \text{in} \quad \Omega.$$

Thus (6.1a) reduces to

$$\sum_{j,k} \langle\langle \widetilde{\lambda}^h, S_{\Gamma_{jk}}(v) \rangle\rangle_{\Gamma_{jk}} = 0, \quad v \in \mathcal{NC}_{-1}^h. \qquad (6.3)$$

Now, for each Ω_j and each $Q \in \Omega_j$ with Q facing the boundary Γ, we can choose $v \in \mathcal{V}^h(\Omega_j)$ with the degrees of freedom chosen such that $S_{\Gamma_{jk}}(v)$ be equal to $\widetilde{\lambda}^h$ at the midpoint m of one edge of Q and zero degrees of freedom at the other three midpoints of Q to show that $\widetilde{\lambda}^h = 0$ at the midpoint m. Repeating the argument for all midpoints of Q and all $Q \in \Omega_j$ whose faces meet $\partial\Omega_j$ for each j yields that $\widetilde{\lambda}^h = 0$. This completes the proof. □

We next notice the validity of the following lemma whose obvious proof is omitted.

Lemma 6.1. *If $\widetilde{u}^h \in \mathcal{V}_{-1}^h$, then $\widetilde{u}^h \in \mathcal{V}^h$ if and only if*

$$\sum_{j,k} \langle\langle \theta, S_{\Gamma_{jk}}(\widetilde{u}^h) \rangle\rangle_{\Gamma_{jk}} = 0, \quad \theta \in \widetilde{\Lambda}_{-1}^h.$$

Remark 6.2. *As a consequence of Theorem 6.1 and Lemma 6.1, $\widetilde{u}^h$ solves Problem* (4.4).

7 The domain decomposition iterative procedures

Consider the decomposition of problem (2.5) and (2.8) over Ω_j as follows: for $j = 1, \ldots, N$, find $u_j(x, \omega)$ satisfying

$$\begin{aligned} -\omega^2 \mathcal{P} u_j + i\omega \mathcal{B} u_j - \mathcal{L}(u_j) &= F, \quad \Omega_j, & (7.1a)\\ \mathcal{G}_{\Gamma_{jk}}(u_j) + i\omega\beta_{jk} S_{\Gamma_{jk}}(u_j) &= \mathcal{G}_{\Gamma_{kj}}(u_k) - i\omega\beta_{jk} S_{\Gamma_{kj}}(u_k), \qquad \Gamma_{jk}, & (7.1b)\\ -\mathcal{G}_{\Gamma_j}(u_j) &= i\omega \mathcal{D} S_{\Gamma_j}(u_j), \qquad \Gamma_j, & (7.1c) \end{aligned}$$

where $\mathcal{G}_{\Gamma_{jk}}$ and $\mathcal{G}_{\Gamma_j}$ are defined as in (2.6)–(2.7). Notice that (7.1b) is equivalent to imposing the two consistency conditions:

$$\begin{aligned} \mathcal{G}_{\Gamma_{jk}}(u_j) &= \mathcal{G}_{\Gamma_{kj}}(u_k), \qquad \Gamma_{jk},\\ \beta_{jk}\left(S_{\Gamma_{jk}}(u_j) + S_{\Gamma_{kj}}(u_k)\right) &= 0, \qquad \Gamma_{jk}. \end{aligned}$$

A weak form of (7.1) at the differential level may be stated as follows: for all j, find $u_j \in \left[H^1(\Omega_j)\right]^2 \times H(\mathrm{div};\Omega_j) \times \left[H^1(\Omega_j)\right]^2$ such that

$$\begin{aligned}
-\omega^2 \left(\mathcal{P}u_j, v\right)_{\Omega_j} + i\omega \left(\mathcal{B}u_j, v\right)_{\Omega_j} + \mathcal{A}_j(u_j, v) + i\omega \left\langle \mathcal{D}\ S_{\Gamma_j}(u_j), S_{\Gamma_j}(v)\right\rangle \\
+ \sum_k \left\langle i\omega\beta_{jk}(\mathcal{S}_{\Gamma_{jk}}(u_j) + \mathcal{S}_{\Gamma_{kj}}(u_k)) - \mathcal{G}_{\Gamma_{jk}}(u_k), v\right\rangle_{\Gamma_{jk}} = (F, v)_{\Omega_j}, \\
v = \left(v^{(1)}, v^{(2)}, v^{(3)}\right)^t \in \left[H^1(\Omega_j)\right]^2 \times H(\mathrm{div};\Omega_j) \times \left[H^1(\Omega_j)\right]^2,
\end{aligned}$$

where $\mathcal{A}_j$ is the restriction to Ω_j of the bilinear form $\mathcal{A}$ defined in (3.3).

We then can define a Jacoby-type iterative procedure at the differential level as follows: given $u_j^{\{0\}} \in \left[H^1(\Omega_j)\right]^2 \times H(\mathrm{div};\Omega_j) \times \left[H^1(\Omega_j)\right]^2$ for all j, iteratively for $n = 1, 2, 3, \cdots$, find $u_j^{\{n\}} \in \left[H^1(\Omega_j)\right]^2 \times H(\mathrm{div};\Omega_j) \times \left[H^1(\Omega_j)\right]^2$ for all j such that

$$\begin{aligned}
-\omega^2 \left(\mathcal{P}u_j^{\{n\}}, v\right)_{\Omega_j} + i\omega \left(\mathcal{B}u_j^{\{n\}}, v\right)_{\Omega_j} + \mathcal{A}_j(u_j^{\{n\}}, v) + i\omega \left\langle \mathcal{D}\ S_{\Gamma_j}(u_j^{\{n\}}), S_{\Gamma_j}(v)\right\rangle_{\Gamma_j} \\
+ \sum_k \left\langle i\omega\beta_{jk}(\mathcal{S}_{\Gamma_{jk}}(u_j^{\{n\}}) + \mathcal{S}_{\Gamma_{kj}}(u_k^{\{n-1\}})) - \mathcal{G}_{\Gamma_{jk}}(u_k^{\{n\}}), v\right\rangle_{\Gamma_{jk}} = (F, v)_{\Omega_j}, \quad (7.2) \\
v = \left(v^{(1)}, v^{(2)}, v^{(3)}\right)^t \in \left[H^1(\Omega_j)\right]^2 \times H(\mathrm{div};\Omega_j) \times \left[H^1(\Omega_j)\right]^2.
\end{aligned}$$

Next we define a hybridized nonconforming domain decomposition procedure motivated by (7.2). For that purpose, we introduce a new set of Lagrange multipliers $\widetilde{\widetilde{\lambda}}_{jk}^h$ associated with $\Gamma_{jk}(u_j)$ at the midpoints ξ_{jk} of face of element $Q \in \Omega_j$ such that $\overline{Q} \cap \Gamma_{jk} \neq \emptyset$ for all the interior interfaces Γ_{jk}. Set

$$\widetilde{\widetilde{\Lambda}}_{-1,j}^h = \{\widetilde{\widetilde{\lambda}}_j^h : \widetilde{\widetilde{\lambda}}_{jk}^h = \widetilde{\widetilde{\lambda}}_j^h|_{\partial Q \cap \Gamma_{jk}} \in [P_0(\partial Q \cap \Gamma_{jk})]^5 \ \forall Q \in \Omega_j \text{ such that } \overline{Q} \cap \Gamma_{jk} \neq \emptyset\},$$

and set

$$\widetilde{\widetilde{\Lambda}}_{-1}^h = \cup_j \widetilde{\widetilde{\Lambda}}_{-1,j}^h$$

Remark 7.1. *Note that we have two copies of $[P_0(\Gamma_{jk})]^5$ on each Γ_{jk}, one from Ω_j to Ω_k and another from Ω_k to Ω_j.*

An iterative procedure corresponding to (7.2) is defined as follows: For all $j = 1, \cdots, N$, choose an initial guess $\left(u_j^{\{h,0\}}, \widetilde{\widetilde{\lambda}}_j^{\{h,0\}}\right) \in \mathcal{V}^h(\Omega_j) \times \widetilde{\widetilde{\Lambda}}_{-1}^h$. Then, for $n = 1, 2, 3, \cdots$, compute

$\left(u_j^{\{h,n\}}, \widetilde{\widetilde{\lambda}}_j^{\{h,n\}}\right) \in \mathcal{V}^h(\Omega_j) \times \widetilde{\widetilde{\Lambda}}_{-1,j}^h$ as the solution of the equations

$$
\begin{aligned}
&\sum_{Q\in\mathcal{T}^h(\Omega_j)} \left[-\omega^2 \left(\mathcal{P}u_j^{\{h,n\}}, v\right)_Q + i\omega \left(\mathcal{B}u_j^{\{h,n\}}, v\right)_Q + \mathcal{A}_{h,Q}(u_j^{\{h,n\}}, v)\right] \\
&\quad + i\omega \left\langle \mathcal{D}\ S_{\Gamma_j}(u_j^{\{h,n\}}), S_{\Gamma_j}(v)\right\rangle_{\Gamma_j} + \sum_k \left\langle i\omega\beta_{jk}(\mathcal{S}_{\Gamma_{jk}}(u_j^{\{h,n\}}), \mathcal{S}_{\Gamma_{jk}}(v)\right\rangle_{\Gamma_{jk}} \\
&= (F, v)_{\Omega_j} - \sum_k \left\langle i\omega\beta_{jk}(\mathcal{S}_{\Gamma_{jk}}(u_k^{\{h,n*\}}), \mathcal{S}_{\Gamma_{jk}}(v)\right\rangle_{\Gamma_{jk}} + \sum_k \left\langle\left\langle \widetilde{\widetilde{\lambda}}_{kj}^{\{h,n*\}}, \mathcal{S}_{\Gamma_{jk}}(v)\right\rangle\right\rangle_{\Gamma_{jk}} \qquad (7.3a) \\
&\qquad\qquad\qquad\qquad\qquad\qquad v \in \mathcal{V}^h(\Omega_j),
\end{aligned}
$$

$$
\widetilde{\widetilde{\lambda}}_{jk}^{\{h,n\}} = \widetilde{\widetilde{\lambda}}_{kj}^{\{h,n*\}} - i\omega\beta_{jk}[S_{\Gamma_{jk}}(u_j^{\{h,n\}} + S_{\Gamma_{kj}}(u_k^{\{h,n*\}})](\xi_{jk}), \quad \text{on } \Gamma_{jk}, \forall k, \qquad (7.3b)
$$

for all $j = 1, \cdots, N$, where n^* is defined according to the iteration type as follows:

Jacobi type	Seidel type	red-black type
$n^* = n-1,$	$n^* = \begin{cases} n-1, & j<k, \\ n, & j>k, \end{cases}$	$n^* = \begin{cases} n-1, & \Omega_j \text{ is red } i.e.\ j \in I_R, \\ n, & \Omega_j \text{ is black } i.e.\ j \in I_B. \end{cases}$

Here for the red-black type, the red and black parts of subdomains are given alternatively such that $\overline{\Omega} = \overline{[\cup_{j\in I_R}\Omega_j]} \cup \overline{[\cup_{j\in I_B}\Omega_j]}$. If, for $\{j,k\} \subset I_R$ or $\{j,k\} \subset I_B$, $\overline{\Omega}_j \cap \overline{\Omega}_k \neq \emptyset$, then $\overline{\Omega}_j \cap \overline{\Omega}_k$ consists of a common vertex (in 2D) or a common edge (in 3D) of Ω_j and Ω_k.

8 Convergence of the iterative procedure

Next, we analyze the convergence of the iterative procedure (7.3). For simplicity in the notation we consider the case $\beta_{jk} = \beta I$ with $\beta = \beta_R > 0$ and I being the identity matrix of suitable size.

It follows immediately from (6.1) that for j,k, $(\widetilde{u}_j^h, \widetilde{\lambda}_{jk}^h) \in \mathcal{V}^h(\Omega_j) \times \widetilde{\widetilde{\Lambda}}_{-1,j}^h$ satisfy the local equations

$$
\begin{aligned}
&\sum_{Q\in\mathcal{T}^h(\Omega_j)} \left[-\omega^2(\mathcal{P}\widetilde{u}^h, v)_Q + i\omega(\mathcal{B}\widetilde{u}^h, v)_Q + \mathcal{A}_{h,Q}(\widetilde{u}^h, v)\right] \\
&\quad - \sum_k \left\langle\left\langle \widetilde{\lambda}_{jk}^h, v\right\rangle\right\rangle_{\Gamma_{jk}} + i\omega\left\langle \mathcal{D}S_{\Gamma_j}(\widetilde{u}^h), S_{\Gamma_j}(v)\right\rangle_{\Gamma_j} = (F, v)_{\Omega_j}, \quad v \in \mathcal{V}^h(\Omega_j). \qquad (8.1)
\end{aligned}
$$

Also, since $\widetilde{\lambda}_{jk}^h = \widetilde{\lambda}_{kj}^h$, (6.1b) is equivalent to

$$
\widetilde{\lambda}_{jk}^h = \widetilde{\lambda}_{kj}^h - i\omega\beta\left[S_{\Gamma_{jk}}(\widetilde{u}_j^h) + S_{\Gamma_{kj}}(\widetilde{u}_k^h)\right](\xi_{jk}), \quad \text{on } \Gamma_{jk}, \forall k. \qquad (8.2)
$$

Since $\widetilde{u}^h$ satisfies the error estimates given in Theorem 5.1, in order to show the convergence of the iteration procedure (7.3) it is sufficient to demonstrate that $u_j^{\{h,n\}} \to \widetilde{u}_j^h$ and $\widetilde{\widetilde{\lambda}}_{jk}^{\{h,n\}} \to \widetilde{\lambda}_{jk}^h$ as $n \to \infty$ for all j,k. For this, set

$$
d_j^n = u_j^{\{h,n\}} - \widetilde{u}_j^h, \quad x \in \Omega_j, \qquad \eta_{jk}^n = \widetilde{\widetilde{\lambda}}_{jk}^{\{h,n\}} - \widetilde{\lambda}_{jk}^h \quad \text{on } \Gamma_{jk}.
$$

Then, from (7.3)–(8.2), we obtain the following iteration error equations:

$$\sum_{Q\in\mathcal{T}^h(\Omega_j)}\left[-\omega^2\left(\mathcal{P}d_j^n,v\right)_Q+i\omega\left(\mathcal{B}d_j^n,v\right)_Q+\mathcal{A}_{h,Q}(d_j^n,v)\right]$$
$$+i\omega\left\langle\mathcal{D}\ S_{\Gamma_j}(d_j^n),S_{\Gamma_j}(v)\right\rangle_{\Gamma_j}-\sum_k\left\langle\left\langle\eta_{jk}^n,\mathcal{S}_{\Gamma_{jk}}(v)\right\rangle\right\rangle_{\Gamma_{jk}}=0,\quad v\in\mathcal{V}^h(\Omega_j),\tag{8.3a}$$
$$\eta_{jk}^n=\eta_{kj}^{n*}-i\omega\beta\left[S_{\Gamma_{jk}}(d_j^n)+S_{\Gamma_{kj}}(d_k^{n*})\right](\xi_{jk}),\quad\text{on }\Gamma_{jk},\forall k,\tag{8.3b}$$

Let us define the pseudo-energy R^n at the n-th iteration step as follows:

$$R^n=R^n(d^n,\eta^n)=\sum_{j,k}|\eta_{jk}^n+i\omega\beta S_{\Gamma_{jk}}(d_j^n)(\xi_{jk})|_{0,\Gamma_{jk}}^2\tag{8.4}$$

As in [22] the following lemma is valid, whose proof is omitted.

Lemma 8.1. *The following recurrence relation holds:*

$$R^n=R^{n-1}-\kappa\omega\beta\,\mathrm{Im}\sum_j\sum_k\left\langle\left\langle\eta_{jk}^{n-1},S_{\Gamma_{jk}}(d_j^{n-1})\right\rangle\right\rangle_{\Gamma_{jk}},\tag{8.5}$$

where $\kappa=4$ for the Jacobi case, and $\kappa=8$ for the Seidel and red-black cases.

Now, we use (8.5) to show the convergence of the iterative procedures (7.3). Choose $v=d_j^n$ in (8.3a), and take the imaginary part in the resulting equation to obtain

$$\mathrm{Im}\sum_k\langle\langle\eta_{jk}^n,S_{\Gamma_{jk}}(d_j^n)\rangle\rangle_{\Gamma_{jk}}=\sum_{Q\in\mathcal{T}^h(\Omega_j)}\left[\omega(\mathcal{B}d_j^n,d_j^n)_Q+(\mathbf{E}_i\widetilde{\varepsilon}(d_j^n),\widetilde{\varepsilon}(d_j^n))_Q\right]\tag{8.6}$$
$$+\omega\left\langle\mathcal{D}S_{\Gamma_j}(d_j^n),S_{\Gamma_j}(d_j^n)\right\rangle_{\Gamma_j}.$$

Summing (8.5) for $n=1,2,\cdots,m$, and $j=1,\cdots,N$, and using (8.6), we have

$$\begin{aligned}R^m\ =\ R^0-\kappa\omega\beta\sum_{n=0}^{m-1}\sum_j\Bigg(&\sum_{Q\in\mathcal{T}^h(\Omega_j)}\left[\omega(\mathcal{B}d_j^n,d_j^n)_Q+(\mathbf{E}_i\widetilde{\varepsilon}(d_j^n),\widetilde{\varepsilon}(d_j^n))_Q\right]\\&+\omega\left\langle\mathcal{D}S_{\Gamma_j}(d_j^n),S_{\Gamma_j}(d_j^n)\right\rangle_{\Gamma_j}\Bigg).\end{aligned}$$

Since β is positive, it follows that (R^m) is a nonnegative, decreasing sequence, which implies that

$$\sum_j\Bigg(\sum_{Q\in\mathcal{T}^h(\Omega_j)}\left[\omega(\mathcal{B}d_j^n,d_j^n)_Q+(\mathbf{E}_i\widetilde{\varepsilon}(d_j^n),\widetilde{\varepsilon}(d_j^n))_Q\right]$$
$$+\omega\left\langle\mathcal{D}S_{\Gamma_j}(d_j^n),S_{\Gamma_j}(d_j^n)\right\rangle_{\Gamma_j}\Bigg)\to 0$$

as n goes to ∞. In this case, the argument given in Theorem 6.1 shows that $d^n\to 0$ in $L^2(\Omega_j)$ and $\eta^n\to 0$ as n goes to ∞, so that the procedures (7.3) converge.

We will show that the iterations approach the fixed point of the operator $T_F : \mathcal{V}^h_{-1} \times \widetilde{\widetilde{\Lambda}}^h_{-1} \to \mathcal{V}^h_{-1} \times \widetilde{\widetilde{\lambda}}^h_{-1}$ defined as follows: for any $(p, \theta) \in \mathcal{V}^h_{-1} \times \widetilde{\lambda}^h_{-1}$, $(u, \eta) = T_F(p, \theta)$ is the solution of the equations:

$$\begin{aligned}
&\sum_{Q \in \mathcal{T}^h(\Omega_j)} \left[-\omega^2 \left(\mathcal{P} u_j, v\right)_Q + i\omega \left(\mathcal{B} u_j, v\right)_Q + \mathcal{A}_{h,Q}(u_j, v) \right] \\
&\quad + i\omega \left\langle \mathcal{D}\ S_{\Gamma_j}(u_j), S_{\Gamma_j}(v) \right\rangle_{\Gamma_j} + \sum_k \left\langle i\omega\beta_{jk}(\mathcal{S}_{\Gamma_{jk}}(u_j), \mathcal{S}_{\Gamma_{jk}}(v) \right\rangle_{\Gamma_{jk}} \qquad (8.7a) \\
&\quad = (F, v)_{\Omega_j} - \sum_k \left\langle i\omega\beta_{jk}(\mathcal{S}_{\Gamma_{jk}}(p), \mathcal{S}_{\Gamma_{jk}}(v) \right\rangle_{\Gamma_{jk}} + \sum_k \left\langle \left\langle \theta_{kj}, \mathcal{S}_{\Gamma_{jk}}(v) \right\rangle \right\rangle_{\Gamma_{jk}}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad v \in \mathcal{V}^h(\Omega_j), \\
&\eta_{jk} = \theta_{kj} - i\omega\beta_{jk} \left[S_{\Gamma_{jk}}(u_j) + S_{\Gamma_{kj}}(p_k) \right] (\xi_{jk}), \quad \text{on } \Gamma_{jk}, \forall k, \qquad (8.7b)
\end{aligned}$$

for all $j = 1, \cdots, N$.

The following lemma follows with an argument similar to that given in [17].

Lemma 8.2. *(u, η) is a solution of (8.1)–(8.2) if and only if it is a fixed point of T_F. If (u, η) is a fixed point of T_F, then $\eta_{jk} = -\eta_{kj}$ for all j, k.*

Next, $T_F(p, \theta) = T_0(p, \theta) + T_F(0, 0)$ and (p, θ) is a fixed point of T_F if and only if

$$T_F(p, \theta) = (p, \theta) = T_0(p, \theta) + T_F(0, 0).$$

Then a fixed point of T_F is a solution of the equation

$$(I - T_0)(p, \theta) = T_F(0, 0).$$

Thus, we study the spectral radius of the operator T_0.

Let δ be an eigenvalue of T_0 with associated eigenvector (u, η), so that $T_0(u, \eta) = \delta(u, \eta)$. Note that, according to (8.4),

$$R(T_0(u, \eta)) = |\delta|^2 R(u, \eta),$$

and from (8.5)

$$R(T_0(u, \eta)) = R(u, \eta) - \kappa\omega\beta \operatorname{Im} \sum_j \sum_k \left\langle \left\langle \eta_{jk}, S_{\Gamma_{jk}}(u_j) \right\rangle \right\rangle_{0, \Gamma_{jk}}, \qquad (8.8)$$

where κ is 4 or 8 as before. Combining (8.8) and (8.8) with (8.6), with η^n_{jk} and d^n_j replaced by η_{jk} and u_j, respectively, we see that

$$\begin{aligned}
|\delta|^2 \quad = \quad 1 - \frac{\kappa\omega\beta}{R(u, \eta)} \sum_j \Bigg(&\sum_{Q \in \mathcal{T}^h(\Omega_j)} \left[\omega(\mathcal{B} u_j, u_j)_Q + (\mathbf{E}_i \widetilde{\varepsilon}(u_j), \widetilde{\varepsilon}(u_j))_Q \right] \\
&+ \omega \left\langle \mathcal{D} S_{\Gamma_j}(u_j), S_{\Gamma_j}(u_j) \right\rangle_{\Gamma_j} \Bigg),
\end{aligned}$$

so that $|\delta| < 1$ provided that

$$\sum_j \Bigg(\sum_{Q\in\mathcal{T}^h(\Omega_j)} [\omega(\mathcal{B}u_j,u_j)_Q + (\mathbf{E}_i\widetilde{\varepsilon}(u_j),\widetilde{\varepsilon}(u_j))_Q] + \omega\left\langle \mathcal{D}S_{\Gamma_j}(u_j),S_{\Gamma_j}(u_j)\right\rangle_{\Gamma_j}\Bigg) > 0,$$

which is ensured by the positive definiteness of $\mathbf{E}_i$ and $\mathcal{D}$ and the nonnegative property of $\mathcal{B}$. Moreover, $\delta = 1$ if and only if

$$\sum_j \Bigg(\sum_{Q\in\mathcal{T}^h(\Omega_j)} [\omega(\mathcal{B}u_j,u_j)_Q + (\mathbf{E}_i\widetilde{\varepsilon}(u_j),\widetilde{\varepsilon}(u_j))_Q] + \omega\left\langle \mathcal{D}S_{\Gamma_j}(u_j),S_{\Gamma_j}(u_j)\right\rangle_{\Gamma_j}\Bigg) = 0,$$

which is identical to (6.2) and an argument similar to that given in the proof of Theorem 4.1 implies that $u_j = 0,\ j = 1,\cdots,J$. Then, (8.7a) immediately shows that

$$\sum_k \left\langle\left\langle \eta_{kj}, S_{\Gamma_{jk}}(v)\right\rangle\right\rangle_{\Gamma_{jk}} = 0, \quad v \in \mathcal{V}_j^h. \tag{8.9}$$

Now (8.9) is identical to (6.3) in the proof of Theorem 6.1 on each subdomain Ω_j, and a repetition of the argument shows that $\eta = 0$. Hence, we have proved the following theorem.

Theorem 8.1. *Let $\rho(T_0)$ be the spectral radius of T_0. Then $\rho(T_0) < 1$ and consequently the iterative procedure* (7.3) *is convergent.*

9 The triangular and the three dimensional cases

9.1 The triangular element case

Let $\overline{\Omega} = \cup_{j=1}^J \overline{Q}_j$ be a quasiregular partition of Ω into triangles Q_j's; here, Ω can be a convex polygon. Let us change the definition of the set $\mathcal{NC}_j^h$ in Section 4 to $\mathcal{NC}_j^h = [P_1(Q_j)]^2$, with the degrees of freedom being the mid point values of the edges of Q_j. Also, change the definition of the space $\mathcal{W}_j$ to be the vector part of the Raviart-Thomas-Nedelec mixed finite element space of zero order based on triangles [36, 32], with the degrees of freedom being the values of the normal component of the fluid displacement vector at the midpoints of the edges of Q_j.

An inspection of the analysis shows that all the conclusions presented for the rectangular case in Theorem 4.1 about the existence and uniqueness of the solution u^h of (4.4), the a priori error estimates in derived in Theorem 5.1 and the convergence of the iterative domain decomposition method in Theorem 8.1 remain valid for the new definition of the space $\mathcal{V}^h$.

9.2 The three dimensional case

Let $Q_j, j = 1,\cdots,J$, be a nonoverlapping partition of Ω. If the Q_j's are tetrahedrons we take $\mathcal{NC}_j^h = [P_1(Q_j)]^3$. If the Q_j's are cubic elements, we set $\widehat{R} = (-1,1)^3$ and

$$\begin{aligned}\widehat{S}(\widehat{R}) &= \mathrm{Span}\Big\{1,\widehat{x},\widehat{y},\widehat{z},\alpha(\widehat{x})-\alpha(\widehat{y}),\alpha(\widehat{x})-\alpha(\widehat{z})\Big\} \\ &= \mathrm{Span}\left\{\frac{1}{2}\widehat{x}\pm\frac{\alpha(\widehat{x})}{2\alpha(1)},\frac{1}{2}\widehat{y}\pm\frac{\alpha(\widehat{y})}{2\alpha(1)},\frac{1}{2}\widehat{z}\pm\frac{\alpha(\widehat{z})}{2\alpha(1)}\right\},\end{aligned}$$

and choose $\mathcal{NC}_j^h = [S(Q_j)]^3$. The four and six degrees of freedom associated with the tetrahedron case and (9.1) are the values at the centers of the faces. Also, take $\mathcal{W}_j$ be the Raviart-Thomas-Nedelec space of order zero over either tetrahedrons or cubic elements depending on Q_j[32].

Next, change the definitions of the spaces $\mathcal{V}^h$, $\mathcal{V}_{-1}^h$, $\widetilde{\Lambda}_s^h$, $\widetilde{\lambda}_{-1,j}^h$ and $\widetilde{\widetilde{\lambda}}_{-1,j}^h$ in the obvious fashion. With these changes in the definitions, all the results derived for the two–dimensional case remain unchanged.

10 Numerical experiments

We performed wave propagation simulation in a sample of water saturated partially frozen Berea sandstone, with an interior plane interface Γ defined by a change in ice content in the pores. The material properties of the system, taken from [9, 12] are given in Table 1. Since we would like to run an experiment in which the slow waves can actually be observed in the low-frequency range, the water viscosity value was taken be of 10^{-6} centipoise. In this case $\Omega^{(1)}$ and $\Omega^{(3)}$ correspond to the sandstone and ice, respectively. The computational domain Ω is a square of side length $L = 3$ Km with a uniform partition of Ω into squares of side length $h = L/261$. The absolute porosity is $\phi^{(a)} = .18$, with the ice content in the pores changing from $S^{(3)'} = 20$ percent in the lower layer to $S^{(3)'} = 82$ percent in the upper layer.

The source function is a point source representing a force applied to the rock frame in the vertical z-direction, located at ($x_s = 1.5$km, $z_s = 1.88$km). It has the form $F = \left(F^{(1)}, F^{(2)}, F^{(3)}\right)^t = \left(F^{(1)}, 0, 0\right)^t$,

$$F^{(1)} = \left(0, \frac{\partial \delta_{(x_s,z_s)}}{\partial z}\right)^t g(\omega),$$

where $\delta_{(x_s,z_s)}$ denotes the Dirac distribution and $g(\omega)$ is the Fourier transform of the waveform of central (dominant) frequency $f_0 = 12$ Hz given by

$$g(t) = -2\xi(t - t_0)e^{-\xi(t-t_0)^2},$$

with $\xi = 8\ f_0^2$, $t_0 = 1.25/f_0$.

For the calculation of the elastic coefficients we need values for the bulk and shear moduli of the two solid (dry) frames, denoted by $K_m^{(s1)}, K_m^{(s3)}, \mu_m^{(s1)}$ and $\mu_m^{(s3)}$, respectively (see Appendix A). Following [25, 12] and [39], it is assumed that $K_m^{(s1)}$= 14.4 GPa and that the modulus $\mu_m^{(s1)}$, $\mu_m^{(s3)}$ and $K_m^{(s3)}$ can be computed using a percolation-type model with critical exponent 3.8 [13] using the relations

$$\mu_m^{(sj)} = \left[\mu_m^{(sj),max} - \mu_m^{(sj),0}\right]\left[\frac{\phi^{(3)}}{1 - \phi^{(1)}}\right]^{3.8} + \mu_m^{(sj),0}, \quad j = 1, 3,$$

$$K_m^{(s3)} = \left[K_m^{(s3),max} - K_m^{(s3),0}\right]\left[\frac{\phi^{(3)}}{1 - \phi^{(1)}}\right]^{3.8} + K_m^{(s3),0},$$

where $\mu_m^{(s1),max}, \mu_m^{(s3),max}$ and $K_m^{(s3),max}$ are computed using the Kuster and Toksöz's model [24], taking the known values of $K^{(s1)}, \mu^{(s1)}, K^{(s3)}, \mu^{(s3)}$ for the background medium with inclusions of air, with properties $K^{(a)}, \mu^{(a)}$ (see Table 1). The moduli $\mu_m^{(s1),0}$, $\mu_m^{(s3),0}$ and $K_m^{(s3),0}$ are appropriate reference values, which we take

$$\mu_m^{(s1),0} = 13.3 \text{ GPa}, \quad K_m^{(s3),0} = \mu_m^{(s3),0} = 0.$$

The viscoelastic parameters describing the dissipative behavior of the saturated sandstone are given as follows (see Appendix A): $T_{1,M} = (2\pi\, 10)^{-1}$ms, $T_{2,M} = (2\pi\, 10^9)^{-1}$ms, and the mean quality factors are taken to be $\widehat{Q}_M = 300$ for $M = K_G^{(1)}, K_G^{(3)}, \mu^{(1)}, \mu^{(3)}, K_{av}$. The value of the Kozeny-Carman constant was taken to be 5 [23]. Also, the coefficient g_{13} in the definition (A.3) for $b_{11}(\omega)$ was taken to be zero.

Solid grain	bulk modulus, $K^{(s1)}$	38.7 GPa
	shear modulus, $\mu^{(s1)}$	39.6 GPa
	density, $\rho^{(1)}$	2650 kg/m^3
	permeability $\kappa^{(1),0}$	$1.07\ 10^{-13}$ m^2
Ice	bulk modulus, $K^{(s3)}$	8.58 GPa
	shear modulus, $\mu^{(s3)}$	3.32 GPa
	density, $\rho^{(3)}$	920 kg/m^3
	permeability $\kappa^{(3),0}$	$5\ 10^{-4}$ m^2
Fluid	bulk modulus, $K^{(f)}$	2.25 GPa
	density, $\rho^{(2)}$	1000 kg/m^3
	viscosity, η	10^{-6} cP
Air	bulk modulus, $K^{(a)}$	$1.5\ 10^{-4}$ GPa
	shear modulus, $\mu^{(a)}$	0 GPa

Table 1: Material properties of the frozen sandstone model

Table 2 displays values of the phase velocity and attenuation factors at 12 Hz for the five different types of waves for the two–layer model used in this experiment.

	Ice content 0.82		Ice content 0.20	
Wave	phase velocity (Km/s)	attenuation (dB)	phase velocity (Km/s)	attenuation (dB)
Fast P1 wave	4.316	$1.872\,10^{-3}$	3.723	$4.282\,10^{-2}$
Slow P2 wave	1.463	1.825	$7.281\,10^{-1}$	$1.151\,10^{1}$
Slow P3 wave	$9.577\,10^{-2}$	$4.053\,10^{1}$	$1.192\,10^{-1}$	6.562
Fast S1 wave	2.946	1.281	2.384	$2.202\,10^{-1}$
Slow S2 wave	$7.104\,10^{-1}$	$5.582\,10^{-2}$	$1.013\,10^{-1}$	$4.605\,10^{-1}$

Table 2: Wave speeds and attenuation factors for all waves at frequency 12 Hz.

The following figures present snapshots of the wave fields for this experiment, generated after solving (7.3) for 110 equally spaced temporal frequencies in the interval $(0, 12\,\text{Hz})$ and using an approximate inverse Fourier transform as explained in [16].

Figures 1, 2 and 3 show respectively snapshots of the vertical component of the particle velocity of the three phases at t = 410 ms where we can observe that after arriving at the interface Γ, the direct P1 wave labeled P1D has generated the transmitted fast P1-wave labeled P1T-P1D and the slow P2 transmitted and reflected waves labeled P2R-P1D and P2T-P1D, respectively. Also, after arriving at Γ, the direct fast shear wave labeled S1D has generated the transmitted and reflected fast shear waves labeled S1T-S1D and S1R-S1D, respectively. In the snapshots for the ice and fluid phases in Figures 2 and 3 we also can observe the direct slow P2 wave front labeled P2D. The relative amplitudes among the snapshots in Figures 1, 2 and 3 are 1, 0.56873, and 0.023708, respectively. We observe that the slow P2 wave is observed better in the ice and fluid phases than in the solid matrix phase.

Appendix

A The elastic coefficients and modifications to viscoelasticity

A.1 The elastic coefficients

Let $K_m^{(s1)}, K_m^{(s3)}, \mu_m^{(s1)}$, and $\mu_m^{(s3)}$ denote the bulk and shear moduli of the two solid (dry) frames, respectively. Also, let $K^{(s1)}, \mu^{(s1)}, K^{(s3)}, \mu^{(s3)}$ denote the bulk and shear moduli of the grains in the two solid phases, respectively, and $K^{(f)}$ the bulk modulus of the fluid phase. Then, following [12, 39], the elastic coefficients can be given by the formulae:

$$
\begin{aligned}
&\mu^{(j)} = [(1-g^{(j)})\phi^{(1)}]^2\mu_{av} + \mu_m^{(sj)}, \quad g^{(j)} = \frac{\mu_m^{(sj)}}{\phi^{(1)}\mu^{(sj)}}, \quad j=1,3,\\
&\mu^{(13)} = (1-g^{(1)})(1-g^{(3)})\phi^{(1)}\phi^{(3)}\mu_{av},\\
&\mu_{av} = [\frac{(1-g^{(1)})\phi^{(1)}}{\mu^{(s1)}} + \frac{\phi}{2\omega\eta} + \frac{(1-g^{(3)})\phi^{(3)}}{\mu^{(s3)}}]^{-1},\\
&K_{av} = \left[(1-c_1)\frac{\phi^{(1)}}{K^{(s1)}} + \frac{\phi}{K^{(f)}} + (1-c^{(3)})\frac{\phi^{(3)}}{K^{(s3)}}\right]^{-1}, \quad c^{(j)} = \frac{K_m^{(sj)}}{\phi^{(1)}K^{(sj)}}, \quad j=1,3,\\
&K_G^{(j)} = K_m^{(sj)} + (\alpha^{(j)})^2 K_{av}, \quad \alpha^{(j)} = S^{(j)} - \frac{K_m^{(sj)}}{K^{(sj)}},\\
&\lambda^{(j)} = K_G^{(j)} - \frac{2}{3}\mu^{(j)} \text{ in 3D}, \quad \lambda^{(j)} = K_G^{(j)} - \mu^{(j)} \text{ in 2D} \quad j=1,3,\\
&B^{(1)} = \frac{S^{(1)}\phi^2 K_{av} + C_{12}}{\phi}, \quad B^{(2)} = \frac{S^{(3)}\phi^2 K_{av} + C_{23}}{\phi},\\
&B^{(3)} = \left(C_{13} + S^{(3)}C_{12} + S^{(1)}C_{23} + S^{(1)}S^{(3)}\phi^2 K_{av}\right),\\
&D^{(3)} = B^{(3)} - \frac{1}{3}\mu^{(13)} \quad \text{in 3D}, \quad D^{(3)} = B^{(3)} - \frac{1}{2}\mu^{(13)} \quad \text{in 2D},
\end{aligned}
$$

where

$$
\begin{aligned}
&C_{12} = \phi K_{av}(\alpha^{(1)} - S^{(1)}\phi), \quad C_{23} = \phi K_{av}(\alpha^{(3)} - S^{(3)}\phi),\\
&C_{13} = K_{av}(\alpha^{(1)} - S^{(1)}\phi)(\alpha^{(3)} - S^{(3)}\phi).
\end{aligned}
$$

The moduli $K_G^{(1)}$ and $K_G^{(3)}$ are the analogues of the Gassmann's moduli [21], while the coefficients $\alpha^{(1)}$ and $\alpha^{(3)}$ correspond to the effective stress coefficients in the classic Biot theory [5, 7].

A.2 Modification of the elastic coefficients to introduce viscoelasticity

In order to introduce viscoelasticity we use the correspondence principle stated by M. Biot [3, 5], *i.e.,* we replace the real poroelastic coefficients in the constitutive relations by complex frequency dependent poroviscoelastic moduli satisfying the same relations as in the elastic case. In this work the linear viscoelastic model presented in [28] is used to make the moduli in (2.3a)–(2.3c) complex and frequency dependent. The set of poroviscoelastic moduli is computed using the following formula:

$$M(\omega) = \frac{M_{re}}{R_M(\omega) - iT_M(\omega)},$$

where the symbol 'M' represents any of the moduli in (2.3a)–(2.3c) and the coefficients M_{re} is the relaxed elastic modulus associated with M [6]. The frequency dependent functions R_M and T_M, associated with a continuous spectrum of relaxation times, characterize the viscoelastic behavior and are given by [28]

$$R_M(\omega) = 1 - \frac{1}{\pi \widehat{Q}_M} \ln \frac{1+\omega^2 T_{1,M}^2}{1+\omega^2 T_{2,M}^2}, \qquad T_M(\omega) = \frac{2}{\pi \widehat{Q}_M} \tan^{-1} \frac{\omega(T_{1,M} - T_{2,M})}{1+\omega^2 T_{1,M} T_{2,M}}.$$

The model parameters $\widehat{Q}_M$, $T_{1,M}$ and $T_{2,M}$ are taken such that the quality factors $Q_M(\omega) = T_M/R_M$ are approximately equal to the constant $\widehat{Q}_M$ in the range of frequencies where the equations are solved, which makes this model convenient for geophysical applications. Values of $\widehat{Q}_M$ range from $\widehat{Q}_M = 20$ for highly dissipative materials to about $\widehat{Q}_M = 1000$ for almost elastic ones.

A.3 Mass and dissipative coefficients

Let $\rho_m, m = 1, 2, 3$, denote the mass density of each solid and fluid constituent in Ω. In [39] the coefficients in the mass matrix $\mathcal{P}$ were taken to be real and frequency independent with $p_{jk} = m_{jk}$ and the m_{jk}-entries defined by the relations

$$\begin{aligned} m_{11} &= \rho^{(2)}\phi\left(1 + (S^{(1)})^2 a_{32} + (S^{(3)})^2 a_{12} - 2S^{(3)} - (S^{(1)})^2\right) \\ &\quad + a^{(13)}\rho^{(1)}\phi^{(1)} + (a_{31} - 1)\rho^{(3)}\phi^{(3)}, \end{aligned} \tag{A.1a}$$

$$m_{12} = \rho^{(2)}\left(S^{(3)}(1 - a_{12}) + S^{(1)} a_{32}\right), \tag{A.1b}$$

$$\begin{aligned} m_{13} &= \rho^{(2)}\phi\left(1 - (S^{(1)})^2 a_{32} - (S^{(3)})^2 a_{12} - S^{(1)}S^{(3)}\right) \\ &\quad + \rho^{(1)}\phi^{(1)}(1 - a^{(13)}) + \rho^{(3)}\phi^{(3)}(1 - a_{31}), \end{aligned} \tag{A.1c}$$

$$m_{22} = \frac{\rho^{(2)}}{\phi}\left(a_{12} + a_{32} - 1\right), \tag{A.1d}$$

$$m_{23} = \rho^{(2)}\left(S^{(1)}(1 - a_{32}) + S^{(3)} a_{12}\right), \tag{A.1e}$$

$$\begin{aligned} m_{33} &= \rho^{(2)}\phi\left(1 + (S^{(1)})^2 a_{32} + (S^{(3)})^2 a_{12} - 2S^{(1)} - (S^{(3)})^2\right) \\ &\quad + a_{31}\rho^{(3)}\phi^{(3)} + (a^{(13)} - 1)\rho^{(1)}\phi^{(1)}. \end{aligned} \tag{A.1f}$$

Here,

$$a_{12} = \frac{\phi^{(1)}\rho}{\phi\rho^{(2)}} r_{12} + 1, \quad a_{32} = \frac{\phi^{(3)}\rho'}{\phi\rho^{(2)}} r_{32} + 1,$$
$$a_{13} = \frac{\phi^{(3)}\rho'}{\phi^{(1)}\rho^{(1)}} r_{13} + 1, \quad a_{31} = \frac{\phi^{(1)}\rho}{\phi^{(3)}\rho^{(3)}} r_{31} + 1,$$

with the r_{jk}'s being the geometrical aspects of the boundaries separating the phases j and k (equal to $\frac{1}{2}$ for spheres) and

$$\rho = \frac{\phi\rho^{(2)} + \phi^{(3)}\rho^{(3)}}{\phi + \phi^{(3)}}, \quad \rho' = \frac{\phi\rho^{(2)} + \phi^{(1)}\rho^{(1)}}{\phi + \phi^{(1)}}.$$

Next, in [39] the elements in the matrix $\mathcal{B}$ were taken to be $b_{11} = f_{11}$, $b_{12} = f_{12}$, and $b_{22} = f_{22}$, with f_{11}, f_{12}, and f_{22} computed as follows. For the case of frozen porous media, let

$$d_{12} = \frac{\phi^2}{\kappa_1}, \quad d_{23} = \frac{\phi^2}{\kappa_3},$$

where and the permeability coefficients κ_1 and κ_3 are defined in terms of the absolute permeabilities $\kappa_{1,0}$ and $\kappa_{3,0}$ of the two solid frames by the relations (see also [25, 26])

$$\kappa_1 = \kappa_{1,0} \frac{\phi^{(3)}}{\left(1 - \phi^{(1)}\right)^3}, \quad \kappa_3 = \kappa_{3,0} \frac{\left(1 - \phi^{(1)}\right)^2}{\left(\phi^{(3)}\right)^2} \left(\frac{\phi}{\phi^{(1)}}\right)^3.$$

For the case of shaley sandstones, following [8], set

$$d_{12} = 45 \left(R^{(s1)}\right)^{-2} \frac{1-\phi}{\phi} \phi^{(1)}, \quad d_{23} = 45 \left(R^{(s3)}\right)^{-2} \frac{1-\phi}{\phi} \phi^{(3)},$$

where $R^{(s1)}$ and $R^{(s3)}$ denote the average radii of the sand and clay particles, respectively. The permeability coefficients are defined in this case by

$$\kappa_1 = 2 \frac{\left(R^{(s1)}\right)^2}{9\phi^{(1)}}, \quad \kappa_3 = 2 \frac{\left(R^{(s3)}\right)^2}{9\phi^{(3)}}.$$

Then the frequency independent friction coefficients f_{11}, f_{12} and f_{22} are determined from the equations

$$g_{11} = d_{12}\left(S^{(3)}\right)^2 + d_{23}\left(S^{(1)}\right)^2, \quad g_{12} = \frac{d_{12}S^{(3)} - d_{23}S^{(1)}}{\phi}, \quad g_{22} = \frac{d_{12} + d_{23}}{\phi^2}, \quad \text{(A.2)}$$
$$f_{11} = \eta\, g_{11} + g_{13}, \quad f_{12} = \eta\, g_{12}, \quad f_{22} = \eta\, g_{22},$$

where η denotes the fluid viscosity and in the case of frozen porous media g_{13} is a friction coefficient between the ice and the solid frames, while g_{13} is taken to be zero in the case of shaley sandstones.

In the high frequency range the set of inertial and friction coefficients $m_{jk}, 1 \leq j, k \leq 3$, and f_{11} f_{12}, and f_{22} need to be modified to include the departure of the relative flow from laminar type above a certain critical frequency depending on the pore radius as explained in [4]. For that purpose, the dissipative terms f_{11}, f_{12} and f_{22} in (A.2) are modified by multiplying the fluid viscosity η by the frequency correction factor $F(\theta)$, where the complex valued frequency dependent function $F(\theta) = F_R(\theta) + iF_I(\theta)$ is the frequency correction function defined by Biot [4]:

$$F(\theta) = \frac{1}{4}\frac{\theta T(\theta)}{1 - \frac{2}{i\theta}T(\theta)}, \quad T(\theta) = \frac{\mathrm{ber}'(\theta) + i\mathrm{bei}'(\theta)}{\mathrm{ber}(\theta) + i\mathrm{bei}(\theta)},$$

with $\mathrm{ber}(\theta)$ and $\mathrm{bei}(\theta)$ being the Kelvin functions of the first kind and zero order. The frequency dependent argument $\theta = \theta(\omega)$ is given in terms of the pore size parameter a_p by the equations:

$$\theta = a_p\sqrt{\omega\,\rho^{(2)}/\eta}, \quad a_p = 2\sqrt{\kappa A_0/\phi},$$

where $\dfrac{1}{\kappa} = \dfrac{1}{\kappa_1} + \dfrac{1}{\kappa_3}$ and A_0 is the Kozeny-Carman constant [2, 23].

Consequently the frequency dependent mass and viscous coupling coefficients are defined in the following fashion

$$p_{11}(\omega) = m_{11} + \frac{\eta F_I(\theta) g_{11}}{\omega}, \quad p_{12}(\omega) = m_{12} - \frac{\eta F_I(\theta) g_{12}}{\omega}, \tag{A.3a}$$

$$p_{13}(\omega) = m_{13} - \frac{\eta F_I(\theta) g_{11}}{\omega}, \quad p_{22}(\omega) = m_{22} + \frac{\eta F_I(\theta) g_{22}}{\omega}, \tag{A.3b}$$

$$p_{23}(\omega) = m_{23} + \frac{\eta F_I(\theta) g_{12}}{\omega}, \quad p_{33}(\omega) = p_{33} + \frac{\eta F_I(\theta) g_{11}}{\omega}, \tag{A.3c}$$

$$b_{11}(\omega) = \eta F_R(\theta) g_{11} + g_{13}, \quad b_{12}(\omega) = \eta F_R(\theta) g_{12}, \quad b_{22}(\omega) = \eta F_R(\theta) g_{22}. \tag{A.3d}$$

The coefficients g_{13} is left as a free parameter chosen so that the condition (2.4) is satisfied.

B A first order absorbing boundary condition

The absorbing boundary condition (2.8) will be first derived in the space-time domain in the 3D elastic case, that is, ignoring dissipation due to viscoelastic behavior of the solid phases and ignoring viscous damping. Thus the entries p_{jk} in $\mathcal{M}$ are taken to be equal to the coefficients m_{jk} defined in (A.1) and the coefficients in the constitutive relations (2.3a)–(2.3c) are taken to be real and equal to their relaxed values M_{re} (see Appendix A.2), where M stands for any of the coefficients in those relations. Also, later the changes for the 2D case will be indicated.

Fix the wave velocity $c > 0$. Denote by $u^{(\iota,c)}, \iota = 1, 2, 3$, the displacements in the phase ι in Ω whose wave fronts arrive normally to any part of the boundary Γ with speed c. According to (2.5) the conservation of momentum on Γ can be written as

$$c\left(m_{11}\dot{u}^{(1,c)} + m_{12}\dot{u}^{(2,c)} + m_{13}\dot{u}^{(3,c)}\right) = -\sigma^{(1)}\nu = -\frac{\partial W}{\partial \epsilon_{jk}(u^{1,c})}, \tag{B.1a}$$

$$c\left(m_{12}\dot{u}^{(1,c)} + m_{22}\dot{u}^{(2,c)} + m_{23}\dot{u}^{(3,c)}\right) = p_f\nu = \frac{\partial W}{\partial \zeta}\nu_j, \tag{B.1b}$$

$$c\left(m_{13}\dot{u}^{(1,c)} + m_{23}\dot{u}^{(2,c)} + m_{33}\dot{u}^{(3,c)}\right) = -\sigma^{(3)}\nu = -\frac{\partial W}{\partial \epsilon_{jk}(u^{3,c})}\nu_k. \tag{B.1c}$$

In the above equations $\dot{u}^{(\iota,c)}$ denotes the time derivative of $u^{(\iota,c)}, \iota = 1, 2, 3$.

Let χ^1 and χ^2 be two unit tangent vectors of Γ so that $\{\nu, \chi^1, \chi^2\}$ forms an orthonormal system on Γ. Taking inner product with χ^1, χ^2 in (B.1), we obtain

$$\dot{u}^{(2,c)} \cdot \chi^l = -\frac{m_{12}\dot{u}^{(1,c)} \cdot \chi^l + m_{23}\dot{u}^{(3,c)} \cdot \chi^l}{m_{22}}, \quad l = 1, 2. \tag{B.2}$$

Let us introduce the variables

$$v_1^c = \frac{1}{c}\dot{u}^{(1,c)} \cdot \nu, \quad v_2^c = \frac{1}{c}\dot{u}^{(1,c)} \cdot \chi^1, \quad v_3^c = \frac{1}{c}\dot{u}^{(1,c)} \cdot \chi^2, \quad v_4^c = \frac{1}{c}\dot{u}^{(2,c)} \cdot \nu,$$
$$v_5^c = \frac{1}{c}\dot{u}^{(3,c)} \cdot \nu, \quad v_6^c = \frac{1}{c}\dot{u}^{(3,c)} \cdot \chi^1, \quad v_7^c = \frac{1}{c}\dot{u}^{(3,c)} \cdot \chi^2.$$

Taking inner product with $\{\nu, \chi^1, \chi^2\}$ in (B.1)-(B.1) and using (B.2), one sees that the following seven equations must hold on Γ:

$$c^2\left(m_{11}v_1^c + m_{12}v_4^c + m_{13}v_5^c\right) = -\sigma^{(1)}\nu \cdot \nu, \tag{B.3a}$$

$$c^2\left(q_1 v_2^c + q_2 v_6^c\right) = -\sigma^{(1)}\nu \cdot \chi^1, \quad c^2\left(q_1 v_3^c + q_2 v_7^c\right) = -\sigma^{(1)}\nu \cdot \chi^2, \tag{B.3b}$$

$$c^2\left(m_{12}v_1^c + m_{22}v_4^c + m_{23}v_5^c\right) = p_f, \tag{B.3c}$$

$$c^2\left(m_{13}v_1^c + m_{23}v_4^c + m_{33}v_5^c\right) = -\sigma^{(3)}\nu \cdot \nu, \tag{B.3d}$$

$$c^2\left(q_2 v_2^c + q_3 v_6^c\right) = -\sigma^{(3)}\nu \cdot \chi^1, \quad c^2\left(q_2 v_3^c + q_3 v_7^c\right) = -\sigma^{(3)}\nu \cdot \chi^2, \tag{B.3e}$$

where

$$q_1 = m_{11} - \frac{(m_{12})^2}{m_{22}}, \quad q_2 = m_{13} - \frac{m_{12}m_{23}}{m_{22}}, \quad q_3 = m_{33} - \frac{(m_{23})^2}{m_{22}}.$$

Next, we write the force

$$\mathcal{F} = \left(\sigma^{(1)}\nu \cdot \nu, \sigma^{(1)}\nu \cdot \chi^1, \sigma^{(1)}\nu \cdot \chi^2, -p_f, \sigma^{(3)}\nu \cdot \nu, \sigma^{(3)}\nu \cdot \chi^1, \sigma^{(3)}\nu \cdot \chi^2\right)^t$$

on Γ associated with the arrival of the wave front traveling with speed c in terms of the new variable $\mathbf{v}^c = (v_1^c, \cdots, v_7^c)^t$. For that purpose, recall that on the surface Γ, the strain tensor $\epsilon_{jk}(u^{(\iota,c)})$ can be written in the form [29, 38]

$$\epsilon_{jk}(u^{(\iota,c)}) = \frac{1}{2}\left(\frac{\partial u_j^{(\iota,c)}}{\partial x_k} + \frac{\partial u_k^{(\iota,c)}}{\partial x_j}\right) = -\frac{1}{2}\left(\nu_k \frac{1}{c}\dot{u}_j^{(\iota,c)} + \nu_j \frac{1}{c}\dot{u}_k^{(\iota,c)}\right), \ \iota = 1, 2, 3. \tag{B.4}$$

By using (B.4) in the constitutive relations (2.3a)–(2.3c) and defining the matrices

$$\mathcal{M}_c = \begin{bmatrix} m_{11} & 0 & 0 & m_{12} & m_{13} & 0 & 0 \\ 0 & q_1 & 0 & 0 & 0 & q_2 & 0 \\ 0 & 0 & q_1 & 0 & 0 & 0 & q_2 \\ m_{12} & 0 & 0 & m_{22} & m_{23} & 0 & 0 \\ m_{13} & 0 & 0 & m_{23} & m_{33} & 0 & 0 \\ 0 & q_2 & 0 & 0 & 0 & q_3 & 0 \\ 0 & 0 & q_2 & 0 & 0 & 0 & q_3 \end{bmatrix},$$

$$\mathcal{E}_c = \begin{bmatrix} \lambda_{re}^{(1)} + 2\mu_{re}^{(1)} & 0 & 0 & B_{re}^{(1)} & D_{re}^{(3)} + \mu_{re}^{(13)} & 0 & 0 \\ 0 & \mu_{re}^{(1)} & 0 & 0 & 0 & \frac{1}{2}\mu_{re}^{(13)} & 0 \\ 0 & 0 & \mu_{re}^{(1)} & 0 & 0 & 0 & \frac{1}{2}\mu_{re}^{(13)} \\ B_{re}^{(1)} & 0 & 0 & K_{av,re} & B_{re}^{(2)} & 0 & 0 \\ D_{re}^{(3)} + \mu_{re}^{(13)} & 0 & 0 & B_{re}^{(2)} & \lambda_{re}^{(3)} + 2\mu_{re}^{(3)} & 0 & 0 \\ 0 & \frac{1}{2}\mu_{re}^{(13)} & 0 & 0 & 0 & \mu_{re}^{(3)} & 0 \\ 0 & 0 & \frac{1}{2}\mu_{re}^{(13)} & 0 & 0 & 0 & \mu_{re}^{(3)} \end{bmatrix},$$

the equations (B.3) can be restated as follows

$$c^2 \mathcal{M}_c \mathbf{v}^c = -\mathcal{F} = \mathcal{E}_c \mathbf{v}^c \quad \text{on } \Gamma. \tag{B.5}$$

On the other hand, applying (B.4) in (2.2), we conclude that the strain energy density $\mathcal{W}$ on Γ can be written in the form

$$\mathcal{W}(\mathbf{v}^c) = \frac{1}{2}(\mathbf{v}^c)^t \; \mathcal{E}_c \; \mathbf{v}^c,$$

so that (B.5) can be stated in the equivalent form

$$c^2 \mathcal{M}_c \mathbf{v}^c = -\mathcal{F} = \frac{\partial \mathcal{W}(\mathbf{v}^c)}{\partial \mathbf{v}^c}. \tag{B.6}$$

Set

$$\mathcal{S}_c = \mathcal{M}_c{}^{-\frac{1}{2}} \mathcal{E}_c \mathcal{M}_c{}^{-\frac{1}{2}}, \quad \text{and } \widetilde{\mathbf{v}}^c = \mathcal{M}_c{}^{\frac{1}{2}} \mathbf{v}^c.$$

Then, in terms of the variable $\widetilde{\mathbf{v}}^c$, (B.5) can be put as an eigenvalue matrix form:

$$\mathcal{S}_c \widetilde{\mathbf{v}}^c = c^2 \widetilde{\mathbf{v}}^c,$$

and the strain energy density $\mathcal{W}$ on Γ can be written as follows:

$$\mathcal{W}(\widetilde{\mathbf{v}}^c) = \frac{1}{2}(\widetilde{\mathbf{v}}^c)^t \; \mathcal{S}_c \; \widetilde{\mathbf{v}}^c. \tag{B.7}$$

The seven eigenvalues $(c_j)^2, j = 1, \cdots, 7$, in the equation

$$\det(\mathcal{S}_c - c^2 I) = 0,$$

are the wave speeds of the system. Four of them correspond to the shear modes of propagation, with only two of them being different, and the other three to the compressional modes. Let $N_j, j = 1, \cdots, 7$, be the set of orthonormal eigenvectors of $\mathcal{S}_c$ associated with the eigenvalues $(c_j)^2, j = 1, \cdots, 7$, $\mathcal{N}$ the matrix containing as columns the eigenvectors N_j, and $\mathcal{C}$ the diagonal matrix containing the eigenvalues $(c_j)^2$. Then we have the following diagonalization form

$$\mathcal{N}^t \mathcal{S}_c \mathcal{N} = \mathcal{C}.$$

Let

$$\mathbf{z} = \left(\dot{u}^{(1)} \cdot \nu, \dot{u}^{(1)} \cdot \chi^1, \dot{u}^{(1)} \cdot \chi^2, \dot{u}^{(2)} \cdot \nu, \dot{u}^{(3)} \cdot \nu, \dot{u}^{(3)} \cdot \chi^1, \dot{u}^{(3)} \cdot \chi^2 \right)^t$$

be a general velocity field on Γ due to the simultaneous arrival of waves with speeds $c_j, j = 1, \cdots, 7$. Setting

$$\widetilde{\mathbf{z}} = \mathcal{M}_c{}^{\frac{1}{2}}\mathbf{z},$$

notice that the orthogonality of the N_j's allows us to represent $\widetilde{\mathbf{z}}$ in the form

$$\widetilde{\mathbf{z}} = \sum_j (N_j \cdot \widetilde{\mathbf{z}})\, N_j = \sum_j \left(N_j \cdot \mathcal{M}_c{}^{\frac{1}{2}}\mathbf{z}\right) N_j.$$

Let us denote

$$\widetilde{\mathbf{z}}^{c_j} = \mathcal{M}_c{}^{\frac{1}{2}}\mathbf{z}^{c_j} := \frac{1}{c_j}\left(N_j \cdot \mathcal{M}_c{}^{\frac{1}{2}}\mathbf{z}\right) N_j. \tag{B.8}$$

Recall that N_j is an eigenvector of $\mathcal{S}_c$, associated with eigenvalue c_j^2. Thus $\widetilde{\mathbf{z}}^{c_j}$ is also an eigenvector of $\mathcal{S}_c$ associated with the same eigenvalue c_j^2:

$$\mathcal{S}_c\widetilde{\mathbf{z}}^{c_j} = c_j^2\widetilde{\mathbf{z}}^{c_j}, \tag{B.9}$$

and therefore, according to (B.7), we get

$$\mathcal{W}(\widetilde{\mathbf{z}}^{c_j}) = \frac{1}{2}(\widetilde{\mathbf{z}}^{c_j})^t\ \mathcal{S}_c\ \widetilde{\mathbf{z}}^{c_j}.$$

Also, using (B.6) and (B.7), we see that the force $\mathcal{F}_j$ associated with $\widetilde{\mathbf{z}}^{c_j}$ satisfies the equation

$$\mathcal{M}^{\frac{1}{2}}\frac{\partial\mathcal{W}(\widetilde{\mathbf{z}}^{c_j})}{\partial\widetilde{\mathbf{z}}^{c_j}} = \mathcal{M}_c{}^{\frac{1}{2}}\mathcal{S}_c\widetilde{\mathbf{z}}^{c_j} = \mathcal{E}_c\mathbf{z}^{c_j} = -\mathcal{F}_j.$$

Assuming that the interaction among the different waves arriving at a given interface Γ is small compared with the total energy involved (see Santos *et al.* [37] for the validity of this assumption in the case of a single solid phase), the total strain energy density on Γ is equal to the sum of the partial energies and the total force $\mathcal{F}$ on Γ is equal to the sum of the forces associated with each type of wave, so that

$$\mathcal{W}(\widetilde{\mathbf{z}}) = \sum_j \mathcal{W}(\widetilde{\mathbf{z}}^{c_j}),$$

and

$$\mathcal{F} = \sum_j \mathcal{F}_j = -\sum_j \mathcal{M}_c{}^{\frac{1}{2}}\mathcal{S}_c\widetilde{\mathbf{z}}^{c_j}. \tag{B.10}$$

Next, decompose $\mathcal{M}_c{}^{-\frac{1}{2}}\mathcal{F}$ in term of the eigenvectors N_j as follows:

$$\mathcal{M}_c{}^{-\frac{1}{2}}\mathcal{F} = \sum_j \left(N_j \cdot \mathcal{M}_c{}^{-\frac{1}{2}}\mathcal{F}\right) N_j. \tag{B.11}$$

In the meanwhile, using (B.8)-(B.9) in (B.10), we have

$$\mathcal{M}_c{}^{-\frac{1}{2}}\mathcal{F} = -\sum_j \mathcal{S}_c\widetilde{\mathbf{z}}^{c_j} = -\sum_j c_j^2\widetilde{\mathbf{z}}^{c_j} = -\sum c_j\left(N_j \cdot \mathcal{M}_c{}^{\frac{1}{2}}\mathbf{z}\right) N_j. \tag{B.12}$$

Thus, from (B.11) and (B.12) we conclude that

$$c_j \left(N_j \cdot \mathcal{M}_c{}^{\frac{1}{2}} \mathbf{z} \right) = - \left(N_j \cdot \mathcal{M}_c{}^{-\frac{1}{2}} \mathcal{F} \right), \; j = 1, \cdots, 7. \tag{B.13}$$

In matrix form, (B.13) can be expressed as follows:

$$\mathcal{C}^{\frac{1}{2}} \mathcal{N}^t \mathcal{M}_c{}^{\frac{1}{2}} \mathbf{z} = - \mathcal{N}^t \mathcal{M}_c{}^{-\frac{1}{2}} \mathcal{F},$$

and multiplying (B.13) to the left by $\mathcal{M}_c{}^{\frac{1}{2}} \mathcal{N}$, we see that under the assumption of small interaction among the different types arriving simultaneously to Γ, the conservation of momentum on Γ can be stated in the form

$$-\mathcal{F} = \mathcal{M}_c{}^{\frac{1}{2}} \mathcal{S}_c{}^{\frac{1}{2}} \mathcal{M}_c{}^{\frac{1}{2}} \mathbf{z} = \left(\mathcal{E}_{\mathbf{c}} \mathcal{M}_{\mathbf{c}}{}^{-\mathbf{1}} \right)^{\frac{1}{2}} \mathcal{M}_{\mathbf{c}} \mathbf{z} \equiv \mathcal{D} \mathbf{z}. \tag{B.14}$$

Note that $\mathcal{D}$ is positive definite. By identifying the Fourier transforms of $\mathcal{F}$ and $\mathbf{z}$ by $\mathcal{G}_\Gamma(u(x,\omega))$ and $S_\Gamma(u(x,\omega))$, respectively, one sees that the boundary condition (2.8) is the Fourier transformation in time of (B.14).

Remark B.1. *In the 2D case the form of the absorbing boundary condition in (B.14) remains identical if the matrices $\mathcal{M}_c$ and $\mathcal{E}_c$ are modified as follows:*

$$\mathcal{M}_c = \begin{bmatrix} m_{11} & 0 & m_{12} & m_{13} & 0 \\ 0 & q_1 & 0 & 0 & q_2 \\ m_{12} & 0 & m_{22} & m_{23} & 0 \\ m_{13} & 0 & m_{23} & m_{33} & 0 \\ 0 & q_2 & 0 & 0 & q_3 \end{bmatrix},$$

$$\mathcal{E}_c = \begin{bmatrix} \lambda_{re}^{(1)} + 2\mu_{re}^{(1)} & 0 & B_{re}^{(1)} & D_{re}^{(3)} + \mu_{re}^{(13)} & 0 \\ 0 & \mu^{(1)} & 0 & 0 & \frac{1}{2}\mu_{re}^{(13)} \\ B_{re}^{(1)} & 0 & K_{av,re} & B_{re}^{(2)} & 0 \\ D_{re}^{(3)} + \mu_{re}^{(13)} & 0 & B_{re}^{(2)} & \lambda_{re}^{(3)} + 2\mu^{(3,re)} & 0 \\ 0 & \frac{1}{2}\mu_{re}^{(13)} & 0 & 0 & \mu_{re}^{(3)} \end{bmatrix}.$$

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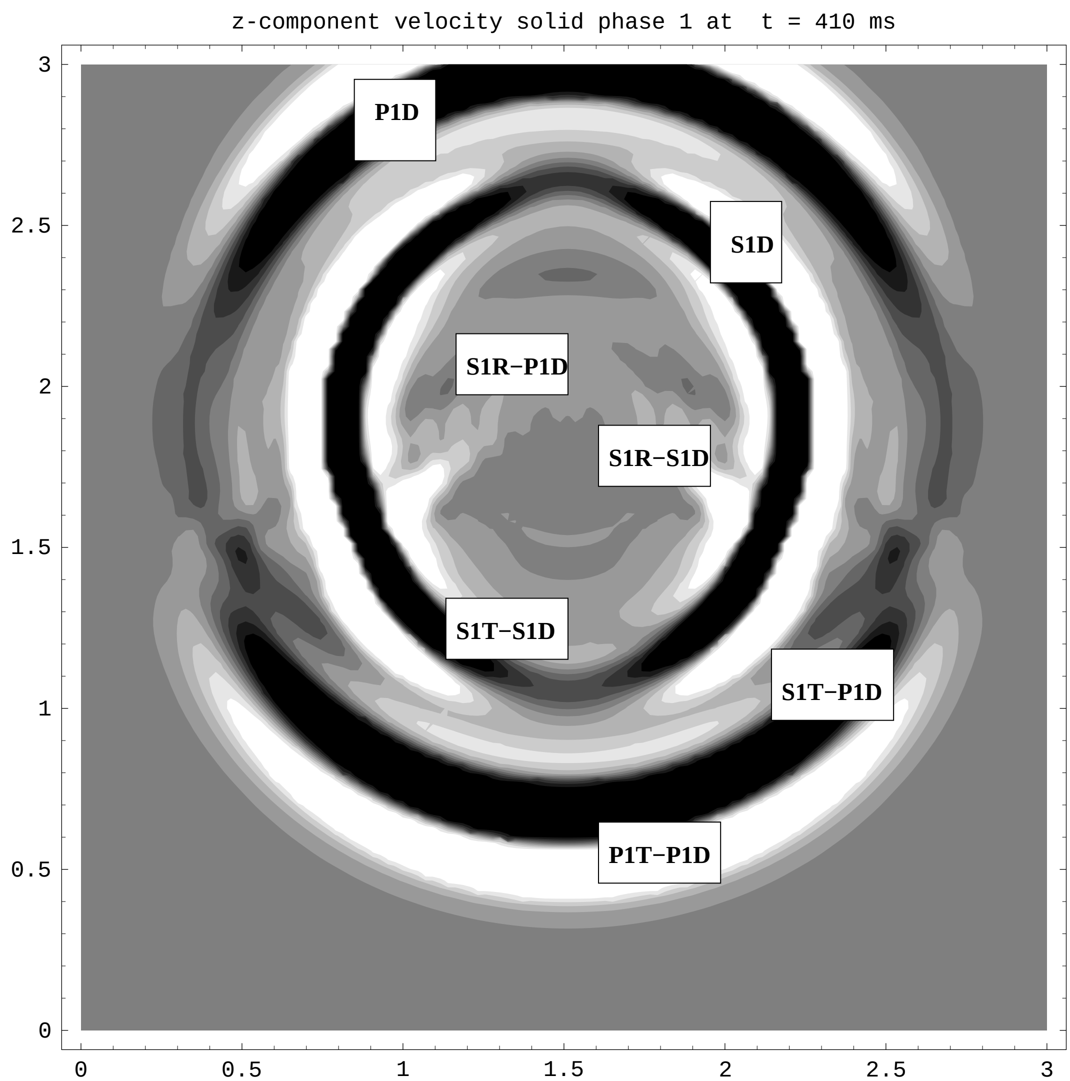

Figure 1. Snapshot of the vertical particle velocity of the solid matrix phase at $t = 410$ ms

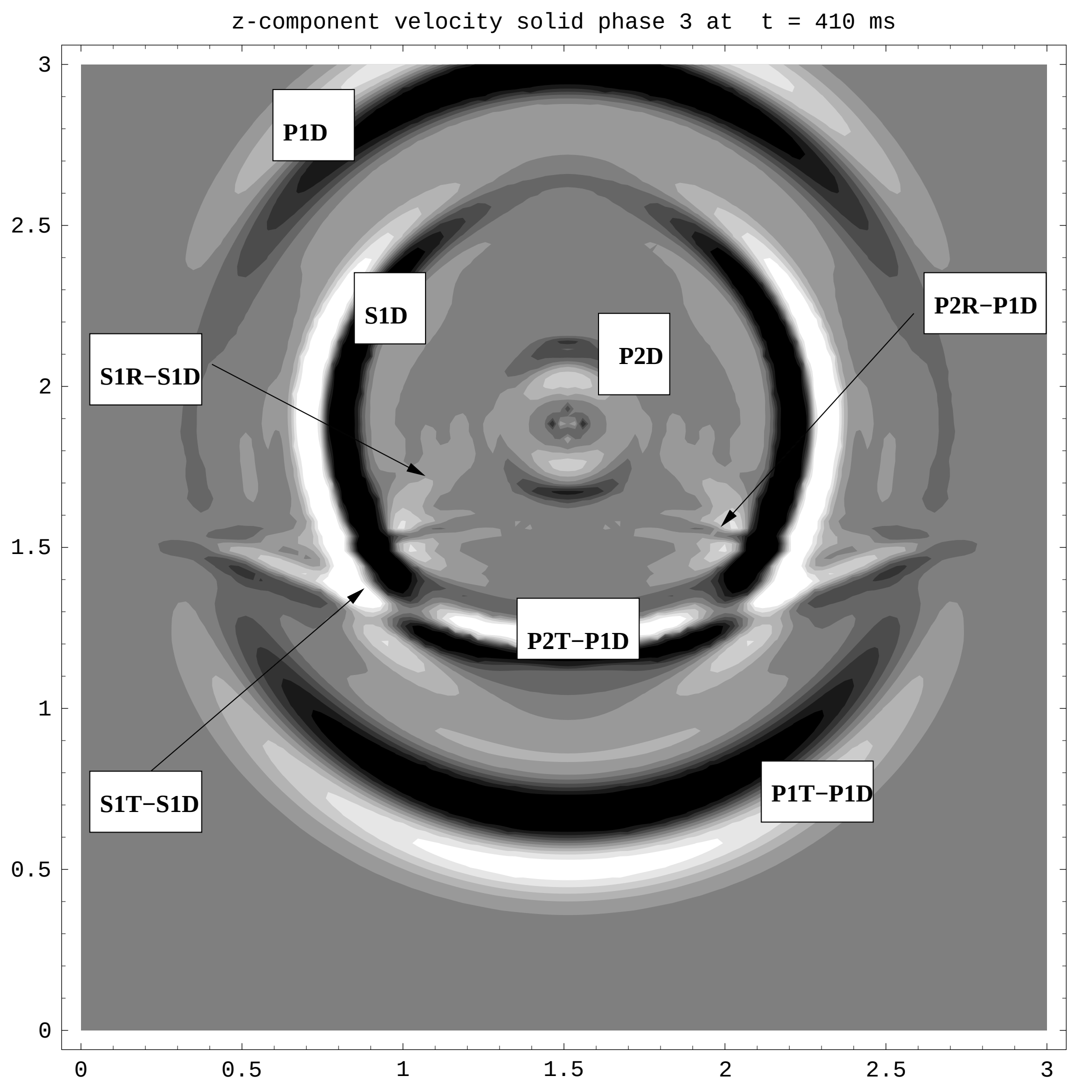

Figure 2. Snapshot of the vertical particle velocity of the ice phase at $t = 410$ ms

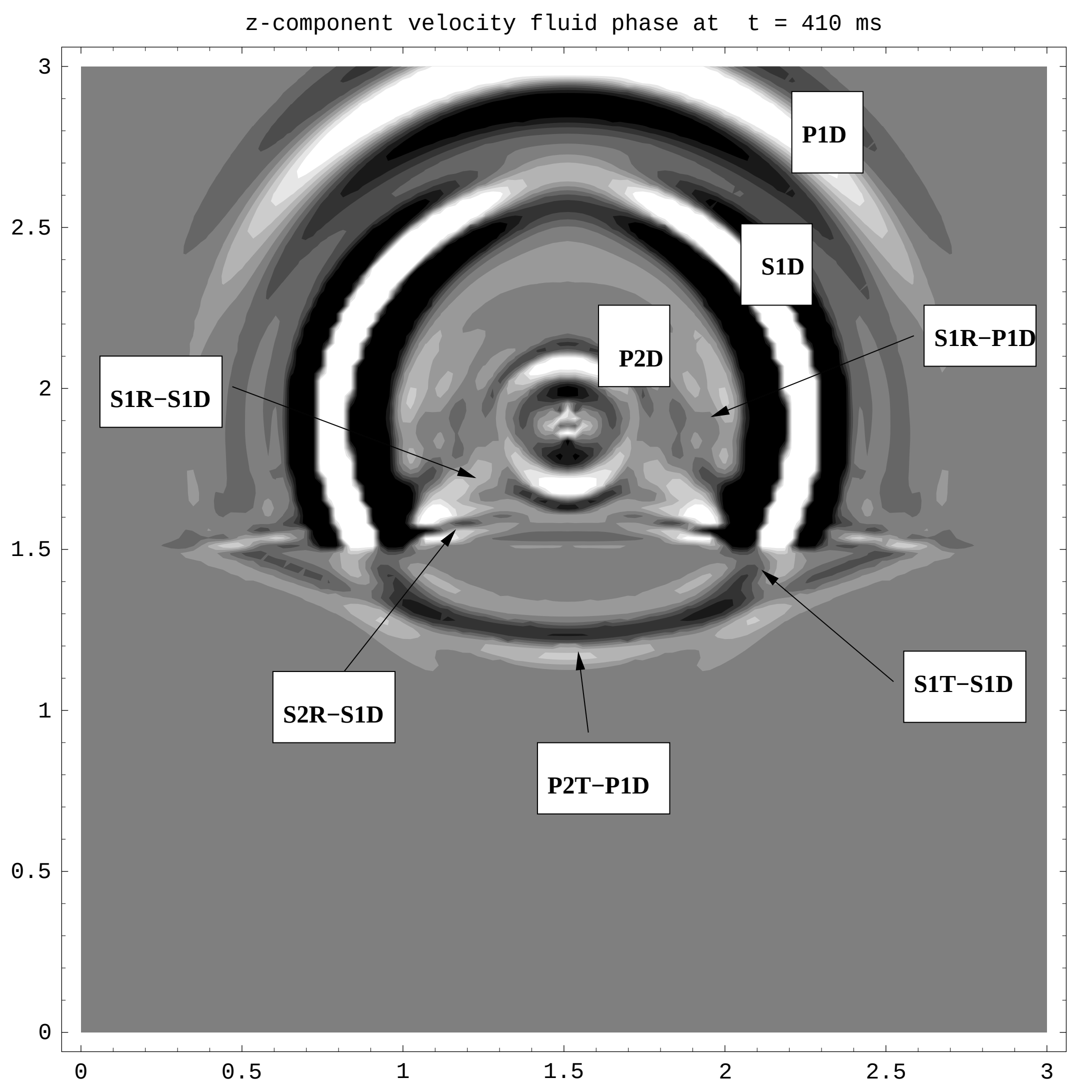

Figure 3. Snapshot of the vertical particle velocity of the fluid phase at $t = 410$ ms