INDIVIDUAL HOMOGENIZATION OF NONLINEAR PARABOLIC OPERATORS

Y. EFENDIEV*, AND L. JIANG †, AND A. PANKOV‡

Abstract.
In this paper, we prove an individual homogenization result for a class of almost periodic nonlinear parabolic operators. The spatial and temporal heterogeneities are almost periodic functions in the sense of Besicovitch. The latter allows discontinuities and suitable for many applications. First, we derive stability and comparison estimates for sequences of G-convergent nonlinear parabolic operators. Further, using these estimates, the individual homogenization result is shown.

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1. Introduction. In the present paper, we consider the homogenization problem for nonlinear parabolic operators of the form

$$\mathcal{L}_\varepsilon(u) = D_t u - div\left(a\left(\frac{x}{\varepsilon^\beta}, \frac{t}{\varepsilon^\alpha}, u, D_x u\right)\right) + a_0\left(\frac{x}{\varepsilon^\beta}, \frac{t}{\varepsilon^\alpha}, u, D_x u\right), \tag{1.1}$$

where the flux functions $a(y,\tau,\cdot,\cdot)$ and $a_0(y,\tau,\cdot,\cdot)$ are almost periodic in $(y,\tau) \in R^{n+1}$ in the sense of Besicovitch.

We are interested in the asymptotic behavior of $\mathcal{L}_\varepsilon$ as $\varepsilon \to 0$. G-convergence theory for parabolic operators guarantees that the limiting operator $\widehat{\mathcal{L}}$ belongs to the same class of parabolic operators. G-convergence of nonlinear parabolic operators has been studied in [20]. To find the form of $\widehat{\mathcal{L}}$ some assumptions on the nature of spatial and temporal heterogeneities of a and a_0 need to be imposed. In the periodic setting, the homogenization of nonlinear parabolic equations is carried out in [20]. In [4], time homogenization of random nonlinear abstract parabolic equations has been studied. The homogenization of linear parabolic operators with almost periodic and random coefficients has been studied in [24, 23]. The homogenization of general nonlinear random parabolic operators is investigated in [11].

Also we would like to mention several results on homogenization of nonlinear elliptic operators [2, 5, 12, 13, 17, 19]. Note that in [17, 19] general elliptic operators in divergence form are considered, including random homogenization, while articles [2, 5, 12, 13] are devoted to the case of monotone second order elliptic operators. As for general references in the field of homogenization, we refer to [1, 3, 6, 7, 15, 20].

We would like to point out that the general result of [11] is of statistical nature, i.e., homogenization takes place for almost all realizations of a random parabolic operator. As it will be shown, any almost periodic operator of the form (1.1) can be considered as a particular realization of certain random homogeneous operator. But this realization is not generic and, therefore, the result of [11] does not apply straightforwardly. Nevertheless, using almost periodicity one can pass from a generic realization to every particular realization. For this purpose, we first derive stability and comparison results for G-convergent sequence of operators. These results allow to estimate the difference between G-limits of two G-convergent sequence of parabolic

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operators. These estimates are of independent interest. Introducing smoothing of almost periodic functions (defined in the sense of Besicovitch), one can derive individual homogenization results. However, to extend these results to almost periodic functions defined in the sense of Besicovitch, we need a careful comparison estimates for sequences of G-convergent parabolic operators. The latter allows us to obtain an individual homogenization result for nonlinear parabolic operators.

Our motivation for considering homogenization of nonlinear parabolic equations comes from applications arisen in flow in porous media for both saturated and unsaturated media, though one encounter nonlinear parabolic equations in many different applications. We refer to [8, 9, 10] for numerical realization of parabolic homogenization in applied problems. In particular, the fluxes that arise in these applications can have discontinuities in space and time. For this purpose, we consider almost periodic functions in the sense of Besicovitch.

The paper is organized as follows. In the next section, we present background material on G-convergence, almost periodic functions and statistical homogenization. In Section 3, we present stability and comparison results, which are essential in deriving individual homogenization results. Finally, in Section 4, we discuss the individual homogenization results.

2. Preliminaries.

2.1. G-convergence. Let $Q_0 \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, $T > 0$, and $Q = Q_0 \times (0, T)$. On Q, we consider parabolic operators of the form

$$\mathcal{L}(u) = D_t u - div\big(a(x, t, u, D_x u)\big) + a_0(x, t, u, D_x u). \tag{2.1}$$

We suppose that the functions $a : Q \times \mathbb{R}^{n+1} \to \mathbb{R}^n$ and $a_0 : Q \times \mathbb{R}^{n+1} \to \mathbb{R}$ satisfy the Carathéodory condition and the following assumptions:

(i) *for every* $\zeta = (\eta, \xi) \in \mathbb{R}^{n+1}$

$$\big|a(x, t, \eta, \xi)\big|^q + \big|a_0(x, t, \eta, \xi)\big|^q \le c_0|\zeta|^p + c \tag{2.2}$$

a. e. on Q, *where* $p > 1$, $c_0 > 0$ *and* $c \ge 0$;

(ii) *for every* $\zeta \in (\eta, \xi)$, $\zeta' = (\eta, \xi') \in \mathbb{R}^{n+1}$

$$\big[a(x, t, \eta, \xi) - a(x, t, \eta, \xi')\big] \cdot (\xi - \xi') \ge \kappa\big[h + |\zeta|^p + |\zeta'|^p\big]^{1-\beta/p}|\xi - \xi'|^\beta \tag{2.3}$$

a. e. on Q, *where* $\kappa > 0$, $\beta \ge \max(p, 2)$, *and* $h \ge 0$;

(iii) *for every* $\zeta = (\eta, \xi)$, $\zeta' = (\eta', \xi') \in \mathbb{R}^{n+1}$

$$\begin{aligned}\big|a(x, t, \eta, \xi)-a(x, t, \eta', \xi')\big|^q + \big|a_0(x, t, \eta, \xi) - a_0(x, t, \eta', \xi')\big|^q \le\\ \le\theta\Big[\big(h + |\zeta|^p + |\zeta'|^p\big)\,\nu\big(|\eta - \eta'|\big) +\\ + \big(h + |\zeta|^p + |\zeta'|^p\big)^{1-s/p}|\xi - \xi'|^s\Big],\end{aligned} \tag{2.4}$$

a. e. on Q, *where* $\theta > 0$, $0 < s \le \min(p, q)$ *and* $\nu(r)$ *is a continuity modulus, i. e. a continuous function* $\nu : [0, +\infty) \to [0, +\infty)$ *such that* $\nu(0) = 0$, $\nu(r) > 0$ *if* $r > 0$ *and* $\nu(r) = 1$ *if* $r \ge 1$.

Here $p > 1$ is fixed and q stands for the conjugate exponent, $q^{-1} + p^{-1} = 1$. In addition, we always assume that $p > 2n/(n + 2)$.

Given $c_0, c, \kappa, h, \theta, \nu, s$ and β, we denote by $\Pi = \Pi(c_0, c, \kappa, h, \theta, \nu, s, \beta)$ the set of all operators of the form (2.1) satisfying (i)–(iii).

Introduce

$$
\begin{gathered}
V = L^p(0,T,W_0^{1,p}(Q_0)), \quad \overline{V} = L^p(0,T,W^{1,p}(Q_0)), \\
W = \{u \in V, D_t u \in L^q(0,T,W^{-1,q}(Q_0))\}, \\
\overline{W} = \{u \in \overline{V}, D_t u \in L^q(0,T,W^{-1,q}(Q_0))\}, \quad W_0 = \{u \in W, u(0) = 0\}.
\end{gathered} \tag{2.5}
$$

For further analysis $X^{'}$ will denote the dual of the space X and $\| \cdot \|_{p,Q}$ denotes the norm in $L^p(Q)$. Any operator $\mathcal{L} \in \Pi$ acts from W_0 into V'.

Now we introduce the notion of G-convergence. For an operator $\mathcal{L}$ of the form (2.1) we set

$$
\mathcal{L}^1(u,v) = D_t u - div\big(a(x,t,v,D_x u)\big).
$$

For every fixed $v \in V$, the operator $u \mapsto \mathcal{L}^1(u,v)$ acts from W_0 into V' and satisfies the coerciveness and strict monotonicity conditions. Therefore, for every $f \in V'$, $v \in V$, the equation

$$
\mathcal{L}^1(u,v) = f
$$

has a unique solution $u \in W_0$ (see, e. g. [16]).

Now let $\mathcal{L}_k \in \Pi$ be a sequence of parabolic operators, with fluxes a^k and a_0^k, and $\mathcal{L} \in \Pi$ of the form (2.1). Given $u \in W_0$ and $v \in V$, we set

$$
\begin{array}{rcl}
\Gamma^k(u,v) & = & a^k(x,t,v,D_x u^k), \\
\Gamma_0^k(u,v) & = & a_0^k(x,t,v,D_x u^k), \\
\Gamma(u,v) & = & a(x,t,v,D_x u),
\end{array}
$$

and

$$
\Gamma_0(u,v) = a_0(x,t,v,D_x u),
$$

where $u_k \in W_0$ is a unique solution of the equation

$$
L_k^1(u_k,v) = L^1(u,v).
$$

The sequence $\mathcal{L}_k$ is called *G-convergent* to $\mathcal{L}$ (in symbols, $\mathcal{L}_k \overset{G}{\Longrightarrow} \mathcal{L}$) if for every $v \in V$ and $u \in W_0$ we have that

$$
\lim u_k = u
$$

weakly in W_0 and

$$
\begin{array}{rcl}
\lim \Gamma^k(u,v) & = & \Gamma(u,v), \\
\lim \Gamma_0^k(u,v) & = & \Gamma_0(u,v)
\end{array}
$$

weakly in $L^q(Q)^n$ and $L^q(Q)$, respectively, as $k \to \infty$. In the following analysis, $k \to \infty$ will be omitted. This notion was introduced in [20], where the term "strong G-convergence" was suggested. In this paper we abbreviate it to "G-convergence" because no other type of such convergence is used here.

The following G-compactness theorem is one of the main results of G-convergence theory (see [20], Theorem 4.1.1).

Theorem 2.1. *Let $\mathcal{L}_k$ be a sequence of parabolic operators of class Π. Then there exist an operator $\mathcal{L}$ of class Π, with possibly another values of parameters $c_0, c, \kappa, h, \theta, s,$ and ν, and a subsequence $\mathcal{L}_{k'}$ that G-converges to $\mathcal{L}$.*

The following theorem is about G-convergence of arbitrary solutions ([20]).

Theorem 2.2. *Assume $\mathcal{L}_k$ G-converges to $\mathcal{L}$, $u_k \in \overline{W}$, $f_k, f \in L^q(0,T,W^{-1,q}(Q_0))$, $\mathcal{L}_k u_k = f_k$, $u_k \to u$ weakly in $\overline{W}$, and $f_k \to f$ strongly in $V^{'}$. Then $\mathcal{L}u = f$, and*

$$a_k(x,t,u_k,D_x u_k) \to a(x,t,u,D_x u),$$
$$a_{0,k}(x,t,u_k,D_x u_k) \to a_0(x,t,u,D_x u)$$

weakly in $L^q(Q)^n$ and $L^q(Q)$, respectively.

2.2. Statistical homogenization results. Let (Ω, Σ, μ) be a probability space and $L^p(\Omega)$ denote the space of all p-integrable functions. Consider $(n+1)$-dimensional dynamical system on Ω, $T(z): \Omega \to \Omega$, $z = (x,t) \in \mathbb{R}^{n+1}$ ($t \in \mathbb{R}$, $x \in \mathbb{R}^n$) that satisfies the following conditions:

1) $T(0) = I$, and $T(x+y) = T(x)T(y)$;

2) $T(z): \Omega \to \Omega$ preserve the measure μ on Ω;

3) For any measurable function $f(\omega)$ on Ω, the function $f(T(z)\omega)$ defined on $R^{n+1} \times \Omega$ is also measurable.

$U(z)f(\omega) = f(T(z)\omega)$ defines a $(n+1)$-parameter group of isometries in the space of $L^p(\Omega)$. $U(z)$ is strongly continuous. Further we assume that the dynamical system T is ergodic, i.e., any measurable T-invariant function on Ω is constant. Denote by $\langle \cdot \rangle$ the mean value over Ω,

$$\langle f \rangle = \int_\Omega f(\omega) d\mu(\omega).$$

For further analysis we will need Birkhoff Ergodic Theorem. Denote

$$M\{f\} = \lim_{s \to \infty} \frac{1}{s^{n+1}|K|} \int_{K_s} f(z) dz,$$

where $K \subset \mathbb{R}^{n+1}$, $|K| \neq 0$, and $K_s = \{z \in \mathbb{R}^{n+1} : s^{-1}z \in K\}$. Let $f(\frac{z}{\epsilon})$ be bounded in $L^p_{loc}(\mathbb{R}^{n+1})$, $1 \leq p < \infty$. Then f has mean value $M\{f\}$ if and only if $f(z/\epsilon) \to M\{f\}$ weakly in $L^p_{loc}(\mathbb{R}^{n+1})$ as $\epsilon \to 0$ ([20], page 134). Birkhoff Ergodic Theorem states that if $f \in L^p(\Omega)$, $1 \leq p < \infty$ then

$$\langle f \rangle = M\{f(T(z)\omega)\} \text{ a.e. on } \Omega.$$

To formulate the auxiliary problem for the homogenization we need the following preliminaries. Following to [23] we define spaces similar to $\overline{V}$ on Ω in the following way. Denote by $\partial_{full} = (\partial_1, \cdots, \partial_{n+1})$ the collection of generators of the group $U(z)$. There is a dense subspace $S \subset L^p(\Omega)$ that contains in the domains of all operators $\partial^\alpha_{full} = \partial_1^{\alpha_1} \cdots \partial_{n+1}^{\alpha_{n+1}}$, $\alpha \in Z^{n+1}_+$.

Further denote by $\mathcal{V}$ the completion of S with respect to the semi-norm

$$\|f\|_{\mathcal{V}} = \left(\sum_{i=1}^n \|\partial_i f\|^p_{L_p(\Omega)} \right)^{1/p}.$$

Note that the completion with respect to a seminorm "cuts off" the kernel of the semi-norm. The operator $\partial = (\partial_1, \ldots, \partial_n) : \mathcal{V} \to L^p(\Omega)^n$ is an isometric embedding. Moreover, the space $\mathcal{V}$ is reflexive, with the dual denoted by $\mathcal{V}^{'}$. By duality we define the operator $\mathbf{div} : L^q(\Omega)^n \to \mathcal{V}^{'}$ by

$$\langle \mathbf{div} u, w \rangle = -\langle u, \partial w \rangle, \ \forall w \in \mathcal{V}. \tag{2.6}$$

We note that the elements of $\mathcal{V}$ in general do not have independent meaning. The space $\mathcal{V}$ contains fields that are not spatially homogeneous. Note that the operators ∂_i may be viewed as derivatives along trajectories of the dynamical system $T(z)$

$$(\partial_i f)(T(z)\omega) = \frac{\partial}{\partial z_i} f(T(z)\omega) \tag{2.7}$$

for a.e. $\omega \in \Omega$ and $f \in Dom(\partial_i, L^p(\Omega))$, [23, 20] (page 138 in [20]).

We set

$$T_1(t) = T(0, \ldots, 0, t), \quad T_2(x) = T(x_1, \ldots, x_n, 0). \tag{2.8}$$

Let

$$M_t\{f_\omega\} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(T_1(\tau)\omega) d\tau, \tag{2.9}$$

$$M_x\{f_\omega\} = \lim_{|K| \to \infty} \frac{1}{|K|} \int_K f(T_2(y)\omega) dy. \tag{2.10}$$

These partial mean values are well-defined for $f \in L^p(\Omega)$, $1 \leq p < \infty$, and for a generic $\omega \in \Omega$. We note that the average of a

$$\overline{a}(\omega, \eta, \xi) = M_t\{a(\omega, \eta, \xi)\}. \tag{2.11}$$

is defined on $L^p(\Omega)$ for each $\eta \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. Consider the subset of S consisting of functions

$$f(\omega) = M_t\{f\}.$$

Denote by $\mathcal{V}_s$ the completion of this set with respect to the norm

$$\|f\| = (\sum_{i=1}^{n} \|\partial_i f\|^p_{L^p(\Omega)})^{1/p}.$$

Now we would like to consider the differentiation with respect to the time along the trajectories as an unbounded operator in appropriate functional spaces. Define an unbounded operator σ from $\mathcal{V}$ into $\mathcal{V}'$ as follows. $\mathcal{V}_1$, defined as the image of operator ∂, is a closed subspace of $L^p(\Omega)^n$ and ∂ maps $\mathcal{V}$ onto $\mathcal{V}_1$ isomorphically. The dual space $\mathcal{V}_1'$ can be identified with the factor space $L^q(\Omega)^n / \mathcal{V}_1^{\perp}$, where $\mathcal{V}_1^{\perp} = ker(\mathbf{div})$ is the orthogonal complement to $\mathcal{V}_1$ in $L^q(\Omega)^n$. Then the operator $\mathbf{div}$ can be considered as an operator $\mathbf{div} : \mathcal{V}_1' \to \mathcal{V}'$. Let $S_1 = \partial(S) \subset \mathcal{V}_1$. It is easy to see that $\partial_i(S_1) \subset S_1$. Now we say that $v \in \mathcal{V}_1$ belongs to the domain $D(\sigma_1)$ if there exists $f \in \mathcal{V}_1'$ such that

$$\langle v, \partial_{n+1}\varphi \rangle = -\langle f, \varphi \rangle, \ \forall \varphi \in S_1$$

and set $\sigma_1 v = f$. The (unbounded) operator σ_1 is a well-defined closed linear operator from $\mathcal{V}_1$ into $\mathcal{V}_1'$ and its domain is dense in $\mathcal{V}_1$. It can be verified [11] that $\sigma_1' = -\sigma_1$, where $\sigma_1' : \mathcal{V}_1 \to \mathcal{V}_1'$ is the adjoint operator to σ_1. Now we set

$$\sigma = \mathbf{div} \circ \sigma_1 \circ \partial\,.$$

Then σ is a closed linear operator from $\mathcal{V}$ into $\mathcal{V}'$, with dense domain $\mathcal{W} = D(\sigma)$, and $\sigma' = -\sigma$. The space $\mathcal{W}$ is endowed with the usual graph norm. As consequence, σ is a maximal monotone operator (see [16], Lemma 1.2 of Ch. 3). Note that in the case $p = 2$ this operator can be defined by means of spectral decomposition theorem [23].

Consider the auxiliary problem

$$\mu\sigma N^{\mu}_{\eta,\xi} - \mathbf{div}\; a(\omega, \eta, \xi + \partial N^{\mu}_{\eta,\xi}) = 0. \tag{2.12}$$

Define the operator A from $\mathcal{W}$ to $\mathcal{W}^{'}$ as

$$\langle Au, v\rangle = \langle a(\omega, \eta, \xi + \partial u), \partial v\rangle.$$

It can be easily verified that A is strongly monotone, continuous, and coercive operator from $\mathcal{W}$ to $\mathcal{W}^{'}$. Since σ is maximal monotone it follows from [16] that the solution of (2.12) in $\mathcal{W}$ exists. Uniqueness follows from the fact that $(\sigma u, u) = 0$ and A is strongly monotone. Thus we have the following lemma [8]

Lemma 2.3. *Equation (2.12) has a unique solution, $N^{\mu}_{\eta,\xi} \in \mathcal{W}$, and*

$$\|N^{\mu}_{\eta,\xi}\|_{\mathcal{V}} \le C. \tag{2.13}$$

The homogenization of nonlinear parabolic equations depends on the ratio between α and β and is presented in [8]. The following cases are distinguished: 1) Self-similar case ($\alpha = 2\beta$); 2) Non self-similar case ($\alpha < 2\beta$); 3) Non self-similar case ($\alpha > 2\beta$); 4) Spatial case ($\alpha = 0$); 5) Temporal case ($\beta = 0$).

Theorem 2.4.
$\mathcal{L}_\epsilon$ G-converges to $\hat{\mathcal{L}}$, where $\hat{\mathcal{L}}$ is given by

$$\hat{\mathcal{L}}u = D_t u - div(\hat{a}(\omega, x, t, u, D_x u)) + \hat{a}_0(\omega, x, t, u, D_x u). \tag{2.14}$$

$\hat{a}$ and $\hat{a}_0$ are defined as follows.

- *For self-similar case ($\alpha = 2\beta$),*

$$\hat{a}(\eta, \xi) = \langle a(\omega, \eta, \xi + \partial N_{\eta,\xi})\rangle,$$
$$\hat{a}_0(\eta, \xi) = \langle a_0(\omega, \eta, \xi + \partial N_{\eta,\xi})\rangle,$$

  where $N_{\eta,\xi} = N^{\mu=1} \in \mathcal{W}$ is the unique solution of

$$\sigma N^{\mu=1} - \mathbf{div}\; a(\omega, \eta, \xi + \partial N^{\mu=1}) = 0. \tag{2.15}$$

- *For non self-similar case ($\alpha < 2\beta$),*

$$\hat{a}(\eta, \xi) = \langle a(\omega, \eta, \xi + \partial N_{\eta,\xi})\rangle,$$
$$\hat{a}_0(\eta, \xi) = \langle a_0(\omega, \eta, \xi + \partial N_{\eta,\xi})\rangle,$$

  where $N_{\eta,\xi} = N^0 \in \mathcal{V}$ is the unique solution of

$$-\mathbf{div}\; a(\omega, \eta, \xi + \partial N^0) = 0. \tag{2.16}$$

- *For non self-similar case ($\alpha > 2\beta$),*

$$\hat{a}(\eta,\xi) = \langle a(\omega,\eta,\xi+\partial N_{\eta,\xi})\rangle,$$
$$\hat{a}_0(\eta,\xi) = \langle a_0(\omega,\eta,\xi+\partial N_{\eta,\xi})\rangle,$$

where $N_{\eta,\xi} = N^\infty \in \mathcal{V}_s$ is the unique solution of

$$-\mathbf{div}\ \overline{a}(\omega,\eta,\xi+\partial N^\infty) = 0. \tag{2.17}$$

$\overline{a}$ is defined in (2.11).

- *For spatial case ($\alpha = 0$),*

$$\hat{a}(T_1(t)\omega,\eta,\xi) = M_x\{a(T_2(x)\omega,\eta,\xi+\partial N_{\eta,\xi}(T_2(x)\omega))\},$$
$$\hat{a}_0(T_1(t)\omega,\eta,\xi) = M_x\{a_0(T_2(x)\omega,\eta,\xi+\partial N_{\eta,\xi}(T_2(x)\omega))\},$$

where $N_{\eta,\xi} = N_x \in \mathcal{V}$

$$-\mathbf{div}\ a(\omega,\eta,\xi+\partial N_x) = 0. \tag{2.18}$$

- *For temporal case ($\beta = 0$), the homogenized fluxes are defined by*

$$\begin{aligned}\hat{a}(\omega,\eta,\xi) &= M_t\{a(\omega,\eta,\xi)\},\\ \hat{a}_0(\omega,\eta,\xi) &= M_t\{a_0(\omega,\eta,\xi)\},\end{aligned} \tag{2.19}$$

where M_t is defined in (2.9).

For temporal case one can also define $N_{\eta,\xi}$ in the following way (see proof of Theorem 4.8 in [8]). Define $F = a(\omega,\eta,\xi) - M_t a(\omega,\eta,\xi)$, and $f = \mathbf{div}\ F$. Then it can be shown that there exists N, such that

$$f = -\sigma N + g, \tag{2.20}$$

where $\|g\|_{\mathcal{V}'} \leq \delta$, for arbitrary small δ. The latter follows from the fact that the range of σ is dense in the orthogonal complement of the kernel of σ, and f belongs to the kernel of σ. The proof of this theorem extensively uses near solutions of (2.12) since $N^\mu_{\eta,\xi}$ is no longer a homogeneous random field.

The theorem on the convergence of arbitrary solutions (Theorem 2.2) for G-convergent sequence of operators allows us not to restrict ourselves to a particular boundary or initial conditions. In particular, from Theorem 2.2 and Theorem 2.4 we have

Theorem 2.5. *Let $u_\epsilon \in \overline{W}$ be a solution of $\mathcal{L}_\epsilon u_\epsilon = f$, $f \in L^q(0,T,W^{-1,q}(Q_0))$, such that $\|u_\epsilon\|_{\overline{W}}$ is bounded. Then u_ϵ converges to u as $\epsilon \to 0$ weakly in $\overline{W}$ (up to a subsequence) where u is a solution of $\hat{\mathcal{L}}u = f$, and $\hat{\mathcal{L}}$ is defined in (2.14).*

Remark 2.1. *We note that the ergodicity assumption is not essential for the proof of the theorem. One can carry out the proof for non-ergodic case essentially in the same manner as that for the ergodic case. The homogenized operators for non-ergodic case will be invariant functions with respect to $T(z)$.*

2.3. Almost periodic functions. Let $C_b(\mathbb{R}^{n+1})$ be the Banach space of all bounded and continuous (complex valued) functions on $\mathbb{R}^{n+1}$, endowed with the standard supremum norm. Denote by $Trig(\mathbb{R}^{n+1})$ the vector space of all trigonometric polynomials, i.e. all finite sums of the form

$$u(z) = \sum u_k \exp(i\xi_k \cdot z), \quad \xi_k \in \mathbb{R}^{n+1},\ u_k \in \mathbf{C}.$$

The closure of the space $Trig(\mathbb{R}^{n+1})$ in $C_b(\mathbb{R}^{n+1})$ is called the space of *Bohr almost periodic* (a.p.) *functions* and is denoted by $CAP(\mathbb{R}^{n+1})$.

Now we recall the concept of Bohr compactification of $\mathbb{R}^{n+1}$ [18]. There exist a compact abelian group $\mathbb{R}^{n+1}_B$ and a continuous group monomorphism

$$i_B : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}_B$$

with the following property:

$f \in C_b(\mathbb{R}^{n+1})$ is almost periodic if and only if there exists a unique function $\tilde{f} \in C(\mathbb{R}^{n+1}_B)$ such that $f(z) = \tilde{f}(i_B z)$.

Such a couple $(\mathbb{R}^{n+1}_B, i_B)$ is unique up to a natural equivalence and is called the *Bohr compactification*. In the following we identity $\mathbb{R}^{n+1}$ with its dense image $i_B(\mathbb{R}^{n+1})$ in $\mathbb{R}^{n+1}_B$. Frequently, we do not distinguish an almost periodic function f and its extension $\tilde{f}$ to $\mathbb{R}^{n+1}_B$. Therefore, $CAP(\mathbb{R}^{n+1}_B)$ may be isometrically identified with $C(\mathbb{R}^{n+1}_B)$. We define a dynamical system $T(z)$ on $\mathbb{R}^{n+1}_B$ by

$$T(z)\omega = \omega + z, \quad \omega \in \mathbb{R}^{n+1}_B,\ z \in \mathbb{R}^{n+1}.$$

Let us denote by μ the Haar measure on $\mathbb{R}^{n+1}_B$ normalized by $\mu(\mathbb{R}^{n+1}_B) = 1$. It is known that

$$M\{f\} = \int_{\mathbb{R}^{n+1}} \tilde{f}(\omega) d\mu(\omega), \tag{2.21}$$

where $\tilde{f}$ is the continuous extension of f to $\mathbb{R}^{n+1}_B$.

Next, we discuss Besicovitch almost periodicity, which is used in the paper. For a function

$$f \in L^p_{loc}(\mathbb{R}^{n+1}), \quad 1 \le p < \infty,$$

we set

$$\|f\|^p_{B^p} = \limsup_{t\to\infty} \frac{1}{|K_t|} \int_{K_t} |f(z)|^p dz, \tag{2.22}$$

where

$$K_t = \{z \in \mathbb{R}^{n+1} : |z_i| \le t,\ i = 1, 2, \dots, n+1\}.$$

A function $f \in L^p_{loc}(\mathbb{R}^{n+1})$ is said to be *Besicovitch almost periodic* with the exponent p if there is a sequence $f_k \in Trig(\mathbb{R}^{n+1})$ such that

$$\lim_{k\to\infty} \|f - f_k\|_{B^p} = 0.$$

In this definition one can replace the space $Trig(\mathbb{R}^{n+1})$ by $CAP(\mathbb{R}^{n+1})$. The space of all such functions is denoted by $B^p(\mathbb{R}^{n+1})$. For any $f \in B^p(\mathbb{R}^{n+1})$ the quantity

$\|f\|_{B^p}$ is finite and defines a semi-norm on $B^p(\mathbb{R}^{n+1})$. With respect to this semi-norm the space $B^p(\mathbb{R}^{n+1})$ possesses a kind of completeness (see [20]). However, $B^p(\mathbb{R}^{n+1})$ is not a Banach space, since the kernel of the semi-norm $\|\cdot\|_{B^p}$ is non-trivial. We say that two functions $f_1,\ f_2 \in B^p(\mathbb{R}^{n+1})$ are *equivalent* if

$$\|f_1 - f_2\|_{B^p} = 0.$$

A vector space formed by equivalence classes of members of $B^p(\mathbb{R}^{n+1})$ will be denoted by $\overline{B}^p(\mathbb{R}^{n+1})$. The semi-norm $\|\cdot\|_{B^p}$ induces a norm on $\overline{B}^p(\mathbb{R}^{n+1})$ and the last space is a Banach space with respect to that norm.

Next, we discuss the mean value of Besicovitch almost periodic functions. Assume that $f \in L^p_{loc}(\mathbb{R}^{n+1})$ and f has a finite norm (2.22). Since the family $f(\varepsilon^{-1}z)$ is bounded in $L^p_{loc}(\mathbb{R}^{n+1})$, the mean value $M\{f\}$, if exists, may be characterized as the weak limit in $L^p_{loc}(\mathbb{R}^{n+1})$:

$$M\{f\} = w\text{-}\lim_{\varepsilon \to 0} f(\varepsilon^{-1}z).$$

Using this statement it is very easy to verify that $M\{f\}$ depends continuously on f with respect to $\|\cdot\|_{B^p}$. More precisely, assume that $f,\ f_k \in L^p_{loc}(\mathbb{R}^{n+1})$, f_k has a mean value, and

$$\|f - f_k\|_{B^p} \to 0.$$

Then f also has a mean value and

$$M\{f_k\} \to M\{f\}.$$

Since any trigonometrical polynomial has a mean value, we see that for each $f \in B^p(\mathbb{R}^{n+1})$ there exists the mean value $M\{f\}$. Moreover,

$$\|f\|_{B^p} = M\{|f|^p\}^{1/p}, \quad f \in B^p(\mathbb{R}^{n+1}). \tag{2.23}$$

Now we invoke the Bohr compactification. Using (2.23) and (2.21) one can extend, by continuity, the isomorphism $f \longmapsto \tilde{f}$ between $CAP(\mathbb{R}^{n+1})$ and $C(\mathbb{R}^{n+1}_B)$ to the map from $B^p(\mathbb{R}^{n+1})$ into $L^p(\mathbb{R}^{n+1}_B)$, the last space being regarded with respect to the measure μ. In fact, the density of $C(\mathbb{R}^{n+1}_B)$ in $L^p(\mathbb{R}^{n+1}_B)$ implies that this map is onto. Moreover,

$$\|\tilde{f}\|_{p,\mathbb{R}^{n+1}_B} = \|f\|_{B^p}. \tag{2.24}$$

Therefore, the map $f \longmapsto \tilde{f}$ induces an isometric isomorphism between $\overline{B}^p(\mathbb{R}^{n+1})$ and $L^p(\mathbb{R}^{n+1}_B)$.

In the analysis, we will employ the approximation of Besicovitch almost periodic functions by Bohr almost periodic functions. Let $\{U_\gamma\}$ be a base of symmetric neighborhoods of zero in $\mathbb{R}^{n+1}_B$, indexed by a directed set Γ in such a way that $U_{\gamma_1} \subset U_{\gamma_2}$ if $\gamma_1 \geq \gamma_2$. By Urysohn's Lemma, for any U_γ there exists an even non-negative function $\tilde{\varphi}_\gamma \in C(\mathbb{R}^{n+1}_B)$ such that $supp\tilde{\varphi}_\gamma \subset U_\gamma$ and

$$\int_{\mathbb{R}^{n+1}_B} \tilde{\varphi}_\gamma(\omega) d\mu(\omega) = 1.$$

Let $\varphi_\gamma \in CAP(\mathbb{R}^{n+1})$ be the restriction of $\tilde{\varphi}_\gamma$ to $\mathbb{R}^{n+1} \subset \mathbb{R}_B^{n+1}$. For any function $\tilde{f} \in L^1(\mathbb{R}_B^{n+1})$ we set

$$(\tilde{S}_\gamma \tilde{f})(\omega) = \tilde{\varphi}_\gamma *_B \tilde{f} = \int_{\mathbb{R}_B^{n+1}} \tilde{\varphi}_\gamma(\omega - \theta)\tilde{f}(\theta)d\mu(\theta), \tag{2.25}$$

where $*_B$ stands for the convolution on $\mathbb{R}_B^{n+1}$. In the similar way, for $f \in B^1(\mathbb{R}^{n+1})$ we set

$$(S_\gamma f)(x) = M_y\{\varphi_\gamma(x-y)f(y)\}, \tag{2.26}$$

where M_y stands for the mean value with respect to the variable y. It is obvious that

$$\widetilde{S_\gamma f} = \tilde{S}_\gamma \tilde{f}. \tag{2.27}$$

Moreover, for $\tilde{f} \in L^p(\mathbb{R}_B^{n+1})$ (resp., $f \in B^p(\mathbb{R}^{n+1})$) we have $\tilde{S}_\gamma \tilde{f} \in C(\mathbb{R}_B^{n+1})$ (resp., $S_\gamma f \in CAP(\mathbb{R}^{n+1})$). The operators S_γ and $\tilde{S}_\gamma$ are uniformly bounded:

$$\|S_\gamma f\|_{B^p} \le \|f\|_{B^p}, \quad f \in B^p(\mathbb{R}^{n+1}), \tag{2.28}$$

$$\|\tilde{S}_\gamma \tilde{f}\|_{p,\mathbb{R}_B^{n+1}} \le \|\tilde{f}\|_{p,\mathbb{R}_B^{n+1}}, \quad \tilde{f} \in L^p(\mathbb{R}_B^{n+1}). \tag{2.29}$$

Directly from the definition of $\tilde{\varphi}_\gamma$ one can deduce that $\tilde{S}_\gamma \tilde{f} \to \tilde{f}$ in $L^p(\mathbb{R}_B^{n+1})$ for any $\tilde{f} \in L^p(\mathbb{R}_B^{n+1})$, $1 \le p < \infty$. Now (2.24) gives rise to the following

Proposition 2.1. *For any $f \in B^p(\mathbb{R}^{n+1})$, $1 \le p < \infty$, we have*

$$\lim_\gamma \|f - S_\gamma f\|_{B^p} = 0. \tag{2.30}$$

Remark 2.2. *Let $f \in \mathcal{F} \subset B^p(\mathbb{R}^{n+1})$. If the image $\tilde{\mathcal{F}}$ of $\mathcal{F}$ in $L^p(\mathbb{R}_B^{n+1})$ is precompact, then the convergence in (2.30) is uniform with respect to $f \in \mathcal{F}$. Moreover, if $\tilde{\mathcal{F}}$ is a separable subset in $L^p(\mathbb{R}_B^{n+1})$, then the net $\{S_\gamma\}$ may be replaced by a subsequence $\{S_m\}$.*

Proposition 2.2. *The map $f \longmapsto \tilde{f}$ is order preserving: if $f_1 \le f_2$, then $\tilde{f}_1 \le \tilde{f}_2$.*

Proof. On $CAP(\mathbb{R}^n)$ this is obvious. Since $\tilde{\varphi}_\gamma \ge 0$, the operator S_γ is order preserving. Therefore, the general statement follows from the previous one by approximation. □

As a consequence, for any $f \in B^1(\mathbb{R}^{n+1}) \cap L^\infty(\mathbb{R}^{n+1})$ we have $\tilde{f} \in L^\infty(\mathbb{R}_B^{n+1})$.

3. Stability and Comparison Results. Let $\mathcal{L}_k$ be a sequence of operators of the class Π such that $\mathcal{L}_k \overset{G}{\Longrightarrow} \mathcal{L}$. Then, $\mathcal{L}_k^{(\eta,\xi)} \overset{G}{\Longrightarrow} \mathcal{L}^{(\eta,\xi)}$ for any $(\eta,\xi) \in \mathbb{R}\times\mathbb{R}^n$. Consider a unique solution, $u_k \in W_0$, of the equation

$$D_t u_k - div\, a^k(t,x,\eta,\xi + D_x u_k) = -div\, a(t,x,\eta,\xi)$$

and set $v_k(t,x) = \xi\cdot x + u_k(t,x)$, then v_k possesses the following properties:

1. *$v_k \to \xi \cdot x$ weakly in $\overline{W}$;*

2. *The sequences* $a^k(t,x,\eta,D_xv_k)$ *and* $a_0^k(t,x,\eta,D_xv_k)$ *are weakly convergent in the spaces* $L^q(Q)^n$ *and* $L^q(Q)$ *respectively*;
3. *the sequence* $D_tv_k - div\, a^k(t,x,\eta,D_xv_k)$ *is precompact in the space* $V^{'}$.

THEOREM 3.1. *Let* $\mathcal{L}_k \in \Pi$. *Assume that, for any* $(\eta,\xi) \in \mathbb{R}\times\mathbb{R}^n$, *there exists* $v_k \in \overline{W}$ *such that the above mentioned properties 1 – 3 are fulfilled. Then the sequence* $\mathcal{L}_k$ *is G-convergent and for the G-limit operator, L, we have*

$$a(t,x,\eta,\xi) = \lim_{k\to\infty} a^k(t,x,\eta,D_xv_k), \tag{3.1}$$

$$a_0(t,x,\eta,\xi) = \lim_{k\to\infty} a_0^k(t,x,\eta,D_xv_k) \tag{3.2}$$

weakly in $L^q(Q)^n$ *and* $L^q(Q)$, *respectively.*

Proof. Let $b(t,x) = b(t,x,\eta,\xi)$ and $b_0(t,x) = b_0(t,x,\eta,\xi)$ be the weak limits of $a^k(t,x,\eta,D_xv_k)$ and $a_0^k(t,x,\eta,D_xv_k)$, respectively. By Theorem 2.1 (see also p.184 of [20]), there exists a subsequence $\sigma(k)$ and an operator $\mathcal{L}$ of the form

$$\mathcal{L}u = D_tu - div\, a(t,x,u,D_xu) + a_0(t,x,u,D_xu),$$

such that $\mathcal{L}_{\sigma(k)} \overset{G}{\Longrightarrow} \mathcal{L}$. To prove the theorem it is sufficient to show that

$$a(t,x,\eta,\xi) = b(t,x) \tag{3.3}$$

and

$$a_0(t,x,\eta,\xi) = b_0(t,x). \tag{3.4}$$

From properties 2 and 3, it follows that

$$-div\, b(t,x) = \lim_{k\to\infty}(D_tv_k - div\, a^k(t,x,\eta,D_xv_k))$$

strongly in $V^{'}$. Because the embedding $L^q(Q)^n \subset V^{'}$ is compact, we also have

$$b_0(t,x) = \lim_{k\to\infty} a_0^k(t,x,\eta,D_xv_k)$$

strongly in $V^{'}$. By parabolic version of Remark 2.3.5 (page 104 of [20]) we obtain (3.3) and (3.4). Parabolic version of Remark 2.3.5 can be readily derived from Theorem 4.1.3 (see [20]).
□

COROLLARY 3.2. *Let* $\mathcal{L}_k \in \Pi$ *be a sequence such that, for any* $(\eta,\xi) \in \mathbb{R}\times\mathbb{R}^n$, *the sequence* $a^k(t,x,\eta,\xi)$ *converges in measure and the sequence* $a_0^k(t,x,\eta,\xi)$ *converges weakly in* $L^1(Q)$. *Then* $\mathcal{L}_k$ *is G-convergent.*

Proof. Inequality (2.2) and the dominated convergence theorem imply that $a^k(t,x,\eta,\xi)$ and $a_0^k(t,x,\eta,\xi)$ converge strongly in $L^1(Q)^n$ and $L^1(Q)$. Now the statement follows directly from Theorem 3.1, with $v_k \equiv \xi\cdot x$. □

Next, we consider the following problem. Given an operator $\mathcal{L} \in \Pi$ and fixed $(t_0,x_0) \in Q$, we define the operator $\mathcal{L}_\rho$, $0 < \rho \le 1$, by the formula

$$\mathcal{L}_\rho u = D_t u - div\, a(t_0+\rho t, x_0+\rho x, u, D_x u) + a_0(t_0+\rho t, x_0+\rho x, u, D_x u). \quad (3.5)$$

We look for the asymptotic behavior of $\mathcal{L}_\rho$, as $\rho \to 0$, assuming that ρ runs a subsequence which tends to 0.

PROPOSITION 3.1. *For any common Lebesgue point* $(t_0, x_0) \in Q$ *of the functions* $a(t,x,\eta,\xi)$ *and* $a_0(t,x,\eta,\xi)$, *i.e. for almost all* $(t_0,x_0) \in Q$, *the sequence* $\mathcal{L}_\rho$ *G-converges, as* $\rho \to 0$, *to the operator*

$$\hat{\mathcal{L}} u = D_t u - div\, a(t_0, x_0, u, D_x u) + a_0(t_0, x_0, u, D_x u). \quad (3.6)$$

Proof. By Lebesgue's differentiation theorem $a(t_0+\rho t, x_0+\rho x, \eta, \xi) \to a(t,x,\eta,\xi)$ and $a_0(t_0+\rho t, x_0+\rho x, \eta,\xi) \to a_0(t, x_0, \eta, \xi)$, as $\rho \to 0$, strongly in $L^1(Q)^n$ and $L^1(Q)$ respectively. Now Corollary 3.2 implies the required statement. ◻

Next, we state another result which provides a criterion for G-convergence and a representation formula for the G-limit operator. Given $\mathcal{L} \in \Pi$, we define the functions

$$\Psi(\eta,\xi,Q_1) = \int_{Q_1} a(t,x,\eta,\xi + D_x v(t,x))dxdt \quad (3.7)$$

and

$$\Psi_0(\eta,\xi,Q_1) = \int_{Q_1} a_0(t,x,\eta,\xi + D_x v(t,x))dxdt, \quad (3.8)$$

for any $(\eta,\xi) \in \mathbb{R} \times \mathbb{R}^n$ and for any open subset $Q_1 = (0,1) \times Q_0^1$, where Q_0^1 is an open subset of Q_0 and the function v is defined as a unique solution $v \in L^p(0,1,W_0^{1,p}(Q_0^1))$ and $v(0,x) = 0$ of the equation

$$D_t v - div\, a(t,x,\eta,\xi + D_x v) = 0 \quad \text{on } Q_1. \quad (3.9)$$

PROPOSITION 3.2. *Let* $\mathcal{L} \in \Pi$. *Then there exists a measurable subset* N *of* Q, *with* $|N| = 0$, *such that for any* $(t_0,x_0) \in (Q \setminus N)$ *and* $(\eta,\xi) \in \mathbb{R} \times \mathbb{R}^n$

$$a(t_0,x_0,\eta,\xi) = \lim_{\rho \to 0} \frac{\Psi(\eta,\xi,U_\rho(t_0,x_0))}{|U_\rho(t_0,x_0)|} \quad (3.10)$$

and

$$a_0(t_0,x_0,\eta,\xi) = \lim_{\rho \to 0} \frac{\Psi_0(\eta,\xi,U_\rho(t_0,x_0))}{|U_\rho(t_0,x_0)|}, \quad (3.11)$$

where $U_\rho(t_0,x_0) = (t_0,x_0) + \rho U$, *with* $U = (0,1) \times U_0$, U_0 *being an open bounded subset of* $\mathbb{R}^n$.

Proof. Let N be the complement of the set of all common Lebesgue points of the family of functions $\{a(t,x,\eta,\xi), a_0(t,x,\eta,\xi)\}_{(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n}$. Given $(t_0,x_0) \in Q \setminus N$, $\rho > 0$, and $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n$, we consider the function $v \in L^p(0, \rho, W_0^{1,p}(U_\rho(x_0)))$ defined to be a unique solution of (3.9), with $Q_0^1 = U_\rho(x_0)$. By performing the change of variables $y = (x - x_0)/\rho$, $\tau = (t - t_0)/\rho$, the equation (3.9) becomes

$$D_\tau u_\rho - div_y a(t_0 + \rho\tau, x_0 + \rho y, \eta, \xi + D_y u_\rho(y)) = 0, \quad y \in U_0,$$

where $u_\rho(\tau, y) = v(t_0 + \rho\tau, x_0 + \rho y)/\rho$. Since $w = 0$ is a unique solution of the equation

$$D_t w - div_y a(t_0, x_0, \eta, \xi + D_y w) = 0 \quad \text{on } U,$$

Proposition 3.1 implies that $u_\rho \to 0$ weakly in $L^p(0, 1, W_0^{1,p}(U_0))$,

$$a(t_0 + \rho\tau, x_0 + \rho y, \eta, \xi + D_y u_\rho) \to a(t_0, x_0, \eta, \xi),$$

weakly in $L^q(Q)^n$, and

$$a_0(t_0 + \rho\tau, x_0 + \rho y, \eta, \xi + D_y u_\rho) \to a_0(t_0, x_0, \eta, \xi)$$

weakly in $L^q(Q)$. Then

$$a(t_0, x_0, \eta, \xi) = \lim_{\rho \to 0} \frac{1}{|U|} \int_0^1 \int_U a(t, x_0 + \rho y, \eta, \xi + D_y u_\rho(y)) dy dt,$$

from where, by the change of variables, we get (3.10). In the similar way, one can derive (3.11).
$\square$

THEOREM 3.3. *Suppose $\mathcal{L}_k$ is a sequence of operators of the class Π. Let Ψ^k and Ψ_0^k be the functions associated with $\mathcal{L}_k$ by (3.7) and (3.8), respectively, U_0 be a bounded open subset in $\mathbb{R}^n$, and $U_\rho(t_0, x_0) = (t_0, x_0) + \rho U$ with $U = (0, 1) \times U_0$. Then the following statements are equivalent:*

(i) *for almost all $x_0 \in Q_0$ and $t_0 > 0$, the limits*

$$\lim_{k \to \infty} \Psi^k(t, x, \eta, \xi, U_\rho(t_0, x_0))$$

and

$$\lim_{k \to \infty} \Psi_0^k(t, x, \eta, \xi, U_\rho(t_0, x_0))$$

exist for any $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and $\rho > 0$ small enough;

(ii) *the sequence $\mathcal{L}_k$ G-converges to an operator $\mathcal{L}$.*

Moreover, if these statements hold true, then, for almost all $x \in Q$ and t,

$$a(t, x, \eta, \xi) = \lim_{\rho \to 0} \lim_{k \to \infty} \frac{\Psi^k(t, x, \eta, \xi, U_\rho(t, x))}{|U_\rho(t, x)|} \tag{3.12}$$

and

$$a_0(t, x, \eta, \xi) = \lim_{\rho \to 0} \lim_{k \to \infty} \frac{\Psi_0^k(t, x, \eta, \xi, U_\rho(t, x))}{|U_\rho(t, x)|} \tag{3.13}$$

for any $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n$, where $\mathcal{L}$ is associated with a and a_0.

Proof. Assume (i).

By G-convergence, we may suppose that a subsequence of $\mathcal{L}_k$ still denoted by $\mathcal{L}_k$ is G-covergent to an operator $\mathcal{L}$. If we prove formula (3.12) and (3.13), we conclude that the initial sequence is G-convergent.

Fix $x_0 \in Q$ and $t_0 > 0$ such that the limits in (i) exist. Given $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and $\rho > 0$ we consider a unique solution $v_k \in L^p(0, \rho, W_0^{1,p}(U_\rho))$, $v_k(t = 0) = 0$, of the equation

$$D_t v_k - div\, a^k(x, \eta, \xi + D_x v_k) = 0,$$

and a unique solution $v \in L^p(0, \rho, W_0^{1,p}(U_\rho))$, $v(t = 0) = 0$, of the equation

$$D_t v - div\, a(x, \eta, \xi + D_x v) = 0.$$

It follows that $v_k \to v$ weakly in $L^p(0, \rho, W^{1,p}(U_\rho(x_0)))$,

$$a^k(t, x, \eta, \xi + D_x v_k) \to a(t, x, \eta, \xi + D_x v) \tag{3.14}$$

weakly in $L^q(U_\rho(t_0, x_0))^n$, and

$$a_0^k(t, x, \eta, \xi + D_x v_k) \to a_0(t, x, \eta, \xi + D_x v) \tag{3.15}$$

weakly in $L^q(U_\rho(t_0, x_0))$. Hence,

$$\lim_{k\to\infty} \Psi^k(t_0, x_0, \eta, \xi, U_\rho(t_0, x_0)) = \int_{U_\rho(t_0,x_0)} a(t, x, \eta, \xi{+}D_x v) dx dt = \Psi(t_0, x_0, \eta, \xi, U_\rho(t_0, x_0))$$

and the similar statement holds for Ψ_0. Now, by Proposition 3.2, we get (3.12) and (3.13).

Assume (ii). Then, for v and v_k defined in the first part of the proof, statements (3.14) and (3.15) hold true and (i) follows immediately.

□

To state the next result, we need the following notations. Let $\mathcal{L}_k \in \Pi$ and $\mathcal{B}_k \in \Pi$. We set

$$g^k(t, x, r) = \sup_{|\xi|, |\eta| \le r} |a^k(t, x, \eta, \xi) - b^k(t, x, \eta, \xi)|,$$

$$g_0^k(t, x, r) = \sup_{|\xi|, |\eta| \le r} |a_0^k(t, x, \eta, \xi) - b_0^k(t, x, \eta, \xi)|.$$

Assuming $\mathcal{L}_k \overset{G}{\Longrightarrow} \mathcal{L}$ and $\mathcal{B}_k \overset{G}{\Longrightarrow} \mathcal{B}$, we introduce also the functions

$$g(t, x, r) = \sup_{|\xi|, |\eta| \le r} |a(t, x, \eta, \xi) - b(t, x, \eta, \xi)|$$

and

$$g_0(t, x, r) = \sup_{|\xi|, |\eta| \le r} |a_0(t, x, \eta, \xi) - b_0(t, x, \eta, \xi)|.$$

Given a bounded open subset $U = (0, 1) \times U_0$, $U_0 \subset \mathbb{R}^n$, with the regular boundary ∂U_0, we set

$$U_\rho(t_0, x_0) = (t_0, x_0) + \rho U.$$

Let us define the functions $\overline{g}(t,x,r)$ and $\overline{g}_0(t,x,r)$ by

$$\overline{g}(t,x,r) = \limsup_{\rho\to 0}\limsup_{k\to\infty}\frac{1}{|U_\rho(t,x)|}\int_{U_\rho(t,x)} g^k(t,y,r)dydt$$

and

$$\overline{g}_0(t,x,r) = \limsup_{\rho\to 0}\limsup_{k\to\infty}\frac{1}{|U_\rho(t,x)|}\int_{U_\rho(t,x)} g_0^k(t,y,r)dydt.$$

For any $r \geq 0$, these functions are well-defined a.e. on Q and belong to $L^\infty(Q)$. Also we shall use the notation

$$\varphi(r) = r^{-p} + r^{-\alpha p/(p+\alpha)}, \quad r > 0,$$

with the constant $\alpha > 0$, which will be specified later.

THEOREM 3.4. *Suppose $\mathcal{L}_k$ and $\mathcal{B}_k$ are sequences of operators of the class Π, $\mathcal{L}_k \overset{G}{\Longrightarrow} \mathcal{L}$, and $\mathcal{B}_k \overset{G}{\Longrightarrow} \mathcal{B}$. There exists $\alpha > 0$ such that given $R > 0$*

$$g(t,x,R) \leq \overline{g}(t,x,r) + K\left[\varphi(r)^{1/q} + \varphi^\gamma(r) + (1+r)^\gamma \overline{g}(t,x,r)^\gamma\right] \tag{3.16}$$

and

$$g_0(t,x,R) \leq \overline{g}_0(t,x,r) + K\left[\varphi(r)^{1/q} + \varphi^\gamma(r) + (1+r)^\gamma \overline{g}(t,x,r)^\gamma\right] \tag{3.17}$$

for a constant $K = K(R)$ and almost all $x \in Q$ and for all $r > 0$, where

$$\gamma = \frac{s}{q^2(\beta - 1)}.$$

Proof. We will prove only (3.16). Inequality (3.17) can be proved in a similar way. For the sake of brevity, we will write U_ρ instead of $U_\rho(x,t)$. Moreover, we will suppress the variable η in the functions a^k and b^k.

Given $R > 0$ we fix $\xi, \eta \in \mathbb{R}$ such that $|\xi| \leq R$, $|\eta| \leq R$. Let $v_k \in L^p(0,\rho,W_0^{1,p}(U_\rho))$, $v_k(t=0) = 0$, be a unique solution of the equation

$$D_t v_k - div\, a^k(t,x,\xi + D_x v_k) = 0$$

and $w_k \in L^p(0,\rho,W_0^{1,p}(U_\rho))$, $w_k(t=0) = 0$, be a unique solution of the equation

$$D_t w_k - div\, b^k(t,x,\xi + D_x w_k) = 0.$$

In view of Theorem 3.3, we need to estimate the quantity

$$\begin{aligned} J = &\int_{U_\rho} |a^k(t,x,\eta,\xi + D_x v_k) - b^k(t,x,\eta,\xi + D_x w_k)|dxdt \leq \\ &\leq \int_{U_\rho} |a^k(t,x,\eta,\xi + D_x v_k) - a^k(t,x,\eta,\xi + D_x w_k)|dxdt + \\ &\quad + \int_{U_\rho} |a^k(t,x,\eta,\xi + D_x w_k) - b^k(t,x,\eta,\xi + D_x w_k)|dxdt = \\ &= J_1 + J_2. \end{aligned} \tag{3.18}$$

In what follows we shall denote by K any constant depending only on R.

First, we note

$$\|\xi + D_x w_k\|^p_{p,U_\rho} \leq K|U_\rho| \leq c\rho^{n+1}. \tag{3.19}$$

Next, we will use Meyers type estimates for the solution of nonlinear parabolic equation [22, 14]. In particular, we assume that $\xi + D_x w_k$ is bounded in $L^{p+\alpha}(U_\rho)$, for some $\alpha > 0$ which depends only on c, h, n and β, i.e.,

$$\|\xi + D_x w_k\|_{p+\alpha,U_\rho} \leq C\rho^{\frac{-(n+1)\alpha}{p(p+\alpha)}}\|\xi + D_x w_k\|_{p,U_\rho}. \tag{3.20}$$

The constant can be easily obtained from rescaling. Similar estimates hold for v_k. Next we define the set

$$A_r = \{(t,x) \in U_\rho : |\xi + D_x w_k(x)| > r\}.$$

Then, we have

$$|A_r| r^p \leq \int_{A_r} |\xi + D_x w_k|^p dxdt.$$

Hence,

$$|A_r| \leq K|U_\rho| r^{-p}.$$

Using the Hölder inequality and (3.20) we obtain

$$\int_{A_r}(1 + |\xi + D_x w_k|^p)dxdt \leq |A_r| + |A_r|^{\alpha/(p+\alpha)}\|\xi + D_x w_k\|^p_{p+\alpha,A_r} \leq$$
$$\leq K|U_\rho|(r^{-p} + r^{-\alpha p/(p+\alpha)}) = K|U_\rho| \cdot \varphi(r). \tag{3.21}$$

To derive an estimate for J, first we consider the integral J_2. Using (3.21), we have

$$\begin{aligned}
J_2 = &\int_{A_r} |a^k(t,x,\eta,\xi + D_x w_k) - b^k(t,x,\eta,\xi + D_x w_k)|dxdt + \\
&+ \int_{U_\rho \setminus A_r} |a^k(t,x,\eta,\xi + D_x w_k) - b^k(t,x,\eta,\xi + D_x w_k)|dxdt \leq \\
&\leq K\int_{A_r}(1 + |\xi + D_x w_k|^{p-1})dxdt + \int_{\mathcal{U}_\rho \setminus A_r} g^k(t,x,r)dxdt \leq \\
&\leq \int_{U_\rho} g^k(t,x,r)dxdt + K|U_\rho|^{1/p}\left[\int_{A_r}(1 + |\xi + D_x w_k|^p)dxdt\right]^{1/q} \leq \\
&\leq \int_{U_\rho} g^k(t,x,r)dxdt + K|U_\rho| \cdot \varphi(r)^{1/q}.
\end{aligned} \tag{3.22}$$

Similarly,

$$\int_{U_\rho} |a^k(t,x,\eta,\xi + D_x w_k) - b^k(t,x,\eta,\xi + D_x w_k)|^q dxdt \leq$$
$$\leq K(1+r)\int_{U_\rho} g^k(t,x,r)dxdt + K|U_\rho| \cdot \varphi(r). \tag{3.23}$$

Here to get the factor $(1+r)$ in the first term of the right-hand part, we have used (2.2) to estimate the integral

$$\int_{U_\rho} g^k(t,x,r)^q dxdt = \int_{U_\rho} g^k(t,x,r)^{q/p} g^k(t,x,r) dxdt.$$

Before to handle J_1, we need an estimate for $D_x v_k - D_x w_k$. Using the Hölder inequality and (3.23), we have

$$\begin{aligned}
0 = &\int_{U_\rho} \left[a^k(t,x,\eta,\xi + D_x v_k) - b^k(t,x,\eta,\xi + D_x w_k)\right] \cdot D_x(v_k - w_k) dxdt + \\
&+ \frac{1}{2}\int_{U_\rho} D_t (v_k - w_k)^2 dxdt = \\
&= \int_{U_\rho} \left[a^k(t,x,\eta,\xi + D_x v_k) - a^k(t,x,\eta,\xi + D_x w_k)\right] \cdot D_x(v_k - w_k) dxdt + \\
&+ \int_{U_\rho} \left[a^k(t,x,\xi + D_x w_k) - b^k(t,x,\xi + D_x w_k)\right] \cdot D_x(v_k - w_k) dxdt + \\
&+ \frac{1}{2}\int_{U_\rho} (v_k - w_k)^2 (t = \rho) dx \geq \\
&\geq K \int_{U_\rho} (1 + |\xi + D_x v_k|^p + |\xi + D_x w_k|^p)^{1-\beta/p} \cdot |D_x(v_k - w_k)|^\beta dxdt - \\
&- J_3^{1/q} \cdot \|D_x(v_k - w_k)\|_{p,U_\rho} \geq \\
&\geq K \left[\int_{U_\rho} (1 + |\xi + D_x v_k|^p + |\xi + D_x w_k|^p) \, dxdt\right]^{1-\beta/p} \cdot \|D_x(v_k - w_k)\|^\beta_{p,U_\rho} - \\
&- J_3^{1/q} \cdot \|D_x(v_k - w_k)\|_{p,U_\rho} \geq \\
&\geq K|U_\rho|^{1-\beta/p} \cdot \|D_x(v_k - w_k)\|^\beta_{p,U_\rho} - J_3^{1/q} \cdot \|D_x(v_k - w_k)\|_{p,U_\rho}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|D_x(v_k - w_k)\|^{\beta-1}_{p,U_\rho} &\leq K|U_\rho|^{\beta/p-1} J_3^{1/q} \leq \\
&\leq K|U_\rho|^{\beta/p-1} \left[(1+r)\int_{U_\rho} g^k(t,x,r) dxdt + |U_\rho| \cdot \varphi(r)\right]^{1/q}.
\end{aligned} \tag{3.24}$$

Hence,

$$\begin{aligned}
J_1 &\leq |U_\rho|^{1/p} \left[\int_{U_\rho} |a^k(t,x,\eta,\xi + D_x v_k) - a^k(t,x,\eta,\xi + D_x w_k)|^q dxdt\right]^{1/q} \leq \\
&\leq |U_\rho|^{1/p} \left[\int_{U_\rho} (1 + |\xi + D_x v_k|^p + |\xi + D_x w_k|^p)^{1-s/p} |D_x(v_k - w_k)|^s dxdt\right]^{1/q} \leq \\
&\leq |U_\rho|^{1/p} \left[\int_{U_\rho} (1 + |\xi + D_x v_k|^p + |\xi + D_x w_k|^p) \, dxdt\right]^{(p-s)/(pq)} \|D_x(v_k - w_k)\|^{s/q}_{p,U_\rho}.
\end{aligned}$$

Using (3.19) and (3.24), we obtain after some calculations

$$J_1 \leq K|U_\rho|^{1-\gamma}\left[(1+r)\int_{U_\rho} g^k(t,x,r)dxdt + |U_\rho|\cdot\varphi(r)\right]^\gamma. \tag{3.25}$$

Inequalities (3.22) and (3.25) imply

$$\begin{aligned} J = \int_{U_\rho} |a^k(t,x,\eta,\xi+D_x v_k) - b^k(t,x,\eta,\xi+D_x w_k)|dxdt \leq \\ \leq K|U_\rho|^{1-\gamma}\left[(1+r)\int_{U_\rho} g^k(t,x,r)dxdt + |U_\rho|\cdot\varphi(r)\right]^\gamma + \\ + \int_{U_\rho} g^k(t,x,r)dxdt + K|U_\rho|\cdot\varphi(r)^{1/q}. \end{aligned} \tag{3.26}$$

From here, dividing both sides by $|U_\rho|$, and taking the limit $\lim_{\rho\to 0}\lim_{k\to\infty}$, we obtain the inequality (3.16). The inequlaity (3.17) can be obtained similarly.
$\square$

Now we state some direct consequences of the last result.

Corollary 3.5. *Let $\mathcal{A}_k$ and $\mathcal{B}_k$ be two sequences of operators of the class Π such that $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$ and $\mathcal{B}_k \overset{G}{\Longrightarrow} \mathcal{B}$. Assume that for any $r \geq 0$*

$$\lim_{k\to\infty} \sup_{|\xi|,|\eta|\leq r} |a^k(t,x,\eta,\xi) - b^k(t,x,\eta,\xi)| =$$

$$= \lim_{k\to\infty} \sup_{|\xi|,|\eta|\leq r} |a_0^k(t,x,\eta,\xi) - b_0^k(t,x,\eta,\xi)| = 0$$

strongly in $L^1(Q)$. Then $\mathcal{A} = \mathcal{B}$.

We shall say that a sequence $\mathcal{A}_k \in \Pi$ converges to $\mathcal{A} \in \Pi$ *component-wise* in L^1 (c.-w. in L^1), if for any $r \geq 0$

$$\lim_{k\to\infty} \sup_{|\xi|,|\eta|\leq r} |a^k(t,x,\eta,\xi) - a(t,x,\eta,\xi)| =$$

$$= \lim_{k\to\infty} \sup_{|\xi|,|\eta|\leq r} |a_0^k(t,x,\eta,\xi) - a_0(t,x,\eta,\xi)| = 0$$

strongly in $L^1(Q)$.

Corollary 3.6. *Let $\mathcal{A}_k^l$ be a double sequence of operators of the class Π such that $\mathcal{A}_k^l \overset{G}{\Longrightarrow} \mathcal{A}^l$ for any $l \in \mathbf{N}$, as $k \to \infty$. Assume that $\mathcal{A}_k^l \to \mathcal{A}_k$ c.-w. in L^1 uniformly with respect to $k \in \mathbf{N}$ and $\mathcal{A}^l \to \mathcal{A}$ c.-w. in L^1, as $l \to \infty$. Then $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$.*

Remark 3.1. *The statement of Corollary 3.6 is still valid without this assumption if we replace c.-w. convergence in L^1 by the following condition:*

$$\lim_{l\to\infty} \ ess\sup_{x\in Q} \sup_{(\eta,\xi)\in\mathbb{R}^{n+1}} \frac{|a_l^k(t,x,\eta,\xi) - a^k(t,x,\eta,\xi)|^q}{(c+|\eta|^p+|\xi|^p)} = 0,$$

and similarly for the differences $a_{0,l}^k(t,x,\eta,\xi) - a_0^k(t,x,\eta,\xi)$, $a_l(t,x,\eta,\xi) - a(t,x,\eta,\xi)$, and $a_{0,l}(t,x,\eta,\xi) - a_0(t,x,\eta,\xi)$.

4. Individual Homogenization. Now we consider almost periodic operators of the class Π. We will prove that homogenization takes place in the individual sense, not only in the statistical one. We consider the case $\alpha > 0$ and $\beta > 0$. More precisely, let us consider a couple of functions

$$(a, a_0) \in \Pi_{\mathbb{R}^{n+1}} = \Pi_{\mathbb{R}^{n+1}}(c_0, c, \kappa, h, \theta, \nu, s, \beta),$$

where c_0, c, κ, h, θ, ν, s, and β are constants subject to the standard assumptions. We consider the family of operators

$$\mathcal{L}_\varepsilon u = D_t u - div\, a(\frac{t}{\epsilon^\alpha}, \frac{x}{\epsilon^\beta}, u, D_x u) + a_0(\frac{t}{\epsilon^\alpha}, \frac{x}{\epsilon^\beta}, u, D_x u), \quad \varepsilon > 0. \tag{4.1}$$

Assume that

$$\begin{gathered}\textit{for any } \zeta = (\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n, \textit{the functions } a(t, x, \eta, \xi) \textit{and } a_0(t, x, \eta, \xi) \\ \textit{are } B^1\textit{-almost periodic with respect to the variable } (t, x) \in \mathbb{R}^{n+1}.\end{gathered} \tag{4.2}$$

Associated with the family $\mathcal{L}_\epsilon$, there is a family of random operators $\mathcal{L}_\epsilon(\omega)$ defined on the probability space $\Omega = \mathbb{R}_B^{n+1}$ in the following way. One can extend the functions $a(t, x, \eta, \xi)$ and $a_0(t, x, \eta, \xi)$ to the functions $\tilde{a}(\omega, \eta, \xi)$ and $\tilde{a}_0(\omega, \eta, \xi)$, respectively, defined on $\mathbb{R}_B^{n+1}$. Proposition 2.2 implies that

$$(\tilde{a}, \tilde{a}_0) \in \Pi_{\mathbb{R}_B^{n+1}}.$$

To simplify the notations, we suppress the tilde here and still denote $\tilde{a}$ and $\tilde{a}_0$ by a and a_0, respectively. Let

$$\mathcal{L}_\varepsilon(\omega)u = D_t u - div\, a(\omega + (\varepsilon^{-\alpha} t, \varepsilon^{-\beta} x), u, D_x u) + a_0(\omega + (\varepsilon^{-\alpha} t, \varepsilon^{-\beta} x), u, D_x u). \tag{4.3}$$

Then, we have formally

$$\mathcal{L}_\varepsilon = \mathcal{L}_\varepsilon(0).$$

Based on statistical homogenization results, for the family $\mathcal{L}_\varepsilon(\omega)$ there exists a homogenized operator $\hat{\mathcal{L}}$. However, since the conclusion of the theorem is fulfilled in the statistical sense, i.e. for almost all $\omega \in \Omega$ only, we cannot conclude directly that $\hat{\mathcal{L}}$ serves the particular operator $\mathcal{L}_\varepsilon = \mathcal{L}_\varepsilon(0)$. In the next theorem, we prove that the individual homogenization takes place.

THEOREM 4.1. *Assume that* $(a, a_0) \in \Pi_{\mathbb{R}^{n+1}}$ *and condition (4.2) is fulfilled. Then for any open bounded subset* $Q \subset \mathbb{R}^{n+1}$ *we have* $\mathcal{L}_\varepsilon \xRightarrow{G} \hat{\mathcal{L}}$.

Proof. First, we prove the statement under a more restrictive assumption than (4.2). Namely, let us assume that for any

$$\zeta = (\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n$$

the functions $a(t, x, \eta, \xi)$ and $a_0(t, x, \eta, \xi)$ are almost periodic in the sense of Bohr with respect to the variable $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$. Then, being extended to $\mathbb{R}_B^{n+1}$, the functions $a(\omega, \eta, \xi)$ and $a_0(\omega, \eta, \xi)$ are continuous with respect to $\omega \in \mathbb{R}_B^{n+1}$. Moreover, since

$$(a, a_0) \in \Pi_{\mathbb{R}_B^{n+1}},$$

it is easy to verify that these functions are equicontinuous in ω if ζ belongs to any bounded subset of $\mathbb{R} \times \mathbb{R}^n$.

Based on statistical homogenization theorem, there exists a subset $\Omega_0 \subset \mathbb{R}_B^{n+1}$, with $\mu(\Omega_0) = 1$, such that $\mathcal{L}_\varepsilon(\omega) \stackrel{G}{\Longrightarrow} \hat{\mathcal{L}}$ for $\omega \in \Omega_0$. We note that any subset $\Omega_0 \subset \mathbb{R}_B^{n+1}$ of full measure is dense in $\mathbb{R}_B^{n+1}$. Hence, there exists a sequence (more precisely, a net) $\omega_l \in \Omega_0$ such that $\omega_l \to 0$ in $\mathbb{R}_B^{n+1}$. Moreover, by Theorem 2.1, for every sequence $\varepsilon' \to 0$ there exists a subsequence ε_k of ε' such that $\mathcal{L}_k = \mathcal{L}_{\varepsilon_k} = \mathcal{L}_{0,\varepsilon_k} \stackrel{G}{\Longrightarrow} \widetilde{\mathcal{L}}$ on Q for some parabolic operator $\widetilde{\mathcal{L}}$ of class Π. Using Corollary 3.6, we conclude that $\widetilde{\mathcal{L}} = \widehat{\mathcal{L}}$. In particular, the passage to a subsequence ε_k is superfluous and we obtain that $\mathcal{L}_\varepsilon \stackrel{G}{\Longrightarrow} \widehat{\mathcal{L}}$ The latter holds for every $\omega \in \mathbb{R}_B^{n+1}$, not only for $\omega = 0$.

Next, we return to the general case. Consider new functions $a^m(t, x, \eta, \xi)$ and $a_0^m(t, x, \eta, \xi)$ defined by

$$a^m(t, x, \eta, \xi) = S_m a(t, x, \eta, \xi),$$

$$a_0^m(t, x, \eta, \xi) = S_m a_0(t, x, \eta, \xi),$$

where S_m is the sequence of "smoothing" operators introduced earlier (see Remark 2.2). Then $a^m(t, x, \eta, \xi)$ and $a_0^m(t, x, \eta, \xi)$ are almost periodic in $(t, x) \in \mathbb{R}^{n+1}$ in the sense of Bohr, and

$$(a^m, a_0^m) \in \Pi_{\mathbb{R}^{n+1}},$$

with the same values of the parameters. The last follows from the fact that the kernel function of S_m is non-negative and has the mean value equals to 1. By definition, for any

$$\zeta = (\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n$$

we have

$$\lim_{m\to\infty} a^m(\cdot, \cdot, \eta, \xi) = a(\cdot, \cdot, \eta, \xi), \tag{4.4}$$

$$\lim_{m\to\infty} a_0^m(\cdot, \cdot, \eta, \xi) = a_0(\cdot, \cdot, \eta, \xi), \tag{4.5}$$

in the B^1-norm. Moreover, since a and a_0 are continuous functions of ζ with values in $B^1(\mathbb{R}^{n+1})$, these limits are uniform with respect to ζ, whenever ζ belongs to any bounded subset of $\mathbb{R} \times \mathbb{R}^n$. By G-compactness, we may assume that

$$\mathcal{L}_\varepsilon \stackrel{G}{\Longrightarrow} \mathcal{B} \in \Pi,$$

where

$$\mathcal{B}u = D_t u - div\, b(t, x, u, D_x u) + b_0(t, x, u, D_x u).$$

Since individual homogenization takes place for the operator

$$\mathcal{L}_\varepsilon^m u = D_t u - div\, a^m(\frac{t}{\epsilon^\alpha}, \frac{x}{\epsilon^\beta}, u, D_x u) + a_0^m(\frac{t}{\epsilon^\alpha}, \frac{x}{\epsilon^\beta}, u, D_x u),$$

we have $\mathcal{L}_\varepsilon^m \overset{G}{\Longrightarrow} \hat{\mathcal{L}}^m$.

Now we apply Theorem 3.4. Consider the functions

$$g(t,x,r) = \sup_{|\eta|,|\xi|\le r} |a^m(t,x,\eta,\xi) - a(t,x,\eta,\xi)|,$$

$$\hat{g}(t,x,r) = \sup_{|\eta|,|\xi|\le r} |\hat{a}^m(\eta,\xi) - b(t,x,\eta,\xi)|,$$

and the functions $g_0(t,x,r)$ and $\hat{g}_0(t,x,r)$ defined similarly in terms of a_0, a_0^m, $\hat{a}_0^m$, and b_0. For simplicity of notations, we suppress here the explicit dependence of the functions g, $\hat{g}$, g_0, and $\hat{g}_0$ on m. Suppose $K = (0,1)\times K_0$, where K_0 is the unit cube in $\mathbb{R}^n$ centered at the origin and

$$K_\rho(t_0,x_0) = (t_0,x_0) + \rho K.$$

We set

$$\overline{g}(t,x,r) = \limsup_{\rho\to 0}\limsup_{\varepsilon\to 0} \frac{1}{|K_\rho(t,x)|}\int_{K_\rho(t,x)} g(\varepsilon^{-\alpha}\tau, \varepsilon^{-\beta}y, r)\,dy d\tau,$$

$$\overline{g}_0(t,x,r) = \limsup_{\rho\to 0}\limsup_{\varepsilon\to 0} \frac{1}{|K_\rho(t,x)|}\int_{K_\rho(t,x)} g_0(\varepsilon^{-\alpha}\tau, \varepsilon^{-\beta}y, r)\,dy d\tau.$$

By Theorem 3.4, we have

$$\hat{g}(t,x,R) \le \overline{g}(t,x,r) + c(R)\left[\varphi(r)^{1/q} + \varphi^\gamma(r) + (1+r)^\gamma \overline{g}(x,r)^\gamma\right], \tag{4.6}$$

$$\hat{g}_0(t,x,R) \le \overline{g}_0(t,x,r) + c(R)\left[\varphi(r)^{1/q} + \varphi^\gamma(r) + (1+r)^\gamma \overline{g}(x,r)^\gamma\right], \tag{4.7}$$

for any $r>0$, where $\gamma>0$ and $\varphi(r)\to 0$, as $r\to\infty$. Taking into account (2.23) and mean value theorem, it follows that

$$\lim_{\varepsilon\to 0} \frac{1}{|K_\rho(t,x)|}\int_{K_\rho(t,x)} g(\varepsilon^{-\alpha}t, \varepsilon^{-\beta}y, r)\,dy = \|g(\cdot,\cdot,r)\|_{B^1}$$

which does not depend on t, x and ρ. By (4.4) and (4.5), for any $r>0$, we have

$$\|g(\cdot,\cdot,r)\|_{B^1} \to 0, \quad \|g_0(\cdot,\cdot,r)\|_{B^1} \to 0, \text{ as } m\to\infty.$$

Passing in (4.6) and (4.7) to the limit as $m\to\infty$ and, then, as $r\to\infty$, we see that

$$b(t,x,\eta,\xi) = \lim_{m\to\infty} \hat{a}^m(\eta,\xi),$$

$$b_0(t,x,\eta,\xi) = \lim_{m\to\infty} \hat{a}_0^m(\eta,\xi).$$

The same argument works for the operators $\mathcal{L}_\varepsilon(\omega)$ and $\mathcal{L}_\varepsilon^m(\omega)$, with $\omega\in\mathbb{R}_B^{n+1}$. For a generic $\omega\in\mathbb{R}_B^{n+1}$, the homogenized operators for $\mathcal{L}_\varepsilon(\omega)$ and $\mathcal{L}_\varepsilon^m(\omega)$ coincide with $\hat{\mathcal{L}}$ and $\hat{\mathcal{L}}^m$, respectively. Therefore,

$$b(t,x,\eta,\xi) = \hat{a}(\eta,\xi),$$

$$b_0(t,x,\eta,\xi) = \hat{a}_0(\eta,\xi).$$

We would like to note that the proof can be easily extended to the case $\alpha = 0$ or $\beta = 0$.
$\square$

Remark 4.1. *By Theorem 3.3, we have the following representation formulas for the homogenized operator* $\hat{L}$:

$$\hat{a}(\eta,\xi) = \lim_{\tau\to\infty} \frac{1}{\tau^{n+1}} \int_{K_\tau} a(t,x,\eta,\xi + D_x v_t^\zeta)dxdt,$$

$$\hat{a}_0(\eta,\xi) = \lim_{\tau\to\infty} \frac{1}{\tau^{n+1}} \int_{K_\tau} a_0(t,x,\eta,\xi + D_x v_t^\zeta)dxdt,$$

where K_τ *is a generic cube, with the side length* τ, *centered at the origin and*

$$v_\tau^\zeta \in L^p(0,\tau,W_0^{1,p}(K_\tau))$$

is a unique solution of the problem

$$D_t v - div\, a(t,x,\eta,\xi + D_x v) = 0 \quad on\ K_\tau.$$

Here, as usual, $\zeta = (\eta,\xi) \in \mathbb{R}\times\mathbb{R}^n$.

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