Error Analysis for a Galerkin Finite Element Method Applied to a Coupled Nonlinear Degenerate System of Advection-diffusion Equations [1]

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Abstract

We consider a standard Galerkin Method applied to both the pressure equation and the saturation equation of a coupled nonlinear system of degenerate advection-diffusion equations modeling two-phase immiscible flow through porous media. After regularizing the problem and establishing some regularity results, we derive error estimates for a semi-discretized Galerkin Method. A decoupled nonlinear scheme is then proposed for a fully discretized (backward in time) Galerkin Method, and error estimates are derived for that method. We also prove existence and uniqueness for the nonlinear operator intervening in the backward time discretization.

1 Introduction

We consider the following coupled nonlinear system modeling two-phase immiscible flow through porous media[2, 5, 13, 22].

$$\begin{cases} \mathbf{u} = -a(S)\nabla p & \text{in } \Omega \times (0,T) \\ \operatorname{div}(\mathbf{u}) = Q & \text{in } \Omega \times (0,T) \\ \mathbf{u}\cdot\mathbf{n} = 0 & \text{on } \partial\Omega \times [0,T] \\ \int_\Omega p\,dx = 0 & \text{for all } t \in [0,T] \\ \phi\dfrac{\partial S}{\partial t} + \nabla\cdot f(S)\mathbf{u} - \nabla\cdot k(S)\nabla S = 0 & \text{in } \Omega \times (0,T) \\ k(S)\dfrac{\partial S}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \times [0,T] \\ S(x,0) = S^0(x) & \text{in } \Omega \end{cases} \tag{1.1}$$

This is a somewhat simplified form of the pressure/saturation system. In particular, we have omitted here the gravitational term in the pressure equation in order to simplify the analysis.

In this problem, $\mathbf{u}$ is the total Darcy's velocity, and p is the global pressure of the two phases. S is the saturation of the invading fluid and k is the conductivity of the medium. The function f is the fractional flow function. The function a is a combination of terms that define the permeabilities of the phases and viscosity of the medium ([5]).

We assume here that $\Omega \subset \mathbf{R}^n$, $n = 1, 2, 3$, is a sufficiently smooth domain or a convex polyhedral domain. In this analysis, we have in mind $n = 2$ and Ω a convex polygonal domain.

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The main goal of this paper is to establish error estimates of an approximation of the solution to (1.1) by a standard Galerkin finite element method applied to both the pressure equation and the saturation equation. In [18], whose line we follow here, the same problem was studied, but in the absence of the pressure equation. The total Darcy velocity $\mathbf{u} = -a(S)\nabla p$ was assumed to be given and to have the regularity needed for the analysis. It has been shown, under various conditions, that problem (1.1) has a unique weak solution (see [1, 6]. Many authors have studied problem (1.1) using mixed finite element methods and deriving error estimates for this problem [7, 9, 27, 8, 28]. The mixed finite element method approximates better the velocity $\mathbf{u}$ and conserve the minimal regularity on $\mathbf{u}$. But, here we choose to work with the standard finite element method(same approximation space for the pressure p and the saturation S) in order to simplify the analysis and focus more on the mathematics.

The mixed finite element method focuses on $\mathbf{u} \in L^\infty(H(div,\Omega))$ [9], whilst the standard finite element method, considered here and focuses rather on the pressure p than on the velocity $\mathbf{u}$. The results obtained here can be compared to the results obtained in [9] where a mixed finite element method is used, and where the problem is formulated differently. In addition, we state or establish, in this paper, results that are not established in [9] (See (4.2) for example.)

The main difference with [9], beside the regularity results and section 3.4, is the formulation of the fully discretized scheme in the last section. In our case, we propose a decoupled implicit scheme for the system. This "semi-implicit" scheme uses the Darcy velocity $\mathbf{u} = -a(S)\nabla p$ calculated at the previous time step n (in lieu of the velocity at the time step $n+1$, as would require a fully backward Euler scheme), thus decoupling, in this the way, the scheme.

These results can also be compared to the ones established in [18] where only the saturation equation was considered. We see that, according to this analysis, the order of convergence in the present paper are roughly one unit less than the ones obtained in [18], if we let $\mu \to 0$ (the nondegenerate case corresponding to $\mu = 0$). An immediate attempt of explanation would be as follows. Since the system is coupled, and because p intervenes only through its gradient, the rate of convergence to ∇p will dominate the process. We get near-optimal results for p, but not for $K(S)$ as we would expect.

The functions k and a are Lipschitz-continuous on the interval $[0,\ 1]$, and $f \in C^2[0,1]$. Most results, in sections 2 and 3, do not use assumption (1.13), except to further give insights on the convergence estimates in terms of β and h. However, this assumption is assumed to hold in the whole section 4. $Q = Q(x,t)$ is a bounded function on $\Omega \times [0,T]$ and continuously differentiable in the time variable t.

We make the following additional assumptions on the data.

$$k(0) = k(1) = 0 \tag{1.2}$$

$$k(s) \geq \begin{cases} c_1 s^\mu & 0 \leq s \leq \alpha_1 < 1 \\ c_2 & \alpha_1 \leq s \leq \alpha_2 < 1 \\ c_3(1-s)^\mu & \alpha_2 \leq s \leq 1 \end{cases} \tag{1.3}$$

with

$$0 < \mu \leq 2, \tag{1.4}$$

and α_1 and α_2 given.

Set

$$\gamma = \frac{2+\mu}{1+\mu} \tag{1.5}$$

Then γ is the conjugate index of $2+\mu$.

For a convex polygonal (polyhedral) domain, we will assume that the maximum angle of the polygon (polyhedron) satisfies the following condition. Let $\theta(\Omega)$ be the maximum angle of Ω, with $\frac{\pi}{2} < \theta(\Omega) < \pi$. Then we assume that the polygonal domain Ω satisfies the condition

$$0 < \mu < \frac{2\pi - 2\theta(\Omega)}{2\theta(\Omega) - \pi}. \tag{1.6}$$

This is to ensure that some inequalities used in this analysis, such as (3.30) and (3.33), for $p = \gamma = \frac{2+\mu}{1+\mu}$, are true for a convex polygonal (polyhedral) domain. In particular, (1.6) ensures that solutions of the poisson equation

$$-\Delta u = f, \tag{1.7}$$

with specified Newman and/or Dirichlet conditions, are in $W^{2,2+\mu}(\Omega)$, when Ω is a polygonal domain (or a domain with corners) of maximum angle $\theta(\Omega)$. See [3, 20, 21].

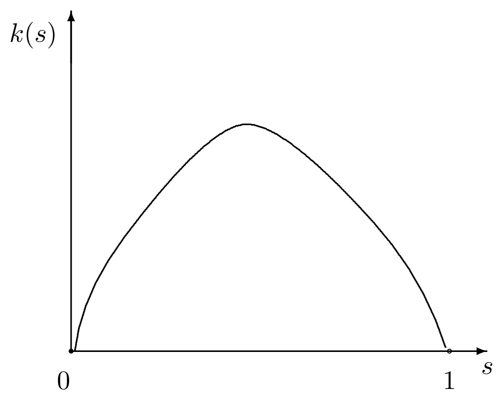

Figure 1: Example of a graph of $k(s)$

Further Assumptions

$$f'(0) = f'(1) = 0. \tag{1.8}$$

$$0 < d_0 \le a(t) \le d_1 < \infty. \tag{1.9}$$

$$0 < \phi_0 \le \phi(t) \le \phi_1 < \infty. \tag{1.10}$$

We define K by

$$K(s) = \int_0^s k(\tau)d\tau \tag{1.11}$$

We also assume that

$$\begin{aligned} |f(s_2) - f(s_1)|^2 \quad &+ \quad |K(s_2) - K(s_1)|^2 \\ &\le \quad C(K(s_2) - K(s_1))(s_2 - s_1) \ \forall s_1, s_2 \in [0,1], \end{aligned} \tag{1.12}$$

and

$$|a(s_2) - a(s_1)|^2 \le C|K(s_2) - K(s_1)||s_2 - s_1|, \ \forall s_1, s_2 \in [0,1]. \tag{1.13}$$

Notice that inequality (1.12) holds under assumption (1.8) and the fact that k is continuous. We also see that (1.3) and (1.8) imply

$$|f'(s)| \le C\sqrt{k(s)} \tag{1.14}$$

(see [14, 17]).

Under assumption (1.3), we have

$$|s_2 - s_1|^{1+\mu} \le C|K(s_2) - K(s_1)| \tag{1.15}$$

(see [14, 17, 24]). We get from (1.15) that

$$\|s_2 - s_1\|^{2+\mu}_{L^{2+\mu}} \le C \int_\Omega |K(s_2) - K(s_1)||s_2 - s_1|dx = C(K(s_2) - K(s_1), s_2 - s_1). \tag{1.16}$$

Also notice that if (1.13) holds, then we have

$$|a'(s)| \le C\sqrt{k(s)}. \tag{1.17}$$

Thus (1.13) implies that $t \to a(t)$ is continuously differentiable.

The presence of σ (or σ_1, σ_2, etc) in front of a term will mean that the term can be hidden in the left handside of the inequality under certain conditions.

The remaining of the paper is structured as follow.

In section 2, we regularize problem (1.1) and give a weak formulation of the regularized problem. We also establish error estimates and regularity results for the regularized problem.

In section 3, we analyze the continuous Galerkin method. Error estimates are established first for a general perturbation, and then for a particular one. Additional error estimates are given in $L^\infty(0,T,L^{2+\mu}(\Omega))$ and in $L^2(0,T,H^1(\Omega))$.

In the last section, we analyze a fully discretized Galerkin Method. A method is proposed which linearizes the pressure equation and decouples the system. Error estimates are established to show the convergence of this method. In the sequel of this paper a method will be proposed that linearizes the saturation equation.

Finally, we set additional notation which will be used throughout the remainder of this paper. We define $(f,g) := (f,g)_\Omega := \int_\Omega fg dx$ when this has a meaning, and in particular we set $f_\Omega := \frac{1}{|\Omega|}(f,1)_\Omega$. We drop the subscript Ω when there is no ambiguity. The notation $\|f\|_{L^p} := \|f\|_{L^p(\Omega)}$ is used for the standard Lebesgue norm of a measurable function, when this quantity is finite. Similarly, we denote by $\|f\|_{L^p(L^q)} := \|f\|_{L^p(0,T,L^q(\Omega))}$ the mixed Lebesgue norm for f, while $\|f\|_{L^p(H^q)} := \|f\|_{L^p(0,T,H^q(\Omega))}$ designates the mixed Sobolev-Lebesgue norm of a function. We use C, c, σ, and η to denote positive constants which may change from line, but which are independent of the parameters β, h and Δt, unless otherwise explicitly specified. Here σ will designate a constant we can control thanks to some classical inequalities.

2 The regularized problem

2.1 Regularization

In [7], Problem (1.1) was approximated without a prior regularization. In the present paper, to solve (1.1), we approximate the following perturbed problem instead.

Let $\beta > 0$, be sufficiently small (intended to tend to 0). Perturb k to k_β in such a way that $k_\beta \to k$ strongly as $\beta \to 0$.

For instance, let $\delta = \min(k(\beta), k(1-\beta))$, and define k_β by

$$\begin{cases} k_\beta(s) = k(s) & \text{if } k(s) \ge \delta \\ \frac{1}{2}\delta \le k_\beta(s) \le \delta & \text{otherwise.} \end{cases} \tag{2.1}$$

Then $k_\beta(t) \ge k(t)$, for all $t \in [0,T]$, and k_β satisfies

$$k_\beta(s) \ge \frac{1}{2}\delta,\ \forall s \in [0,1],$$

thus is bounded away from 0.

Another possible perturbation of k is given by

$$k_\beta(s) = \max(k(s), \beta^\mu) \tag{2.2}$$

In general, define

$$m(\beta) = \inf\{k_\beta(s), 0 \le s \le 1\} \tag{2.3}$$

Substitute k_β to k in problem (1.1) to get the nondegenerate system

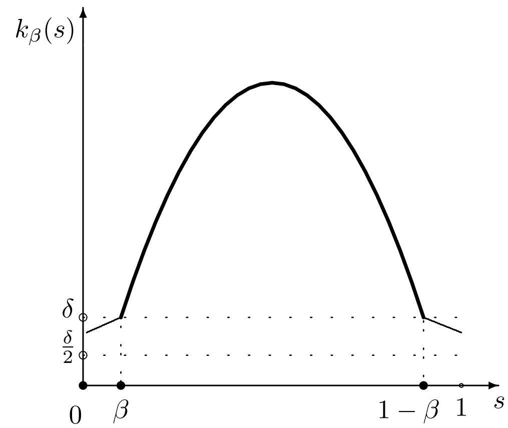

Figure 2: Un example of a perturbation of k

$$
\begin{cases}
\mathbf{u}_\beta = -a(S_\beta)\nabla p_\beta & \text{in } \Omega \times (0,T) \\
\operatorname{div}(\mathbf{u}_\beta) = Q & \text{in } \Omega \times (0,T) \\
\mathbf{u}_\beta \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times [0,T] \\
\int_\Omega p_\beta dx = 0 & \text{for all } t \in [0,T] \\
\phi \dfrac{\partial S_\beta}{\partial t} + \nabla \cdot f(S_\beta)\mathbf{u}_\beta - \nabla \cdot k_\beta(S_\beta)\nabla S_\beta = 0 & \text{in } \Omega \times (0,T) \\
k_\beta(S_\beta)\dfrac{\partial S_\beta}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \times [0,T] \\
S_\beta(x,0) = S^0(x) & \text{in } \Omega
\end{cases}
\tag{2.4}
$$

Define K_β by

$$K_\beta(s) = \int_0^s k_\beta(\tau)d\tau. \tag{2.5}$$

Then K_β also satisfies conditions (1.12)–(1.16).

Note: The fourth equations in (1.1) and (2.4) are to ensure uniqueness only, since $\mathbf{u}$ defines p up to an additive constant. In fact, if $\int_\Omega p(x)dx$ is not 0, then set $\tilde{p} = p - \int_\Omega p dx$ to get $\mathbf{u} = -a(S)\nabla\tilde{p} = -a(S)\nabla p$, and $\int_\Omega \tilde{p} dx = 0$.

2.2 Weak formulation

In the remaining of this paper, because of (1.10), we assume, without lost of generality, that

$$\phi = 1.$$

We assume that (2.4) has a unique solution in the following sense. There exists a couple of functions (p_β, S_β) satisfying

$$p_\beta \in L^\infty(0,T,H^1(\Omega)), \tag{2.6}$$

$$S_\beta \in L^\infty(0,T,L^2(\Omega)) \cap L^2(0,T,H^1(\Omega)) \tag{2.7}$$

and such that
$$\begin{cases} (a(S_\beta)\nabla p_\beta, \nabla\psi) = (Q,\psi) & \forall\psi\in H^1(\Omega),\ \forall t\in[0,T] \\ \int_\Omega p_\beta dx = 0 & \forall t\in[0,T] \\ \left(\dfrac{\partial S_\beta}{\partial t},\psi\right) - (f(S_\beta)(-a(S_\beta)\nabla p_\beta),\nabla\psi) & \\ \qquad\qquad + (\nabla K_\beta(S_\beta),\nabla\psi) = 0 & \forall\psi\in H^1(\Omega),\ \forall t\in(0,T] \\ S_\beta(x,0) = S^0(x) & \forall x\in\Omega \end{cases} \tag{2.8}$$

We also assume that the initial problem (1.1) has a unique solution in the sense of (2.8), at the exception that we replace (2.7) by
$$S\in L^\infty(0,T,L^2(\Omega)) \tag{2.9}$$
and
$$K(S)\in L^2(0,T,H^1(\Omega)). \tag{2.10}$$

Remark 2.1 If k is bounded away from zero, i.e. if $k(s)\geq k_0>0$ for some positive constant k_0, then
$$K(S)\in L^2(0,T,H^1(\Omega)) \Longrightarrow S\in L^2(0,T,H^1(\Omega))\cap L^\infty(0,T,L^2(\Omega)).$$
In fact, we have $\nabla K(S) = k(S)\nabla S$, so that
$$\int_\Omega |\nabla K(S)|^2 dx = \int_\Omega |k(S)\nabla S|^2 dx \geq k_0^2\|\nabla S\|^2_{L^2(\Omega)}.$$

Therefore (2.9) and (2.10) generalize (2.7) for the case of a degenerate problem.

2.3 Convergence and Regularity Results for the Perturbed Problem

The following theorem gives convergence results for the regularized problem. We give a sketch of the proof of Theorem 2.1 in Appendix B.

Theorem 2.1 *Under the above conditions on f, k, k_β, p and a, the following is true.*
$$\|\sqrt{a(S)}\nabla(p_\beta-p)\|_{L^2(L^2)} \leq C\|\nabla p\|_{L^\infty(L^\infty)}\|a(S_\beta)-a(S)\|_{L^2(L^2)} \tag{2.11}$$
and
$$\begin{aligned} \|S_\beta - S\|^2_{L^\infty((H^1)^*)} &+ \eta\int_0^T (K_\beta(S_\beta)-K_\beta(S), S_\beta - S)(\tau)d\tau \\ &\leq C\Big\{\|K_\beta(\cdot)-K(\cdot)\|^\gamma_{L^\infty(0,1)} \\ &+ \sigma\|\nabla p\|_{L^\infty(L^\infty)}\|a(S_\beta)-a(S)\|^2_{L^2(L^2)}\Big\} \end{aligned} \tag{2.12}$$
where $\gamma=\frac{\mu+2}{\mu+1}$, and μ defined as in (1.4), and where σ can be an arbitrary positive number thanks to the arithmetic-geometric inequality.

Using condition (1.13) we get the following immediate consequence.

Corollary 2.1 *Under the hypotheses of Theorem 2.1 and condition (1.13), we have*
$$\|\sqrt{a(S)}\nabla(p_\beta-p)\|^2_{L^2(L^2)} \leq C\int_0^T (K_\beta(S_\beta)-K_\beta(S), S_\beta-S)(\tau)d\tau \tag{2.13}$$
and
$$\begin{aligned} \|S_\beta - S\|^2_{L^\infty((H^1)^*)} &+ \eta\int_0^T (K_\beta(S_\beta)-K_\beta(S), S_\beta-S)(\tau)d\tau \\ &\leq C\|K_\beta(\cdot)-K(\cdot)\|^\gamma_{L^\infty(0,1)} \end{aligned} \tag{2.14}$$

This Corollary yields the following.

Corollary 2.2 *Under the hypotheses of theorem 2.1 and condition (1.13) we have*

$$\|S_\beta - S\|^{2+\mu}_{L^{2+\mu}(L^{2+\mu})} + \left\|\sqrt{a(S)}\nabla(p_\beta - p)\right\|^2_{L^2(L^2)} \leq C\{\|K_\beta(\cdot) - K(\cdot)\|^\gamma_\infty \tag{2.15}$$

and

$$\|K_\beta(S_\beta) - K(S)\|^2_{L^2(L^2} \leq \|K_\beta(\cdot) - K(\cdot)\|^\gamma_\infty. \tag{2.16}$$

Note that the constants appearing in Corollaries 2.1 and 2.2 are functions of $\|\nabla p\|_{L^\infty(L^\infty)}$, but are independent of β.

We can prove the following two regularity results by modifying slightly the proofs of Theorem 3.7 and Lemma 4.3 of [17], respectively.

Theorem 2.2 *If S_β is a solution to Problem (2.4), then we have*

$$\|S_\beta\|^2_{L^\infty(L^2)} + \eta\left\|\sqrt{k(S_\beta)}\nabla S_\beta\right\|^2_{L^2(L^2)} \leq C\|Q\|_{L^1(L^1)} + \|S^0\|^2_{L^2} \tag{2.17}$$

Proof.

In the second equation of (2.8), let $\psi = S_\beta$ to get

$$\frac{1}{2}\frac{d}{dt}\|S_\beta\|^2_{L^2} + \left\|\sqrt{k_\beta(S_\beta)}\nabla S_\beta\right\|^2_{L^2} = (f(S_\beta)\mathbf{u}_\beta, \nabla S_\beta) \tag{2.18}$$

As in [17], define F by

$$F(s) = \int_0^s f(\tau)d\tau \tag{2.19}$$

Then

$$\begin{aligned}(f(S_\beta)\mathbf{u}_\beta, \nabla S_\beta) &= \int_\Omega \mathbf{u}_\beta \cdot \nabla F(S_\beta(x,t))dx \\ &= \int_{\partial\Omega} F(S_\beta(x,t))\mathbf{u}_\beta \cdot \mathbf{n}d\sigma - \int_\Omega F(S_\beta(x,t))\nabla \cdot \mathbf{u}_\beta dx\end{aligned} \tag{2.20}$$

Now the first term on the righthand side of (2.20) vanishes by (2.4). So we get

$$\begin{aligned}(f(S_\beta)\mathbf{u}_\beta, \nabla S_\beta) &\leq C\|F(\cdot)\|_{L^\infty}\|\nabla \cdot \mathbf{u}_\beta\|_{L^1} \\ &\leq C\|Q\|_{L^1}\end{aligned} \tag{2.21}$$

Hence, combining (2.18) and (2.21), and integrating over the interval $[0,T]$, we get the Theorem.
$\square$

Theorem 2.3 *Let S_β be the solution to Problem (2.4), then we have*

$$\begin{aligned}&\left\|\sqrt{k_\beta(S_\beta)}S_{\beta t}\right\|^2_{L^2(L^2)} + \eta\|\nabla K_\beta(S_\beta)\|^2_{L^\infty(L^2)} \\ &\quad\leq C\{\|\mathbf{u}_\beta\|^2_{L^\infty(L^\infty)}(\|Q|_{L^1(L^1)} + \|S^0\|^2_{L^2}) + \|Q\|^2_{L^2(L^2)}\} \\ &\quad+ \|\nabla K_\beta(S^0)\|^2_{L^2}\end{aligned} \tag{2.22}$$

for some $\eta > 0$.

Proof.

We multiply the fifth equation of (2.4) by $(K_\beta(S_\beta))_t$, integrate over Ω, and use the sixth equation of (2.4) to get

$$
\begin{aligned}
&\left\|\sqrt{k_\beta(S_\beta)}S_{\beta t}\right\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\nabla K_\beta(S_\beta)\|_{L^2}^2 \\
&\qquad = -(\nabla(f(S_\beta)\mathbf{u}_\beta), (K_\beta(S_\beta))_t) \\
&\qquad = -(f'(S_\beta)\nabla S_\beta \cdot \mathbf{u}_\beta + f(S_\beta)\nabla\cdot\mathbf{u}_\beta, (K_\beta(S_\beta))_t) \\
&\qquad \le \frac{1}{2}\left\|\sqrt{k_\beta(S_\beta)}S_{\beta t}\right\|_{L^2}^2 + \frac{1}{2}\left\{\left\|f'(S_\beta)\sqrt{k_\beta(S_\beta)}\nabla S_\beta\cdot\mathbf{u}_\beta\right\|_{L^2}^2\right. \\
&\qquad \left. + \left\|\sqrt{k_\beta(S_\beta)}f(S_\beta)\nabla\cdot\mathbf{u}_\beta\right\|_{L^2}^2\right\} \\
&\qquad \le \frac{1}{2}\left\|\sqrt{k_\beta(S_\beta)}S_{\beta t}\right\|_{L^2}^2 + \frac{1}{2}\|f'(\cdot)\|_{L^\infty}^2\|\mathbf{u}_\beta\|_{L^\infty}^2\left\|\sqrt{k_\beta(S_\beta)}\nabla S_\beta\right\|_{L^2}^2 \\
&\qquad + \left\|\sqrt{k_\beta(\cdot)}f(\cdot)\right\|_{L^\infty}^2\|\nabla\cdot\mathbf{u}_\beta\|_{L^2}^2
\end{aligned}
\tag{2.23}
$$

In (2.23), we have used the following.

$$
(v, (K_\beta(S_\beta))_t) = (v, k_\beta(S_\beta)S_{\beta t}) = \left(\sqrt{k_\beta(S_\beta)}v, \sqrt{k_\beta(S_\beta)}S_{\beta t}\right) \tag{2.24}
$$

for any $v \in L^2(\Omega)$.

After hiding the first term on the righthand side in the like term in the lefthand side, and integrating over the interval $[0,T]$, we get

$$
\begin{aligned}
&\left\|\sqrt{k_\beta(S_\beta)}S_\beta t\right\|_{L^2(L^2)}^2 + \eta\|\nabla K_\beta(S_\beta)\|_{L^\infty(L^2)}^2 \\
&\qquad \le C\left\{\|\mathbf{u}_\beta\|_{L^\infty(L^\infty)}^2\left\|\sqrt{k_\beta(S_\beta)}\nabla S_\beta\right\|_{L^2(L^2)}^2\right. \\
&\qquad \left. + \|\nabla\cdot\mathbf{u}_\beta\|_{L^2(L^2)}^2\right\} + \|\nabla K_\beta(S^0)\|_{L^2}^2
\end{aligned}
\tag{2.25}
$$

for some $\eta > 0$.

Finally use (2.17) and (2.4) to get the theorem. $\square$

We also have:

Theorem 2.4 *Under the hypotheses on problems (1.1) and (2.4), we have*

$$
\begin{aligned}
\sup_{0\le t\le T}(K_\beta(S_\beta) - K(S), S_\beta - S) \quad &+ \quad \eta\|\nabla(K_\beta(S_\beta) - K(S))\|_{L^2(0,T,L^2(\Omega))}^2 \\
&\le \quad C(m(\beta) + \beta) \\
&+ \quad \sigma\|a(S_\beta) - a(S)\|_{L^2(0,T,L^2(\Omega))}^2
\end{aligned}
\tag{2.26}
$$

where $m(\beta)$ is defined by (2.3).

The proof of this theorem is a combination of the proofs of Theorem 4.6 of [17] and Theorem 2.1 above.

In view of these results, to approximate problem 1.1, we need only approximate problem 2.4, provided that the constants appearing in the estimates do not depend on β.

The following estimate is found in Appendix B of this paper, Theorem B.1.

$$
\|\mathbf{u}_\beta\|_{L^\infty(L^2)} < C, \tag{2.27}
$$

with C independent of β. Assumption (2.28), below, is used only in the proof of Corollary 4.1.

$$\|\mathbf{u}_{\beta t}\|_{L^\infty(L^2)} \le Cm(\beta)^{-\frac{1}{2}}. \tag{2.28}$$

Note that relation (2.3) implies:

$$\left\|\sqrt{k_\beta(\cdot)}\right\|_{L^\infty} \ge m(\beta)^{\frac{1}{2}}. \tag{2.29}$$

Now we get from (2.17), (2.22), and (2.29),

$$\|S_{\beta t}\|_{L^2(0,T,L^2(\Omega))} + \|\nabla S_\beta\|_{L^2(0,T,L^2(\Omega))} \le Cm(\beta)^{-\frac{1}{2}}. \tag{2.30}$$

3 The Continuous Galerkin Method

3.1 The Finite Element Space

As in [18], let $\{M_h\}_{0<h<1}$ be a family of finite dimensional spaces, with $M_h \subset H^1(\Omega)$. We assume that M_h has the approximation property:

$$\inf_{\chi\in M_h} \|f-\chi\|_{L^p(\Omega)} \le Ch^2\|f\|_{W^{2,p}} \qquad \text{for all } f \in W^{2,p},\ p \ge 1. \tag{3.1}$$

We are also going to need the inverse estimate assumption:

$$\|\chi\|_{H^1} \le Ch^{-1}\|\chi\|_{L^2} \tag{3.2}$$

for all $\chi \in M_h$.

If (3.2) holds, then we have

$$\|\chi\|^2_{L^2} = (\chi,\chi) \le \|\chi\|_{H^1}\|\chi\|_{(H^1)^*} \le Ch^{-1}\|\chi\|_{L^2}\|\chi\|_{(H^1)^*}.$$

Hence

$$\|\chi\|_{L^2(\Omega)} \le Ch^{-1}\|\chi\|_{(H^1)^*} \tag{3.3}$$

for all $\chi \in M_h$.

3.2 The Discretized Problem

Because of possible numerical oscillations, we extend the functions defined on $[0,1]$ as follow.

$$k_\beta(s) = \begin{cases} k_\beta(-s) & \text{if } s \le 0 \\ k_\beta(1) & \text{if } s \ge 1, \end{cases} \tag{3.4}$$

$$f(s) = \begin{cases} 0 & \text{if } s \le 0 \\ f(1) & \text{if } s \ge 1 \end{cases} \tag{3.5}$$

and

$$a(s) = \begin{cases} a(0) & \text{if } s \le 0 \\ a(1) & \text{if } s \ge 1 \end{cases} \tag{3.6}$$

Notice that if $f \in C^1([0,1])$, then (extended) $f \in C^1(\mathbf{R})$, by (1.8). The same remark holds for $a(s)$.

Let K_β be as before, i.e.

$$K_\beta(s) = \int_0^s k(\tau)d\tau.$$

Then $K'_\beta(s) = k_\beta(s) \ge m(\beta) > 0$; thus K_β is strictly increasing on $\mathbf{R}$. Hence K_β has un inverse which we call H_β:

$$s = H_\beta(K_\beta(s)) \tag{3.7}$$

for all $s \in \mathbf{R}$.

With this in mind the discrete version of Problem (2.4) is defined as follow.

Let $h > 0$, sufficiently small, be given.

Find $(p_h, K_h) \in M_h \times M_h$, such that

$$\begin{cases} (a(H_\beta(K_h))\nabla p_h, \nabla\chi) = (Q, \chi) & \forall \chi \in M_h \\ \int_\Omega p_h dx = 0 & \forall t \in [0, T] \\ (H_\beta(K_h)_t, \chi) - (f(H_\beta(K_h))(-a(H_\beta(K_h))\nabla p_h), \nabla\chi) & \\ \qquad\qquad + (\nabla K_h, \nabla\chi) = 0 & \forall \chi \in M_h \\ \mathcal{P}_h H_\beta(K_h^0) = \mathcal{P}_h S^0 & \end{cases} \tag{3.8}$$

where S^0 is as in (1.1), and $\mathcal{P}_h$ is the L^2-projection onto M_h.

Since M_h is a finite dimensional space and because of the coupling, (3.8) consists of a nonlinear algebraic system of equations (defined by the first equation of (3.8)), coupled with a system of coupled ordinary differential equations in t (defined by the third equation of (3.8). Since the parameters a, k, and f are assumed Lipschitz, the general theory on ordinary differential equations guarantees existence and uniqueness for the system, for some $T > 0$.

Remark 3.1 *1.* For a given fixed K_h, the system of algebraic equations, defined by the first equation of (3.8), becomes linear and is well-defined, since

$$(a(H_\beta(K_h))\nabla v, \nabla v) = \left\| \sqrt{a(H_\beta(K_h))}\nabla v \right\|_{L^2}^2 \geq 0, \ \forall v \in M_h, \textit{ with } v_\Omega = 0 \tag{3.9}$$

and

$$\left\| \sqrt{a(H_\beta(K_h))}\nabla v \right\|_{L^2}^2 = 0$$

only if $v = 0$ *(*using the fact that $v_\Omega = 0$*).*

2. K_h^0 is well-defined on Ω since, by [24], $\mathcal{P}_h H_\beta$ is bijective.

3.3 Error Analysis for the Continuous Galerkin Method

We want to estimate the error $(p, K(S)) - (p_h, K_h)$ in terms of h and β. This will yield an estimate of a rate of convergence of this method (regularizing then approximating by a standard finite element method).

For convenience we set

$$S_h = H_\beta(K_h), \tag{3.10}$$

where H_β is defined by (3.7).

Then

$$K_h = K_\beta(S_h).$$

By (1.9), (2.4), and (B.60), we have

$$\|\nabla p_\beta\|_{L^\infty(L^2)} \leq C, \tag{3.11}$$

but we will need, in this analysis, the following stronger assumption:

$$\|\nabla p_\beta\|_{L^\infty(L^\infty)} \leq C. \tag{3.12}$$

Assumption (3.13) (below) is used only in Lemma 4.1, and Lemma 4.1 is not used in any other result of the present paper.

$$\|\nabla p_h\|_{L^\infty(L^\infty)} \leq C. \tag{3.13}$$

We derive the main results of this section via two lemmas. Some of the results obtained here are known in the literature (see, for instance, [9]), but, for completeness and to be consistent with the next section, we state and sketch the proofs of these results using different approaches.

Lemma 3.1 *Let (p_β, S_β) be the solution to problem (2.8), and (p_h, K_h) be the solution to problem (3.8), with $S_h = H_\beta(K_h)$. Then*

$$\begin{aligned}\left\|\sqrt{a(S_h)}\nabla(p_\beta - p_h)\right\|^2_{L^2(\Omega)} &\le Ch^2\|p_\beta\|^2_{H^2(\Omega)} \\ &+ C\|\nabla p_\beta\|_{L^\infty(L^\infty)}\|a(S_\beta) - a(S_h)\|^2_{L^2(\Omega)} \\ & \quad t \in [0,T]\end{aligned} \tag{3.14}$$

Proof.

We first notice that

$$\begin{aligned}\left\|\sqrt{a(s_\beta)}\nabla(p_\beta - p_h)\right\|_{L^2} \le & \left\|\sqrt{a(s_\beta)}\nabla\mathcal{P}_h(p_\beta - p_h)\right\|_{L^2} \\ &+ \left\|\sqrt{a(s_\beta)}\nabla(I - \mathcal{P}_h)(p_\beta - p_h)\right\|_{L^2}.\end{aligned} \tag{3.15}$$

Obviously, we get from (2.8) and (3.8):

$$(a(S_\beta)\nabla p_\beta - a(S_h)\nabla p_h, \nabla\chi) = 0. \ \forall \chi \in M_h \tag{3.16}$$

Now set $\chi = \mathcal{P}_h(p_\beta - p_h)$ in (3.16) to get

$$(a(S_h)\nabla(p_\beta - p_h), \nabla\mathcal{P}_h(p_\beta - p_h)) = -\left((a(S_\beta) - a(S_h))\nabla p_\beta, \nabla\mathcal{P}_h(p_\beta - p_h)\right). \tag{3.17}$$

The last inequality can be rewritten as

$$\begin{aligned}&(a(S_h)\nabla\mathcal{P}_h(p_\beta - p_h), \nabla\mathcal{P}_h(p_\beta - p_h)) \\ &\qquad = -\left((a(S_\beta) - a(S_h))\nabla p_\beta, \nabla\mathcal{P}_h(p_\beta - p_h)\right. \\ &\qquad + \left(a(S_h)\nabla(\mathcal{P}_h - I)(p_\beta - p_h), \nabla\mathcal{P}_h(p_\beta - p_h)\right).\end{aligned} \tag{3.18}$$

Estimating the righthand side of (3.18), we get

$$\begin{aligned}\left\|\sqrt{a(S_h)}\nabla\mathcal{P}_h(p_\beta - p_h)\right\|^2_{L^2(\Omega)} \le & \frac{1}{2}\left\|\sqrt{a(S_h)}\nabla\mathcal{P}_h(p_\beta - p_h)\right\|^2_{L^2(\Omega)} \\ &+ C\left\|\sqrt{a(\cdot)}\right\|^2_{L^\infty}\|\nabla(I - \mathcal{P}_h)(p_\beta - p_h)\|^2_{L^2(\Omega)} \\ &+ C\frac{1}{\left\|\sqrt{a(\cdot)}\right\|^2_{L^\infty}}\|\nabla p_\beta\|^2_{L^\infty(\Omega)}\|a(S_\beta) - a(S_h)\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.19}$$

Using the approximation property [3, 10]:

$$\|(I - \mathcal{P}_h)v\|_{H^1(\Omega)} \le Ch\|v\|_{H^2(\Omega)}, \ \forall v \in H^2(\Omega), \tag{3.20}$$

we get

$$\begin{aligned}\left\|\sqrt{a(S_\beta)}\nabla\mathcal{P}_h(p_\beta - p_h)\right\|^2_{L^2(\Omega)} &\le C\{h^2\|p_\beta\|^2_{H^2(\Omega)} \\ &+ \|\nabla p_\beta\|_{L^\infty(L^\infty)}\|a(S_\beta) - a(S_h)\|^2_{L^2(\Omega)}\} \\ &\quad t \in [0,T],\end{aligned} \tag{3.21}$$

where C depends on $a(\cdot)$, but is independent of β and h by (1.9).

Now, since

$$\|\nabla(I - \mathcal{P}_h)(p_\beta - p_h)\|_{L^2(\Omega)} \le Ch\|p_\beta\|_{H^2}, \tag{3.22}$$

by (3.20), we obtain the Lemma thanks to (3.15).

Lemma 3.2 *Let (p_β, S_β) be the solution to problem (2.8), and (p_h, K_h) be the solution to problem (3.8), with $S_h = H_\beta(K_h)$. Then*

$$
\begin{aligned}
\|\mathcal{P}_h(S_\beta - S_h)\|^2_{L^\infty((H^1(\Omega))^*)} &+ \eta \int_0^T (K_\beta(S_\beta) - K_\beta(S_h), S_\beta - S_h) d\tau \\
&\leq \sigma_0 \|f(S_\beta) - f(S_h)\|^2_{L^2(L^2)} \\
&+ \sigma_2 \|a(S_\beta) - a(S_h)\|^2_{L^2(0,T,L^2(\Omega))} \\
&+ \sigma_3 \left\| \sqrt{a(S_h)} \nabla(p_\beta - p_h) \right\|^2_{L^2(0,T,L^2(\Omega))} \\
&+ C \left(\max \left(1, \|\mathbf{u}_h\|^2_{L^\infty(L^\infty)}, \|f(S_\beta)\nabla p_\beta\|^2_{L^\infty(L^\infty)} \right) \right) \times \\
& \quad h^{2\gamma} \|K_\beta(S_\beta)\|^\gamma_{W^{2,\gamma}},
\end{aligned}
\tag{3.23}
$$

where

$$
\gamma = \frac{2+\mu}{1+\mu}, \tag{3.24}
$$

with μ defined by (1.4), and η some positive number.

Proof.

From (2.8) and (3.8) one gets:

$$
\begin{aligned}
(S_{\beta t} - S_{ht}, \chi) &- (f(S_\beta)(-a(S_\beta)\nabla p_\beta) - f(S_h)(-a(S_h)\nabla p_h), \nabla\chi) \\
&+ (\nabla(K_\beta(S_\beta) - K_\beta(S_h)), \nabla\chi) = 0, \ \forall \chi \in M_h.
\end{aligned}
\tag{3.25}
$$

That is

$$
\begin{aligned}
(S_{\beta t} - S_{ht}, \chi) &+ (\nabla(K_\beta(S_\beta) - K_\beta(S_h)), \nabla\chi) = \\
&- ((f(S_\beta) - f(S_h))a(S_h)\nabla p_h, \nabla\chi) \\
&- ((a(S_\beta) - a(S_h))f(S_\beta)\nabla p_\beta, \nabla\chi) \\
&- (\nabla(p_\beta - p_h)a(S_h)f(S_\beta), \nabla\chi), \ \forall \chi \in M_h.
\end{aligned}
\tag{3.26}
$$

Now set $\chi = T_h^0(S_\beta - S_h) \in M_h$ in (3.26) (see the definitions and properties of T^0, T_h^0 and E_h in appendix **A**) to get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathcal{P}_h(S_\beta - S_h)\|^2_{(H^1(\Omega))^*} &+ (K_\beta(S_\beta) - K_\beta(S_h), S_\beta - S_h) \\
&= -((I - E_h)(K_\beta(S_\beta) - K_\beta(S_h)), S_\beta - S_h) \\
&- ((f(S_\beta) - f(S_h))(a(S_h)\nabla p_h), \nabla T_h^0(S_\beta - S_h)) \\
&- ((a(S_\beta) - a(S_h))f(S_\beta)\nabla p_\beta, \nabla T_h^0(S_\beta - S_h)) \\
&- (\nabla(p_\beta - p_h)a(S_h)f(S_\beta), \nabla T_h^0(S_\beta - S_h)),
\end{aligned}
\tag{3.27}
$$

where we have used the fact that $(v_t, T_h^0 v) = \frac{1}{2}\frac{d}{dt}\|v\|^2_{(H^1)^*}$, for all $v \in (H^1)^*$, $T_h^0 v = T_h^0 \mathcal{P}_h v$, for all $v \in L^2$, by [14] and (A.51). Next, use the Hölder inequality and the arithmetic-geometric mean inequality to get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathcal{P}_h(S_\beta - S_h)\|^2_{(H^1(\Omega))^*} \quad &+ \quad (K_\beta(S_\beta) - K_\beta(S_h), S_\beta - S_h) \\
&\leq \quad \sigma_0\|f(S_\beta) - f(S_h)\|^2_{L^2(\Omega))} \\
&+ \quad \sigma_1\|S_\beta - S_h\|^{2+\mu}_{L^{2+\mu}(\Omega)} \\
&+ \quad \sigma_2\|a(S_\beta) - a(S_h)\|^2_{L^2(\Omega)} \\
&+ \quad \sigma_3\left\|\sqrt{a(S_h)}\nabla(p_\beta - p_h)\right\|^2_{L^2(\Omega)} \\
&+ \quad C\|(I - E_h)K_\beta(S_\beta)\|^\gamma_{L^\gamma(\Omega)} \\
&+ \quad C\left(\max\left(1, \|\mathbf{u}_h\|^2_{L^\infty(L^\infty)}, \|f(S_\beta)\nabla p_\beta\|^2_{L^\infty(L^\infty)}\right)\right) \times \\
&\quad\quad \|\nabla T^0_h(S_\beta - S_h)\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.28}$$

By (A.51), we have

$$\|\nabla T^0_h(S_\beta - S_h)\|_{L^2(\Omega)} = \|\mathcal{P}_h(S_\beta - S_h)\|_{H^{-1}_h}, \tag{3.29}$$

since $(S_\beta - S_h)_\Omega = 0$ (set $\chi = 1$ in (3.25) then use the fact that $\mathcal{P}_h(S^0_\beta - S^0_h) = 0$). Also by [3, 12, 23], we have

$$\|(I - E_h)v\|_{L^p} \leq Ch^2\|v\|_{W^{2,p}}, \ \forall v \in W^{2,p}, \tag{3.30}$$

for a smooth domain. For a convex polygonal (polyhedral) domain, (3.30) is true for $p = 2$. For $p = \gamma = \frac{2+\mu}{1+\mu}$, we assume that the maximum angle of the polygonal domain $\theta(\Omega)$ satisfies (1.6) [3, 20, 21].

Using (A.52) and applying the Grönwall Lemma to (3.28), after hiding the second term on the righthand side of (3.28) in its lefthand side (Choose σ_1 sufficiently small) thanks to (1.16), we get the Lemma.
□

Lemmas 3.1 and 3.2, together with conditions (1.12) and (1.15), give

Theorem 3.1 *Let (p_β, S_β) be the solution to problem (2.8), and (p_h, K_h) be the solution to problem (3.8), with $S_h = H_\beta(K_h)$. Then*

$$\begin{aligned}
\|\mathcal{P}_h(S_\beta - S_h)\|^2_{L^\infty((H^1)^*)} \quad &+ \quad \eta\left\{\|K_\beta(S_\beta) - K_h\|^2_{L^2(L^2)} + \left\|\sqrt{a(S_\beta)}\nabla(p_\beta - p_h)\right\|^2_{L^2(L^2)}\right. \\
&+ \quad \left.\|S_\beta - S_h\|^{2+\mu}_{L^{2+\mu}(L^{2+\mu})}\right\} \\
&\leq \quad C\{h^{2\gamma}\|K_\beta(S_\beta)\|^\gamma_{L^\gamma(W^{2,\gamma})} + h^2\|p_\beta\|^2_{L^2(H^2)} \\
&+ \quad \|K_\beta(\cdot) - K(\cdot)\|^\gamma_{L^\infty} + \|\nabla p_\beta\|^2_{L^\infty(L^\infty)}\|a(S_\beta) - a(S_h)\|^2_{L^2(L^2)}\}
\end{aligned} \tag{3.31}$$

for some $\eta > 0$.

In the above theorem, we have two terms which we need to make more precise:

$$\|K_\beta(S_\beta)\|_{L^\gamma(W^{2,\gamma})} \text{ and } \|p_\beta\|_{L^2(H^2)} \tag{3.32}$$

To see what the theorem implies in terms of β and h, we make the following additional assumptions on $K_\beta(S_\beta)$ and p_β.

$$\|K_\beta(S_\beta)\|_{W^{2,p}} \leq C\left\{\|\Delta K_\beta(S_\beta)\|_{L^p} + \|\nabla K_\beta(S_\beta)\|_{L^p}\right\}, \ 1 < p < \infty \tag{3.33}$$

and

$$\|p_\beta\|_{H^2} \leq C\left\{\|\Delta p_\beta\|_{L^2} + \|\nabla p_\beta\|_{L^2}\right\} \tag{3.34}$$

We notice that the above assumptions are true for a smooth domain ([3, 20]). For a convex polygonal (polyhedral) domain, and for $p = \gamma = \frac{2+\mu}{1+\mu}$, we assume that (1.6) holds. See [18] and also inequality (4.1.2) and Theorem 4.3.2.4 of [20].

But, by (2.4), (2.5), (2.22), (2.30) and (3.33), we get

$$\|K_\beta(S_\beta)\|_{L^\gamma(W^{2,\gamma})} \le Cm(\beta)^{-\frac{1}{2}}, \tag{3.35}$$

since

$$\begin{aligned}\|\Delta K_\beta(S_\beta)\|_{L^2} &= \|\nabla\cdot k_\beta(S_\beta)\nabla S_\beta\|_{L^2} \\ &= \|S_{\beta t} + \nabla\cdot f(S_\beta)u_\beta\|_{L^2} \le Cm(\beta)^{-\frac{1}{2}}\end{aligned} \tag{3.36}$$

and since $\gamma \le 2$.

We can reasonably assume that

$$\|\Delta p_\beta\|_{L^2} \le Cm(\beta)^{-\frac{1}{2}}. \tag{3.37}$$

Then

$$\|p_\beta\|_{H^2} \le Cm(\beta)^{-\frac{1}{2}} \tag{3.38}$$

We even have better under assumption (1.13). Indeed, from

$$-\nabla\cdot a(S_\beta)\nabla p_\beta = Q,$$

we get

$$\begin{aligned}-a(S_\beta)\Delta p_\beta &= Q + a'(S_\beta)\nabla S_\beta\cdot\nabla p_\beta \\ &= Q + \frac{a'(S_\beta)}{\sqrt{k_\beta(S_\beta)}}\sqrt{k_\beta(S_\beta)}\nabla S_\beta\cdot\nabla p_\beta\end{aligned} \tag{3.39}$$

Hence, by (1.17)

$$\|\Delta p_\beta\|_{L^2(\Omega)} \le C\left\{\|Q\|_{L^2(\Omega)} + \|\nabla p_\beta\|_{L^\infty(\Omega)}\left\|\sqrt{k_\beta(S_\beta)}\nabla S_\beta\right\|_{L^2(\Omega)}\right\} \tag{3.40}$$

Therefore, with the help of (2.17) and (3.12), we get

$$\|\Delta p_\beta\|^2_{L^2(L^2)} \le C \tag{3.41}$$

We then get

$$\|p_\beta\|_{L^2(H^2)} \le C, \tag{3.42}$$

by (3.34).

Under these additional assumptions, we can reformulate Theorem 3.1 as follows.

Corollary 3.1 *Under the hypotheses of Theorem 3.1 and in view of conditions (3.33)–(3.37), we have*

$$\begin{aligned}\left\|\sqrt{a(S_\beta)}\nabla(p_\beta - p_h)\right\|^2_{L^2(L^2)} &+ \|K_\beta(S_\beta) - K_h\|^2_{L^2(L^2)} \\ &+ \|S_\beta - S_h\|^2_{L^\infty((H^1)^*)} \\ &+ \|S_\beta - S_h\|^{2+\mu}_{L^{2+\mu}(L^{2+\mu})} \\ &\le C\{h^{2\gamma}m(\beta)^{-\frac{\gamma}{2}} + h^2 m(\beta)^{-1} \\ &+ \|K_\beta(\cdot) - K(\cdot)\|^\gamma_{L^\infty}\}\end{aligned} \tag{3.43}$$

We would like to make more precise Theorem 3.1, or Corollary 3.1, in terms of convergence. For this reason, we consider the particular regularization (2.2). We then have

$$\|K_\beta(\cdot) - K(\cdot)\|^\gamma_{L^\infty} \le C\beta^{2+\mu} \tag{3.44}$$

and

$$c_1\beta^\mu \le m(\beta) \le c_2\beta^\mu \tag{3.45}$$

by (1.3). Furthermore, we choose β so that

$$\beta = \beta_0 h^{\lambda}, \tag{3.46}$$

with β_0 a given positive constant and

$$\lambda = \frac{4}{2+3\mu}, \tag{3.47}$$

as in [9]. Then

$$2\gamma - \frac{\mu\gamma\lambda}{2} > 2 - \mu\lambda = \frac{4+2\mu}{2+3\mu}.$$

Hence we have

Corollary 3.2 *Under the hypotheses of Corollary 3.1, regularization (2.2), and in view of (3.47), we have*

$$\begin{aligned}\left\|\sqrt{a(S_\beta)}\nabla(p_\beta - p_h)\right\|^2_{L^2(L^2)} &+ \|K_\beta(S_\beta) - K_h\|^2_{L^2(L^2)} \\ &+ \|S_\beta - S_h\|^2_{L^\infty((H^1)^*)} \\ &+ \|S_\beta - S_h\|^{2+\mu}_{L^{2+\mu}(L^{2+\mu})} \\ &\le Ch^{\frac{4+2\mu}{2+3\mu}} = Ch^{\frac{(2+\mu)\lambda}{2}}\end{aligned} \tag{3.48}$$

Notice the result when $\mu \to 0$ (which corresponds to the nondegenerate case): The best accuracy corresponds to the nondegenerate case ($\mu = 0$). Also we have less and less accuracy as μ moves away from 0, the worse accuracy case corresponding to the case $\mu = 2$. These observations denote the fact that the solution to the initial Problem 1.1 is less and less smooth as μ moves away from 0.

If we assume that (1.13) holds, then (1.17 holds, and, consequently, (3.41) and (3.42) hold. Therefore, we get the following.

Corollary 3.3 *Under the conditions of Theorem 3.1, in view of conditions (3.33)–(3.35), regularization (2.2), condition (3.47) and (3.42), we have*

$$\begin{aligned}\left\|\sqrt{a(S_\beta)}\nabla(p_\beta - p_h)\right\|^2_{L^2(L^2)} &+ \|K_\beta(S_\beta) - K_h\|^2_{L^2(L^2)} \\ &+ \|S_\beta - S_h\|^2_{L^\infty((H^1)^*)} \\ &+ \|S_\beta - S_h\|^{2+\mu}_{L^{2+\mu}(L^{2+\mu})} \\ &\le Ch^2\end{aligned} \tag{3.49}$$

We notice that the estimate of the rate of convergence in the lemma above is better than the one gotten in Corollary 3.2, for any value of μ, the two being the same for $\mu \to 0$. When $\mu \to 0$, we get

$$\|S_\beta - S_h\|_{L^{2+\mu}(L^{2+\mu})}|_{\mu\to 0} = O(h),$$

which is the same as in Corollary 3.2.

We also notice that the approximation of the pressure p_β by p_h is near-optimal (optimal for $\mu \to 0$) in either case. We get

$$\|p_\beta - p_h\|_{L^2(H^1)} = O(h)$$

for Corollary 3.3, and

$$\|p_\beta - p_h\|_{L^2(H^1)}|_{\mu\to 0} = O(h)$$

for Corollary 3.2.

3.4 Other Estimates

The following theorem gives additional estimates for $S - S_h$ in $L^\infty(0, T, L^{2+\mu}(\Omega))$, and for $K(S) - K_h$ in $L^2(0, T, H^1(\Omega))$. Compare to the rates of convergence for the same quantities in $L^{2+\mu}(0, T, L^{2+\mu}(\Omega))$, and in $L^2(0, T, L^2(\Omega))$, respectively, given by Corollary 3.2.

Theorem 3.2 *Assume $1 \le \mu \le 2$. Then, under the hypotheses of Corollary 3.2, we have:*

$$\|S_\beta - S_h\|_{L^\infty(L^{2+\mu})} \le C h^{\frac{\lambda}{2(2+\mu)}} \tag{3.50}$$

$$\|K_\beta(S_\beta) - K_h\|_{L^2(H^1)} \le C h^{\frac{\lambda}{4}} \tag{3.51}$$

where γ is defined by (3.47), and where we have used (1.12) and (1.16).

By conditions (1.12) through (1.16), it suffices to establish the estimates for

$$\sup_{0 \le t \le T} (K_\beta(S_\beta) - K_\beta(S_h), S_\beta - S_h) + \|\nabla(K_\beta(S_\beta) - K_\beta(S_h))\|^2_{L^2(L^2)} \tag{3.52}$$

The proof goes as in [18] and [14], except that there are additional terms intervening here because of the coupling. These terms can be handled as in the proofs of the previous theorems, so we omit the proof.

4 The Discrete Galerkin Method

The continuous Galerkin Method analyzed in the previous section gives qualitative estimates without giving a computable scheme, since the time variable remains continuous. In this section, the method is further discretized to get a scheme usable for computing "effectively" the numerical solution. But the fully discretized scheme proposed here is implicit, and yields a nonlinear algebraic equation at each time step. If the scheme were linear, we would just have to find a way of inverting a matrix at each time step. In that case one uses one of the direct Gaussian methods, or an iterative method to solve the system. Here instead, it is a nonlinear operator which intervenes at each time step. Thus the method analyzed here is still a theoretical one. For a really effective method, one has to linearize further in some way this method, though this method is already partially linearized. A fully linearized scheme will be proposed in a forthcoming work (also see [16]).

Notice that the proposed scheme below is decoupled. We believe this is one of the particularities of this work.

Unlike in the previous sections, we assume that (1.13) holds all the way through this section.

4.1 On the existence of a solution

We consider the following fully discretized problem. Given a positive integer N, let $t_0 = 0 < t_1 < ... < t_{N-1} < t_N$ be a (regular) subdivision of the interval $[0, T]$, with $\Delta t = t_n - t_{n-1} = T/N$ and let $h > 0$ be sufficiently small. Let M_h be defined as in the previous section, and H_β be defined by (3.7).

We want to find a sequence of couples of functions $(p_h^n, K_h^n) \in M_h \times M_h$, $0 \le n \le N$, such that

$$\begin{cases} (a(H_\beta(K_h^n))\nabla p_h^n, \nabla\chi) = (Q^n, \chi), & \forall \chi \in M_h \\ \int_\Omega p_h^n dx = 0 & \\ \left(\dfrac{H_\beta(K_h^{n+1}) - H_\beta(K_h^n)}{\Delta t}, \chi\right) + (\nabla K_h^{n+1}, \nabla\chi) & \\ \qquad - (f(H_\beta(K_h^{n+1}))(-a(H_\beta(K_h^n))\nabla p_h^n), \nabla\chi) = 0, & \forall \chi \in M_h \\ \mathcal{P}_h H_\beta(K_h^0) = \mathcal{P}_h S^0 & \end{cases} \tag{4.1}$$

We notice first the decoupling of the system: The velocity at the previous time step n is used instead of the velocity at the time step $n+1$ as would require the fully implicit scheme. We also notice the linearity of the first equation: Since S^0 is given, we get K_h^0 through the last equation of (4.1), then p_h^0 by solving a linear equation, i.e. the first equation of (4.1). We then plug this value of p in the third equation (which is however still nonlinear) to get the value of K_h^1, and so on. The next proposition shows that this scheme is well defined, at least for a sufficiently small time step. Since, at each time step, we are solving the same nonlinear algebraic equation, it is enough to show that the scheme is well defined at the first time step.

Theorem 4.1 *Let $K_h^{0,1}$ and $K_h^{0,2}$ be obtained through the initial conditions $S^{0,1}$ and $S^{0,2}$ respectively, thanks to the last equation of (4.1). Let $K_h^{1,1}$ and $K_h^{1,2}$ be the corresponding first step solutions. By the implicit nature of the scheme, let $\mathcal{F}$ be defined, from M_h into M_h, by $K_h^{0,1} = \mathcal{F}K_h^{1,1}$ and $K_h^{0,2} = \mathcal{F}K^{1,1}$. Then*

$$\begin{aligned}(\mathcal{P}_h H_\beta \mathcal{F} K_h^{1,2} - \mathcal{P}_h H_\beta \mathcal{F} K_h^{1,1}, K_h^{1,2} - K_h^{1,1}) \\ +\eta\Delta t \left(\mathcal{P}_h H_\beta \mathcal{F} K_h^{1,2} - \mathcal{P}_h H_\beta \mathcal{F} K_h^{1,1}, \mathcal{F}K_h^{1,2} - \mathcal{F}K_h^{1,1}\right) \\ \geq \frac{1}{2}\Delta t \|\nabla(K_h^{1,2} - K_h^{1,1})\|^2_{L^2(\Omega)} \\ +c_1(1 - c_2\Delta t)\|K_h^{1,2} - K_h^{1,1}\|^2_{L^2(\Omega)},\end{aligned} \tag{4.2}$$

for Δt sufficiently small.

Note: The last equation of (4.1) has a meaning since $\mathcal{P}_h H_\beta$ is known to be bijective ([24]). The above proposition states that the operator $\mathcal{P}_h H_\beta \mathcal{F}$ is bijective by [4], provided the time step Δt is sufficiently small. Hence the nonlinear operator $\mathcal{F}$ is bijective. Thus our scheme is well defined, at least for Δt small.

Proof.

Subtract system (4.1) corresponding to the initial data $K^{0,1}$ from the same system corresponding to $K^{0,2}$ to get the estimate for the pressure (set $\chi = p_h^{0,2} - p_h^{0,1}$)

$$\left\|\sqrt{a(H_\beta(K_h^{0,2}))}\nabla(P_h^{0,2} - P_h^{0,1})\right\|^2_{L^2(\Omega)} \leq C\|a(H_\beta(K_h^{0,2})) - a(H_\beta(K_h^{0,1}))\|^2_{L^2(\Omega)}, \tag{4.3}$$

and, for the saturation (set $\chi = K_h^{1,2} - K_h^{1,1}$, and rearrange the terms),

$$\begin{aligned}\left(\frac{\mathcal{P}_h H_\beta \mathcal{F} K_h^{1,2} - \mathcal{P}_h H_\beta \mathcal{F} K_h^{1,1}}{\Delta t}, K_h^{1,2} - K_h^{1,1}\right) &= \|\nabla(K_h^{1,2} - K_h^{1,1})\|^2_{L^2(\Omega)} \\ &+((f(H_\beta(K_h^{1,1})) - f(H_\beta(K_h^{1,2})))a(H_\beta(K_h^{0,1}))\nabla p_h^{0,1}, \nabla(K_h^{1,2} - \nabla K_h^{1,1})) \\ &+(f(H_\beta(K_h^{1,2}))(a(H_\beta(K_h^{0,1})) - a(H_\beta(K_h^{0,2}))\nabla p_h^{0,1}, \nabla(K_h^{1,2} - K_h^{1,1})) \\ &+(f(H_\beta(K_h^{1,2}))a(H_\beta(K_h^{0,2}))\nabla(p_h^{0,1} - p_h^{0,2}), \nabla(K_h^{1,2} - K_h^{1,1})) \\ &+\left(\frac{\mathcal{P}_h H_\beta K_h^{1,2} - \mathcal{P}_h H_\beta K_h^{1,1}}{\Delta t}, K_h^{1,2} - K_h^{1,1}\right)\end{aligned} \tag{4.4}$$

Now use the Hölder inequality followed by the arithmetic-geometric mean inequality on the second, third, and fourth terms of the righthand side of (4.4) and hide the appropriate terms to get

$$
\begin{aligned}
\left(\frac{\mathcal{P}_h H_\beta \mathcal{F} K_h^{1,2} - \mathcal{P}_h H_\beta \mathcal{F} K_h^{1,1}}{\Delta t}, K_h^{1,2} - K_h^{1,1}\right) \quad &\geq \quad \frac{1}{2}\|\nabla(K_h^{1,2} - K_h^{1,1})\|_{L^2(\Omega)}^2 \\
&- \quad C\left\{\|f(H_\beta(K_h^{1,2})) - f(H_\beta(K_h^{1,1}))\|_{L^2(\Omega)}^2\right. \\
&+ \quad \|a(H_\beta(K_h^{0,2})) - a(H_\beta(K_h^{0,1}))\|_{L^2(\Omega)}^2 \\
&+ \quad \left.\left\|\sqrt{a(H_\beta(K_h^{0,2}))}\nabla(p_h^{0,2} - p_h^{0,1})\right\|_{L^2(\Omega)}^2\right\} \\
&+ \quad \left(\frac{\mathcal{P}_h H_\beta K_h^{1,2} - \mathcal{P}_h H_\beta K_h^{1,1}}{\Delta t}, K_h^{1,2} - K_h^{1,1}\right) \qquad (4.5)
\end{aligned}
$$

In the above estimates, C is a function of $\|a(H_\beta(K_h^{0,1}))\nabla p_h^{0,1}\|_{L^\infty}$, $\|f(H_\beta(K_h^{1,2}))\nabla p_h^{0,1}\|_{L^\infty}$ and $\|a(H_\beta(K_h^{0,2}))f(H_\beta(K_h^{1,2}))\|_{L^\infty}$, but is independent of Δt.

Note: From (4.4) to (4.5), we have used the inequality

$$
(v, w) \geq -\|v\|_{L^2}\|w\|_{L^2} \geq -\left(\frac{\epsilon}{2}\|v\|_{L^2}^2 + \frac{1}{2\epsilon}\|w\|_{L^2}^2\right) \qquad (4.6)
$$

for $\epsilon > 0$.

Also by (1.12), (1.13), and (3.7), we get

$$
\begin{aligned}
\|a(H_\beta(K_h^{0,2})) - a(H_\beta(K_h^{0,1}))\|_{L^2(\Omega)}^2 &\leq C\left(H_\beta(K_h^{0,2}) - H_\beta(K_h^{0,1}), K_h^{0,2} - K_h^{0,1}\right) \\
&= C\left(\mathcal{P}_h H_\beta(K_h^{0,2}) - \mathcal{P}_h H_\beta(K_h^{0,1}), K_h^{0,2} - K_h^{0,1}\right) \\
&= C\left(\mathcal{P}_h H_\beta \mathcal{F} K_h^{1,2} - \mathcal{P}_h H_\beta \mathcal{F} K_h^{1,1}, \mathcal{F} K_h^{1,2} - \mathcal{F} K_h^{1,1}\right) \qquad (4.7)
\end{aligned}
$$

and

$$
\begin{aligned}
\|f(H_\beta(K_h^{1,2})) - f(H_\beta(K_h^{1,1}))\|_{L^2(\Omega)}^2 &\leq C\left(H_\beta(K_h^{1,2}) - H_\beta(K_h^{1,1}), K_h^{1,2} - K_h^{1,1}\right) \\
&= C\left(\mathcal{P}_h H_\beta(K_h^{1,2}) - \mathcal{P}_h H_\beta(K_h^{1,1}), K_h^{1,2} - K_h^{1,1}\right), \qquad (4.8)
\end{aligned}
$$

since $K_h^{1,2} - K_h^{1,1} \in M_h$.

Finally, multiply (4.5) by Δt, use (4.3), (4.7) and (4.8) to get the Theorem. $\square$

4.2 Error analysis

We set $S_\beta^n = S_\beta(t_n)$, and for notational convenience, $S_h^n = H_\beta(K_h^n)$, and thus $K_h^n = K_\beta(S_h^n)$. Then (4.1) becomes

$$
\begin{cases}
(a(S_h^n)\nabla p_h^n, \nabla\chi) = (Q^n, \chi), & \forall \chi \in M_h \\
\int_\Omega p_h^n dx = 0 & \\
\left(\dfrac{S_h^{n+1} - S_h^n}{\Delta t}, \chi\right) + (\nabla K_\beta(S_h^{n+1}), \nabla\chi) & \\
\qquad - (f(S_h^{n+1})(-a(S_h^n)\nabla p_h^n), \nabla\chi) = 0, & \forall \chi \in M_h \\
\mathcal{P}_h S_h^0 = \mathcal{P}_h S^0 &
\end{cases} \qquad (4.9)
$$

for $0 \leq n < N$. Here $Q^n := Q(\cdot, t^n)$.

Lemma 4.1 *Let (p_h^n, K_h^n) be the solution to problem (4.1), with $S_h^n = H_\beta(K_h^n)$. Assume Q is C^1 in the time variable t. Then, for $0 \le n < N$, we have*

$$\left\| \sqrt{a(S_h^{n+1})} \nabla (p_h^{n+1} - p_h^n) \right\|_{L^2}^2 \le C\{\|a(S_h^{n+1}) - a(S_h^n)\|_{L^2}^2 + (\Delta t)^2 \|Q'(\cdot)\|_{L^\infty}^2\} \tag{4.10}$$

Proof.

Subtract the first equation of (4.9) for n from the same equation for $n+1$ to get

$$(a(S_h^{n+1})\nabla p_h^{n+1} - a(S_h^n)\nabla p_h^n, \nabla \chi) = (Q^{n+1} - Q^n, \chi), \ \forall \chi \in M_h. \tag{4.11}$$

Rewrite (4.11), then set $\chi = p_h^{n+1} - p_h^n$ to get

$$\begin{aligned} \left\| \sqrt{a(S_h^{n+1})} \nabla (p_h^{n+1} - p_h^n) \right\|_{L^2}^2 &= (\nabla p_h^n (a(S_h^n) - a(S_h^{n+1})), \nabla (p_h^{n+1} - p_h^n)) \\ &\quad + (Q^{n+1} - Q^n, p_h^{n+1} - p_h^n) \end{aligned} \tag{4.12}$$

Next, using Hölder inequality, the arithmetic geometric inequality, (1.9), the Poincaré inequality (B.62), and the second equation of (4.1), we get

$$\begin{aligned} \left\| \sqrt{a(S_h^{n+1})} \nabla (p_h^{n+1} - p_h^n) \right\|_{L^2}^2 &\le \frac{1}{4} \frac{\|\nabla p_h\|_{L^\infty(L^\infty)}^2}{\left\| \sqrt{a(\cdot)} \right\|_{L^\infty}} \|a(S_h^{n+1}) - a(S_h^n)\|_{L^2}^2 \\ &\quad + \frac{1}{2} \left\| \sqrt{a(S_h^{n+1})} \nabla (p_h^{n+1} - p_h^n) \right\|_{L^2}^2 \\ &\quad + \frac{1}{4} (\Delta t)^2 \left\| \frac{Q^{n+1} - Q^n}{\Delta t} \right\|_{L^2}^2 \end{aligned} \tag{4.13}$$

Finally, hiding the second term of the righthand side of (4.13) in its lefthand side, using (3.13), (1.9), and assuming Q is C^1 in t, we get the Lemma □

Since time is not explicitly involved in the pressure equation, Lemma 3.1 is still valid in its discrete-time version, and we have:

Lemma 4.2 *Let (p_β^n, S_β^n) be the solution to problem (2.8), and (p_h^n, K_h^n) be the solution to problem (4.1), with $S_h^n = H_\beta(K_h^n)$. Then, for $0 \le n \le N$,*

$$\begin{aligned} \left\| \sqrt{a(S_\beta^n)} \nabla (p_\beta^n - p_h^n) \right\|_{L^2(\Omega)}^2 &\le Ch^2 \|p_\beta^n\|_{H^2(\Omega)}^2 \\ &+ C\|a(S_\beta^n) - a(S_h^n)\|_{L^2(\Omega)}^2, \\ &\quad 0 \le n \le N \end{aligned} \tag{4.14}$$

Next, we have to establish the main result of this section.

Theorem 4.2 *Let (p_β^n, S_β^n) be the solution to problem (2.8), and (p_h^n, K_h^n) be the solution to problem (4.1),* *with* $S_h^n = H_\beta(K_h^n)$. *Then*

$$\begin{aligned}
\max_{0\le n\le N-1} \|\mathcal{P}_h(S_\beta^{n+1} - S_h^{n+1})\|^2_{(H^1)^*} &+ \eta \max_{0\le n\le N-1} \Delta t(K_\beta(S_\beta^{n+1}) - K_\beta(S_h^{n+1}), S_\beta^n - S_h^n) \\
&\le C\{h^{2\gamma} \sum_{0\le n\le N} \Delta t\|K_\beta(S_\beta^n)\|^\gamma_{W^{2,\gamma}} \\
&+ h^2 \sum_{0\le n\le N} \Delta t\|p_\beta^n\|^2_{H^2} \\
&+ (\Delta t)^{\frac{\gamma+2}{2}} \|E_h K_\beta(S_\beta)_t\|^\gamma_{L^2(L^2)} \\
&+ (\Delta t)^2 \|(f(S_\beta)\mathbf{u}_\beta)_t\|^2_{L^2(L^2)} \\
&+ (\Delta t)^{\frac{3}{2}} \{\|S_{\beta t}|_{L^2(L^2)} + \|\nabla(p_{\beta t})\|_{L^2(L^2)}\} \\
&+ \Delta t(K_\beta(S_\beta^0) - K_\beta(S_h^0), S_\beta^0 - S_h^0)\} \qquad (4.15)
\end{aligned}$$

Proof.

Subtract the third equation of (4.9) from the third equation of (2.8) after setting $\psi = \chi \in M_h$; rewrite the terms to get

$$\begin{aligned}
\left(\frac{S_\beta^{n+1} - S_\beta^n}{\Delta t} - \frac{S_h^{n+1} - S_h^n}{\Delta t}, \nabla\chi\right) &+ (\nabla(K_\beta(S_\beta^{n+1}) - K_\beta(S_h^{n+1}), \nabla\chi)) = \\
&- ((f(S_\beta^{n+1}) - f(S_h^{n+1}))a(S_h^n)\nabla p_h^n, \nabla\chi) \\
&- (f(S_\beta^{n+1})\nabla p_h^n(a(S_\beta^n) - a(S_h^n)), \nabla\chi) \\
&- (f(S_\beta^{n+1})\nabla p_h^n(a(S_\beta^n) - a(S_\beta^{n+1})), \nabla\chi) \\
&- (f(S_\beta^{n+1})a(S_\beta^n)\nabla(p_\beta^n - p_h^n), \nabla\chi) \\
&- (f(S_\beta^{n+1})a(S_\beta^n)\nabla(p_\beta^n - p_\beta^{n+1}), \nabla\chi) \\
&- \left(\frac{\partial S_\beta^{n+1}}{\partial t} - \frac{S_\beta^{n+1} - S_\beta^n}{\Delta t}, \chi\right) \qquad (4.16)
\end{aligned}$$

For the treatment of the last term of the righthand side of (4.16), we refer to the proof of Theorem 4.1 of [18]. We treat the third and the fifth terms as follows.

First, for the third term, we have

$$\begin{aligned}
|a(s_2) - a(s_1)| = \left|\int_{s_1}^{s_2} \frac{d}{ds} a(s)ds\right| &\le |(s_2 - s_1)|\|a'(\cdot)\|_{L^\infty} \\
&\le C\sqrt{|k(\cdot)\|_{L^\infty}}|s_2 - s_1|, \qquad (4.17)
\end{aligned}$$

by (1.17), and

$$\left|S_\beta^{n+1} - S_\beta^n\right| = \left|\int_{t_n}^{t_{n+1}} S_{\beta t} d\tau\right|. \qquad (4.18)$$

These two inequalities yield, after using Hölder,

$$\begin{aligned}
\|a(S_\beta^{n+1}) - a(S_\beta^n)\|_{L^2} &\le \|a'(\cdot)\|_{L^\infty} (t^{n+1} - t^n)^{\frac{1}{2}} \|S_{\beta t}\|_{L^2(t^n, t^{n+1}, L^2)} \\
&\le C(\Delta t)^{\frac{1}{2}} \|S_{\beta t}\|_{L^2(t^n, t^{n+1}, L^2)}. \qquad (4.19)
\end{aligned}$$

For the fifth term:

$$|\nabla(p_\beta^{n+1} - p_\beta^n)| = \left|\nabla\left(\int_{t^n}^{t_{n+1}} p_{\beta t} d\tau\right)\right| = \left|\left(\int_{t^n}^{t_{n+1}} \nabla(p_{\beta t}) d\tau\right)\right|. \qquad (4.20)$$

So that, using Hölder inequality, we get

$$\|\nabla(p_\beta^{n+1}-p_\beta^n)\|_{L^2} \leq C(\Delta t)^{\frac{1}{2}}\|\nabla(p_{\beta t})\|_{L^2(t_n,t_{n+1},L^2)} \tag{4.21}$$

After setting $\chi = T_h^0(S_\beta^{n+1}-S_h^{n+1})$ in (4.16) (see appendix **A** for the properties of T_h^0 and of the norm $\|.\|_{H_h^{-1}}$ on M_h), using (A.51), (4.19), (4.21), Lemma 4.2, the proof of Theorem 5.1 of [18], and hiding the appropriate terms by the usual technics, we get

$$\begin{aligned}
\frac{1}{2\Delta t}\|\mathcal{P}_h(S_\beta^{n+1}-S_h^{n+1})\|^2_{H_h^{-1}} &- \frac{1}{2\Delta t}\|\mathcal{P}_h(S_\beta^{n}-S_h^{n})\|^2_{H_h^{-1}}\\
&+ \frac{1}{4}(K_\beta(S_\beta^{n+1})-K_\beta(S_h^{n+1}),S_\beta^{n+1}-S_h^{n+1})\\
&\leq C\|\mathcal{P}_h(S_\beta^{n+1}-S_h^{n+1})\|^2_{H_h^{-1}}\\
&+ Ch^{2\gamma}\|K_\beta(S_\beta^{n+1})\|^\gamma_{W^{2,\gamma}}\\
&+ C\Delta t\|(f(S_\beta)\mathbf{u}_\beta)_t\|^2_{L^2(t_n,t_{n+1},L^2)}\\
&+ C(\Delta t)^{\frac{\gamma}{2}}\|E_hK_\beta(S_\beta)_t\|^\gamma_{L^2(t_n,t_{n+1},L^2)}\\
&+ C(\Delta t)^{\frac{1}{2}}\{\|S_{\beta t}|_{L^2(t^n,t^{n+1},L^2)}+\|\nabla(p_{\beta t})\|_{L^2(t_n,t_{n+1},L^2)}\}\\
&+ Ch^2\|p_\beta^n\|^2_{H^2}\\
&+ \sigma\|a(S_\beta^n)-a(S_h^n)\|^2_{L^2}.
\end{aligned} \tag{4.22}$$

Now we can choose σ so small that

$$\sigma\|a(S_\beta^n)-a(S_h^n)\|^2_{L^2} \leq \frac{1}{4}(K_\beta(S_\beta^n)-K_\beta(S_h^n),S_\beta^n-S_h^n) \tag{4.23}$$

by (1.13). Then (4.22) becomes:

$$\begin{aligned}
\frac{1}{2\Delta t}\|\mathcal{P}_h(S_\beta^{n+1}-S_h^{n+1})\|^2_{H_h^{-1}} &- \frac{1}{2\Delta t}\|\mathcal{P}_h(S_\beta^{n}-S_h^{n})\|^2_{H_h^{-1}}\\
&+ \frac{1}{4}(K_\beta(S_\beta^{n+1})-K_\beta(S_h^{n+1}),S_\beta^{n+1}-S_h^{n+1})\\
&- \frac{1}{4}(K_\beta(S_\beta^{n})-K_\beta(S_h^{n}),S_\beta^{n}-S_h^{n})\\
&\leq C\|\mathcal{P}_h(S_\beta^{n+1}-S_h^{n+1})\|^2_{H_h^{-1}}\\
&+ Ch^{2\gamma}\|K_\beta(S_\beta^{n+1})\|^\gamma_{W^{2,\gamma}}\\
&+ C\Delta t\|(f(S_\beta)\mathbf{u}_\beta)_t\|^2_{L^2(t_n,t_{n+1},L^2)}\\
&+ C(\Delta t)^{\frac{\gamma}{2}}\|E_hK_\beta(S_\beta)_t\|^\gamma_{L^2(t_n,t_{n+1},L^2)}\\
&+ C(\Delta t)^{\frac{1}{2}}\{\|S_{\beta t}|_{L^2(t^n,t^{n+1},L^2)}+\|\nabla(p_{\beta t})\|_{L^2(t_n,t_{n+1},L^2)}\}\\
&+ Ch^2\|p_\beta^n\|^2_{H^2}
\end{aligned} \tag{4.24}$$

Finally, multiply (4.24) by Δt, sum from $n = 0$ to $n = m-1$, with $0 < m < N+1$, use the fact that $\mathcal{P}_h(S_\beta^0 - S_h^0) = 0$, and then apply the discrete Grönwall Lemma (see [14]) to get

$$
\begin{aligned}
\max_{0\le n\le N-1} \|\mathcal{P}_h(S_\beta^{n+1} - S_h^{n+1})\|^2_{H_h^{-1}} \quad &+ \quad \eta \max_{0\le n\le N-1} \Delta t(K_\beta(S_\beta^{n+1}) - K_\beta(S_h^{n+1}), S_\beta^{n+1} - S_h^{n+1}) \\
&\le \quad Ch^{2\gamma} \sum_{0\le n\le N-1} \Delta t \|K_\beta(S_\beta^{n+1})\|^\gamma_{W^{2,\gamma}} \\
&+ \quad C(\Delta t)^2 \|(f(S_\beta)\mathbf{u}_\beta)_t\|^2_{L^2(L^2)} \\
&+ \quad C(\Delta t)^{\frac{\gamma+2}{2}} \|E_h K_\beta(S_\beta)_t\|^\gamma_{L^2(L^2)} \\
&+ \quad C(\Delta t)^{\frac{3}{2}} \{\|S_{\beta t}|_{L^2(L^2)} + \|\nabla(p_{\beta t})\|_{L^2(L^2)}\} \\
&+ \quad Ch^2 \sum_{0\le n\le N} \Delta t \|p_\beta^n\|^2_{H^2} \\
&+ \quad \Delta t(K_\beta(S_\beta^0) - K_\beta(S_h^0), S_\beta^0 - S_h^0) \qquad (4.25)
\end{aligned}
$$

□

Remark 4.1 In the proof of Theorem 4.2, Lemma 4.1 could have been used to deal with the fifth term of the right handside of (4.16), but, then this would be at the price of using the rather strong hypothesis (3.13).

Remark 4.2 The use of the discrete Grönwall Lemma needs some justification here. After multiplying (4.24) by Δt and summing from $n = 0$ to $n = m-1$, the first term of the right handside of (4.24) becomes

$$
\begin{aligned}
C\Delta t \sum_{0\le n\le m-1} \|\mathcal{P}_h(S_\beta^{n+1} - S_h^{n+1})\|^2_{H_h^{-1}} &= C\Delta t \|\mathcal{P}_h(S_\beta^m - S_h^m)\|^2_{H_h^{-1}} \\
&+ C\Delta t \sum_{0\le n\le m-2} \|\mathcal{P}_h(S_\beta^{n+1} - S_h^{n+1})\|^2_{H_h^{-1}} \qquad (4.26)
\end{aligned}
$$

We can now bring the first term of (4.26) on the left handside of the inequality obtained after summation. For Δt sufficiently small, $1 - C\Delta t > 0$, so we can apply the discrete Grönwall Lemma.

Remark 4.3 The error estimates obtained in Theorem 4.2 are not clearly expressed in terms of β, h, and Δt only. To get a much clearer idea on these estimates, we need more information on the terms $\|K_\beta(S_\beta^n)\|_{W^{2,\gamma}}$, $\|p_\beta^n\|_{H^2}$, and $\|\Delta(p_{\beta t})\|_{L^2(L^2)}$, among others.

In what follows, to get a better insight on these estimates, we make additional assumptions on these terms (some of which are justified in some way). Note that condition (2.28) is used only in the results below, so does not affect Theorem 4.2 or any other result above.

Since

$$\sum_{0\le n\le N} \Delta t \|K_\beta(S_\beta^n)\|^\gamma_{W^{2,\gamma}} \to \|K_\beta(S_\beta)\|^\gamma_{L^\gamma(W^{2,\gamma})}$$

and

$$\sum_{0\le n\le N} \Delta t \|p_\beta^n\|^2_{H^2} \to \|p_\beta\|^2_{L^2(H^2)}$$

as $\Delta t \to 0$ (or $N \to +\infty$), we have

$$\sum_{0\le n\le N} \Delta t \|K_\beta(S_\beta^n)\|^\gamma_{W^{2,\gamma}} + \sum_{0\le n\le N} \Delta t \|p_\beta^n\|^2_{H^2} \le Cm(\beta)^{\frac{-1}{2}} \qquad (4.27)$$

under conditions (3.35) and (3.38).

We have

$$
\begin{aligned}
(f(S_\beta)\mathbf{u}_\beta)_t &= f'(S_\beta)(S_\beta)_t\mathbf{u}_\beta + f(S_\beta)(\mathbf{u}_{\beta t}) \\
&= \frac{f'(S_\beta)}{\sqrt{k_\beta(S_\beta)}}\sqrt{k_\beta(S_\beta)}S_{\beta t}\mathbf{u}_\beta \\
&+ f(S_\beta)\mathbf{u}_{\beta t}
\end{aligned} \tag{4.28}
$$

Now, using (1.14), (2.22, and (2.28), we see, through (4.28), that

$$
\|(f(S_\beta)\mathbf{u}_\beta)_t\|_{L^2(L^2)} \leq Cm(\beta)^{-\frac{1}{2}} \tag{4.29}
$$

We may assume that

$$
\|\nabla(p_{\beta t})\|_{L^2} \leq Cm(\beta)^{\frac{-1}{2}} \tag{4.30}
$$

In fact, this is the case under condition (1.17). In this case, since

$$
\nabla p_\beta = -\frac{1}{a(S_\beta)}(\mathbf{u}_\beta), \tag{4.31}
$$

we get by differentiating with respect to t:

$$
\nabla(p_{\beta t}) = -\frac{a(S_\beta)\mathbf{u}_{\beta t} - a'(S_\beta)S_{\beta t}\mathbf{u}_\beta}{(a(S_\beta))^2}. \tag{4.32}
$$

Then, using (1.9), (1.17), (2.28), and Theorem 2.3, we see that (4.30) is verified.

Thus, under condition (3.47), and if we assume that

$$
\|E_h K_\beta(S_\beta)_t\|^\gamma_{L^2(L^2)} \leq C, \tag{4.33}
$$

we obtain the following.

Corollary 4.1 *Under conditions (2.28), (2.30), (3.35), (3.42), (3.47), (4.30), (4.29), and (4.33) we have*

$$
\begin{aligned}
\max_{0\leq n\leq N-1}\|S^{n+1}_\beta - S^{n+1}_h\|^2_{(H^1)^*} &+ \max_{0\leq n\leq N-1}\Delta t(K_\beta(S^{n+1}_\beta) - K_\beta(S^{n+1}_h), S^n_\beta - S^n_h) \\
&+ \sum_{0\leq n\leq N}\Delta t\left\|\sqrt{a(S^n_\beta)}\nabla(p^n_\beta - p^n_h)\right\|^2_{L^2} \\
&\leq C\{h^{\frac{4+2\mu}{2+3\mu}} + (\Delta t)^{\frac{\gamma+2}{2}} + (\Delta t)^{\frac{3}{2}}h^{-\frac{2\mu}{2+3\mu}} \\
&+ \Delta t(K_\beta(S^0_\beta) - K_\beta(S^0_h), S^0_\beta - S^0_h)\}.
\end{aligned} \tag{4.34}
$$

From this, and thanks to (1.12) and (1.16), we get the following.

Corollary 4.2 *Under the conditions of Corollary 4.1, we have*

$$
\begin{aligned}
\max_{0\leq n\leq N}\Delta t\|S^n_\beta - S^n_h\|^{2+\mu}_{L^{2+\mu}} &+ \max_{0\leq n\leq N}\Delta t\|K_\beta(S^n_\beta) - K_\beta(S^n_h)\|^2_{L^2} \\
&\leq C\{h^{\frac{4+2\mu}{2+3\mu}} + (\Delta t)^{\frac{\gamma+2}{2}} + (\Delta t)^{\frac{3}{2}}h^{-\frac{2\mu}{2+3\mu}} \\
&+ \Delta t(K_\beta(S^0_\beta) - K_\beta(S^0_h), S^0_\beta - S^0_h)\}.
\end{aligned} \tag{4.35}
$$

We notice from [18] that Corollary 4.2 does not need the inverse estimate assumption (3.2).

Finally, a triangle inequality argument shows the convergence of S^n_h to $S(t^n)$, and the convergence of p^n_h to $p(t^n)$, as $N \to +\infty$ and $h \to 0^+$, if we choose

$$
\Delta t = C_0 h^\nu,
$$

with $\nu > \frac{4}{3}\frac{\mu}{2+3\mu}$.

We can compare Theorem 4.2 to Theorem 5.2 of [9], where the term $(K_\beta(S^n_\beta) - K_\beta(S^n_h), S^n_\beta - S^n_h)$ is estimated in $L^2(L^2)$, and, for our case, in $L^\infty(L^2)$.

5 Conclusion

The problem considered here has three main difficulties: It is nonlinear, coupled, and degenerate. The present work has tried to look at it in a different way than what is already done in the literature. Without pretending to solve the problem, we hope to bring a modest contribution towards its solution.

Some key results have been obtained under the rather strong condition (3.12). Because the first equation of (2.4) does not involve the time variable explicitly, our attempt to establish (3.12) (or (3.13)) has failed so far. However, the L^2 version, (3.11), has been obtained thanks to Theorem B.1.

Another assumption used often here is (1.13). We can see that if a is not a function of S, then (1.13) clearly holds. But, then the problem would no longer be coupled, and one difficulty would be eliminated. Also by [14] (page 20, Lemma 2.1) and [17], if

$$a'(0) = a'(1) = 0, \tag{5.36}$$

then (1.13) holds. Physically, if S is the saturation of the invading fluid (for instance, water injected in an oil reservoir), $S = 1$ corresponds to the absence of oil (only water), and $S = 0$ corresponds to the absence of water. So, (5.36) would mean that the permeabilities of the phases tend to level off near $S = 1$ (only water) and $S = 0$ (only oil). In particular, if we make the assumption that a is independent of S near $S = 1$ and $S = 0$, then assumption (5.36) would hold, so would (1.13).

A The Poisson Solution Operator

In [14, 17], properties of the Solution Operator T^0 were given which are useful here. A summary is given here. We define the Mean-Value Preserving Elliptic Projection, and the discrete version of the Solution Operator, and give some of their properties that are useful for our analysis.

A.1 The Poisson Solution Operator

Consider the elliptic boundary value problem:

$$\begin{cases} -\Delta\omega = f - f_\Omega & \text{in } \Omega \\ \dfrac{\partial\omega}{\partial\mathbf{n}} = 0 & \text{on } \partial\Omega \\ \omega_\Omega = f_\Omega \end{cases} \tag{A.37}$$

Then (see [10, 19]) problem (A.37) has a unique weak solution $\omega \in H^1$.

We define the solution operator $T^0 :(H^1)^* \to H^1$ by $T^0(f) = \omega$, where $\omega \in H^1$ is the unique weak solution to (A.37), and $f \in (H^1)^*$. Then

$$(\nabla(T^0 f), \nabla\phi) = (f, \phi) - f_\Omega\phi_\Omega, \text{ for all } f \in (H^1)^* \tag{A.38}$$

We also have

$$\|\nabla T^0 f\|_{L^2}^2 = (f, T^0 f) - (f_\Omega)^2 = (f, T^0 f) - (T^0 f)_\Omega^2. \tag{A.39}$$

and

$$|||f|||_{(H^1)^*} := (T^0 f, f)^{\frac{1}{2}} = \left(\|\nabla T^0 f\|_{L^2}^2 + (f_\Omega)^2\right)^{\frac{1}{2}} \tag{A.40}$$

Proposition A.1 *Suppose f belongs to $(H^1)^*$, then*

$$(T^0 f, f)^{\frac{1}{2}} = \|f\|_{(H^1)^*}. \tag{A.41}$$

From [14, 17, 24], we also have the following results.

Proposition A.2 *1. For $f \in H^1$, T and $\frac{\partial}{\partial t}$ commute, i.e*

$$\frac{\partial}{\partial t}(T^0 f) = T^0(\frac{\partial f}{\partial t}) \tag{A.42}$$

2. Let $f \in H^1(\Omega)$, and suppose

$$\frac{\partial f}{\partial n} = 0 \ \text{on} \ \partial\Omega, \tag{A.43}$$

then

$$T^0(\Delta f) = \Delta(T^0 f), \ \text{in the weak sense.} \tag{A.44}$$

A.2 The Mean-Value Preserving Projection

Let $\{M_h\}_{h>0}$ be a family of finite dimensional spaces such that $M_h \subset H^1(\Omega)$. M_h is defined more accurately in section 3. Let $f \in H^1$, and consider the problem of finding $f_h \in M_h$ such that

$$\begin{cases} (\nabla f_h, \nabla\chi) = & (\nabla f, \nabla\chi), \ \forall\chi \in M_h \\ (f_h)_\Omega = & f_\Omega \end{cases} \tag{A.45}$$

Then Problem (A.45) has a unique solution in M_h (See [14, 24]). We define the mean–value preserving operator E_h by $E_h(f) := f_h$, where f_h is the unique solution to (A.45), and denote

$$E_h : \quad H^1(\Omega) \longrightarrow M_h \qquad\qquad f \longrightarrow f_h. \tag{A.46}$$

Proposition A.3 *Suppose $f \in H^1(\Omega)$, then*

$$\|\nabla E_h f\|_{L^2} \le \|\nabla f\|_{L^2}. \tag{A.47}$$

and

$$\|E_h f\|_{H^1} \le \|f|_{H^1}, \ \text{for all} \ f \in H^1 \tag{A.48}$$

A.3 The discrete analogue of the Poisson Operator

We define the discrete analogue

$$T_h^0 : \quad (H^1)^* \to M_h,$$

of T^0, by

$$T_h^0 f := E_h(T^0 f) = (E_h \circ T^0) f \qquad \forall f \in (H^1)^*. \tag{A.49}$$

Then

$$(\nabla T_h^0 f, \nabla\chi) = (f - f_\Omega, \chi), \ \forall\chi \in M_h \tag{A.50}$$

By [14, 24, 26], we have $\chi \to (T_h^0\chi, \chi)^{\frac{1}{2}}$ is a norm on M_h (but only a semi–norm on $(H^1)^*$). We thus define on M_h the norm

$$\|\chi\|_{H_h^{-1}} = (T_h^0\chi, \chi)^{\frac{1}{2}} = \left(\|\nabla T_h^0\chi\|_{L^2}^2 + (\chi_\Omega)^2\right)^{\frac{1}{2}}. \tag{A.51}$$

Theorem A.1

$$\forall\chi \in M_h \qquad \|\chi\|_{H_h^{-1}} \le \|\chi\|_{(H^1)^*}. \tag{A.52}$$

where $\|\chi\|_{(H^1)^*}$ is defined by (A.41).

See proof in [15].

B Additional Proofs

B.1 Proof of Theorem 2.1

Proof.

First subtract the corresponding equations for the pressure in system (1.1) from the one in system (2.4), and rewrite to get

$$\nabla \cdot ((a(S) - a(S_\beta))\nabla p + \nabla \cdot (a(S_\beta)\nabla(p - p_\beta)) = 0$$

Integrate over Ω against $p - p_\beta$ and use the divergence theorem to get

$$\left\| \sqrt{a(S_\beta)}\nabla(p - p_\beta) \right\|^2_{L^2(\Omega)} = -\int_\Omega (a(S) - a(S_\beta))\nabla p \cdot \nabla(p - p_\beta)dx \tag{B.53}$$

Now use Hölder's inequality, the arithmetic-geometric mean inequality, Poincaré's inequality for H^1 (B.62), and the fact that a is bounded away from 0 to get (2.11).

To derive inequality (2.12), we proceed similarly to obtain

$$\begin{aligned} &\frac{\partial(S_\beta - S)}{\partial t} - \Delta(K_\beta(S_\beta) - K_\beta(S)) \\ &\qquad = \Delta(K_\beta(S) - K(S)) - \nabla \cdot (f(S_\beta) - f(S))\mathbf{u}_\beta - \nabla \cdot f(S)(\mathbf{u}_\beta - \mathbf{u}). \end{aligned} \tag{B.54}$$

Integrate (B.54) against $T^0(S_\beta - S)$ over Ω, where T^0 is the Poisson Solution operator defined in the subsection A.1. Use the divergence theorem and the boundary conditions to get

$$\begin{aligned} &\frac{d}{dt}\|S_\beta - S\|^2_{(H^1(\Omega))^*} + (K_\beta(S_\beta) - K_\beta(S), S_\beta - S) \\ &\qquad = (K(S) - K_\beta(S), S_\beta - S) + ((f(S_\beta) - f(S))\mathbf{u}_\beta, \nabla T^0(S_\beta - S)) \\ &\qquad + (f(S)(\mathbf{u}_\beta - \mathbf{u}), \nabla T^0(S_\beta - S)). \end{aligned} \tag{B.55}$$

Since $(S_\beta - S)_\Omega = 0$, we see, by (A.40), that

$$\|\nabla T^0(S_\beta - S)\|^2_{L^2} = \|S_\beta - S\|^2_{(H^1(\Omega))^*}. \tag{B.56}$$

.

We get, by Hölder and the arithmetic-geometric inequalities,

$$\begin{aligned} &\frac{d}{dt}\|S_\beta - S\|^2_{(H^1(\Omega))^*} + (K_\beta(S_\beta) - K_\beta(S), S_\beta - S) \\ &\qquad \leq \sigma_1\|S_\beta - S\|^{2+\mu}_{L^{2+\mu}} + \sigma_2\|\mathbf{u}_\beta\|_{L^\infty}\|f(S_\beta) - f(s)\|_{L^2} \\ &\qquad + C_1\|K_\beta(\cdot) - K(\cdot)\|^\gamma_{L^\gamma(0,1)} + \sigma_3 C_2\|\mathbf{u}_\beta - \mathbf{u}\|^2_{L^2} \\ &\qquad + C_3\|S_\beta - S\|^2_{(H^1(\Omega))^*}, \end{aligned} \tag{B.57}$$

where the positive numbers σ_1, σ_2 and σ_3 can be chosen arbitrary by the arithmetic-geometric inequality.

We also have

$$\mathbf{u}_\beta - \mathbf{u} = -a(S_\beta)\nabla p + a(S)\nabla p = (a(S) - a(S_\beta))\nabla p + a(S_\beta)(\nabla p - \nabla p_\beta). \tag{B.58}$$

Thus

$$\|\mathbf{u}_\beta - \mathbf{u}\|_{L2} \leq C\|\nabla p\|_{L^\infty(L^\infty)}\|a(S_\beta) - a(S)\|_{L^2}, \tag{B.59}$$

by (2.11).

Finally after hiding the first and second terms of the right handside of (B.57) (choose σ_1 and σ_2 sufficiently small) by (1.12) and (1.16), and using the Grönwall Lemma, we see that (2.12) is established.

B.2 $\mathbf{u}_\beta$ is bounded independently of β

Theorem B.1 *Under the hypotheses on problems (1.1) and (2.4), we have*

$$\|\mathbf{u}_\beta\|_{L^\infty(L^2)} \leq C \tag{B.60}$$

where C is independent of β.

Proof of Theorem B.1: A weak formulation for the pressure part of the regularized problem (2.4) is

$$\int_\Omega a(S_\beta)\nabla p_\beta \cdot \nabla\psi dx = \int_\Omega Q\psi dx,\ \forall \psi \in H^1(\Omega).$$

Now choose $\psi = p_\beta$ to obtain

$$\begin{aligned} \int_\Omega a(S_\beta)|\nabla p_\beta|^2 dx &= \int_\Omega Q p_\beta dx \\ &\leq \frac{C^{*2}}{2d_0}\|Q\|^2_{L^2(\Omega)} + \frac{d_0}{2C^{*2}}\|p_\beta\|^2_{L^2(\Omega)} \\ &\leq \frac{C^{*2}}{2d_0}\|Q\|^2_{L^2(\Omega)} + \frac{d_0}{2}\|\nabla p_\beta\|^2_{L^2(\Omega)} \end{aligned} \tag{B.61}$$

where we have used Hölder, and then the arithmetic-geometric mean inequality, and where d_0 is as in(1.9). We have also made use of the Poincaré inequality for H^1:

$$\|f\|_{L^2(\Omega)} \leq C^* \left\{ \|\nabla f\|^2_{L^2(\Omega)} + \left(\int_\Omega f dx\right)^2 \right\}^{\frac{1}{2}} \tag{B.62}$$

for all $f \in H^1(\Omega)$ [11, 25], and the fact that $\int_\Omega p_\beta dx = 0$. Therefore, after hiding the second term of the righthand side of (4.2) (thanks to (1.9)) in its left side, we have

$$\left\| \sqrt{a(S_\beta)}\nabla p_\beta \right\|_{L^2(\Omega)} \leq C\|Q\|^2_{L^2}. \tag{B.63}$$

Now

$$\begin{aligned} \|\mathbf{u}_\beta\|_{L^2(\Omega)} &= \|a(S_\beta)\nabla p_\beta\|_{L^2(\Omega)} \\ &\leq \left\|\sqrt{a(S_\beta)}\right\|_{L^\infty(\Omega)} \left\|\sqrt{a(S_\beta)}\nabla p_\beta\right\|_{L^2(\Omega)}. \end{aligned} \tag{B.64}$$

Hence the theorem. □

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