Postprocessing and Higher Order Convergence of Mixed Finite Element Approximations of Biharmonic Eigenvalue Problems ⋆

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Abstract

A new procedure for accelerating the convergence of mixed finite element approximations of the eigenpairs and of the biharmonic operator is proposed. It is based on a postprocessing technique that involves an additional solution of a source problem on an augmented finite element space. This space could be obtained either by substantially refining the grid, the two-grid method, or by using the same grid but increasing the order of polynomials by one, the two-space method. The numerical results presented and discussed in the paper illustrate the efficiency of the postprocessing method.

Key words: biharmonic eigenvalue problem, mixed method, finite element approximation, postprocessing.

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1 Introduction

In this paper we consider the following biharmonic eigenvalue problem: for a given bounded domain $\Omega \in \mathbf{R}^2$ with Lipschitz boundary Γ, find $u(x) \neq 0$ and

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$\lambda \in \mathbf{R}$ satisfying the differential equation

$$\Delta^2 u(x) \equiv \frac{\partial^4 u}{\partial x_1^4} + 2\frac{\partial^4 u}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 u}{\partial x_2^4} = \lambda\ u(x), \quad x \in \Omega, \tag{1}$$

subject to homogeneous Dirichlet boundary conditions

$$u(x) = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu}(x) = 0, \quad x \in \Gamma. \tag{2}$$

Here ν is the outward unit normal vector to the boundary Γ. The problem (1), (2) describes the eigenmodes of a vibrating homogeneous isotropic plate with constant thickness and clamped boundary.

The differential equation (1) can be recast in mixed form as a system of equations of second order (often refered to as problem with two unknown fields, cf., e.g., [4,15,17]):

$$-\Delta u = \sigma \quad \text{and} \quad -\Delta\sigma = \lambda\ u, \tag{3}$$

subject to the boundary conditions (2). It is well known that if the domain Ω has smooth boundary or Ω is convex polygonal domain then the eigenvalue problem (3) has infinitely many solutions $(\lambda_i, (\sigma_i, u_i))$ such that $\sigma_i = -\Delta u_i, i = 1, 2, \ldots$ (see, e.g. [7]) and $0 < \lambda_1 \leq \lambda_2 \leq \ldots \nearrow \infty$.

If $(\lambda, (\sigma, u))$ is an eigenpair of (3) then (λ, u) is an eigenpair of (1), (2) and $\sigma = -\Delta u$. Hence the regularity of (σ, u) can be inferred from the regularity properties of the problem (1), (2). For regularity results for such problems we refer to Grisvard [16] (Section 7, p. 301).

Canuto [1] and Ishihara [2] considered the finite element approximaton of eigenvalues of the mixed problem (3) and derived estimates for the error of the eigenvalue and eigenvector approximations by using the analysis of Brezzi [4] and Miyoshi [5].

Further, Mercier, Osborn, Rapaz, and Raviart [6] developed an abstract analysis of the approximate eigenpairs using mixed/hybrid finite element methods based on the general theory of compact operator (see also [8] and [7]). Finally, Canuto [9] and Rannacher [10] considered eigenvalue approximation for fourth-order self-adjoint eigenvalue problems by non-conforming and hybrid finite elements.

In recent years a variety of effective advanced procedures that control the error and enhance the accuracy have been developed and analyzed (see, e.g., [11–13]). Larson [11] has combined a-posteriori error estimates with a-priori residual estimate. Xu and Zhou [12] have presented a two-mesh discretization technique that uses two finite element subspaces for solving second-order eigenvalue problems. Racheva and Andreev [13] have proposed a postprocessing strategy that increases the convergence rate for the numerical solution of $2m-$order self-adjoint eigenvalue problems.

In this paper, using the ideas developed in [13], we extend the results applied to the mixed finite element formulation of the biharmonic eigenvalue problem (3). We derive and justify a postprocessing algorithm that allows us to get higher order convergence for the postprocessed eigenvalues. The essence of the new method is the following: (1) for a given finite element spaces solve the mixed finite element eigenvalue problem and (2) solve one additional source problem on an augmented space with a right hand side the obtained eigefunctions. In our opinion, this new procedure is quite accurate and computationally inexpensive since we replace the solution of eigenvalue problem on a finer mesh or space of higher degree polynomials by an additional solution of a source problem.

The paper is organized as follows. In Section 2 we introduce the necessary notation and the weak mixed formulation of the boundary value problem (3). Following Babuška and Osborn [7] we recast the weak form in an abstract saddle-point form by introducing the real Hilbert spaces V, Σ and H. Further, we introduce the finite element approximation of the weak mixed formulation and review the main known results regarding error analysis. In Section 3 we first give motivation and introduce the main ideas for our method for postprocessing. In Section 4 we present the postprocessing method and discuss its implementation. Finally, in Section 7 we give the results of numerical experiments on two model problems.

2 Preliminaries and notations

The standard $L^2(\Omega)$-norm is denoted by $\|\cdot\|_{0,\Omega} \equiv \|\cdot\| \equiv \|\cdot\|_{L^2(\Omega)}$. Also we use the Sobolev spaces with integer k, $H^k(\Omega)$ and $H_0^k(\Omega)$. The norms in these spaces are denoted by $\|\cdot\|_{k,\Omega} \equiv \|\cdot\|_{H^k(\Omega)}$ (see, e.g., [14,15]). The space $H^k(\Omega)$ is the closure of all infinitely smooth functions defined on Ω in the H^k-norm. Similarly the space $H_0^k(\Omega)$ is the closure in the H^k-norm of all infinitely smooth functions with compact support in Ω. Finally, the Sobolev spaces with non-integer k are defined by the real method of interpolation [14]. If X, Y are two normed spaces, for an operator $A : X \to Y$ its norm is defined in a standard way: $\|A\| = \sup_{w\in X}\{\|Aw\|_Y/\|w\|_X\}$ (see, e.g. [7,8]).

The weak form of (3) is derived by multiplying the first equation of (3) by $\psi \in H^1(\Omega)$, the second equation by $v \in H_0^1(\Omega)$, and integrating by parts over Ω so that

$$\int_\Omega \nabla\sigma\cdot\nabla v\,dx = \lambda\int_\Omega uv\,dx,\;\; \int_\Omega \nabla u\cdot\nabla\psi\,dx = \int_\Omega \sigma\psi\,dx. \tag{4}$$

These identities are obviously well defined for $u \in H_0^1(\Omega)$ and $\sigma \in H^1(\Omega)$ they represent the week mixed form of the biharmonic eigenvalue problem.

Following Babuška and Osborn [7] we shall consider (4) as a particular case of an abstract eigenvalue saddle point problem. The abstract form of an eigenvalue saddle point problem is related to three real Hilbert spaces V, Σ, and H with inner products $(\cdot,\cdot)_V$, $(\cdot,\cdot)_\Sigma$, $(\cdot,\cdot)_H$, and norms $||\cdot||_V$, $||\cdot||_\Sigma$, $||\cdot||_H$, and two bilinear forms $a(\sigma,\psi)$ and $b(\psi,v)$ defined on $H \times H$ and $\Sigma \times V$, respectively. We assume that $V \subset H$ and $\Sigma \subset H$. Babuška and Osborn (see e.g. [7] p. 752) have studied the following problem: find $(\sigma,u) \in \Sigma \times V$, $(\sigma,u) \neq (0,0)$ and $\lambda \in \mathbf{R}$ such that

$$\begin{aligned} -a(\sigma,\psi) + b(\psi,u) &= 0, \qquad \forall\ \psi \in \Sigma, \\ b(\sigma,v) \qquad\qquad &= \lambda a(u,v), \ \forall\ v \in V. \end{aligned} \tag{5}$$

This problem is studied in [7] under the following assumptions:
(A1): $b(\psi,v)$ is defined on $\Sigma \times V$ and satisfies

$$|b(\psi,v)| \leq C_1||\psi||_\Sigma\ ||v||_V, \quad \forall \psi \in \Sigma, \quad \forall v \in V, \tag{6}$$

$$\sup_{\psi\in\Sigma} |b(\psi,u)| > 0, \quad \forall\ 0 \neq u \in V; \tag{7}$$

(A2): the bilinear form $a(\sigma,\psi)$ is symmetric on $H \times H$ and satisfies

$$a(\sigma,\psi) \leq C_2||\sigma||_H\ ||\psi||_H, \quad \forall \psi,\sigma \in H, \tag{8}$$

$$a(\sigma,\sigma) > 0, \ \forall\ 0 \neq \sigma \in H. \tag{9}$$

If we identify as $H = L^2(\Omega)$, $\Sigma = H^1(\Omega)$, and $V = H_0^1(\Omega)$ and

$$b(\sigma,v) = \int_\Omega \nabla\sigma \cdot \nabla v\, dx \quad \text{and} \quad a(\sigma,\psi) = \int_\Omega \sigma\ \psi\, dx,$$

then the weak form of (4) could be rewritten in the abstract form (5): find $(\sigma,u) \in \Sigma \times V$, $(\sigma,u) \neq (0,0)$ and $\lambda \in \mathbf{R}$ such that

$$-a(\sigma,\psi) + b(\psi,u) + b(\sigma,v) = \lambda a(u,v), \quad \forall (\psi,v) \in \Sigma \times V. \tag{10}$$

The inner products (and the corresponding norms) in V, Σ, and H are:

$$(u,v)_V = \int_\Omega \nabla u \cdot \nabla v\, dx, \ (\sigma,\psi)_\Sigma = \int_\Omega (\nabla\sigma \cdot \nabla\psi + \sigma\psi)\, dx, \ (u,v)_H = \int_\Omega uv\, dx.$$

It follows from the definition that the bilinear form $a(\cdot,\cdot)$ coincides with the inner product in H and therefore is symmetric and coercive on H. Similarly, the bilinear form $b(\cdot,\cdot)$ is symmetric and coercive on V and therefore defines an inner product that is equivalent to the inner product on V (i.e. on $H_0^1(\Omega)$).

In our case the conditions (6), (8) and (9) of the abstract problem are obviously satisfied. We only need to check the condition (7). In fact, since $V \subset \Sigma$ we have even stronger inf-sup type inequality for $b(\cdot,\cdot)$:

$$\sup_{0 \neq \psi \in \Sigma} \frac{|b(\psi,u)|}{\|\psi\|_\Sigma} \geq \sup_{0 \neq \psi \in V} \frac{|b(\psi,u)|}{\|\psi\|_V} \geq \frac{|b(u,u)|}{\|u\|_V} = \|u\|_V, \quad \forall\, u \in V. \tag{11}$$

If $(\lambda,(\sigma,u))$ is an eigenpair of (5) then (λ,u) is an eigenpair of (1), (2) and $\sigma = -\Delta u$. Hence the regularity of (σ,u) can be inferred from the regularity of the solution of the problem (1), (2) (see, e.g. Grisvard [16], Section 7, p. 301).

Remark 1 *One can consider also other boundary conditions. For example, the deformations of an isotropic plate with simply supported boundary will be governed by the equation (1) with the boundary conditions*

$$u(x) = 0 \quad \textit{and} \quad \frac{\partial^2 u}{\partial \nu^2}(x) = 0, \quad x \in \Gamma. \tag{12}$$

If Ω is a polygonal domain then the second boundary condition in (12) essentially reduces to $\Delta u = 0$ on Γ and one can recast this problem in the following weak form: find $(\sigma,u) \in V \times V$, $(\sigma,u) \neq (0,0)$ and $\lambda \in \mathbf{R}$ such that

$$-a(\sigma,\psi) + b(\psi,u) + b(\sigma,v) = \lambda a(u,v), \quad \forall (\psi,v) \in V \times V. \tag{13}$$

This formulation falls into the class of problems we consider. In fact, this is a much simpler problem, since now both forms $a(\sigma,\psi)$ and $b(\psi,u)$ are symmetric on $H \times H$ and $V \times V$, respectively.

Now we shall consider the finite element approximation of the problem (5). Let $\mathcal{T}_h$ be a splitting of Ω into a finite number of finite elements (triangles or quadrilaterals), which is quasiuniform and has characteristic size h. We assume that $\mathcal{T}_h$ satisfies the conditions of finite element triangulation (cf. Ciarlet [15], p. 38). Associated with the triangulation $\mathcal{T}_h$ we define the finite element spaces $V_h \subset V$ and $\Sigma_h \subset \Sigma$ of piece wise polynomials of degree n (see, for example, [7], p. 758). Since the finite element spaces are subspaces of $H^1(\Omega)$ the functions in V_h and Σ_h need to be continuous so $n \geq 1$. Further, we shall need only the approximation properties of these spaces. Namely, we assume that

$$\inf_{v \in V_h} \{||u-v||_{0,\Omega} + h||\nabla(u-v)||_{0,\Omega}\} \leq Ch^{n+1}||u||_{n+1,\Omega},$$

and

$$\inf_{\tau \in \Sigma_h} \{||\sigma-\tau||_{0,\Omega} + h||\nabla(\sigma-\tau)||_{0,\Omega}\} \leq Ch^{n+1}||\sigma||_{n+1,\Omega}.$$

It is well known that the rate of convergence of a finite element approximation to the eigenvalues and the eigenfunctions depends on the smoothness of the exact eigenfunctions. For general domains, the eigenfunctions of the biharmonic problem belong to the space $H^2(\Omega)$. Additional smoothness could be ensured for domains with smooth boundaries. In this case we need to use either isoparametric finite elements that fit the domain exactly or finite element subspaces $V_h \not\subset V$ and $\Sigma_h \not\subset \Sigma$ (see, e.g. [15,18]).

The goal of this paper is to design and justify a post-processing technique that would allow us to achieve higher order convergence for both the eigenvalues and eigenfunctions. The assumption that Ω is a convex polygonal domain on one hand simplifies the exposition and makes the presentation of the main ideas more transparent. On the other hand, its limits the regularity of the eigenfunctions and makes the investigation of the convergence rates much more difficult. It is well known (see, e.g. [16]) that for a given $f \in L^2(\Omega)$ the solution w of the corresponding biharmonic boundary value problem

$$\Delta^2 w = f \quad \text{in } \Omega, \quad w = \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \tag{14}$$

belongs to $H^3(\Omega)$. More refined results regarding the smoothness for the solutions of the first biharmonic boundary value problem could be obtained either by employing weighted Sobolev spaces (see, e.g. [20]) or by using Sobolev spaces of fractional order (e.g. [3,16]). For example, in [20] it has been shown that if the largest interior angle of the boundary $\partial\Omega$ is less than 126.28^o and $f \in L^2(\Omega)$, then $w \in H^4(\Omega)$. Using interpolation of Banach spaces it has been shown in [16] that if $f \in L^2(\Omega)$ and Ω is a convex polygonal domain then $w \in H^{3+s}(\Omega)$, where $0 < s \leq 1$ is a constant that depends on the largest interior angle of $\partial\Omega$. For an easy procedure for finding s through the angle of the domain Ω we refer to [3]. The regularity results for the source problem (14) will lead to regularity of the eigenfunctions.

We define the approximation of the eigenpair $(\lambda, (\sigma, u))$ by the mixed finite element method as $\lambda_h \in \mathbf{R}, (\sigma_h, u_h) \in \Sigma_h \times V_h$ such that $a(u_h, u_h) = 1$ and

$$-a(\sigma_h, \psi) + b(\psi, u_h) + b(\sigma_h, v) = \lambda_h a(u_h, v), \;\; \forall(\psi, v) \in \Sigma_h \times V_h. \tag{15}$$

It has been shown (cf. [7], Theorem 11.4, p. 763, see also [1,6,17]) that if the finite element spaces contain polynomials of degree $n \geq 2$ and $u \in H^{n+1}(\Omega)$, then

$$|\lambda - \lambda_h| \leq C\ h^{2n-2}\|u\|_{n+1,\Omega}, \quad \|u - u_h\|_{0,\Omega} \leq C\ h^n\|u\|_{n+1,\Omega} \tag{16}$$

and

$$\|u - u_h\|_{1,\Omega} + h\|\sigma - \sigma_h\|_{0,\Omega} \leq C\ h^n\|u\|_{n+1,\Omega}. \tag{17}$$

For polygonal domains the solution does not have the required in (16), (17) regularity for $n \geq 3$ and the estimate does not remain valid. For Ω a convex polygonal domain and $n \geq 3$ instead of (16) and (17) we have (see, e.g. [6])

$$\begin{aligned} |\lambda - \lambda_h| &\leq C\ h^{2+2s}\|u\|_{3+s,\Omega}, \\ \|u - u_h\|_{1,\Omega} + h\|\sigma - \sigma_h\|_{0,\Omega} &\leq C\ h^{2+s}\|u\|_{3+s,\Omega}. \end{aligned} \tag{18}$$

Here $0 < s \leq 1$ depends on the maximal interior angle of the boundary $\partial\Omega$ (see, e.g. [16,20]). For rectangular domains $s = 1$ and the error has optimal convergence rate.

The case $n = 1$ is considered by Ishihara [19] who proved a convergence rate for the eigenvalues and eigenfunctions of the form $|\lambda - \lambda_h| + \|u - u_h\|_{1,\Omega} = \mathcal{O}(h^{\frac{1}{2}})$.

Remark 2 *The inequalities (16) and (17) show that for $n = 2$ the error estimates of the mixed finite element approximation of the eigenvalues and eigenfunctions of a fourth order problem are optimal with respect to both the regularity of the solution and the order of approximation so that the convergence rate for the eigenvalues is twice that of the finite element approximation error in the energy norm. However, in the case of polygonal domains and finite elements of degree higher or equal to three the convergence rate is limited due to the limited regularity of the solution.*

3 Postprocessing Technique: Motivation

We now present a relatively simple postprocessing procedure that gives better accuracy for both eigenvalues and eigenfunctions. This postprocessing technique involves solving the original finite element eigenvalue problem using piece wise polynomials of degree n and one additional source problem using an enriched finite element space. We consider two possibilities, namely spaces based on a much finer grid and spaces of piece wise polynomials of higher degree, e.g., $n + 1$.

To motivate our approach we shall first study the corresponding source problem, namely, the mixed form of the elliptic problem with right-hand side $f \in L^2(\Omega)$: find $(\tau, w) \in \Sigma \times V$ such that

$$-a(\tau, \psi) + b(\psi, w) + b(\tau, v) = a(f, v), \quad \forall (\psi, v) \in \Sigma \times V. \tag{19}$$

The solution (τ, w) of this problem defines two *component solution operators*:

$$S : L^2(\Omega) \to \Sigma, \quad Sf = \tau, \quad T : L^2(\Omega) \to V, \quad Tf = w.$$

The solution $(\lambda, (\sigma, u))$ of the eigenvalue problem (5) will satisfy the following relations expressed through the operators T and S: $\sigma = \lambda\ Su$ and $u = \lambda\ Tu$. Indeed, consider the problem (19) with $f = \lambda u$, where $(\lambda, (\sigma, u))$ is the solution of (5). In this case the solution of (19) is $(S(\lambda u), T(\lambda u)) = (\lambda Su, \lambda Tu)$, i.e.,

$$-a(\lambda Su, \psi) + b(\psi, \lambda Tu) + b(\lambda Su, v) = a(\lambda u, v), \quad \forall (\psi, v) \in \Sigma \times V. \tag{20}$$

Comparing (20) and (10) we see that $(\lambda Su, \lambda Tu)$ and (σ, u) are solutions to the same source problem. Because of uniqueness they coincide, i.e. $\sigma = \lambda\, Su$ and $u = \lambda\, Tu$.

Similarly, the corresponding finite element approximation: find $(\tau_h, w_h) \in \Sigma_h \times V_h$ such that

$$-a(\tau_h, \psi) + b(\psi, w_h) + b(\tau_h, v) = a(f, v), \quad \forall (\psi, v) \in \Sigma_h \times V_h,$$

defines the *discrete component solution operators*:

$$S_h : L^2(\Omega) \to \Sigma_h, \quad S_h f = \tau_h, \quad T_h : L^2(\Omega) \to V_h, \quad T_h f = w_h.$$

Obviously, the operators $S,\ T,\ S_h$, and T_h satisfy the identities

$$-a(Sf, \psi) + b(\psi, Tf) + b(Sf, v) = a(f, v), \quad \forall\ \psi \in \Sigma,\ \forall\ v \in V,$$

and

$$-a(S_h f, \psi) + b(\psi, T_h f) + b(S_h f, v) = a(f, v), \quad \forall\ \psi \in \Sigma_h,\ \forall\ v \in V_h.$$

In fact, $(S_h f,\ T_h f)$ is the "Ritz projection" of $(Sf,\ Tf)$ onto the finite element space $\Sigma_h \times V_h$ and it satisfies the orthogonality condition

$$-a(Sf - S_h f, \psi) + b(\psi, Tf - T_h f) + b(Sf - S_h f, v) = 0, \quad \forall\ \psi \in \Sigma_h,\ \forall\ v \in V_h.$$

Now consider the operator T and the family of operators $\{T_h\}$ on the space $L^2(\Omega)$. It follows (see, e.g. Falk and Osborn [17]) that

$$\|Tf - T_h f\|_{0,\Omega} = \|w - w_h\|_{0,\Omega} \le C\, h^2 \|Tf\|_{3,\Omega}.$$

The regularity result gives $\|Tf\|_{3,\Omega} \le C\|f\|_{0,\Omega}$ so we get

$$\|T - T_h\| = \sup_{f \in L^2(\Omega)} \frac{\|(T - T_h)f\|_{0,\Omega}}{\|f\|_{0,\Omega}} \le C\ h^2,$$

and consequently $\|T - T_h\| \to 0$ as $h \to 0$.

Since the operator T_h has a finite dimensional range, i.e. $\dim \mathcal{R}(T_h) < \infty$, T_h is compact. The last convergence implies that T is also compact. Thus, the eigenpairs $(\lambda, (\sigma, u))$ of (5) can be characterized in terms of operator T. This means, that if $(\lambda, (\sigma, u))$ is an eigenpair of (5), then $\lambda T u = u,\ u \ne 0$, and conversely if $\lambda T u = u,\ u \ne 0$, then there is a $\sigma = S(\lambda u),\ \sigma \in \Sigma$, such that $(\lambda, (\sigma, u))$ is an eigenpair of (5). Thus λ is an eigenvalue of (5) if and only if λ^{-1} is an eigenvalue of T.

The operators T and T_h are symmetric in the inner product defined by the bilinear form $a(\cdot, \cdot)$. Indeed, for any $u,\ v \in V$ we have the following sequence of equalities:

$$\begin{aligned} a(u,Tv) &= b(Su,Tv) \text{ (by the definition of the operator } S) \\ &= a(Su,Sv) \text{ (by the definition of the operator } T) \\ &= a(Sv,Su) \text{ (by the symmetry of the form } a(\cdot,\cdot)) \\ &= b(Sv,Tu) \text{ (by the definition of the operator } T) \\ &= a(v,Tu) \quad \text{(by the definition of the operator } S) \\ &= a(Tu,v). \end{aligned}$$

A similar argument can be applied to the discrete problem. Thus, T_h is symmetric in the inner product $a(\cdot,\cdot)$ and the approximate eigenvalues defined by (15) can be characterized in terms of the eigenvalues of T_h. Namely, λ_h is an eigenvalue of (15) if and only if λ_h^{-1} is an eigenvalue of T_h. As shown by Falk and Osborn [17] we also have

$$\|(T-T_h)f\|_{1,\Omega} + h\|(S-S_h)f\|_{0,\Omega} + h^2\|\nabla(S-S_h)f\|_{0,\Omega} \le Ch^n\|Tf\|_{n+1,\Omega}. \tag{21}$$

Assume that a solution $(\lambda_h,(\sigma_h,u_h))$ of the mixed finite element problem (15) is already found. We may then consider the elliptic problem (19) with a right-hand side u_h and solution $(\tilde{\tau},\tilde{w}) \in \Sigma \times V$:

$$-a(\tilde{\tau},\psi) + b(\psi,\tilde{w}) + b(\tilde{\tau},v) = a(u_h,v), \quad \forall\, \psi \in \Sigma, \forall\, v \in V. \tag{22}$$

Using the operators S and T the solution of the problem (22) could be written as $(\tilde{\tau},\tilde{w}) = (Su_h,Tu_h)$.

For the moment, assume that the solution $(\tilde{\tau},\tilde{w})$ of (22) is available so we can evaluate the number

$$\tilde{\lambda} = \frac{1}{a(u_h,Tu_h)} = \frac{1}{a(u_h,\tilde{w})}. \tag{23}$$

In the next theorem we show that $\tilde{\lambda}$ provides a good approximation to λ:

Theorem 3.1 *Let $(\lambda,(\sigma,u))$ be an eigenpair of problem (5), and let also $(\lambda_h,(\sigma_h,u_h)) \in \mathbf{R}\times\Sigma_h\times V_h$ be its finite element approximation obtained from (15) assuming that the eigenfunctions are normalized by $\|u\|_{0,\Omega} = \|u_h\|_{0,\Omega} = 1$. Let $\tilde{\lambda}$ be computed by (23), where $\tilde{w}$ is the solution of (22). Then*

$$|\lambda - \tilde{\lambda}| \le C\ \|u-u_h\|^2_{0,\Omega}. \tag{24}$$

P r o o f. By taking into account the symmetry of the operator T in the inner product defined by the form $a(\cdot,\cdot)$, the equality $u = \lambda Tu$, and the normalization of the eigenfunctions $a(u,u) = a(u_h,u_h) = 1$ we easily get

$$\begin{aligned}
\frac{1}{\lambda} - \frac{1}{\widetilde{\lambda}} &= a(u,Tu) - a(u_h,Tu_h) \\
&= a(u,Tu) - a(u_h,Tu_h) + a(u-u_h,T(u-u_h)) - a(u-u_h,T(u-u_h)) \\
&= 2a(u,Tu) - 2a(u_h,Tu) - a(u-u_h,T(u-u_h)) \\
&= \frac{1}{\lambda}a(u-u_h,u-u_h) - a(u-u_h,T(u-u_h)).
\end{aligned}$$

Since

$$a(u-u_h,T(u-u_h)) \le \|u-u_h\|_{0,\Omega}\|T(u-u_h)\|_{0,\Omega} \le \|T\|\|u-u_h\|^2_{0,\Omega}$$

we get

$$|\lambda - \widetilde{\lambda}| \le \lambda\widetilde{\lambda}\left(\frac{1}{\lambda} + \|T\|\right)\|u-u_h\|^2_{0,\Omega} \le C\|u-u_h\|^2_{0,\Omega}.$$

The boundness of the operator T implies (24). ∎

As a corollary of the above estimate (24) and the error estimates (16) we can conclude that for $n \ge 2$

$$|\lambda - \widetilde{\lambda}| \le C\,h^{2n} \quad \text{if} \quad u \in H^{n+1}(\Omega),$$

which is a substantial improvement compared with the estimate (16).

For a convex polygonal domain Ω we get a slightly worst result for $n = 3$ (since we need to use (18))

$$|\lambda - \widetilde{\lambda}| \le C\,h^{2(2+s)} \quad \text{since} \quad u \in H^{3+s}(\Omega).$$

Recall that $0 < s \le 1$ and depends on the maximal interior angle of the boundary $\partial\Omega$.

4 Postprocessing Technique: Algorithm

The above theorem is very useful from a theoretical point of view. However, it is not very practical since the exact solution of the source problem (22) is hardly ever available. To make it useful for computational practice we need to appropriately approximate $\widetilde{\lambda}$. Here we shall present and discuss two possible approaches. The first approach is the "two-grid method" of Xu and Zhou introduced and studied in [12] for second order differential equations and integral equations. The second approach proposed and studied by Andreev and Racheva in [13] uses the same grid but finite elements of higher degree.

The first approach uses a finer grid (with mesh size h^2) to get an approximation of $\widetilde{\lambda}$ with an error $O(h^{2n})$. The advantage of this approach is that it uses the same finite element spaces and does not require higher regularity of the solution. The disadvantage is that we have to generate an order of magnitude finer mesh. The second approach is based on the same finite element partition $\mathcal{T}$ but using piece-wise polynomials of degree $n+1$. Here we need to generate the corresponding finite element matrices for higher degree polynomials. Also, to get an approximation of $\widetilde{\lambda}$ with an error $O(h^{2n})$ in this case, we need higher regularity of the solution u. For polygonal domains this approach could be used for $n = 2, 3$ only.

We shall treat both approaches in the same abstract manner. Namely, we introduce an additional finite element spaces of continuous functions $\widetilde{\Sigma}_h \times \widetilde{V}_h$ such that $\Sigma_h \times V_h \subset \widetilde{\Sigma}_h \times \widetilde{V}_h \subset \Sigma \times V$ and consider the following discrete elliptic problem (source problem): find $(\widetilde{\tau}_h, \widetilde{w}_h) \in \widetilde{\Sigma}_h \times \widetilde{V}_h$ such that

$$-a(\widetilde{\tau}_h, \psi) + b(\psi, \widetilde{w}_h) + b(\widetilde{\tau}_h, v) = a(u_h, v), \quad \forall \psi \in \widetilde{\Sigma}_h, \ \forall v \in \widetilde{V}_h. \tag{25}$$

The solution $(\widetilde{\tau}_h, \widetilde{w}_h)$ of this problem can be expressed as $\widetilde{\tau}_h = \widetilde{S}_h u_h$ and $\widetilde{w}_h = \widetilde{T}_h u_h$, where $\widetilde{S}_h$ and $\widetilde{T}_h$ are solution operators related to the finite element space $\widetilde{\Sigma}_h \times \widetilde{V}_h$.

Now we present a postprocessing algorithm which will give improved approximations of the eigenvalues and eigenfunctions of the mixed problem (10).

Algorithm 4.1:

(1) Solve the eigenvalue problem (15) for $\lambda_h \in \mathbf{R}$ *and* $(\sigma_h, u_h) \in \Sigma_h \times V_h$.
(2) Solve the sorce problem (25) and find $(\widetilde{\tau}_h, \widetilde{w}_h) \in \widetilde{\Sigma}_h \times \widetilde{V}_h$.
(3) Compute

$$\widetilde{\lambda}_h = a(u_h, \widetilde{w}_h)^{-1}. \tag{26}$$

(4) Evaluate $\widetilde{u}_h = \widetilde{\lambda}_h \ \widetilde{w}_h$ *and* $\widetilde{\sigma}_h = \widetilde{\lambda}_h \ \widetilde{\tau}_h$.

The pair $(\widetilde{\lambda}_h, (\widetilde{\sigma}_h, \widetilde{u}_h))$ *represents a new (and better) approximations to* $(\lambda, (\sigma, u))$.

In the next two sections we shall study the error in the eigenvalues and eigenfunctions defined by this algorithm for two particular choices of the spaces $\widetilde{\Sigma}_h \times \widetilde{V}_h$ outlined above.

5 Error estimates for the recovered eigenvalues

Below we establish the main results of this paper, namely estimates for the approximate eigenvalues computed by Algorithm 4.1.

Theorem 5.1 *Let $(\lambda, (\sigma, u))$ be an eigenpair of the problem (5) and $(\widetilde{\tau}_h, \widetilde{w}_h))$ and let $\widetilde{\lambda}_h, (\widetilde{\sigma}_h, \widetilde{u}_h))$ be found by the Algorithm 4.1. If the eigenfunctions are normalized by $\|u\|_{0,\Omega} = \|u_h\|_{0,\Omega} = 1$, then*

$$|\lambda - \widetilde{\lambda}_h| \leq C \left(\|u - u_h\|_{0,\Omega}^2 + \|\widetilde{\tau} - \widetilde{\tau}_h\|_{1,\Omega} \|\widetilde{w} - \widetilde{w}\|_{1,\Omega} + \|\widetilde{\tau} - \widetilde{\tau}_h\|_{0,\Omega}^2 \right). \quad (27)$$

The constant C may depend on λ but is independent of h.

Proof. We first note that $|\lambda - \widetilde{\lambda}_h| \leq |\lambda - \widetilde{\lambda}| + |\widetilde{\lambda} - \widetilde{\lambda}_h|$. The first term in this inequality has already been estimated in (24), so to complete the proof we only need to estimate the second term.

Using the definition of $\widetilde{\lambda}$ and $\widetilde{\lambda}_h$ and the properties of the operators T and $\widetilde{T}_h$ we have

$$\begin{aligned}
\frac{1}{\widetilde{\lambda}} - \frac{1}{\widetilde{\lambda}_h} &= a(u_h, \widetilde{w}) - a(u_h, \widetilde{w}_h) \\
&= [2b(\widetilde{\tau}, \widetilde{w}) - a(\widetilde{\tau}, \widetilde{\tau})] - [2b(\widetilde{\tau}_h, \widetilde{w}_h) - a(\widetilde{\tau}_h, \widetilde{\tau}_h)] \\
&= 2b(\widetilde{\tau} - \widetilde{\tau}_h, \widetilde{w} - \widetilde{w}_h) - a(\widetilde{\tau} - \widetilde{\tau}_h, \widetilde{\tau} - \widetilde{\tau}_h) \\
&\quad + 2\left[b(\widetilde{\tau} - \widetilde{\tau}_h, \widetilde{w}_h) - a(\widetilde{\tau} - \widetilde{\tau}_h, \widetilde{\tau}_h) + b(\widetilde{\tau}_h, \widetilde{w} - \widetilde{w}_h)\right] \\
&= 2b(\widetilde{\tau} - \widetilde{\tau}_h, \widetilde{w} - \widetilde{w}_h) - a(\widetilde{\tau} - \widetilde{\tau}_h, \widetilde{\tau} - \widetilde{\tau}_h).
\end{aligned}$$

Here we have used the following orthogonality condition for the finite element problem (25):

$$b(\widetilde{\tau} - \widetilde{\tau}_h, v) - a(\widetilde{\tau} - \widetilde{\tau}_h, \psi) + b(\psi, \widetilde{w} - \widetilde{w}_h) = 0, \quad \forall\ (\psi, v) \in \widetilde{\Sigma}_h \times \widetilde{V}_h$$

by choosing $\psi = \widetilde{\tau}_h$ and $v = \widetilde{w}_h$.

The above equality then leads to

$$\begin{aligned}
\left|\frac{1}{\widetilde{\lambda}} - \frac{1}{\widetilde{\lambda}_h}\right| &\leq 2|b(\widetilde{\tau} - \widetilde{\tau}_h, \widetilde{w} - \widetilde{w}_h)| + |a(\widetilde{\tau} - \widetilde{\tau}_h, \widetilde{\tau} - \widetilde{\tau}_h)| \\
&\leq 2\|\widetilde{\tau} - \widetilde{\tau}_h\|_{1,\Omega} \|\widetilde{w} - \widetilde{w}_h\|_{1,\Omega} + \|\widetilde{\tau} - \widetilde{\tau}_h\|_{0,\Omega}^2,
\end{aligned}$$

which together with (24) completes the proof of (27). ■

A key point in Algorithm 4.1 is the construction of appropriate finite element spaces $\widetilde{\Sigma}_h$ and $\widetilde{V}_h$ for solving the discrete problem (25). Below we present two practical approaches to this problem.

Method 1 ("two-grid method" of Xu and Zhou [12]): Let $\widetilde{\Sigma}_h$ and $\widetilde{V}_h$ be spaces of continuous functions that are piece wise polynomials of degree n on a mesh $\widetilde{\mathcal{T}}_h$ with characteristic grid-size h^β with $\beta > 1$ (which will be chosen later). This is a finer grid that could be generated by multilevel refinement of the original grid $\mathcal{T}_h$ (see, e.g. [12]).

First, we consider the case when the problem (14) allows smooth solutions. Our analysis is restricted to $n \leq 4$ since $H^5(\Omega)$ is the maximum regularity of the solution $\widetilde{w}$ of the problem (14) with a right hand side in $H^1(\Omega)$. Choosing $\beta = n/(n-1)$ and applying Theorem 5.1 and the error estimate (21) for $\widetilde{w}_h$ we get

$$|\lambda - \widetilde{\lambda}_h| \leq C(\|u - u_h\|_{0,\Omega}^2 + \|\widetilde{\tau} - \widetilde{\tau}_h\|_{1,\Omega}\|\widetilde{w} - \widetilde{w}_h\|_{1,\Omega} + \|\widetilde{\tau} - \widetilde{\tau}_h\|_{0,\Omega}^2)$$

$$\leq Ch^{2n}(\|Tu\|_{n+1,\Omega}^2 + \|Tu_h\|_{n+1,\Omega}^2), \quad n \leq 4.$$

The above estimate is valid also for convex polygonal domains and spaces involving polynomials of degree $n = 2$. In this case the solution is in $H^3(\Omega)$ and we can take $\beta = 2$ so that the rate of convergence in the eigenvalues is $\mathcal{O}(h^4)$. This is a significant improvement compared with the estimate (16), which ensures a convergence rate of $|\lambda - \widetilde{\lambda}_h| = \mathcal{O}(h^2)$.

For $n = 3, 4$ and Ω a convex polygonal domain, we use the estimate (18) to get

$$|\lambda - \widetilde{\lambda}_h| \leq C(\|u - u_h\|_{0,\Omega}^2 + \|\widetilde{\tau} - \widetilde{\tau}_h\|_{1,\Omega}\|\widetilde{w} - \widetilde{w}_h\|_{1,\Omega} + \|\widetilde{\tau} - \widetilde{\tau}_h\|_{0,\Omega}^2)$$

$$\leq C(h^{2(2+s)}\|Tu\|_{3+s,\Omega}^2 + h^{2\beta(1+s)}\|Tu_h\|_{3+s,\Omega}^2).$$

The parameter β is chosen appropriately in order to balance the terms in the above inequality. We know that the solution is in $H^{3+s}(\Omega)$, $0 < s \leq 1$. For a known value of s we can choose $\beta = (2+s)/(1+s)$. We can always choose $\beta = 2$ (this is the worst case that leads to some extra work since the mesh is finer) to get

$$|\lambda - \widetilde{\lambda}_h| = \mathcal{O}(h^{4+2s}) \quad \text{for} \quad u \in H^{3+s}(\Omega), \quad 0 < s \leq 1.$$

This estimate is an improvements of the error estimate for the eigenvalues, namely, we get convergence rates $\mathcal{O}(h^{4+2s})$ instead of $\mathcal{O}(h^{2+2s})$. This improvement is at the cost of solving an additional source problem on a finer mesh with characteristic mesh-size h^2. Although this involves significant additional work, the solution of the source problem is much cheaper than solving the eigenvalue problem on a finer mesh that will ensure the same convergence rate.

Method 2 ("two space" method of Andreev and Racheva [13]): Let $\widetilde{\Sigma}_h$ and $\widetilde{V}_h$ be spaces of continuous functions that are piece wise polynomials of degree $n+1$ on the same mesh $\mathcal{T}_h$. If the problem (14) has solution in $H^{n+2}(\Omega)$, then the error estimate (21) for the approximation (26) gives

$$|\lambda - \widetilde{\lambda}_h| \leq Ch^{2n}(\|Tu\|_{n+1,\Omega}^2 + \|Tu_h\|_{n+2,\Omega}^2), \quad n = 2, 3.$$

The restriction $n \leq 3$ represents the maximum regularity we can get for the solution of the source problem (22) with the right hand side $u_h \in H^1$. This is an improvement over (16) which ensures convergence rate of $\mathcal{O}(h^{2n-2})$.

For Ω a convex polygonal domain it only makes sense to apply this approach for $n = 2$. Then the spaces V_h, Σ_h, and $\widetilde{V}_h, \widetilde{\Sigma}_h$ contain polynomials of degree 2 and 3, respectively. Taking into account (18) we get

$$|\lambda - \widetilde{\lambda}_h| \leq C(h^4 \|Tu\|^2_{3,\Omega} + h^{2+2s} \|Tu_h\|^2_{3+s,\Omega}).$$

Here $0 < s \leq 1$ depends on the maximal interior angle of the boundary Γ. The estimate (16) for piece wise quadratic finite elements ensures $|\lambda - \widetilde{\lambda}_h| = \mathcal{O}(h^2)$. The improvement that we get, an error estamte $|\lambda - \widetilde{\lambda}_h| = \mathcal{O}(h^{2+2s})$, costs solving one source problem on the same grid with piece wise cubic elements.

6 Error estimates for the recovered eigenfunctions

We now present a postprocessing technique for biharmonic eigenfunctions. First, we define an approximation $(\widetilde{u}, \widetilde{\sigma})$ of the exact eigenfunctions (u, σ) by

$$\widetilde{u} = \widetilde{\lambda}_h \, \widetilde{w} := \widetilde{\lambda}_h \, Tu_h, \quad \widetilde{\sigma} = \widetilde{\lambda}_h \, \widetilde{\tau} := \widetilde{\lambda}_h \, Su_h. \tag{28}$$

Theorem 6.1 *Let the assumptions of Theorem 3.1 be satisfied. Then the following estimate is valid for $\widetilde{u}$ and $\widetilde{\sigma}$ defined by (28):*

$$|u - \widetilde{u}|^2_{1,\Omega} \leq \|\sigma - \widetilde{\sigma}\|^2_{0,\Omega} = |\widetilde{\lambda}_h - \lambda| + \|u - u_h\|^2_{0,\Omega} + \frac{\widetilde{\lambda}_h}{\widetilde{\lambda}} |\widetilde{\lambda}_h - \widetilde{\lambda}| \tag{29}$$

P r o o f. We begin with the identity:

$$\begin{aligned} a(\sigma - \widetilde{\sigma}, \sigma - \widetilde{\sigma}) &= -a(\sigma - \widetilde{\sigma}, \sigma - \widetilde{\sigma}) + 2b(\sigma - \widetilde{\sigma}, u - \widetilde{u}) \\ &= [-a(\sigma, \sigma) + 2b(\sigma, u)] \\ &\quad +2[a(\sigma, \widetilde{\sigma}) - b(\widetilde{\sigma}, u) - b(\sigma, \widetilde{u})] \\ &\quad [-a(\widetilde{\sigma}, \widetilde{\sigma}) + 2b(\widetilde{\sigma}, \widetilde{u})]. \end{aligned} \tag{30}$$

Now we transform the terms in the brackets. First, using (10) with $(\psi, v) = (\sigma, u)$ we get

$$-a(\sigma, \sigma) + 2b(\sigma, u) = \lambda a(u, u) = \lambda. \tag{31}$$

Taking into account the definition of $(\widetilde{\sigma}, \widetilde{u})$ by (28) we also obtain

$$a(\sigma, \widetilde{\sigma}) - b(\widetilde{\sigma}, u) - b(\sigma, \widetilde{u}) = \widetilde{\lambda}_h \, [a(\sigma, \widetilde{\tau}) - b(\widetilde{\tau}, u) - b(\sigma, \widetilde{u})] = -\widetilde{\lambda}_h \, a(u_h, u) \tag{32}$$

and

$$-a(\widetilde{\sigma},\widetilde{\sigma})+2b(\widetilde{\sigma},\widetilde{u})=\widetilde{\lambda}_h^2\left[-a(\sigma,\widetilde{\tau})+2b(\widetilde{\tau},\widetilde{w})\right]=\frac{\widetilde{\lambda}_h^2}{\widetilde{\lambda}}. \tag{33}$$

Inserting (31), (33), and (32) into (30) we get the equality:

$$\begin{aligned} a(\sigma-\widetilde{\sigma},\sigma-\widetilde{\sigma}) &= \lambda-2\widetilde{\lambda}_h a(u_h,u)+\tfrac{\widetilde{\lambda}_h^2}{\widetilde{\lambda}} \\ &= [\lambda-\widetilde{\lambda}_h]+\widetilde{\lambda}_h[2-2a(u_h,u)]+\left[\tfrac{\widetilde{\lambda}_h^2}{\widetilde{\lambda}}-\widetilde{\lambda}_h\right] \\ &= [\lambda-\widetilde{\lambda}_h]+\widetilde{\lambda}_h\|u-u_h\|_{0,\Omega}^2+\tfrac{\widetilde{\lambda}_h}{\widetilde{\lambda}}[\widetilde{\lambda}_h-\widetilde{\lambda}], \end{aligned} \tag{34}$$

which gives the estimate for $\|\sigma-\widetilde{\sigma}\|_{0,\Omega}$.

To show the estimate for $u-\widetilde{u}$ we use the identity $b(\psi,u-\widetilde{u})=a(\sigma-\widetilde{\sigma},\psi)$ for $\psi\in\Sigma$, the inf-sup condition (11), and the estimate for $\|\sigma-\widetilde{\sigma}\|_{0,\Omega}$ to obtain

$$\|u-\widetilde{u}\|_{1,\Omega}\le\sup_{\psi\in\Sigma}\frac{|b(\psi,u-\widetilde{u})|}{\|\psi\|_\Sigma}\le\sup_{\psi\in\Sigma}\frac{|a(\sigma-\widetilde{\sigma},\psi)|}{\|\psi\|_\Sigma}\le\|\sigma-\widetilde{\sigma}\|_{0,\Omega}. \qquad \blacksquare$$

As a corollary or this theorem we can get estimate for the eigenfunctions. Namely, we consider $\widetilde{u}_h=\widetilde{\lambda}_h\widetilde{w}_h=\widetilde{\lambda}_h\widetilde{T}_hu_h$ and $\widetilde{\sigma}_h=\widetilde{\lambda}_h\widetilde{\tau}_h=\widetilde{\lambda}_h\widetilde{S}_hu_h$ as new approximations of the eigenfunctions (u,σ). If in $\widetilde{\Sigma}_h\times\widetilde{V}_h$ we choose polynomials one degree higher than those of the spaces $\Sigma_h\times V_h$ or a spaces on a finer grid in case of smooth solutions we get the estimate (after applying triangle inequality and the estimate (29))

$$\|\sigma-\widetilde{\sigma}_h\|_{0,\Omega}^2\le\|\sigma-\widetilde{\sigma}\|_{0,\Omega}+\|\widetilde{\sigma}-\widetilde{\sigma}_h\|_{0,\Omega}\le Ch^n\|u\|_{n+1,\Omega}.$$

Since this is an improvement for σ only we shall not elaborate further on this.

7 Numerical Results

The efficiency of the postprocessing algorithm is illustrated on two simple model problems. The exact eigenvalues and eigenfunctions are know and the eigenfunctions in both examples are smooth. Therefore, there are no restrictions concerning the regularity.

To find the approximate eigenpairs $(\lambda_{j,h},(\sigma_{j,h},u_{j,h}))$, $j=1,2,3,4$ we have used the method of subspace iteration (e.g. [21], p. 288).

Example 1. The first example represents a model problem of a long thin bar of length l with unit flexural rigidity and density which is simply supported at its endpoints. The natural frequencies of the bar are determined by the eigenvalues of the following problem:

$$u^{IV}(x) = \lambda\, u(x), \quad x \in (0,l), \quad u(0) = u''(0) = 0, \quad u(l) = u''(l) = 0. \qquad (35)$$

When $l = 1$, the exact solutions are:

$$\lambda_j = \pi^4 j^4,\ u_j(x) = \sqrt{2}\sin \pi j x,\ \sigma_j(x) = -\sqrt{2}(\pi j)^2 \sin \pi j x,\ j = 1, 2, \ldots,$$

where the eigenfunctions are normalized $a(u_j, u_j) = 1$. For convenience we give the first four eigenvalues with 8 significant digits:

$$\lambda_1 = 97.409091, \quad \lambda_2 = 1558.5454, \quad \lambda_3 = 7890.1363, \quad \lambda_4 = 24936.727.$$

The numerical results presented in Table 1 and Table 3 have been obtained by using the mixed finite element method on uniform partitions consisting of one-dimensional (beam) elements. The spaces V_h and Σ_h contained C^0 piecewise quadratic polynomials, i.e. $n = 2$, while the solution of the corresponding elliptic source problem used finite element spaces $\widetilde{V}_h$ and $\widetilde{\Sigma}_h$ defined on the same mesh and contained continuous piece wise cubic polynomials.

Table 1: Error for $|\lambda_j - \lambda_{j,h}|$ for Problem (35):

# elements	$j = 1$	$j = 2$	$j = 3$	$j = 4$
16	4.01×10^{-4}	1.02×10^{-1}	2.58	25.6
32	2.51×10^{-5}	6.42×10^{-3}	1.64×10^{-1}	1.63
64	1.57×10^{-6}	4.02×10^{-4}	1.03×10^{-2}	1.03×10^{-1}
128	9.83×10^{-8}	2.51×10^{-5}	6.44×10^{-4}	6.43×10^{-3}

Table 2: Error for $|\lambda_j - \widetilde{\lambda}_{j,h}|$ for Problem (35):

# elements	$j = 1$	$j = 2$	$j = 3$	$j = 4$
16	5.47×10^{-7}	5.44×10^{-4}	2.98×10^{-2}	4.97×10^{-1}
32	8.63×10^{-9}	8.76×10^{-6}	4.99×10^{-4}	8.69×10^{-3}
64	1.36×10^{-10}	1.38×10^{-7}	7.93×10^{-6}	1.40×10^{-4}
128	2.13×10^{-12}	5.42×10^{-9}	4.93×10^{-8}	7.25×10^{-5}

In Table 2 and Table 4 we present the error of the first four eigenvalues and eigenfunctions respectively using the mixed method with the postprocessing procedure. It is readily seen that a considerable acceleration of convergence due to the postprocessing arises on the coarse mesh, i.e. on the mesh with 16 or 32 finite elements.

Table 3: Error $\left(\|u_j - u_{j,h}\|^2_{1,\Omega} + \|\sigma_j - \sigma_{j,h}\|^2_{0,\Omega}\right)^{\frac{1}{2}}$ for Problem (35):

# elements	$j=1$	$j=2$	$j=3$	$j=4$
16	4.29×10^{-3}	1.37×10^{-2}	1.04×10^{-1}	4.37×10^{-1}
32	1.13×10^{-3}	9.18×10^{-3}	3.31×10^{-2}	1.03×10^{-1}
64	2.82×10^{-4}	2.27×10^{-3}	7.79×10^{-3}	2.34×10^{-2}
128	7.05×10^{-5}	5.65×10^{-4}	1.92×10^{-3}	5.54×10^{-3}

Table 4: Error $\left(\|u_j - \widetilde{u}_{j,h}\|^2_{1,\Omega} + \|\sigma_j - \widetilde{\sigma}_{j,h}\|^2_{0,\Omega}\right)^{\frac{1}{2}}$ for Problem (35):

# elements	$j=1$	$j=2$	$j=3$	$j=4$
16	1.36×10^{-4}	2.28×10^{-3}	1.37×10^{-2}	5.57×10^{-2}
32	1.70×10^{-5}	2.76×10^{-4}	1.56×10^{-3}	6.37×10^{-2}
64	2.27×10^{-6}	1.01×10^{-4}	8.19×10^{-4}	2.33×10^{-3}
128	1.53×10^{-7}	2.25×10^{-5}	3.17×10^{-5}	2.05×10^{-4}

Example 2. We consider the two-dimensional biharmonic eigenvalue problem

$$\Delta^2 u = \lambda u \text{ in } \Omega, \quad u = \Delta u = 0 \text{ on } \Gamma, \tag{36}$$

where $\Omega = (0,1)\times(0,1)$ and $\Gamma \equiv \partial\Omega$. This problem has been discussed in Remark 2.1 and its mixed weak formulation is presented in (13). The exact eigenvalues of this problem can be calculated by the formula $\lambda = (l^2+m^2)^2\pi^4, \quad l,m\in\mathbf{N}$, while the corresponding eigenfunctions, normalized by $a(u,u)=1$, are $u(x,y) = 4\sin l\pi x \sin m\pi y$.

Thus the first four eigenvalues and corresponding eigenfunctions are

$$\begin{aligned}
\lambda_1 &= \ 4\pi^4 = \ 389.6363641, \quad u_1 = 4\sin\pi x\sin\pi y, \quad (l=1, m=1),\\
\lambda_2 &= 25\pi^4 = 2435.227276, \quad u_2 = 4\sin\pi x\sin 2\pi y, \quad (l=1, m=2),\\
\lambda_3 &= 25\pi^4 = 2435.227276, \quad u_3 = 4\sin 2\pi x\sin\pi y, \quad (l=2, m=1),\\
\lambda_4 &= 64\pi^4 = 6234.181826, \quad u_4 = 4\sin 2\pi x\sin 2\pi y, \ (l=2, m=2).
\end{aligned}$$

In Table 5 we show the eigenvalues, calculated by the mixed method on uniform rectangular mesh. Σ_h and V_h are the space of continuous functions that are biquadratic over the finite elements. For the postprocessing method given by Algorithm 4.1 we use as $\widetilde{\Sigma}_h$ and $\widetilde{V}_h$ the space of continuous functions that are bicubic over the rectangular finite elements. For solving the algebraic eigenvalue probelm we use the method of subspace iterations [21]. Based on Table

6 one may conclude that it is not very reasonable to use postprocessing for the first eigenvalue and first eigenfunction. Also, if we are going to use postprocessing, it is sufficient the mixed method to obtain an approximate solution on a coarse mesh with small number of iterations. Note that the postprocessing decreases the values of $\lambda_{j,h}$.

Table 5: The approximate eigenvalues of Problem (36) computed by the mixed FEM

elements	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
9	391.6563	2522.890	2522.890	6510.286
16	390.8524	2465.127	2465.127	6416.349
25	390.0355	2447.942	2447.942	6324.614

Table 6: The approximate eigenvalues of Problem (36) computed by postprocessing Algorithm 4.1

elements	$\widetilde{\lambda}_{1,h}$	$\widetilde{\lambda}_{2,h}$	$\widetilde{\lambda}_{3,h}$	$\widetilde{\lambda}_{4,h}$
9	390.1023	2450.109	2464.251	6412.786
16	389.8916	2441.264	2444.357	6328.817
25	389.6961	2434.270	2435.012	6244.637

8 Remarks and Conclusions

Comparing the results proved in the previous sections as well as the numerical result for the eigenvalues and corresponding eigenfunctions of the mixed variant of the first biharmonic eigenvalue problem so-called "optimal" case (see [1,5–7]) we see that higher order accuracy could be extracted using a relatively simple postporcessing technique.

Our approach is easily extended to various other problems, such as:

- One dimensional problems with various boundary conditions, for example: $u^{IV}(x) = \lambda\ u(x),\ x \in (0,l),\ u(0) = u'(0) = 0,\ u''(l) = u'''(l) = 0$. This problem easily falls into the class of problems considered in this paper.
- Plates with variable density. Namely, the term λu in (1) is replaced by $\lambda\rho u$, where ρ is a strictly positive and bounded function on Ω.
- Eigenvalue value problem $\Delta^2 u = \lambda\Delta u,$ in $\Omega,\ u = \frac{\partial u}{\partial n} = 0,$ on Γ, which has been considered in [6].

We note that linear finite elements (i.e. $n = 1$) are rarely applied to mixed biharmonic problems (see Ishihara [2,19]). The rate of convergence for the eigenfunctions in any Sobolev norm $\| \cdot \|_{k,\Omega}$, $k = 0, 1, 2$ is $\mathcal{O}(h^{\frac{1}{2}})$ (see, e.g., [19]). Our postprocessing method could be applied to this case as well and will be discussed in a separate paper.

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