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ERROR ESTIMATES FOR A FINITE VOLUME ELEMENT METHOD FOR ELLIPTIC PDEs IN NONCONVEX POLYGONAL DOMAINS*

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Abstract. We consider standard finite volume piecewise linear approximations for second order elliptic boundary value problems on a nonconvex polygonal domain. Based on sharp shift estimates, we derive error estimations in H^1-, L_2- and L_∞-norms, taking into consideration the regularity of the data. Numerical experiments and counterexamples illustrate the theoretical results.

Key words. finite volume element method, nonconvex polygons, error estimations

AMS subject classifications. 65N15, 65N30

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1. Introduction.

We analyze the standard finite volume element method for the discretization of second order linear elliptic PDEs on a nonconvex polygonal domain $\Omega \subset \mathbb{R}^2$. Namely, for a given function f, we seek u such that

$$Lu \equiv -\operatorname{div}(A\nabla u) = f \quad \text{in } \Omega, \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega \tag{1.1}$$

with $A = (a_{ij})_{i,j=1}^2$ a given symmetric matrix function with real-value entries $a_{ij} \in W_\infty^1$, $1 \le i, j \le 2$. We assume that the matrix A is uniformly positive definite in Ω, i.e., there exists a positive constant α_0 such that

$$\xi^T A(x)\xi \ge \alpha_0 \xi^T \xi \quad \forall \xi \in \mathbb{R}^2, \, \forall x \in \bar{\Omega}. \tag{1.2}$$

The class of finite volume methods is based on some approximation of the balance relation

$$-\int_{\partial b} A\nabla u \cdot n \, ds = \int_b f \, dx, \tag{1.3}$$

which is valid for any subdomain $b \subset \Omega$. Here n denotes the outer unit normal vector to the boundary of b.

There are various approaches to the finite volume method. One, the finite volume element method, uses a finite element partition of Ω, where the solution space consists of continuous piecewise linear functions, a collection of vertex-centered control volumes, and a test space of piecewise constant functions over the control volumes (cf., e.g., [6, 10, 25, 28]). A second approach, usually called the finite volume difference method, uses cell-centered grids and approximates the derivatives in the balance equation by finite differences (cf., e.g., [22, 29, 33]). Another approach uses mixed reformulation of the problem [12, 16]. The first approach is quite close to the finite element method but nevertheless has some new properties that make it attractive for the applications [1, 20]. The second approach is closer to the classical finite difference method and extends it to more general than rectangular meshes. It is used mostly on perpendicular bisection or Voronoi type meshes. Approximations on such rectangular

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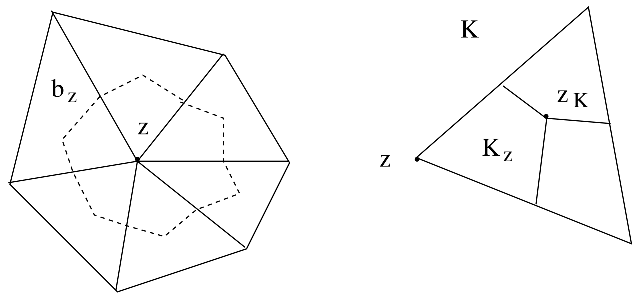

FIG. 1.1. *Left-hand side: A sample region with dotted lines indicating the corresponding box b_z. Right-hand side: A triangle K partitioned into three subregions K_z.*

and triangular meshes were studied, for example, in [34] and [26], respectively. The third approach is close to mixed and hybrid finite element methods and can deal, for example, with irregular quadrilateral and hexahedral cells [12, 30]. Finite volume discretizations for more general convection-diffusion-reaction problems were studied by many authors. For a comprehensive presentation and more references of existing results we refer to the monographs on the finite volume difference method [22] and on the finite volume element method [28], and for various applications on the special issue [21].

We shall consider a finite volume element discretization of (1.1), in the standard conforming space of piecewise linear functions,

$$X_h = \{\chi \in C(\Omega) : \ \chi|_K \text{ is linear } \forall K \in T_h \text{ and } \chi|_{\partial\Omega} = 0\}$$

with $\{T_h\}_{0<h<1}$ a given family of triangulations of Ω with h denoting the maximum diameter of the triangles of T_h. For simplicity we shall assume that T_h is a quasi-uniform triangulation. However, this assumption is only required to show L_∞-norm error estimates. For L_2- and H^1-norm error estimations, *nondegenerate* triangulations [9, equation (4.4.16)] are sufficient.

The finite volume problem will satisfy a relation similar to (1.3) for b in a finite collection of subregions of Ω called control volumes, the number of which will be equal to the dimension of the finite element space X_h. These control volumes are constructed in the following way: Let z_K be the barycenter of $K \in T_h$. We connect z_K with line segments to the midpoints of the edges of K, thus partitioning K into three quadrilaterals K_z, $z \in Z_h(K)$, where $Z_h(K)$ are the vertices of K. Then with each vertex $z \in Z_h = \cup_{K \in T_h} Z_h(K)$ we associate a control volume (also called a box) b_z, which consists of the union of the subregions K_z, sharing the vertex z (see Figure 1.1). We denote the set of interior vertices of Z_h by Z_h^0.

The finite volume element method is then to find $u_h \in X_h$ such that

$$-\int_{\partial b_z} (A\nabla u_h) \cdot n \, ds = \int_{b_z} f \, dx, \quad \forall z \in Z_h^0. \tag{1.4}$$

Before we start our description of this work we introduce some notation. We will use the standard notation for the Sobolev spaces W_p^s and $H^s = W_2^s$ (cf. [2]). Namely, $L_p(V)$, $1 \le p < \infty$, denotes the space of p-integrable real functions over $V \subset \mathbb{R}^2$, $(\cdot,\cdot)_V$ the inner product in $L_2(V)$, $|\cdot|_{H^s(V)}$ and $\|\cdot\|_{H^s(V)}$ the seminorm and norm, respectively, in $H^s(V)$, $|\cdot|_{W_p^s(V)}$ and $\|\cdot\|_{W_p^s(V)}$ the seminorm and norm, respectively, in $W_p^s(V)$, $p \geq 1$, and $s \in \mathbb{R}$. In addition, if $V = \Omega$ we suppress the index V, and if $p = 2$ and $s = 0$ we also suppress these indexes and denote $\|\cdot\|_{W_2^0} = \|\cdot\|$. Further, we shall denote with p' the adjoint of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$, $p > 1$.

We begin with some comments. It is well known that in the case of a polygonal Ω, if $f \in L_p$, $1 < p < \infty$, then the solution u of (1.1) is not always in W_p^2 (cf., e.g., [24] and section 2). However, it is always in $W_{\bar{p}}^2$ or in a fractional order space H^{1+s} for some $0 < s < 1$, where s and $\bar{p}$, given in section 2, depend on both the maximal interior angle of Ω and p. In short, for p large, s and $\bar{p}$ depend on the maximal interior angle, while for p close to 1, they depend on p.

In this paper we study the influence of the corner singularities imposed by the nonconvex polygonal domain Ω and the possible insufficient regularity of the right-hand side f, say, $f \in L_p(\Omega)$, $p < 2$, or $f \in H^{-\ell}(\Omega)$, $0 \leq \ell < 1/2$, on the convergence rate of the finite volume element method. For domains with smooth boundary and convex polygonal domains, H^1- and L_2-norm error estimates were derived in [15] and [23], respectively, taking into account the regularity of f.

Note that we use the conservative version of the method, namely the right-hand side of the scheme is computed by the L_2–inner product of f with the characteristic functions of the finite volumes (or equivalently by the duality between H^ℓ and $H^{-\ell}$ for $0 \leq \ell < 1/2$). The reason for $\ell < 1/2$ is that (1.4) makes sense for at least $f \in L_1$. For results concerning finite volume schemes for problems with more singular f, i.e., $f \in H^{-1}$, we refer to [19], where an approximation of $\int_b f$ is considered.

As a model for our analysis we shall consider the corresponding Galerkin finite element method, which is to find $\underline{u}_h \in X_h$ such that

$$a(\underline{u}_h, \chi) = (f, \chi), \quad \forall \chi \in X_h, \tag{1.5}$$

with $a(\cdot,\cdot)$ the bilinear form defined by

$$a(v, w) = \int_\Omega A\nabla v \cdot \nabla w \, dx.$$

It is known that $\underline{u}_h$ satisfies (cf., e.g., [3, 8] and [9, Chapter 12])

$$\|u - \underline{u}_h\| + h^\delta \|u - \underline{u}_h\|_{H^1} \leq Ch^{s+\delta} \begin{cases} \|u\|_{H^{1+s}}, \\ \|u\|_{W_{\bar{p}}^2}, \end{cases} \quad \text{any } \delta < \pi/\omega, \tag{1.6}$$

where s is given by (2.4) or (2.6), $\bar{p}$ by (2.3), and ω denotes the biggest interior angle of Ω (cf. section 2). Note that the convergence rate of the finite element method (1.5) is optimal in the H^1-norm and suboptimal in the L_2-norm, since X_h has the following approximation properties (cf., e.g., [9, p. 285]):

$$\inf_{\chi \in X_h} (\|v - \chi\| + h\|v - \chi\|_{H^1}) \leq \begin{cases} Ch^{1+s}\|v\|_{H^{1+s}}, & \forall v \in H^{1+s} \cap H_0^1, \ 0 < s < 1, \\ Ch^{3-2/p}\|v\|_{W_p^2}, & \forall v \in W_p^2 \cap H_0^1, \ 1 \leq p \leq 2. \end{cases} \tag{1.7}$$

In the literature there are various techniques for improving the convergence rate of a finite element method in nonconvex domains, e.g., mesh refinement, augmenting the basis functions with appropriate singular functions (cf., e.g., [8, 11]). Also, recently in [18] such a method was analyzed for some finite volume element methods. Here, we are interested in the analysis of (1.4) in a mesh T_h, which does not have any prior knowledge of the singularity imposed by the domain.

TABLE 1.1

Theoretical convergence rate of the finite volume element method versus the finite element method in a nonconvex polygonal domain, when the exact solution u of problem (1.1) is in H^{1+s}, where s is defined by (2.4) or (2.6), and any $\delta < \pi/\omega$.

$p_\omega = 2/(2-\pi/\omega)$, $s_0 = 1-\pi/\omega$		H^1-norm		L_2-norm		L_∞-norm	
$\tilde{p}_\omega = 2p_\omega/(3p_\omega-2)$		FVE	FE	FVE	FE	FVE	FE
$p_\omega < 2$	$1 < p < \tilde{p}_\omega$	s		$s+\delta$	$s+\delta$	$\approx s$	
$\tilde{p}_\omega < p_\omega$	$\tilde{p}_\omega < p < p_\omega$			$\min(1, s+\delta)$			
$f \in L_p$	$p_\omega \leq p$			1			
	$1 < \alpha < \tilde{p}_\omega$			$s+\delta$			
$f \in W_\alpha^t$	$\tilde{p}_\omega \leq \alpha \leq 2$			$\min(s+\delta, 1+t)$			
$s_0 < 1/2$	$\ell < s_0$			$1-\ell$			
$f \in H^{-\ell}$	$s_0 < \ell < 1/2$			$1-\ell$			

In Theorems 4.3 and 5.2, we show optimal order H^1-norm error estimates for the finite volume element method (1.4), if $f \in L_p$, $p > 1$, and $f \in H^{-\ell}$, $\ell \in (0, 1/2)$. Thus, the finite element (1.5) and finite volume element method (1.4) converge with the same rate in the H^1-norm.

However, as in the convex case (cf., e.g., [13, 27]), the situation in the L_2-norm error estimate is quite different. The convergence rate in the L_2-norm of the finite volume element method (1.4) is suboptimal and lower than the corresponding finite element method. In Theorem 4.3, for $f \in L_p$, $p > 1$, we show L_2-norm error estimations where the order cannot be higher than 1. However, assuming additional regularity for f, namely, $f \in W_\alpha^t$, $t \in (0, 1]$, $\alpha \in (1, 2]$, we are able to show, in Theorem 4.6, L_2-norm error estimations that, depending on α and t, could be of the same order as the finite element method. For example, this is true for α or t sufficiently close to 1. Also, in Theorem 4.8 we derive almost optimal order L_∞-norm error estimates.

In section 5, we consider the case where $f \in H^{-\ell}$, $\ell \in (0, 1/2)$ with $A = I$ and show optimal order H^1-norm, suboptimal L_2-norm, and almost optimal L_∞-norm error estimates. In Theorem 5.2, we show again that the convergence rate of the finite volume element method (1.4) in the L_2-norm is suboptimal and lower than the corresponding suboptimal rate of the finite element method.

In Table 1.1, we summarize the theoretical results concerning the convergence rate of the finite volume element method in the H^1-, L_2-, and L_∞-norms obtained in sections 4 and 5 and compare them with the corresponding known results for the finite element method. According to (1.6) the rate of the finite element method in the H^1-norm and L_2-norm is s and $s + \delta$, respectively, for any $\delta < \pi/\omega$ and s given by either (2.4) or (2.6), depending on whether $f \in L_p$ or $f \in H^{-\ell}$. Note that if we assume that $f \in W_\alpha^t$, with $t \in (0, 1]$ and $\alpha \in (1, 2]$, both methods give the same convergence rate, if $\alpha < \tilde{p}_\omega = 2p_\omega/(3p_\omega - 2)$ with $p_\omega = 2/(2 - \pi/\omega)$. Otherwise, this is determined by $\min(s + \pi/\omega, 1 + t)$.

Also, in section 7 we present some numerical results for Poisson's equation on a Γ-shaped domain. The particular examples we consider justify the theoretical results of Theorems 4.3, 4.6, and 4.8. However, these do not show the lower convergence rate in the L_2-norm of Theorem 4.3, which occurs if $f \in L_p$, $p > 1$, and $f \notin W_\alpha^t$, for any $\alpha \in (1, 2]$ and $t \in (0, 1)$. To show that the L_2-norm estimates of Theorems 4.3 and 5.2 are sharp, following [27], we consider two counterexamples.

A short presentation of parts of this work can be found in [14]. For simplicity we choose not to include convection terms in the differential equation (1.1). But they

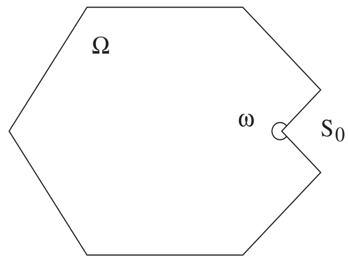

Fig. 2.1. *A nonconvex domain* Ω.

can be included provided they are bounded and the diffusion term is dominating. A brief description of this paper is the following: In section 2 we give, in short, known sharp regularity estimates for the exact solution of problems (1.1) and (2.1), based on [24, 4]. In section 3 we present the finite volume element method. In sections 4 and 5, we analyze the finite volume element method (1.4) and derive error estimates in the H^1-, L_2- and L_∞-norms. The approach follows the one developed in [13] and uses known sharp regularity results for the solutions of elliptic boundary value problems (cf. [24]). In section 6, we derive some auxiliary results, needed in proving Theorems 4.3, 4.6, 4.8, 5.2, 5.4, and 5.5. Finally in section 7, we present numerical examples that illustrate the theoretical results of section 4.

2. Preliminaries. Let us first consider the Dirichlet problem for Poisson's equation: Given $f \in L_p$, $p > 1$, find a function $u : \Omega \to \mathbb{R}^2$ such that

$$-\Delta u = f, \quad \text{in } \Omega, \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega \tag{2.1}$$

with Ω a bounded, nonconvex, polygonal domain in $\mathbb{R}^2$. For simplicity we assume that Ω has only one interior angle greater than π, namely $\omega \in (\pi, 2\pi)$ (cf. Figure 2.1). It is well known that for such domains there exists a unique solution $u \in H_0^1$ of (2.1).

The solution u could be represented in the form $u = u_S + u_R$, where $u_R \in W_p^2 \cap H_0^1$, and $u_S = cr^{\lambda_m} \frac{1}{\sqrt{\omega}\lambda_m} \sin(\lambda_m \theta)\eta(re^{i\theta})$, expressed in polar coordinates (r, θ) with respect to the vertex S_0 with angle ω (cf. [24]). Here c is a constant, $\lambda_m = \frac{m\pi}{\omega}$, $m \in \mathbb{N}$, and η is a cutoff function which is one near S_0 and zero away from S_0. A crucial role in determining the regularity of the solution u of (1.1) is played by the constant $p_\omega \equiv \frac{2}{2-\pi/\omega}$. According to [24, p. 233],

$$\text{if } f \in L_p,\ p > 1, \quad \text{then } u \in W_{\bar{p}}^2, \tag{2.2}$$

where

$$\bar{p} = \begin{cases} p, & p < p_\omega, \\ \gamma, \quad \text{any } \gamma < p_\omega, & p \geq p_\omega, \end{cases} \qquad p_\omega \equiv \frac{2}{2 - \pi/\omega}. \tag{2.3}$$

Using also the imbedding $W_{\bar{p}}^2 \subset H^{1+s}$, for $s = 2 - 2/\bar{p}$ (cf., e.g., [24, p. 34]), we obtain the following:

$$\text{if } f \in L_p,\ p > 1, \quad \text{then } u \in H^{1+s},\ \text{for } s = 2 - \frac{2}{\bar{p}}. \tag{2.4}$$

Also, for problem (2.1) we have (cf., e.g., [4]),

$$\text{if } f \in H^{-\ell},\ 0 \leq \ell \leq 1, \quad \text{then } u \in H^{1+s}, \tag{2.5}$$

where

$$s = \begin{cases} 1-\ell, & s_0 < \ell \le 1, \\ \delta, \quad \text{any } \delta < \pi/\omega, & 0 \le \ell \le s_0, \end{cases} \qquad s_0 = \frac{2}{p_0} - 1 = 1 - \frac{\pi}{\omega}. \tag{2.6}$$

For the more general problem (1.1), similar results hold. Let S be a vertex of Ω, and denote the corresponding interior angle of Ω by $\omega(S)$. Let $\mathcal{A}$ and $\mathcal{T}$ be matrices such that $\mathcal{A} = (a_{ij}(S))_{i,j=1}^2$ and $-\mathcal{T}\mathcal{A}\mathcal{T}^T = I$, and let $\omega_A(S)$ be the angle at the vertex $\mathcal{T}\mathcal{S}$ of the transformed domain $\mathcal{T}\Omega = \{\mathcal{T}x : x \in \Omega\}$. Define

$$\omega = \max_S \omega_A(S) \quad \text{and} \quad p_\omega = \frac{2}{2-\pi/\omega}.$$

3. The finite volume element method. In order to analyze the finite volume element method (1.4) we shall need to rewrite it in a variational form resembling the one for the finite element problem (1.5) (cf., e.g., [13]). For this purpose we introduce the space

$$Y_h = \{\eta \in L_2(\Omega) : \ \eta|_{b_z} \text{ is constant}, \quad z \in Z_h^0, \ \eta|_{b_z} = 0 \text{ if } z \in \partial\Omega\}.$$

For an arbitrary $\eta \in Y_h$ we multiply the integral relation (1.4) by $\eta(z)$ and sum over all $z \in Z_h^0$. Thus we obtain the following Petrov–Galerkin formulation of the finite volume element method: Find $u_h \in X_h$ such that

$$a_h(u_h, \eta) = (f, \eta), \quad \forall \eta \in Y_h, \tag{3.1}$$

where the bilinear form $a_h(\cdot,\cdot) : X_h \times Y_h \to \mathbb{R}$ is defined by

$$a_h(v, \eta) = -\sum_{z\in Z_h^0} \eta(z) \int_{\partial b_z} (A\nabla v)\cdot n \, ds, \quad v \in X_h, \ \eta \in Y_h. \tag{3.2}$$

Further, we consider the interpolation operator $I_h : C(\Omega) \to Y_h$, defined by

$$I_h v = \sum_{z\in Z_h^0} v(z)\varphi_z, \tag{3.3}$$

where φ_z is the characteristic function of b_z. Then, we can rewrite (1.4) as

$$a_h(u_h, I_h\chi) = \sum_{z\in Z_h^0} \chi(z) \int_{b_z} f\,dx, \quad \forall \chi \in X_h. \tag{3.4}$$

Note that for every $f \in L_p$ and $\chi \in X_h$,

$$(f, I_h\chi) = \sum_{z\in Z_h} \chi(z) \int_\Omega f\varphi_z\,dx = \sum_{z\in Z_h^0} \chi(z) \int_{b_z} f\,dx. \tag{3.5}$$

Thus (3.4) can be written equivalently in the form

$$a_h(u_h, I_h\chi) = (f, I_h\chi), \quad \forall \chi \in X_h. \tag{3.6}$$

Existence of u_h follows from the fact that a_h is coercive, for h sufficiently small (cf., e.g., [13] or [28, Theorem 3.2.1]),

$$\exists\, c_0 > 0 : \quad c_0|\chi|_{H^1}^2 \le a_h(\chi, I_h\chi), \quad \forall \chi \in X_h.$$

Then this, the local stability of I_h,

$$\|I_h\chi\|_{L_p(K)} \le C\|\chi\|_{L_p(K)}, \quad \forall \chi \in X_h,\ K \in T_h,\ p > 1,$$

and the Sobolev imbedding

$$\|\chi\|_{L_p} \le C\|\chi\|_{H^1}, \quad \forall \chi \in X_h,\ p > 1,$$

give the stability of the finite volume scheme (3.6),

$$\|u_h\|_{H^1} \le C\|f\|_{L_p}, \quad p > 1. \tag{3.7}$$

Also, note that if $A(x)$ is a constant matrix over each finite element $K \in T_h$, then $a_h(\chi, I_h\psi) = a(\chi, \psi)$, $\forall \chi, \psi \in X_h$ (cf., e.g., [27]). In particular, if $A = I$, we have

$$a_h(\chi, I_h\psi) = a(\chi, \psi) = \int_\Omega \nabla\chi \cdot \nabla\psi\, dx, \quad \forall \chi, \psi \in X_h \tag{3.8}$$

(cf., e.g., [6]). Thus, (3.6) takes the form

$$a(u_h, \chi) = (f, I_h\chi), \quad \forall \chi \in X_h. \tag{3.9}$$

In the case of general matrix $A(x)$, the identity (3.8) is not valid. However, following [13], we are able to rewrite a_h in a form similar to a. Indeed, we transform the left-hand side of (1.4) using integration by parts to get, for $z \in Z_h^0$ and $K \in T_h$,

$$\int_{K_z} L\chi\, dx + \int_{\partial K_z \cap \partial K} A\nabla\chi \cdot n\, ds = -\int_{\partial K_z \cap \partial b_z} A\nabla\chi \cdot n\, ds, \quad \forall \chi \in X_h.$$

Thus, multiplying by $\psi(z)$, $\psi \in X_h$, and summing over the triangles having z as a vertex and then over the vertices $z \in Z_h^0$, we obtain

$$a_h(\chi, I_h\psi) = \sum_K \{(L\chi, I_h\psi)_K + (A\nabla\chi \cdot n, I_h\psi)_{\partial K}\}, \quad \forall \chi, \psi \in X_h. \tag{3.10}$$

This is similar to

$$a(\chi, \psi) \equiv (A\nabla\chi, \nabla\psi) = \sum_K \{(L\chi, \psi)_K + (A\nabla\chi \cdot n, \psi)_{\partial K}\}, \quad \forall \chi, \psi \in X_h.$$

Due to this similarity and for convenience, in what follows we shall use (3.10) as a definition of the bilinear form a_h.

4. Nonsmooth data: L_p case. In this section we shall derive H^1-, L_2-, and L_∞-norm estimates of the error $u - u_h$ for $f \in L_p$, $p > 1$. First, we shall demonstrate that the finite element method (1.5) and finite volume element method (3.6) have the same convergence rate in the H^1-norm. The L_2-norm error estimate is quite different, and we derive two separate results. First, we will show suboptimal order L_2-norm error estimates for $f \in L_p$, $p > 1$, where the order is less than in the corresponding order for the finite element scheme (1.5). Next, assuming higher regularity for f, namely $f \in W_\alpha^t$, $t \in (0, 1]$, $\alpha \in (1, 2]$, we will show again suboptimal order L_2-norm error estimates, but now depending on α and t, these could be of the same order as the corresponding estimates of the finite element scheme. Finally, we show almost optimal L_∞-norm estimates of the error $u - u_h$.

For the analysis of the finite volume element method (3.6) we shall need to estimate the errors ε_h and ε_a defined by

$$\begin{aligned}\varepsilon_h(f,\chi) &= (f,\chi) - (f, I_h\chi), && \forall f \in L_p,\ \chi \in X_h,\\ \varepsilon_a(\chi,\psi) &= a(\chi,\psi) - a_h(\chi, I_h\psi), && \forall \chi,\psi \in X_h.\end{aligned}$$

In section 6 we will give the proof of the following two lemmas.

LEMMA 4.1. *There exists a constant C such that for every $\chi \in X_h$,*

$$|\varepsilon_h(f,\chi)| \le Ch\|f\|_{L_p} |\chi|_{W^1_{p'}}, \qquad \forall f \in L_p,\ \frac{1}{p}+\frac{1}{p'} = 1, \tag{4.1}$$

$$|\varepsilon_h(f,\chi)| \le Ch^{1+t}\|f\|_{W^t_p} |\chi|_{W^1_{p'}}, \quad \forall f \in W^t_p,\ 0 < t \le 1. \tag{4.2}$$

LEMMA 4.2. *Assume that $A \in W^2_\infty$. Then there exists a positive constant $C = C(A)$ such that*

$$|\varepsilon_a(\psi,\chi)| \le Ch|\psi|_{W^1_p} |\chi|_{W^1_{p'}}, \qquad \forall \chi,\psi \in X_h, \tag{4.3}$$

$$|\varepsilon_a(u_h,\chi)| \le Ch\big(\|\nabla(u-u_h)\|_{L_2} + h\|u\|_{W^2_{\bar p}}\big)|\chi|_{W^1_{\bar p'}}, \quad \forall \chi \in X_h. \tag{4.4}$$

Next, we derive H^1- and L_2-norm error estimates for the finite volume element method (1.4).

THEOREM 4.3. *Let u and u_h be the solutions of* (1.1) *and* (1.4)*, respectively, with $f \in L_p$, $p > 1$. Then, there exists a constant C, independent of h, such that*

$$\|u-u_h\|_{H^1} \le C\big(h^s\|u\|_{W^2_{\bar p}} + h^{\min(1,2-2/p)}\|f\|_{L_p}\big) \le Ch^s\|f\|_{L_p}, \tag{4.5}$$

$$\|u-u_h\| \le C\big(h^{s+\delta}\|u\|_{W^2_{\bar p}} + h^{\min(1,s+\delta)}\|f\|_{L_p}\big), \quad \textit{for any } \delta < \pi/\omega, \tag{4.6}$$

with $\bar p$ and s given by (2.3) *and* (2.4)*, respectively.*

Remark 4.4. The H^1-norm error estimation (4.5) is of optimal order (cf. (1.7)). However, the L_2-norm error estimation is not of the same order as the finite element approximation (cf. (1.6)) for every p. For example, for p sufficiently close to 1, $s+\delta < 1$, thus, $\|u-u_h\| = O(h^{s+\delta})$. However, for $p \ge 2$, $s = 2-2/\bar p \approx \pi/\omega$. Therefore, since $s+\delta \approx 2\pi/\omega > 1$, $\|u-u_h\| = O(h)$. The most interesting outcome of this theorem is that the convergence rate for the L_2-norm is suboptimal and lower than the rate of the finite element method (1.5). This estimate is sharp, as first demonstrated by a counterexample in [27], for convex domains. Later in section 7 we give a similar example to the one in [27], which shows the sharpness of the L_2-error estimate (4.6).

Proof. In view of (2.2), $u \in W^2_{\bar p}$, with $\bar p$ defined by (2.3). Using the triangle inequality,

$$\|u-u_h\|_{H^1} \le \|u-\chi\|_{H^1} + \|u_h-\chi\|_{H^1}, \quad \forall \chi \in X_h, \tag{4.7}$$

and the approximation properties (1.7) of X_h, it suffices to consider the last term of (4.7). The positive definiteness of A, (1.2), gives

$$\alpha_0\|u_h-\chi\|^2_{H^1} \le a(u_h-\chi, u_h-\chi), \quad \forall \chi \in X_h. \tag{4.8}$$

Thus, in view of

$$\begin{aligned} a(u_h-\chi,u_h-\chi) &= a(u-u_h,u_h-\chi)+a(u-\chi,u_h-\chi)\\ &\le a(u-u_h,u_h-\chi)+C\|u-\chi\|_{H^1}\,\|u_h-\chi\|_{H^1}, \quad \forall\chi\in X_h, \end{aligned}$$

and (4.8), we get for every $\chi \in X_h$,

$$\|u_h-\chi\|^2_{H^1} \le C|a(u-u_h,u_h-\chi)|+C\|u-\chi\|^2_{H^1}. \tag{4.9}$$

In addition, using the definitions of ε_h and ε_a, we have

$$\begin{aligned} a(u-u_h,u_h-\chi) &= a(u,u_h-\chi)-a_h(u_h,I_h(u_h-\chi))-\varepsilon_a(u_h,u_h-\chi)\\ &= \varepsilon_h(f,u_h-\chi)-\varepsilon_a(u_h,u_h-\chi), \quad \forall\chi\in X_h. \end{aligned} \tag{4.10}$$

Applying then, to this relation, (4.1), (4.3), and the inverse inequality

$$|\chi|_{W^1_{p'}} \le Ch^{2/p'-1}|\chi|_{H^1}, \quad p'>2,\ \forall\chi\in X_h,$$

we obtain

$$|a(u-u_h,u_h-\chi)| \le C(h^{\min(1,2-2/p)}\|f\|_{L_p}+h\|u_h\|_{H^1})\|u_h-\chi\|_{H^1}. \tag{4.11}$$

Thus, for h sufficiently small, this estimate, (3.7) and (4.9) yield

$$\|u_h-\chi\|_{H^1} \le C\|u-\chi\|_{H^1}+Ch^{\min(1,2-2/p)}\|f\|_{L_p}, \quad \forall\chi\in X_h, \tag{4.12}$$

which combined with (1.7) and (4.7) gives

$$\|u-u_h\|_{H^1} \le C\big(h^s\|u\|_{W^2_{\bar p}}+h^{\min(1,2-2/p)}\|f\|_{L_p}\big).$$

Using now the fact that for $p<p_\omega$, $s=2-2/p$ and for $p\ge p_\omega$, $s<\min(1,2-2/p)$, we get

$$\|u-u_h\|_{H^1} \le Ch^s\big(\|u\|_{W^2_{\bar p}}+\|f\|_{L_p}\big). \tag{4.13}$$

Finally, employing the a priori regularity estimation of u, (2.2), we obtain the desired estimate (4.5).

We now prove (4.6) by using a duality argument. We consider the following auxiliary problem: Seek $\varphi\in H^1_0$ such that

$$L\varphi = u-u_h \quad \text{in } \Omega \quad \text{and} \quad \varphi=0 \quad \text{on } \partial\Omega. \tag{4.14}$$

In view of (2.2) and the fact that $u-u_h\in L_2$, we have $\varphi\in W^2_\gamma$, where $\gamma<p_\omega$, i.e. $2/\gamma = 2-\pi/\omega+\varepsilon$, with arbitrary small $\varepsilon>0$, and satisfies the a priori estimate

$$\|\varphi\|_{W^2_\gamma} \le C\|u-u_h\|, \quad \gamma<p_\omega. \tag{4.15}$$

Now let $\Pi_h : W^2_\gamma\cap H^1_0 \to X_h$ denote the standard nodal interpolation operator. It is well known that Π_h has the following approximation property (cf., e.g., [17, Theorem 3.1.6] and [2, Theorem 5.4]),

$$\|\Pi_h v-v\|_{H^1} \le Ch^{\pi/\omega-\varepsilon}\|v\|_{W^2_\gamma}, \quad \forall v\in W^2_\gamma\cap H^1_0, \tag{4.16}$$

and Π_h is bounded in $\|\cdot\|_{W_q^1}$ (cf., e.g., [17, Theorem 3.1.6] and [2, Theorem 5.4]),

$$\|\Pi_h v\|_{W_q^1} \leq C\|v\|_{W_\gamma^2}, \quad \forall v \in W_\gamma^2 \cap H_0^1, \quad q \leq p'_\gamma = 2\gamma/(2-\gamma), \tag{4.17}$$

where $p_\gamma = 2\gamma/(3\gamma - 2)$.

Using (4.14) and Green's formula, we easily obtain

$$\begin{aligned} \|u - u_h\|^2 &= -(u - u_h, L\varphi) = a(u - u_h, \varphi) \\ &= a(u - u_h, \varphi - \Pi_h\varphi) + a(u - u_h, \Pi_h\varphi) := I + II. \end{aligned} \tag{4.18}$$

The first term, I, can obviously be bounded in the following way by using (4.13) and (4.16):

$$|I| \leq C\|u - u_h\|_{H^1} \|\varphi - \Pi_h\varphi\|_{H^1} \leq Ch^{s+\pi/\omega-\varepsilon}\big(\|u\|_{W_{\bar p}^2} + \|f\|_{L_p}\big) \|\varphi\|_{W_\gamma^2}. \tag{4.19}$$

Also, in view of (4.10), the second term, II, can be written in the form

$$II = a(u - u_h, \Pi_h\varphi) = \varepsilon_h(f, \Pi_h\varphi) - \varepsilon_a(u_h, \Pi_h\varphi). \tag{4.20}$$

Then using (4.1) and (4.4), II can be estimated by

$$|II| \leq Ch\|f\|_{L_p} |\Pi_h\varphi|_{W_{p'}^1} + h\big(\|\nabla(u - u_h)\|_{L_2} + h\|u\|_{W_{\bar p}^2}\big) |\Pi_h\varphi|_{W_{\bar p'}^1}. \tag{4.21}$$

In order to bound $|\Pi_h\varphi|_{W_{p'}^1}$ and $|\Pi_h\varphi|_{W_{\bar p'}^1}$ in (4.21) we consider two different cases for p: (1) $p \geq p_\gamma = 2\gamma/(3\gamma - 2)$, and (2) $1 < p < p_\gamma$. We can easily see that $p_\gamma < p_\omega$. Thus, in view of the definition of $\bar p$ (cf. (2.3)), for $p \geq p_\gamma$ we also have $\bar p \geq p_\gamma$ and for $1 < p < p_\gamma$, $\bar p < p_\gamma$.

Let us first consider the case $p \geq p_\gamma$. Then we have $p' \leq p'_\gamma$ and $\bar p' \leq p'_\gamma$ so for the respective norms of $\Pi_h\varphi$ in (4.21) we can apply the estimate (4.17). Using also (4.13) we get

$$\begin{aligned} |II| &\leq C(h\|f\|_{L_p} + h^{1+s}(\|u\|_{W_{\bar p}^2} + \|f\|_{L_p}))\|\varphi\|_{W_\gamma^2} \\ &\leq C(h\|f\|_{L_p} + h^{1+s}\|u\|_{W_{\bar p}^2})\|\varphi\|_{W_\gamma^2}. \end{aligned} \tag{4.22}$$

Combining now this estimation with (4.19), (4.15), and (4.18), we obtain the desired estimate (4.6), for $p \geq p_\gamma$.

In the remaining case $1 < p < p_\gamma$ we cannot directly employ (4.17) for the estimation of II. However, the inverse inequality

$$|\chi|_{W_q^1} \leq Ch^{2/q - 2/p'_\gamma} |\chi|_{W_{p'_\gamma}^1}, \quad q > p'_\gamma, \quad \forall \chi \in X_h,$$

and (4.17), give

$$|\Pi_h v|_{W_q^1} \leq Ch^{2/q - 1 + \pi/\omega - \varepsilon}\|v\|_{W_\gamma^2}, \quad \forall v \in W_\gamma^2 \cap H_0^1,\ q > p'_\gamma. \tag{4.23}$$

Using now this estimation in (4.21) and the fact that for $1 < p < p_\gamma$, $2/p' = 2 - 2/p = s$, we get

$$\begin{aligned} |II| &\leq Ch^{2/p' + \pi/\omega - \varepsilon}\big(\|f\|_{L_p} + h\|u\|_{W_{\bar p}^2}\big) \|\varphi\|_{W_\gamma^2} \\ &\leq Ch^{s + \pi/\omega - \varepsilon}\big(\|f\|_{L_p} + h^s\|u\|_{W_{\bar p}^2}\big) \|\varphi\|_{W_\gamma^2}. \end{aligned} \tag{4.24}$$

Then, combining this estimation with (4.19), (4.15), and (4.18), we obtain

$$\|u - u_h\| \leq Ch^{s+\delta}(\|u\|_{W^2_{\bar{p}}} + \|f\|_{L_p}).$$

Finally, (4.6) follows from the fact that for $1 < p < p_\gamma$, $s = 2 - \frac{2}{p} < 2 - \frac{2}{p_\gamma} = \frac{2}{\gamma} - 1 = 1 - \frac{\pi}{\omega} + \varepsilon$. □

Remark 4.5. For the proof of Theorems 4.3 and 4.6 it is not necessary to assume a quasi-uniform mesh T_h. This is done in order to simplify the proof, and it is only required for the validity of the inverse inequalities that are used. This assumption can be avoided by applying local inverse inequalities which hold in more general triangulations.

Next, we shall demonstrate that under some additional assumptions on the smoothness of the data the convergence rate in the L_2-norm can be improved and be equal to the rate of the corresponding finite element method.

THEOREM 4.6. *Let u and u_h be the solutions of* (1.1) *and* (1.4)*, respectively. Assume that $f \in W^t_\alpha$, $1 < \alpha \leq 2$, $0 < t \leq 1$, and $A \in W^2_\infty$. Then there exists a constant C, independent of h, such that*

$$\|u - u_h\| \leq C(h^{s+\delta}\|f\|_{L_p} + h^{1+t+\min(0,1-2/\alpha+\delta)}\|f\|_{W^t_a}), \quad \textit{for any } \delta < \pi/\omega, \tag{4.25}$$

with $p = 2\alpha/(2 - t\alpha)$ and $\bar{p}$ and s given by (2.3) *and* (2.4)*, respectively.*

Remark 4.7. For $\alpha < \tilde{p}_\omega = \frac{2p_\omega}{3p_\omega - 2}$, we have $2/\alpha > 1 + \pi/\omega$. Thus $1 + t + \min(0, 1 - 2/\alpha + \delta) = 2 + t - 2/\alpha + \delta = 2 - 2/p + \delta \geq s + \delta$. Therefore, $\|u - u_h\| = O(h^{s+\delta})$, i.e., in this case the L_2-norm error estimate of the finite volume element method has the same convergence rate as the corresponding finite element method. If $\alpha \geq \tilde{p}_\omega$, i.e., $2/\alpha \leq 1 + \delta$, then the order of $\|u - u_h\|$ is $\min(s + \delta, 1 + t)$.

Proof. The proof will be similar to the one for (4.6). First, let us note that since $f \in W^t_\alpha$, we have by imbedding (cf. [2, Theorem 7.57]) that $f \in L_p$, with $p = 2\alpha/(2 - t\alpha)$. Thus, in view of (2.2), $u \in W^2_{\bar{p}}$ with $\bar{p}$ given by (2.3).

Let again $\gamma < p_\omega$, such that $2/\gamma = 2 - \pi/\omega + \varepsilon$, with arbitrary small $\varepsilon > 0$, and let $\varphi \in W^2_\gamma \cap H^1_0$ be the solution of the auxiliary problem (4.14). Obviously, in order to show a higher order L_2-norm error estimation of $u - u_h$, we need to derive "better" bounds for I and II of (4.18). It is obvious that the estimation of I, (4.19), derived in Theorem 4.3 is of the desired order. Thus, it suffices to show a better estimate for II than the ones derived in Theorem 4.3 (cf. (4.22) and (4.24)).

Using (4.2) and (4.4) in (4.20), we get

$$|II| \leq C(h^{1+t}\|f\|_{W^t_\alpha}|\Pi_h\varphi|_{W^1_{\alpha'}} + h^{1+s}\|f\|_{L_p}|\Pi_h\varphi|_{W^1_{\bar{p}'}}). \tag{4.26}$$

Similarly, as in Theorem 4.3 we need to derive bounds for

$$|\Pi_h\varphi|_{W^1_{\alpha'}} \quad \text{and} \quad |\Pi_h\varphi|_{W^1_{\bar{p}'}},$$

and we will need to consider various cases for α and p with respect to $p_\gamma = 2\gamma/(3\gamma - 2)$.

Since $p = 2\alpha/(2 - t\alpha)$, we can easily see that $p > \alpha$; thus we have the following three cases: (1) $p > \alpha \geq p_\gamma$, (2) $p \geq p_\gamma > \alpha$, and (3) $p_\gamma > p > \alpha$.

First, we consider the case $p > \alpha \geq p_\gamma$. For such p, according to (2.3), we have $\bar{p} > p_\gamma$. Thus, using (4.17) in (4.26), and the fact that $1/2 < \pi/\omega < 1$, we get

$$|II| \leq C(h^{1+t}\|f\|_{W^t_\alpha} + h^{s+\pi/\omega-\varepsilon}\|f\|_{L_p})\|\varphi\|_{W^2_\gamma}. \tag{4.27}$$

Therefore, combining this estimation, (4.19), (4.15), (4.18), and the fact that if $\alpha > p_\gamma$, then $2/\alpha < 2/p_\gamma = 1 + \pi/\omega - \varepsilon$, we obtain the desired result, (4.25), in the case $p > \alpha \geq p_\gamma$.

Now let $p \geq p_\gamma > \alpha$. Again we can easily see that $\bar{p} \geq p_\gamma$. Therefore, applying (4.23) and (4.17) in (4.26) and using the fact that $1/2 < \pi/\omega < 1$, we obtain

$$\begin{aligned} |II| &\leq C(h^{t+2/\alpha'+\pi/\omega-\varepsilon}\|f\|_{W^t_\alpha} + h^{1+s}\|f\|_{L_p})\|\varphi\|_{W^2_\gamma} \\ &\leq C(h^{1+t+1-2/\alpha+\pi/\omega-\varepsilon}\|f\|_{W^t_\alpha} + h^{s+\pi/\omega-\varepsilon}\|f\|_{L_p})\|\varphi\|_{W^2_\gamma}. \end{aligned} \tag{4.28}$$

Therefore, combining this estimation, (4.19), (4.15), (4.18), and the fact that if $\alpha \leq p_\gamma$, then $2/\alpha > 1 + \pi/\omega - \varepsilon$, we obtain the desired result, (4.25), if $p > p_\gamma \geq \alpha$.

In the remaining case $p_\gamma > p > \alpha$, we have $\bar{p} = p < p_\gamma$. Thus, applying (4.23) in (4.26) and using the fact that $1/2 < \pi/\omega < 1$, we have

$$\begin{aligned} |II| &\leq C(h^{t+2/\alpha'+\pi/\omega-\varepsilon}\|f\|_{W^t_\alpha} + h^{2/p'+\pi/\omega-\varepsilon+s}\|f\|_{L_p})\|\varphi\|_{W^2_\gamma} \\ &\leq C(h^{1+t+1-2/\alpha+\pi/\omega-\varepsilon}\|f\|_{W^t_\alpha} + h^{s+\pi/\omega-\varepsilon}\|f\|_{L_p})\|\varphi\|_{W^2_\gamma}. \end{aligned} \tag{4.29}$$

Therefore, combining this estimation, (4.19), (4.15), and (4.18) we obtain the desired result, (4.25), for the remaining case $p_\gamma \geq p > \alpha$. ∎

Finally, we will show an almost optimal L_∞-norm error estimate.

THEOREM 4.8. *Let u and u_h be the solutions of* (1.1) *and* (1.4)*, respectively, with $f \in L_p$, $p > 1$, and $A \in W^2_\infty$. Then there exists a constant C, independent of h, such that*

$$\|u - u_h\|_{L_\infty} \leq Ch^s \log\frac{1}{h}\|f\|_{L_p}. \tag{4.30}$$

Proof. We split the error $u - u_h$ by adding and subtracting the Galerkin finite element approximation $\underline{u}_h$ (cf. (1.5)); thus $u - u_h = (u - \underline{u}_h) + (\underline{u}_h - u_h)$. The estimation of $\|u - \underline{u}_h\|_{L_\infty}$ is well known (cf., e.g., [32]). However, we shall briefly demonstrate it.

In view of [32, equation (0.8)] and the standard imbedding $W^2_{\bar{p}} \subset C^{0,2-2/\bar{p}}$ (cf., e.g., [24, Theorem 1.4.5.2]), we have

$$\|u - \underline{u}_h\|_{L_\infty} \leq Ch^s \log\frac{1}{h}\|u\|_{C^{0,s}} \leq Ch^s \log\frac{1}{h}\|u\|_{W^2_{\bar{p}}},$$

where $s = 2 - 2/\bar{p}$ and $C^{m,\ell}$ is the space of m times continuously differentiable functions whose mth order derivative fulfills a uniform Hölder condition of order ℓ.

Then, combining this with the elliptic regularity estimate,

$$\|u\|_{W^2_{\bar{p}}} \leq C_{\bar{p}}\|f\|_{L_p} \tag{4.31}$$

(cf. [24, Theorem 5.2.7]), we obtain

$$\|u - \underline{u}_h\|_{L_\infty} \leq C_{\bar{p}} h^s \log\frac{1}{h}\|f\|_{L_p}, \quad p > 1. \tag{4.32}$$

We turn now to the estimation of $\|\underline{u}_h - u_h\|_{L_\infty}$. Let $x_0 \in K_0 \in \mathcal{T}_h$ such that $\|\underline{u}_h - u_h\|_{L_\infty} = |(\underline{u}_h - u_h)(x_0)|$ and $\delta_{x_0} = \delta \in C^\infty_0(\Omega)$ a regularized Dirac δ-function satisfying

$$(\delta, \chi) = \chi(x_0), \quad \forall \chi \in X_h.$$

For such a function δ (cf., e.g., [9]) we have

$$\operatorname{supp}\delta \subset B = \{x \in \Omega : |x - x_0| \le h/2\}, \quad \int_\Omega \delta = 1, \quad 0 \le \delta \le Ch^{-2},$$
$$\|\delta\|_{L_p} \le Ch^{2(1-p)/p}, \quad 1 < p < \infty.$$

Also let us consider the corresponding regularized Green's function $G \in H_0^1$, defined by

$$a(G, v) = (\delta, v), \quad \forall v \in H_0^1. \tag{4.33}$$

Then, we have

$$\begin{aligned} \|\underline{u}_h - u_h\|_{L_\infty} &= (\delta, \underline{u}_h - u_h) = a(G, \underline{u}_h - u_h) = a(G_h, \underline{u}_h - u_h) \\ &= a(u - u_h, G_h) = \varepsilon_h(f, G_h) - \varepsilon_a(u_h, G_h), \end{aligned} \tag{4.34}$$

where $G_h \in X_h$ is the finite element approximation of G, i.e.,

$$a(G, \chi) = a(G_h, \chi), \quad \forall \chi \in X_h.$$

Further, using (4.1), (4.3), and the inverse inequality

$$|\chi|_{W_q^1} \le Ch^{2/q-1}|\chi|_{H^1}, \quad \forall \chi \in X_h,\ q > 2,$$

in (4.34) we obtain

$$\begin{aligned} \|\underline{u}_h - u_h\|_{L_\infty} &\le C\big\{h\big(\|\nabla(u - u_h)\|_{L_2} + h\|u\|_{W_{\bar{p}}^2}\big)|G_h|_{W_{\bar{p}'}^1} + h\|f\|_{L_p}|G_h|_{W_{p'}^1}\big\} \\ &\le C\big\{h^{2-2/\bar{p}}\big(\|\nabla(u - u_h)\|_{L_2} + h\|u\|_{W_{\bar{p}}^2}\big) \\ &\quad + h^{\min(1,2-2/p)}\|f\|_{L_p}\big\}\|G_h\|_{H^1}, \end{aligned} \tag{4.35}$$

with $p > 1$. In addition, in view of [31, Lemma 3.1] we get

$$\|G_h\|_{H^1} \le C\|\nabla G\|_{L_2} \le C\frac{1}{(q-1)^{1/2}}\,\|\delta\|_{L_q} \tag{4.36}$$

with $q \downarrow 1$. Choosing now $q = 1 + (\log \frac{1}{h})^{-1}$ we have

$$\|G_h\|_{H^1} \le C\left(\log\frac{1}{h}\right)^{1/2}. \tag{4.37}$$

Combining now (4.34)–(4.37) and Theorem 4.3, we obtain

$$\|\underline{u}_h - u_h\|_{L_\infty} \le Ch^{2s}\left(\log\frac{1}{h}\right)^{1/2}\|u\|_{W_{\bar{p}}^2} + Ch^{\min(1,2-2/p)}\left(\log\frac{1}{h}\right)^{1/2}\|f\|_{L_p}. \tag{4.38}$$

From this, (4.31), and (4.32) we get the desired estimation (4.30). $\square$

Remark 4.9. Assuming $f \in L_\infty$ will not improve the convergence rate in (4.30), we can easily see that in this case (4.38) does not contribute terms of order higher than 1. However, (4.32) gives terms of order almost $2-2/\bar{p}$, which is less than 1. Also, if we assume $f \in W_\alpha^t$, then similarly as in Theorem 4.6 we can show $\|\underline{u}_h - u_h\|_{L_\infty} = O(h^{1+t})$, but again the error $\|u - \underline{u}_h\|_{L_\infty}$ will be at most of order $2 - 2/\bar{p}$.

5. Nonsmooth data: $\boldsymbol{H^{-\ell}}$ case. In this section we will consider problem (2.1), i.e., $A = I$, and we shall derive H^1-, L_2- and L_∞-norm estimates of the error $u - u_h$ for $f \in H^{-\ell}$, $\ell \in (0, 1/2)$. We will show optimal H^1-, suboptimal L_2-, and almost optimal L_∞-norm error estimates. The H^1- and L_∞-norm estimations are of the same order with the corresponding estimations for the finite element scheme, whereas the L_2-norm estimates are smaller.

This time for the analysis of the finite volume element method (3.6) we shall need in addition the following lemma, which we prove in section 6.

LEMMA 5.1. *There exists a constant C such that for every $\chi \in X_h$,*

$$|\varepsilon_h(f,\chi)| \le Ch^{1-\ell}\|f\|_{H^{-\ell}}\,|\chi|_{H^1}, \quad \forall f \in H^{-\ell},\ 0 < \ell < 1/2. \tag{5.1}$$

THEOREM 5.2. *Let u and u_h be the solutions of* (2.1) *and* (1.4)*, respectively, with $f \in H^{-\ell}$, $0 \le \ell < 1/2$. Then there exists a constant C, independent of h, such that*

$$\|u - u_h\|_{H^1} \le C\big(h^s\|u\|_{H^{1+s}} + h^{1-\ell}\|f\|_{H^{-\ell}}\big) \le Ch^s\|f\|_{H^{-\ell}}, \tag{5.2}$$

$$\|u - u_h\| \le C\big(h^{s+\delta}\|u\|_{H^{1+s}} + h^{1-\ell}\|f\|_{H^{-\ell}}\big), \quad \textit{any } \delta < \pi/\omega. \tag{5.3}$$

Remark 5.3. The convergence rate of the H^1-norm is of optimal order (cf. (1.7)). However, since $s + \delta > 1 \ge 1 - \ell$, for δ arbitrarily close to π/ω and $\delta < \pi/\omega$, the convergence rate in the L_2-norm is suboptimal and lower than the rate of the corresponding finite element method (cf. (1.5)). Later in section 7 we give an example similar to the one in [27], which shows the sharpness of the L_2-error estimate (5.3).

Proof. The proof is similar as in Theorem 4.3, thus it suffices to estimate the first term of the right-hand side of (4.9). If $f \in H^{-\ell}$, with $0 < \ell < 1/2$, then in view of (2.5), $u \in H^{1+s}$, with s defined by (2.6). Since $A = I$, $\varepsilon_a \equiv 0$. Therefore, using (5.1) in (4.10) we obtain

$$|a(u - u_h, u_h - \chi)| \le Ch^{1-\ell}\|f\|_{H^{-\ell}}|u_h - \chi|_{H^1}, \quad \forall \chi \in X_h. \tag{5.4}$$

Then, in view of the approximation property (1.7) of X_h we get

$$\|u - u_h\|_{H^1} \le C\big(h^s\|u\|_{H^{1+s}} + h^{1-\ell}\|f\|_{H^{-\ell}}\big).$$

Using now the fact that for $s_0 < \ell < 1/2$, $s = 1 - \ell$ (cf. (2.6)), and for $0 \le \ell \le s_0$, $s < 1 - \ell$, and the a priori regularity estimate (2.5), we obtain the desired estimate (5.2).

We now turn to (5.3). Using again the same arguments as in Theorem 4.3 it suffices to estimate term II of (4.18). Let again $\gamma < p_\omega$, such that $2/\gamma = 2 - \pi/\omega + \varepsilon$, with arbitrarily small $\varepsilon > 0$, and let $\varphi \in W^2_\gamma \cap H^1_0$ be the solution of the auxiliary problem (4.14). Combining (4.10) and (5.1), we have

$$|II| \le Ch^{1-\ell}\|f\|_{H^{-\ell}}|\Pi_h\varphi|_{H^1}. \tag{5.5}$$

Finally, since, $p_\gamma < 2$, we can employ (4.17) in the estimation above, and then combining (4.18), (4.16), (5.2), and (4.15), we obtain the desired estimate (5.3). $\square$

In Theorem 5.2 we demonstrated that $\|u - u_h\|_{H^1} \approx Ch^s$, for $u \in H^{1+s}$, $s < \pi/\omega$. In general, we know that $u \notin H^{1+\pi/\omega}$, even if f is smooth. In Theorem 5.4, we will show that for $f \in H^{-\ell}$, with $\ell \in (0, s_0)$, $\|u - u_h\|_{H^1} \approx C_\ell h^{\pi/\omega}$, where the constant C_ℓ blows up when $\ell \to s_0$. This is a slight improvement of the result of Theorem 5.2, which in this case gives $\|u - u_h\|_{H^1} \approx Ch^{\pi/\omega-\varepsilon}$ with $\varepsilon > 0$ arbitrarily small. Here we use the technique developed in [5].

THEOREM 5.4. *Let u and u_h be the solutions of* (2.1) *and* (1.4)*, respectively, with $f \in H^{-\ell}$, $0 \le \ell < s_0$. Then there exists a constant C, independent of h, such that*

$$\|u - u_h\|_{H^1} \le C \frac{1}{s_0 - \ell} h^{\pi/\omega} \|f\|_{H^{-\ell}}. \tag{5.6}$$

Proof. Obviously, if $f \in H^{-\ell}$, with $0 \le \ell < s_0$, then $f \in H^{-\tilde{\ell}}$, $\tilde{\ell} \in (s_0, 1/2)$. Then according to Theorem 5.2, we have that

$$\|u - u_h\|_{H^1} \le C h^{1-\tilde{\ell}} \big(\|u\|_{H^{2-\tilde{\ell}}} + \|f\|_{H^{-\tilde{\ell}}} \big). \tag{5.7}$$

Also, since $u \in H^{2-\tilde{\ell}}$, we have

$$(f, v) = a(u, v) \le \|u\|_{H^{2-\tilde{\ell}}} \|v\|_{H^{\tilde{\ell}}}, \quad \forall v \in H_0^1; \tag{5.8}$$

thus, $\|f\|_{H^{-\tilde{\ell}}} \le \|u\|_{H^{2-\tilde{\ell}}}$, which in view of (5.7) gives

$$\|u - u_h\|_{H^1} \le C h^{1-\tilde{\ell}} \|u\|_{H^{2-\tilde{\ell}}}. \tag{5.9}$$

In addition, we can easily see that if $u \in H^2 \cap H_0^1$,

$$\|u - u_h\|_{H^1} \le C h \|u\|_{H^2}. \tag{5.10}$$

Then, by interpolation between (5.9) and (5.10), we get

$$\|u - u_h\|_{H^1} \le C h^{1-s_0} \|u\|_X, \tag{5.11}$$

where $X = [H^2 \cap H_0^1, H^{2-\tilde{\ell}} \cap H_0^1]_{s_0/\tilde{\ell}, \infty}$. Here $[V, W]_{\theta,q}$, $0 \le \theta \le 1$, $1 \le q \le \infty$, denote the Banach spaces intermediate between V and W defined by the K-functional, which are used in interpolation theory (cf., e.g., [7, Chapter 5]). Denote now with $L_{2,\psi}$ the orthogonal space with respect to the L_2-inner–product to the space spanned by the function $\psi = \varphi + u_R$, where $\varphi = r^{-\pi/\omega} \sin(\vartheta\pi/\omega)\eta$, and $u_R \in H_0^1$ the variational solution of $-\Delta u_R = \Delta\varphi$. Then, in view of [5, Theorem 4.1], we have

$$\|u\|_X \le C \|f\|_Y \tag{5.12}$$

with $Y = [L_{2,\psi}, H^{-\tilde{\ell}}]_{s_0/\tilde{\ell}, \infty}$; thus

$$\|u - u_h\|_{H^1} \le C h^{1-s_0} \|f\|_Y. \tag{5.13}$$

Further, since $\tilde{\ell} > s_0$, $[L_{2,\psi}, H^{-1}]_{\tilde{\ell},2} = [L_2, H^{-1}]_{\tilde{\ell},2} = H^{-\tilde{\ell}}$ (cf., e.g., [5, equation (3.16)]). Therefore, in view of the reiteration theorem for the interpolation of spaces (cf., e.g., [7, Chapter 5]), we get

$$Y = [L_{2,\psi}, H^{-1}]_{s_0, \infty}. \tag{5.14}$$

In addition, in view of [5, Theorem 3.1 and Remark 3.1], we have

$$\|f\|_{[L_{2,\psi}, H^{-1}]_{s_0,\infty}} \le C \frac{1}{s_0 - \ell} \|f\|_{H^{-\ell}}, \quad \forall f \in H^{-\ell}. \tag{5.15}$$

Thus, combining (5.13)–(5.15) we get the desired estimate. $\square$

Finally, we will show an almost optimal L_∞-norm error estimate.

THEOREM 5.5. *Let u and u_h be the solutions of* (2.1) *and* (1.4)*, respectively, with $f \in H^{-\ell}$, $0 < \ell < 1/2$. Then there exists a constant C, independent of h, such that*

$$\|u - u_h\|_{L_\infty} \le Ch^s \log \frac{1}{h} \|f\|_{H^{-\ell}}. \tag{5.16}$$

Proof. The proof is similar to the one for Theorem 4.8. Hence, we will derive bounds for $\|u - \underline{u}_h\|_{L_\infty}$ and $\|\underline{u}_h - u_h\|_{L_\infty}$.

This time using [32, equation (0.8)] and the standard imbedding $H^{1+s} \subset C^{0,s}$ (cf., e.g., [24, Theorem 1.4.5.2]), we have

$$\|u - \underline{u}_h\|_{L_\infty} \le Ch^s \log \frac{1}{h} \|u\|_{C^{0,s}} \le Ch^s \log \frac{1}{h} \|u\|_{H^{1+s}}.$$

Then, combining this with the elliptic regularity estimate,

$$\|u\|_{H^{1+s}} \le C_\ell \|f\|_{H^{-\ell}}$$

(cf. [4]), we obtain

$$\|u - \underline{u}_h\|_{L_\infty} \le C_\ell h^s \log \frac{1}{h} \|f\|_{H^{-\ell}}, \quad 0 \le \ell < 1/2. \tag{5.17}$$

We turn now to the estimation of $\|\underline{u}_h - u_h\|_{L_\infty}$. Since $A = I$, (4.34) gives

$$\|\underline{u}_h - u_h\|_{L_\infty} = \varepsilon_h(f, G_h), \tag{5.18}$$

where $G_h \in X_h$ is the finite element approximation of the regularized Green function G (cf. (4.33)). Then, using Lemma 5.1 and (4.37), we obtain

$$\|\underline{u}_h - u_h\|_{L_\infty} \le Ch^{1-\ell} \left(\log \frac{1}{h} \right)^{1/2} \|f\|_{H^{-\ell}}.$$

From this and (5.17) we get the desired estimation (5.16). $\square$

6. Auxiliary results. In this section we shall prove Lemmas 4.1, 4.2, and 5.1 of the previous sections.

Proof of Lemma 4.1. We can easily see that the interpolation operator I_h satisfies the property

$$\begin{aligned} \|\chi - I_h \chi\|^q_{L_q(K)} &= \sum_{z \in Z_h(K} \int_{K_z} (\chi - \chi(z))^q \, dx \\ &\le h_K^q |\chi|^q_{W^1_q(K)}, \quad \forall \chi \in X_h, \ q > 1, \end{aligned} \tag{6.1}$$

with $Z_h(K)$ the set of the vertices of K. Also, since in the construction of the control volumes we choose z_K to be the barycenter of K, we have

$$\int_K \chi \, dx = \int_K I_h \chi \, dx, \quad \forall K \in T_h, \ \forall \chi \in X_h. \tag{6.2}$$

In view of (6.1), (4.1) follows easily. Let now $\bar{f}_K$ be the mean value of f in K. Thus,

$$\|f - \bar{f}_K\|_{L_p(K)} \le Ch_K |f|_{W^1_p(K)}, \quad \forall f \in W^1_p(K), \ p > 1. \tag{6.3}$$

Then, by interpolation of this estimate and $\|f - \bar{f}_K\|_{L_p(K)} \le C\|f\|_{L_p(K)}$, we get, for $f \in W_p^t(K)$, $p > 1$, and $0 < t \le 1$

$$\|f - \bar{f}_K\|_{L_p(K)} \le C h_K^t |f|_{W_p^t(K)}. \tag{6.4}$$

Since $\bar{f}_K$ is constant over K, due to (6.2), we have

$$(f, \chi - I_h\chi)_K = (f - \bar{f}_K, \chi - I_h\chi)_K, \quad \forall \chi \in X_h.$$

Thus, due to this, (6.4), and (6.1), we get for every $\chi \in X_h$,

$$|(f, \chi - I_h\chi)_K| = |(f - \bar{f}_K, \chi - I_h\chi)_K| \le C h_K^{1+t} |f|_{W_p^t(K)} \, |\chi|_{W_{p'}^1(K)},$$

which concludes the proof of (4.2). ◻

We now turn to the proof of Lemma 4.2. For this we shall need the following auxiliary result.

Lemma 6.1. *Let K be a triangle and e a side of K. Then for $\varphi \in W_p^1(K)$, $p > 1$, there exists a constant C independent of K such that*

$$\left| \int_e \varphi(\chi - I_h\chi)\, ds \right| \le Ch|\varphi|_{W_p^1(K)} |\chi|_{W_{p'}^1(K)}, \quad \forall \chi \in \mathbb{P}_1(K).$$

Proof of Lemma 6.1. It is obvious that, for c constant, $I_h c = c$ and

$$\int_e I_h\chi\, ds = \int_e \chi\, ds, \quad \forall \chi \in X_h,\ \forall e \in E_h. \tag{6.5}$$

Thus, we have for every $\chi \in \mathbb{P}_1(K)$ and $\varphi \in L_2(e)$,

$$\int_e \varphi(\chi - I_h\chi)\, ds = \int_e (\varphi - c_1)(\chi - c_2 - I_h(\chi - c_2))\, ds,$$

for all constants $c_1, c_2 \in \mathbb{R}$, $K \in T_h$, and $e \in E_h(K)$. Using now in the relation above the fact that $\|I_h\chi\|_{L_\infty(e)} \le \|\chi\|_{L_\infty(e)}$ and a local inverse inequality, we get for all constants $c_1, c_2 \in \mathbb{R}$, $\chi \in \mathbb{P}_1(K)$, and $\varphi \in W_p^1(K)$,

$$\begin{aligned} \left| \int_e \varphi(\chi - I_h\chi)\, ds \right| &\le \|\varphi - c_1\|_{L_p(e)} \|\chi - c_2 - I_h(\chi - c_2)\|_{L_{p'}(e)} \\ &\le h_e^{1/p'} \|\varphi - c_1\|_{L_p(e)} \, \|\chi - c_2 - I_h(\chi - c_2)\|_{L_\infty(e)} \\ &\le C h_e^{1/p'} \|\varphi - c_1\|_{L_p(e)} \|\chi - c_2\|_{L_\infty(e)} \\ &\le C \|\varphi - c_1\|_{L_p(e)} \, \|\chi - c_2\|_{L_{p'}(e)} \end{aligned} \tag{6.6}$$

with $h_e = |e|$. In view of the Bramble–Hilbert lemma and a standard homogeneity argument, we can easily show

$$\inf_{c \in \mathbb{R}} \|\varphi - c\|_{L_p(e)} \le C h_e^{1-1/p} |\varphi|_{W_p^1(K)}, \quad \forall \varphi \in W_p^1(K),\ p > 1.$$

Finally, combining this with (6.6) we obtain the desired estimate. ◻

We now turn to the proof of Lemma 4.2.

Proof of Lemma 4.2. First we will show (4.3). In view of Green's formula, we have

$$\varepsilon_a(\psi,\chi) = \sum_K (L\psi, \chi - I_h\chi)_K + \sum_K (A\nabla\psi\cdot n, \chi - I_h\chi)_{\partial K} = I + II. \tag{6.7}$$

For the first term we have from (6.1),

$$|I| \le C \sum_K \|L\psi\|_{L_p(K)} \|\chi - I_h\chi\|_{L_{p'}(K)} \le C \sum_K h_K |\psi|_{W_p^1(K)} |\chi|_{W_{p'}^1(K)}.$$

The bound for II follows at once from Lemma 6.1 since $|A\nabla\psi\cdot n|_{W_p^1(K)} \le C|\psi|_{W_p^1(K)}$.

We now turn to (4.4). Let $\psi = u_h$ in (6.7) and $(\nabla A)_K$ be the average over K. Then in view of (6.1)–(6.3) we have for every $\chi \in X_h$,

$$(Lu_h, \chi - I_h\chi)_K = ([\nabla A - (\nabla A)_K]\nabla u_h, \chi - I_h\chi))_K \le Ch_K^2 |u_h|_{W_{\bar p}^1(K)} |\chi|_{W_{\bar p'}^1(K)}$$

with $\bar p$ given by (2.3). From the estimation above we easily obtain the desired bound for I. Let now $E_h(K)$ be the set of edges of $K \in T_h$ and $\bar A_e = A(m_e)$, where m_e is the midpoint of the edge e. We will show that for every $\chi \in X_h$,

$$II = \sum_K \sum_{e\in E_h(K)} ((A - \bar A_e)\nabla(u_h - u)\cdot n, \chi - I_h\chi)_e. \tag{6.8}$$

Provided that this holds, we may apply Lemma 6.1 and the estimate

$$|(A - \bar A_e)\nabla(u_h - u)|_{W_{\bar p}^1(K)} \le C\big(\|\nabla(u - u_h)\|_{L_2(K)} + h\|u\|_{W_{\bar p}^2(K)}\big) \tag{6.9}$$

to obtain

$$|II| \le Ch\big(\|\nabla(u - u_h)\|_{L_2} + h\|u\|_{W_{\bar p}^2}\big) |\chi|_{W_{\bar p'}^1}, \quad \forall\chi \in X_h,$$

which gives the desired estimate for II. Therefore, it remains to prove (6.8). We will show, for every $\psi \in X_h$,

$$\sum_K (A\nabla u\cdot n, \psi - I_h\psi)_{\partial K} = \sum_K \sum_{e\in E_h(K)} (\bar A_e \nabla u\cdot n, \psi - I_h\psi)_e = 0. \tag{6.10}$$

In the first sum we have by Green's formula for every $\psi \in X_h$,

$$\sum_K (A\nabla u\cdot n, \psi)_{\partial K} = \sum_K (A\nabla u, \nabla\psi)_K - (Lu, \psi)_K = (A\nabla u, \nabla\psi) - (Lu, \psi) = 0.$$

In addition, $\sum_K (A\nabla u\cdot n, I_h\psi)_{\partial K} = 0$ because $I_h\psi$ is piecewise constant on each interior edge e and $A\nabla u\cdot n$ is continuous across e (in the trace sense), and $I_h\psi = 0$ on $\partial\Omega$. Since the first sum in (6.10) vanishes for each smooth A and is continuous in A on $L_1(\cup\partial K)$, the second sum is the limit of sums with a smooth A and, therefore, also vanishes. Finally, since $\bar A_e\nabla u_h\cdot n$ is constant on each e, in view of (6.5) we have

$$\sum_K \sum_{e\in E_h(K)} (\bar A_e\nabla u_h\cdot n, \chi - I_h\chi)_e = 0, \quad \forall\chi \in X_h. \qquad \square$$

It remains now to prove Lemma 5.1.

Proof of Lemma 5.1. In view of the definition of ε_h, it suffices to show

$$|\chi - I_h\chi|_{H^\ell} \le Ch^{1-\ell}\|\nabla\chi\|_{L_2}, \quad 0 < \ell < 1/2.$$

The fractional order seminorm $|\cdot|_{H^\ell}$ is given by

$$|w|^2_{H^\ell} = \int_\Omega\int_\Omega \frac{|w(x) - w(y)|^2}{|x-y|^{2(1+\ell)}}\, dy\, dx; \tag{6.11}$$

therefore,

$$\begin{aligned}
|\chi - I_h\chi|^2_{H^\ell} &= \sum_{z,w\in Z_h}\int_{b_z}\int_{b_w} \frac{|(\chi - I_h\chi)(x) - (\chi - I_h\chi)(y)|^2}{|x-y|^{2(1+\ell)}}\, dy\, dx \\
&\le 4\sum_{\substack{z,w\in Z_h\\ z\ne w}}\int_{b_z}\int_{b_w} |\nabla\chi(z)|^2 \frac{|x-z|^2}{|x-y|^{2(1+\ell)}}\, dy\, dx \\
&\quad + \sum_{z\in Z_h}\int_{b_z}\int_{b_z} |\nabla\chi(z)|^2 |x-y|^{-2+2(1-\ell)}\, dy\, dx = 4I + II.
\end{aligned}$$

For the estimation of II we rewrite the integral with respect to the y variable in polar coordinates (r,θ) having as center x; thus $|x-y| = r$ and

$$\int_{b_z} |x-y|^{-2+2(1-\ell)}\, dy \le C\int_0^h r^{(1-\ell)p-2+1}\, dr = Ch^{2(1-\ell)}.$$

Therefore,

$$\int_{b_z}\int_{b_z} |\nabla\chi|^2 |x-y|^{-2+2(1-\ell)}\, dy\, dx \le Ch^{2(1-\ell)}\|\nabla\chi\|^2_{L_2(b_z)}, \tag{6.12}$$

which gives the desired estimate for II. Let us consider now $z \ne w$ and fix temporarily an $x \in K_z$. Using again polar coordinates with center x we estimate the integral with respect to y,

$$\int_{b_w} |x-y|^{-2(1+\ell)}\, dy \le C\int_{r_0(x)}^\infty r^{1-2(1+\ell)}\, dr \le Cr_0^{-2\ell}(x) = III,$$

where $r_0(x) = \operatorname{dist}(x, b_w)$. Let us assume that vertices z and w are in a different triangle and $|r_0(x)| > kh$; therefore, $|III| \le C|r_0(x)|^{-2\ell} \le Ch^{-2\ell}$. Thus,

$$\int_{b_z}\int_{b_w} |\nabla\chi(z)|^2 |x-z|^2 |x-y|^{-2(1+\ell)}\, dy\, dx \le Ch^{2(1-\ell)}\|\nabla\chi\|^2_{L_2(b_z)}. \tag{6.13}$$

Finally, let us consider the case that $z \ne w$ and are vertices of the same triangle K. Then $r_0(x)$ could be arbitrarily small and in order for

$$\int_{b_z} r_0(x)^{-2\ell}(x)\, dx < +\infty,$$

we need to assume that $\ell < 1/2$. In a such case, we have

$$\int_{b_z}\int_{b_w} |\nabla\chi(z)|^2 |x-z|^2 |x-y|^{-2(1+\ell)}\, dy\, dx \le Ch^2\int_{b_z} |\nabla\chi(z)|^2 r_0^{2\ell}(x)\, dx. \tag{6.14}$$

Next we will estimate the right-hand side of the relation above. For this, it suffices to bound $\int_{K_z} |\nabla\chi(z)|^2 r_0^{2\ell}$. Let us denote with x_1 and x_2 the two coefficients of a point x in K_z and introduce a rotation and translation of the (x_1, x_2)-coordinate system to $(\tilde{x}_1, \tilde{x}_2)$, where $\tilde{x}_1$-axis is the common edge $K_z \cap K_w$. We can easily see that for any point in $x \in K_z$,

$$r_0(x) = \mathrm{dist}(x, b_w) \geq \mathrm{dist}((\tilde{x}_1, 0), b_w) = \tilde{x}_1.$$

Therefore, $r_0^{-2\ell}(x) \leq \tilde{x}_1^{-2\ell}$, $\forall x \in K_z$. Then

$$\begin{aligned} h^2 \int_{K_z} |\nabla\chi(z)|^2 r_0^{-2\ell}(x)\, dx &\leq Ch^2 |\nabla\chi(z)|^2 \int_0^h \int_0^h \tilde{x}_1^{-2\ell}\, d\tilde{x}_1\, d\tilde{x}_2 \\ &\leq Ch^{2(1-\ell)} \|\nabla\chi\|^2_{L_2(K_z)}, \end{aligned}$$

assuming $\ell < 1/2$. Hence, the relation above and (6.14) give

$$\int_{b_z} \int_{b_w} |\nabla\chi(z)|^2 |x - z|^2 |x - y|^{-2(1+\ell)}\, dy\, dx \leq Ch^{2(1-\ell)} \|\nabla\chi\|^2_{L_2(b_z)}.$$

Combining this with (6.12) and (6.13), we obtain the desired estimate. □

7. Numerical results. In this section we will illustrate on several numerical examples the theoretical results of section 4. Our examples are similar to the ones considered in [8, 27].

First, we will show that the theoretical L_2-norm convergence rate of Theorem 4.6 is satisfied for the model Dirichlet boundary value problem for the Poisson equations in a Γ-shaped domain (cf. Figure 7.1), with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, $(-1, 1)$, $(-1, -1)$, and $(0, -1)$. As in [8], we consider the following two singular functions for this Γ-shaped domain:

$$S_1(r, \theta) = \phi(r) r^{2/3} \sin\left(\frac{2}{3}\theta\right), \qquad S_2(r, \theta) = \phi(r) r^{\beta} \sin\left(\frac{2}{3}\theta\right),$$

where $\beta \in (0, 1)$ and ϕ is a cutoff function defined by

$$\phi(r) = \begin{cases} 1 & 0 \leq r \leq 1/4, \\ -192r^5 + 480r^4 - 440r^3 + 180r^2 - \frac{135}{4}r + \frac{27}{8}, & 1/4 \leq r \leq 3/4, \\ 0 & 3/4 \leq r. \end{cases}$$

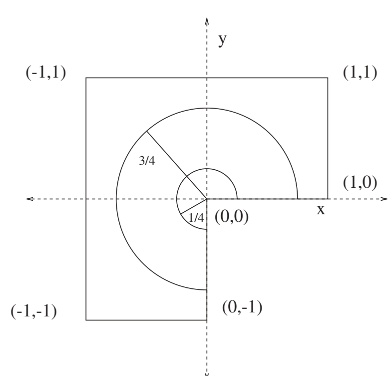

FIG. 7.1. *A Γ-shaped domain.*

Table 7.1
Approximate theoretical convergence rate for exact solution $u = S_1 + S_2 + (x - x^3)(y^2 - y^4)$.

$p_\omega = 3/2, \quad \tilde{p}_\omega = 6/5$		β			
		1/3	1/2	2/3	3/4
f $(-\Delta u)$ almost in W_α^t		$W_{11/10}^{10/66}$	$W_{6/5}^{1/6}$	$W_{6/5}^{4/3}$	$W_{6/5}^{5/12}$
rate in H^1-norm $= s$		1/3	1/2	2/3	2/3
rate in L_2–norm $\approx$	$s + 2/3, \ (\alpha < \tilde{p}_\omega)$	$s + \frac{2}{3} = 1$			
	$\min(s + 2/3, 1 + t), (\alpha \geq \tilde{p}_\omega)$		$s + \frac{2}{3} = \frac{7}{6}$	$s + \frac{2}{3} = \frac{4}{3}$	$s + \frac{2}{3} = \frac{4}{3}$
rate in L_∞-norm $\approx s$		1/3	1/2	2/3	2/3

For $f = -\Delta(S_1 + S_2) + 6x(y^2 - y^4) + (x - x^3)(12y^2 - 2)$ the exact solution is $u = S_1 + S_2 + (x - x^3)(y^2 - y^4)$. We can easily see that

$$\Delta(S_1 + S_2) = \phi(r)(\beta^2 - (2/3)^2)r^{\beta-2} \sin\left(\frac{2}{3}\theta\right) + (2\beta + 1)\phi'(r)r^{\beta-1} \sin\left(\frac{2}{3}\theta\right)$$
$$+ \phi''(r)r^{\beta} \sin\left(\frac{2}{3}\theta\right) + \frac{7}{3}\phi'(r)r^{-1/3} \sin\left(\frac{2}{3}\theta\right) + \phi''(r)r^{2/3} \sin\left(\frac{2}{3}\theta\right).$$

Since ϕ is a smooth cutoff function and $6x(y^2-y^4)+(x-x^3)(12y^2-2)$ is a polynomial, the nonsmoothness of f results from $-\Delta(S_1 + S_2)$ and for $\beta \in (0, 1)$ this is dictated from the term $r^{\beta-2}$, except in the case $\beta = 2/3$, where the leading term is $r^{-1/3}$.

According to [24, Theorem 1.4.5.3], if a function g can be written as $g = r^\gamma \varphi(\vartheta)$, in polar coordinates, where φ is smooth function, then $g \in W_\alpha^t$, with $t > 0$ and $\alpha > 1$, for $\gamma > t - 2/\alpha$. Thus, applying this to f, we have that f is almost in $W_\alpha^{\beta-2+2/\alpha}$, with $\alpha \in (1, 2/(2-\beta))$, for $\beta \neq 2/3$, and $f \in W_\alpha^{-1/3+2/\alpha}$, with $\alpha \in (1, 6)$, for $\beta = 2/3$. In addition, in view of the imbedding $L_p \subset W_a^t$, with $p = 2\alpha/(2 - t\alpha)$, for $\beta \neq 2/3$, then $f \in L_p$, with $p = 2/(2 - \beta)$, and for $\beta = 2/3$, $f \in L_6$.

Since we have considered a Γ-shaped domain, the largest interior angle is $3\pi/2$; therefore, $p_\omega = 2/(2 - (\pi/\frac{3\pi}{2})) = 3/2$. Thus, in view of (2.2), the solution u of the Poisson problem is almost in $W_{\bar{p}}^2$, with $\bar{p} = \min(2/(2-\beta), 3/2)$, or else u is almost in H^{1+s} with $s = \min(\beta, 2/3)$.

For example, we consider $\beta = 1/3$, $1/2$, $2/3$, and $3/4$. Then f is almost in the Sobolev spaces $W_{11/10}^{10/66}$, $W_{6/5}^{1/6}$, $W_{6/5}^{4/3}$, and $W_{6/5}^{5/12}$ for $\beta = 1/3$, $1/2$, $2/3$, and $3/4$, respectively. In Table 7.1 we present the theoretical and in Tables 7.2 and 7.3 the computed rates of convergence of the finite volume element method which illustrate the results of Theorem 4.6. The computation is done in the following way: For a given triangulation with number of nodes N and stepsize $2h$, we compute the finite volume solution and the norms of the errors $\|u - u_{2h}\|_T$, where $T = H^1, L_2, L_\infty$. Then we split each triangle into four similar triangles and compute the solution u_h and the corresponding norms of the errors, $\|u - u_h\|_T$. Then the computed rates are given by $\log_2 \frac{\|u-u_{2h}\|_T}{\|u-u_h\|_T}$. This procedure is repeated up to seven levels of refinement. The integrals in the finite volume formulation were approximated with a 13-point Gaussian quadrature. For the solution of the corresponding linear system we used a multigrid preconditioner.

One may argue that the suboptimal order of the L_2-norm error estimates in Theorems 4.3 and 5.4 of the finite volume element method might be an artifact of the proof and expect the same rate as in the finite element method. However, this is not correct. In what follows, we consider a counterexample which is based on a

Table 7.2
Experimental convergence rate for $\beta = \frac{1}{3}$ and $\beta = \frac{1}{2}$.

# of nodes	$\beta = 1/3$			$\beta = 1/2$		
	H^1	L_2	L_∞	H^1	L_2	L_∞
225	0.75	1.26	0.28	0.86	1.54	0.48
833	0.68	1.14	0.38	0.83	1.39	0.54
3201	0.59	1.09	0.38	0.81	1.32	0.54
12545	0.49	1.06	0.37	0.75	1.26	0.54
49665	0.42	1.04	0.36	0.69	1.23	0.54
197633	0.38	1.03	0.36	0.63	1.22	0.53
788481	0.37	1.02	0.35	0.60	1.21	0.53
Theoretical ≈	0.33	1	0.33	0.5	1.17	0.5

Table 7.3
Experimental convergence rate for $\beta = \frac{2}{3}$ and $\beta = \frac{3}{4}$.

# of nodes	$\beta = 2/3$			$\beta = 3/4$		
	H^1	L_2	L_∞	H^1	L_2	L_∞
225	0.89	1.76	0.92	0.90	1.84	1.15
833	0.89	1.60	0.66	0.91	1.66	0.69
3201	0.91	1.55	0.67	0.93	1.63	0.70
12545	0.90	1.46	0.67	0.93	1.54	0.69
49665	0.87	1.41	0.67	0.91	1.47	0.69
197633	0.83	1.37	0.67	0.88	1.42	0.69
788481	0.79	1.36	0.67	0.85	1.40	0.69
Theoretical ≈	0.66	1.33	0.66	0.66	1.33	0.66

similar argument given in [27]. The following arguments can easily be modified and apply to a model problem in a convex domain. This can then be used to illustrate the theoretical convergence rates derived in [14, 23].

First we will show that the L_2-norm estimate in Theorem 4.3 is sharp. We consider the model problem

$$-\Delta u = f \quad \text{in } \Omega, \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega, \tag{7.1}$$

where $f \in L_2$ and Ω is the Γ-shaped domain with vertices $(0,0)$, $(2,0)$, $(2,2)$, $(-2,2)$, $(-2,-2)$, and $(0,-2)$. Since $\pi/\omega = 2/3$, according to Theorem 4.3 we know that

$$\|u - u_h\|_{H^1} \le Ch^s \|f\|_{L_2}$$

with $s = 2/3 - \varepsilon$, $\varepsilon > 0$ arbitrarily small. Let us assume then that (4.6) is not true and the finite volume and finite element methods converge in the L_2-norm with the same rate, i.e.,

$$\|u - u_h\|_{L_2} \le Ch^{2s} \|f\|_{L_2}.$$

Obviously,

$$\|u - u_h\|_{L_2} = \sup_{\phi \in L_2 \setminus \{0\}} \frac{(u - u_h, \phi)}{\|\phi\|_{L_2}}.$$

Hence, our assumption leads to

$$|(u - u_h, \phi)| \le Ch^{2s} \|\phi\|_{L_2} \|f\|_{L_2}. \tag{7.2}$$

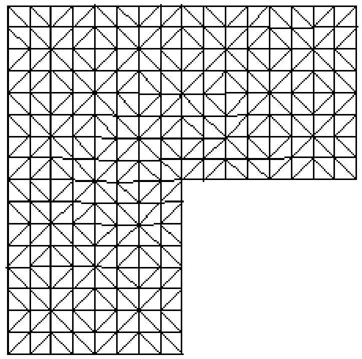

Fig. 7.2. *An example of a triangulation. The successive uniform refinement occurs by splitting the triangles into four.*

Next, let us denote $\psi \in H^{1+s} \cap H_0^1$ the solution of the auxiliary problem

$$-\Delta\psi = \phi \quad \text{in } \Omega, \quad \text{and} \quad \psi = 0 \quad \text{on } \partial\Omega. \tag{7.3}$$

Thus,

$$(u - u_h, \phi) = a(u - u_h, \psi) = a(u - u_h, \psi - \Pi_h\psi) + a(u - u_h, \Pi_h\psi), \tag{7.4}$$

where $\Pi_h\psi$ is the interpolant of ψ in X_h. Obviously, then

$$a(u - u_h, \Pi_h\psi) = (f, \Pi_h\psi - I_h\Pi_h\psi). \tag{7.5}$$

We can easily see that

$$a(u - u_h, \psi - \Pi_h\psi) \leq Ch^{2s}\|\phi\|_{L_2}\|f\|_{L_2}.$$

Thus combining (7.2)–(7.5), we get

$$(f, \Pi_h\psi - I_h\Pi_h\psi) \leq Ch^{2s}\|\phi\|_{L_2}\|f\|_{L_2}.$$

Since f is an arbitrary function of L_2, this leads to

$$\|\Pi_h\psi - I_h\Pi_h\psi\|_{L_2} \leq Ch^{2s}\|\phi\|_{L_2}.$$

Hence,

$$\|\psi - I_h\Pi_h\psi\|_{L_2} \leq Ch^{2s}\|\phi\|_{L_2}.$$

Then, since ϕ is also an arbitrary function, this should be true for any function $\psi \in H^{1+s} \cap H_0^1$. Therefore, let us consider a function $\psi \in H^{1+s} \cap H_0^1$ such that

$$\psi(x_1, x_2) = x_1(1 - x_1), \quad (x_1, x_2) \in \Omega_1 = [1/2, 3/2] \times [1/2, 3/2]. \tag{7.6}$$

For this ψ we should get

$$\|\psi - I_h\Pi_h\psi\|_{L_2(\Omega_1)} \leq Ch^{2s}. \tag{7.7}$$

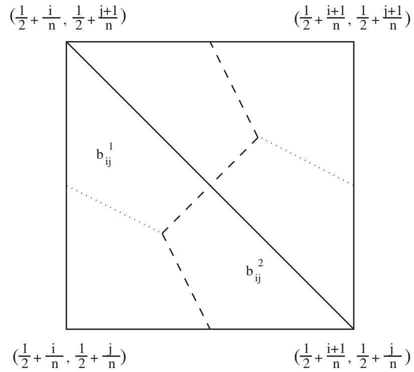

FIG. 7.3. *A sample square K_{ij}. The two regions b_{ij}^1 and b_{ij}^2 are separated with the dashed line.*

We discretize Ω_1 into n^2 equal size squares with length $h = 1/n$, and each square is divided further into two right triangles in the same direction. Next, we construct the relative control volumes by connecting the barycenter of its triangle with the middle of the edges. Let us denote z_{ij} the vertices $(1/2+i/n, 1/2+j/n)$, $i, j = 0, \dots, n-1$. Also, let K_{ij} be the square $[1/2+i/n, 1/2+(i+1)/n] \times [1/2+j/n, 1/2+(j+1)/n]$, $i, j = 1, \dots, n$, and $b_{ij}^1 = K_{ij} \cap (b_{ij} \cup b_{i(j+1)})$ and $b_{ij}^2 = K_{ij} \cap (b_{(i+1)j} \cup b_{(i+1)(j+1)})$ (cf. Figure 7.3). Then, since ψ depends only on x, $I_h\psi_I$ has the same value on the control volumes b_{ij}, $j = 0, \dots, n-1$, for every $i = 0, \dots, n-1$. For this reason, on the square K_{ij}, $I_h\psi = \psi(z_{ij})$ on b_{ij}^1 and $I_h\psi = \psi(z_{(i+1)j})$ on b_{ij}^2. Then we have

$$\begin{aligned}
\|\psi - I_h\Pi_h\psi\|_{L_2(\Omega_1)}^2 &= \int_{\Omega_1} \psi^2\, dx_1\, dx_2 + \int_{\Omega_1} (I_h\Pi_h\psi)^2\, dx_1\, dx_2 \\
&\quad - 2\int_{\Omega_1} \psi I_h\Pi_h\psi\, dx_1\, dx_2 \\
&= \int_{1/2}^{3/2} x_1^2(1-x_1)^2\, dx_1 + \sum_{i,j=0}^{n-1} (\psi^2(z_{ij})|b_{ij}^1| + \psi^2(z_{(i+1)j})|b_{ij}^2|) \\
&\quad - 2\sum_{i,j=0}^{n-1} \Bigg(\psi(z_{ij}) \int_{b_{ij}^1} x_1(1-x_1)\, dx_1\, dx_2 \\
&\qquad\qquad + \psi(z_{(i+1)j}) \int_{b_{ij}^2} x_1(1-x_1)\, dx_1\, dx_2 \Bigg) \\
&= \frac{10}{81n^2} + \frac{1}{405n^4}.
\end{aligned}$$

Finally, we have

$$\|\psi - I_h\Pi_h\psi\|_{L_2(\Omega_1)} = \frac{\sqrt{10}}{9}\frac{1}{n} + o\left(\frac{1}{n}\right) = O(h).$$

Combining this with (7.7) we get a contradiction, since $2s \approx 4/3$.

Similar arguments can be used in order to show now the sharpness of the L_2-norm error estimate in Theorem 5.2. Thus, let us consider this time the model problem (7.1), with $f \in H^{-1/3}$.

According to Theorem 5.2 we have that

$$\|u - u_h\|_{H^1} \le Ch^s \|f\|_{H^{-1/3}},$$

with $s = 2/3 - \varepsilon$, $\varepsilon > 0$, arbitrarily small. Let us assume that the L_2-norm error estimate in Theorem 5.4 does not hold and the finite volume and finite element methods converge in L_2-norm with the same rate, i.e.,

$$\|u - u_h\|_{L_2} \le Ch^{2s} \|f\|_{H^{-1/3}}.$$

Repeating similar arguments as in the previous counterexample, the function $\psi \in H^{1+s} \cap H_0^1$ that satisfies (7.6) and (7.3) should also satisfy

$$\|\Pi_h \psi - I_h \Pi_h \psi\|_{H^{1/3}} \le Ch^{2s} \|\phi\|_{L_2}. \tag{7.8}$$

We discretize again Ω_1 in the same way as before, into n^2 equal size squares with length $h = 1/n$, and each square is divided further into two right triangles in the same direction. We construct the control volumes b_{ij} in the same manner as before and denote z_{ij} the vertices $(1/2 + i/n, 1/2 + j/n)$, $i, j = 0, \dots, n-1$. Then, using the definition of $|\cdot|_{H^\ell}$ (6.11), we can estimate $\|\Pi_h \psi - I_h \Pi_h \psi\|_{H^{1/3}(\Omega_1)}$ from below by

$$\|\Pi_h \psi - I_h \Pi_h \psi\|^2_{H^{1/3}(\Omega_1)} \ge \sum_{i,j=1}^{n-1} \int_{b_{z_{ij}}} \int_{b_{z_{ij}}} \frac{|\nabla \Pi_h \psi(z_{ij}) \cdot (x-y)|^2}{|x-y|^{2(1+1/3)}} \, dy \, dx. \tag{7.9}$$

Also, let $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $\tilde{K}_{ij} = [1/2 + i/n, 1/2 + i/n + 1/3n] \times [1/2 + j/n, 1/2 + j/n + 1/3n]$, and since ψ is invariant in the x_2-direction and $|x - y| \le \sqrt{2}/3n$, (7.9) gives

$$\begin{aligned} \|\Pi_h \psi - I_h \Pi_h \psi\|^2_{H^{1/3}(\Omega_1)} & \\ \ge \sum_{i,j=1}^{n-1} \int_{\tilde{K}_{ij}} \int_{\tilde{K}_{ij}} \frac{|\nabla \Pi_h \psi(z_{ij}) \cdot (x-y)|^2}{|x-y|^{2(1+1/3)}} \, dy \, dx & \\ \ge \left(\frac{3n}{\sqrt{2}} \right)^{2(1+1/3)} (n-1) \sum_{i=1}^{n-1} \int_{\tilde{K}_{i1}} \int_{\tilde{K}_{i1}} \left| \frac{\partial \Pi_h \psi(z_{i1})}{\partial x_1} \right|^2 (x_1 - y_1)^2 \, dy \, dx. & \end{aligned} \tag{7.10}$$

Next, we can easily see that $|\frac{\partial \Pi_h \psi(z_{i1})}{\partial x_1}| = (2i+1)/n$ and

$$\int_{\tilde{K}_{i1}} \int_{\tilde{K}_{i1}} (x_1 - y_1)^2 \, dy \, dx = \frac{1}{2 \cdot 3^7 n^6}.$$

Thus,

$$\sum_{i=1}^{n-1} \int_{\tilde{K}_{i1}} \int_{\tilde{K}_{i1}} \left| \frac{\partial \Pi_h \psi(z_{i1})}{\partial x_1} \right|^2 (x_1 - y_1)^2 \, dy \, dx \ge \frac{1}{2 \cdot 3^7 n^8} \sum_{i=1}^{n-1} i^2 = \frac{n(n-1)(2n-1)}{4 \cdot 3^8 n^8}.$$

Finally, employing this in (7.10) we get

$$\|\Pi_h \psi - I_h \Pi_h \psi\|_{H^{-1/3}(\Omega_1)} \ge \frac{C}{n^{2/3}} + o\left(\frac{1}{n^{2/3}} \right) = O(h^{2/3}).$$

Combining this with (7.8) we get a contradiction, since $2s \approx 4/3$.

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