

ERROR ESTIMATES FOR THE FINITE VOLUME ELEMENT METHOD FOR PARABOLIC EQUATIONS IN CONVEX POLYGONAL DOMAINS

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ABSTRACT. We analyze the spatially semidiscrete piecewise linear finite volume element method for parabolic equations in a convex polygonal domain in the plane. Our approach is based on the properties of the standard finite element Ritz projection and also of the elliptic projection defined by the bilinear form associated with the variational formulation of the finite volume element method. Because the domain is polygonal special attention has to be paid to the limited regularity of the exact solution. We give sufficient conditions in terms of data which yield optimal order error estimates in L_2 and H^1 . The convergence rate in the L_∞ norm is suboptimal, the same as in the corresponding finite element method, and almost optimal away from the corners. We also briefly consider the lumped mass modification and the backward Euler fully discrete method.

1. INTRODUCTION

We consider the model initial–boundary value problem

$$(1.1) \quad \begin{aligned} u_t + Lu &= f, & \text{in } \Omega, & \quad t \geq 0, \\ u &= 0, & \text{on } \partial\Omega, & \quad t \geq 0, \\ u(0) &= u^0, & \text{in } \Omega, & \end{aligned}$$

where Ω is a bounded, convex, polygonal domain in \mathbb{R}^2 and $Lu \equiv -\operatorname{div}(A\nabla u)$, with $A = (a_{ij})_{i,j=1}^2$ a real–valued positive definite smooth matrix function.

We shall study spatially semidiscrete approximations of (1.1) by the finite volume element method, which for brevity we will refer to as the finite volume method below. The approximate solution will be sought in the piecewise linear finite element space

$$X_h \equiv X_h(\Omega) = \{\chi \in C(\Omega) : \chi|_K \text{ linear}, \forall K \in T_h; \chi|_{\partial\Omega} = 0\},$$

where $\{T_h\}_{0 < h < 1}$ is a family of regular triangulations of Ω , with h denoting the maximum diameter of the triangles of T_h .

We begin by recalling the semidiscrete Galerkin finite element method, which is to find $\underline{u}_h(t) \in X_h$ such that, with (\cdot, \cdot) the inner product in $L_2(\Omega)$,

$$(1.2) \quad \begin{aligned} (\underline{u}_{h,t}, \chi) + a(\underline{u}_h, \chi) &= (f, \chi), & \forall \chi \in X_h, \quad t \geq 0, \\ \underline{u}_h(0) &= u_h^0, \end{aligned}$$

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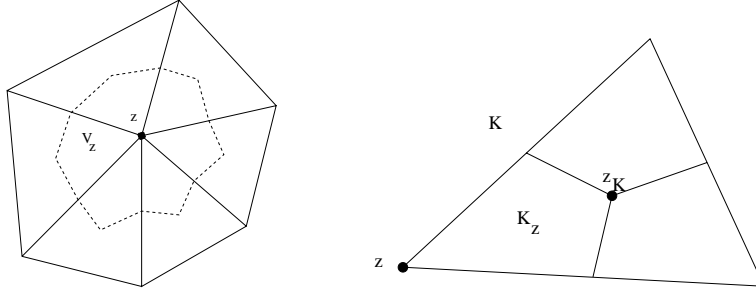


FIGURE 1. *Left:* A union of triangles that have a common vertex z ; the dotted line shows the boundary of the corresponding control volume V_z . *Right:* A triangle K partitioned into the three subregions K_z .

where $u_h^0 \in X_h$ is a given approximation of u^0 and the bilinear form $a(\cdot, \cdot)$ is defined by

$$a(v, w) = \int_{\Omega} A \nabla v \cdot \nabla w \, dx, \quad \text{for } v, w \in H_0^1(\Omega).$$

The finite volume method relies on a local conservation property associated with the differential equation. Namely, integrating (1.1) over any region $V \subset \Omega$ and using Green's formula, we obtain

$$(1.3) \quad \int_V u_t \, dx - \int_{\partial V} (A \nabla u) \cdot n \, ds = \int_V f \, dx, \quad t \geq 0,$$

where n denotes the unit exterior norm to ∂V .

The semidiscrete finite volume problem will satisfy a relation similar to (1.3) for V in a finite collection of subregions of Ω called control volumes, the number of which will be equal to the dimension of the finite element space X_h . These control volumes are constructed in the following way. Let z_K be the barycenter of $K \in T_h$. We connect z_K with line segments to the midpoints of the edges of K , thus partitioning K into three quadrilaterals K_z , $z \in Z_h(K)$, where $Z_h(K)$ are the vertices of K . Then with each vertex $z \in Z_h = \cup_{K \in T_h} Z_h(K)$ we associate a control volume V_z , which consists of the union of the subregions K_z , sharing the vertex z (see Figure 1). We denote the set of interior vertices of Z_h by Z_h^0 .

The semidiscrete finite volume method is then to find $u_h(t) \in X_h$ for $t \geq 0$ such that

$$(1.4) \quad \int_{V_z} u_{h,t} \, dx - \int_{\partial V_z} (A \nabla u_h) \cdot n \, ds = \int_{V_z} f \, dx, \quad \forall z \in Z_h^0, \, t \geq 0,$$

$$u_h(0) = u_h^0,$$

with $u_h^0 \in X_h$ a given approximation of u^0 .

This version of the finite volume method is also referred to as the vertex centered finite volume method. Similar discretization techniques have been analyzed for a various linear and nonlinear evolution problems, cf., e.g., [2, 11, 18].

The finite volume problem (1.4) can be rewritten in a variational form similar to the finite element problem (1.2). For this purpose we introduce the finite dimensional space

$$Y_h = \{\eta \in L_2(\Omega) : \eta|_{V_z} = \text{constant}, \forall z \in Z_h^0; \eta|_{V_z} = 0, \forall z \in \partial\Omega\}.$$

For an arbitrary $\eta \in Y_h$, we multiply the integral relation in (1.4) by $\eta(z)$ and sum over all $z \in Z_h^0$ to obtain the Petrov–Galerkin formulation, to find $u_h(t) \in X_h$ for $t \geq 0$ such that

$$(1.5) \quad \begin{aligned} (u_{h,t}, \eta) + a_h(u_h, \eta) &= (f, \eta), \quad \forall \eta \in Y_h, \quad t \geq 0, \\ u_h(0) &= u_h^0, \end{aligned}$$

where the bilinear form $a_h(\cdot, \cdot) : X_h \times Y_h \rightarrow \mathbb{R}$ is defined by

$$(1.6) \quad a_h(v, \eta) = - \sum_{z \in Z_h^0} \eta(z) \int_{\partial V_z} (A \nabla v) \cdot n \, ds, \quad v \in X_h, \quad \eta \in Y_h.$$

Obviously, $a_h(v, \eta)$ may be defined by (1.6) also for $v \in H^2(\Omega)$, and using Green's formula we easily see that

$$(1.7) \quad a_h(v, \eta) = (Lv, \eta), \quad \text{for } v \in H^2(\Omega), \quad \eta \in Y_h.$$

In the same way as for the finite element method, the finite volume method (1.5) may be written as a system of ordinary differential equations. In fact, let $\{\Phi_z\}_{z \in Z_h^0}$ be the standard ‘‘pyramid’’ basis of X_h , with $\Phi_z(w) = 1$ if $w = z$ and $\Phi_z(w) = 0$ if $w \neq z$, $w \in Z_h^0$, and let $\{\Psi_z\}_{z \in Z_h^0}$ be the associated basis of Y_h consisting of the characteristic functions of the control volumes V_z . Writing $u_h(t) = \sum_{z \in Z_h^0} \alpha_z(t) \Phi_z$, (1.5) then takes the form

$$(1.8) \quad \mathcal{M} \alpha'(t) + \mathcal{S} \alpha(t) = \tilde{f}(t), \quad \forall t \geq 0, \quad \text{with } \alpha(0) = \tilde{\alpha},$$

where $\mathcal{M} = (m_{zw})$ and $\mathcal{S} = (s_{zw})$ are the associated mass and stiffness matrices, respectively, with $m_{zw} = (\Phi_z, \Psi_w)$ and $s_{zw} = a_h(\Phi_z, \Psi_w)$, and where $\alpha(t)$, $\tilde{\alpha}$, and $\tilde{f}(t)$ have elements $\alpha_z(t)$, (u^0, Ψ_z) , and $(f(t), \Psi_z)$. It is easy to see that \mathcal{M} is symmetric and that both \mathcal{M} and \mathcal{S} are positive definite, see Section 3 below.

We shall denote by $L_p(V)$ the p -integrable real-valued functions over $V \subset \mathbb{R}^2$, $(\cdot, \cdot)_V$ the inner product in $L_2(V)$, and $\|\cdot\|_{W_p^s(V)}$ the norm in the Sobolev space $W_p^s(V)$, $s \geq 0$. If $V = \Omega$ we suppress the index V , and if $p = 2$ we write $H^s = W_2^s$ and $\|\cdot\| = \|\cdot\|_{L_2}$. Finally, we denote by $L_q(0, T; W_p^s)$, $1 \leq p, q \leq \infty$, $s \geq 0$, the space of functions $v(t) : [0, T] \mapsto W_p^s$ such that $\|u(t)\|_{W_p^s} \in L_q(0, T)$ (see, e.g. [12, p. 285]).

The above approach was perhaps first formulated in the Petrov-Galerkin framework, employing two different meshes to define the solution space and the test space, for elliptic equations of second order, by Bank and Rose [3], Cai [6], and Süli [22]. This approach allows us to analyze the error by using some of the tools developed in the finite element method theory. In the existing literature the error estimates for the finite volume method are usually derived for solutions that are sufficiently smooth, cf., e.g., [2, 9, 11, 14, 18]. However, in many applications the solutions are not smooth due to, e.g., nonsmooth diffusion coefficient (this is often called an interface problem), nonsmooth right-hand side, incompatible initial data, and domains with nonsmooth boundaries, e.g., polygonal domains.

Recently, Ewing et al. [13], Droniou and Galouët [10], and two of the present authors [8] considered finite volume schemes for elliptic problems with nonsmooth right-hand side, namely $f \in H^{-\ell}(\Omega)$ with $\ell > 0$. In this paper we present an error analysis of the finite volume method for parabolic problems in convex polygonal domains.

We begin with some comments about error estimates for the finite element method (1.2); details will be given in Section 2 below. It is well known, cf., e.g., [23, Chapter 1], that

$$(1.9) \quad \|\underline{u}_h(t) - u(t)\| \leq \|u_h^0 - u^0\| + Ch^2(\|u^0\|_{H^2} + \int_0^t \|u_t\|_{H^2} d\tau), \quad t \geq 0,$$

where we assume that u is smooth enough for the right-hand side to be finite. This error bound is shown in [23] for domains with smooth boundary, but the proof is valid also for a polygonal domain. If u_h^0 is chosen so that $\|u_h^0 - u^0\| \leq Ch^2\|u^0\|_{H^2}$, the first term on the right may be bounded by the second. In the sequel we shall often make such particular choices, to avoid having to account for this term.

In the proof of (1.9) one splits the error as $\underline{u}_h - u = (\underline{u}_h - R_h u) + (R_h u - u)$ where R_h denotes the standard Ritz projection $R_h : H_0^1 \rightarrow X_h$ defined by

$$(1.10) \quad a(R_h u, \chi) = a(u, \chi), \quad \forall \chi \in X_h.$$

It is well known that R_h has the approximation property

$$(1.11) \quad \|R_h v - v\|_{H^j} \leq Ch^{\ell-j}\|v\|_{H^\ell}, \quad \forall v \in H^\ell \cap H_0^1, \quad j = 0, 1, \ell = 1, 2.$$

Before we continue with the known results for the finite volume method we remark that error bounds in this area are normally given in terms of norms of the exact solution of the problem, such as is the case in (1.9). In the case of a domain with smooth boundary it is well known that the regularity required of the exact solution can be attained by assuming enough regularity of the data. Thus there is no great need to express the error bounds in terms of data, which would be an unnecessary complication in the analysis.

In the case of a polygonal domain the situation is different. The corners of the domain then give rise to singularities in the solution that are present regardless of the smoothness of the data. It then becomes an essential part of the analysis to show that the norms appearing in a proposed optimal order error estimate can be guaranteed to be finite under the appropriate smoothness and possible additional assumptions on data. Such an additional assumption could be a compatibility relation between f and u^0 on $\partial\Omega$ for $t = 0$. We have therefore chosen here to express our error estimates in terms of norms of data, sometimes supplemented with compatibility conditions.

As an example, in order for (1.9) to show a $O(h^2)$ error estimate we need to show that $u_t \in L_1(0, T; H^2)$ for some $T > 0$. We recall that since the domain is convex, we have the elliptic regularity estimate (see, e.g., [15])

$$(1.12) \quad \|u\|_{H^2} \leq C\|Lu\|.$$

Using this together with the differential equation (1.1) we shall show

$$(1.13) \quad \int_0^t \|u_t\|_{H^2} d\tau \leq C_T \varepsilon^{-1}(\|g_0\|_{H^\varepsilon} + \int_0^t \|f_t\|_{H^\varepsilon} d\tau), \quad t \leq T,$$

for $\varepsilon \in (0, 1/2)$, where $g_0 = u_t(0) = f(0) - Lu^0$. Thus the L_2 error in (1.9) is of order $O(h^2)$ for $t \leq T$ if $u^0 \in H^2$ and the right-hand side of (1.13) is finite. We note that no artificial boundary conditions need to be satisfied by f , when $\varepsilon < 1/2$.

In Section 3 we turn to the finite volume method (1.5). It was shown in Ewing, Lazarov, and Lin [14] and Li, Chen, and Wu [16] that, for $p > 1$ and u_h^0 appropriately chosen,

$$(1.14) \quad \|u_h(t) - u(t)\| \leq Ch^2(\|u^0\|_{W_p^3} + \int_0^t \|u_t\|_{W_p^3} d\tau), \quad t \geq 0.$$

This time the proof uses a splitting of the error based on the elliptic projection operator $\tilde{R}_h : H^2 \cap H_0^1 \rightarrow X_h$ defined by

$$(1.15) \quad a_h(\tilde{R}_h v, \eta) = a_h(v, \eta), \quad \forall \eta \in Y_h.$$

In Chou and Li [9] and Li, Chen, and Wu [16] it was shown that

$$(1.16) \quad \|\tilde{R}_h v - v\| \leq Ch^2 \|v\|_{W_p^3}, \quad \text{for } p > 1.$$

The right-hand side here is bounded for p close to 1, which is expressed by the elliptic regularity estimate, cf. [15, Chapter 5],

$$(1.17) \quad \|v\|_{W_p^3} \leq C_p \|Lv\|_{W_p^1}, \quad 1 < p < p_0 \leq 2, \quad v \in W_p^3 \cap H_0^1,$$

where p_0 is defined in the following way: Let S be a vertex of Ω , and denote the corresponding interior angle of Ω by $\omega(S)$. Let \mathcal{A} and \mathcal{T} be matrices such that $\mathcal{A} = (a_{ij}(S))_{i,j=1}^2$ and $\mathcal{T}\mathcal{A}\mathcal{T}^T = I$, and let $\omega_A(S)$ be the angle at the vertex $\mathcal{T}S$ of the transformed domain $\mathcal{T}\Omega = \{\mathcal{T}x : x \in \Omega\}$. Define

$$(1.18) \quad \omega = \max_S \omega_A(S), \quad \beta = \pi/\omega, \quad \text{and} \quad p_0 = 2/(3 - \beta).$$

We remark that $\omega < \pi$, and that $\omega_A(S) = \omega(S)$ if $L = -\Delta$. Note that for a general convex polygonal domain (1.17) does not hold for $p = 2$. To obtain a $O(h^2)$ error bound for the finite volume method (1.5), we shall demonstrate this time that, with $g_1 = u_{tt}(0) = f_t(0) - Lg_0$, we have for $p < p_0$,

$$(1.19) \quad \int_0^t \|u_t\|_{W_p^3} d\tau \leq C(\|g_1\|_{L_p} + \int_0^t \|f_{tt}\|_{L_p} d\tau), \quad t \leq T.$$

Section 4 is devoted to an alternative way to obtain a $O(h^2)$ error bound for the finite volume method (1.5), under slightly different regularity assumptions, which is to base the analysis on the standard Ritz projection R_h defined in (1.10) rather than \tilde{R}_h from (1.15). In this case we are able to show

$$\|u_h(t) - u(t)\| \leq Ch^2 \left(\|u^0\|_{H^2} + \left(\int_0^t (\|u_t\|_{H^2}^2 + \|f\|_{H^1}^2) d\tau \right)^{1/2} \right), \quad t \leq T.$$

After establishing the *a priori* estimate

$$\int_0^t \|u_t\|_{H^2}^2 d\tau \leq C(\|g_0\|_{H^1}^2 + \int_0^t \|f_t\|^2 d\tau), \quad t \leq T,$$

under the compatibility condition $g_0 = 0$ on $\partial\Omega$, we obtain the desired $O(h^2)$ error bound, under slightly different conditions than those in (1.19).

In Section 5 we apply the technique of Section 4 to derive error bounds in H^1 and L_∞ norms. In the latter case we show a bound of the form

$$(1.20) \quad \|u_h(t) - u(t)\|_{L_\infty} \leq C_{T,\gamma}(u^0, f) h^\gamma, \quad t \leq T, \quad \text{for any } \gamma < \beta.$$

We remark that a $O(h^2 \log \frac{1}{h})$ uniform bound for the error was shown in [9] under the assumption that $u \in L_\infty(0, T; H^3)$ for $T > 0$, in the case of a smooth boundary. However, as indicated above, this is not a realistic assumption for a polygonal domain. However, the significant loss of convergence rate expressed in (1.20) only takes place near the corners of Ω , and away from these we are able to show a $O(h^2 \log \frac{1}{h})$ maximum–norm error bound.

In Section 6 we consider a lumped mass finite volume element method. In this case the method loses the property of being locally conservative. For a constant coefficient matrix A , it reduces to a linear system with the same left-hand side as the corresponding lumped mass finite element method. We analyze the lumped mass method using the elliptic projection R_h and obtain the same error bounds as for (1.5).

Finally, in Section 7, we show that our approach also applies to fully discrete schemes. As an example we analyze the backward Euler finite volume element method.

2. THE FINITE ELEMENT METHOD

In this section, as a guide to the proofs of our subsequent error bounds for the finite volume method, we shall show the following result for the standard finite element method.

Theorem 2.1. *Let \underline{u}_h and u be the solutions of (1.2) and (1.1), respectively, and assume that $u^0 = 0$ on $\partial\Omega$ and $\varepsilon \in (0, 1/2)$. Then, if $u_h^0 = R_h u^0$,*

$$(2.1) \quad \|\underline{u}_h(t) - u(t)\| \leq C_{T,\varepsilon}(u^0, f)h^2, \quad t \leq T,$$

where

$$C_{T,\varepsilon}(u^0, f) = C_T \left(\|u^0\|_{H^2} + \varepsilon^{-1} (\|g_0\|_{H^\varepsilon} + \int_0^t \|f_t\|_{H^\varepsilon} d\tau) \right).$$

Proof. In a standard way we split the error $\underline{u}_h - u$ using the Ritz projection $R_h u$, defined in (1.10), as

$$\underline{u}_h - u = (\underline{u}_h - R_h u) + (R_h u - u) = \underline{\varrho} + \varrho.$$

In view of (1.11) we have

$$(2.2) \quad \|\varrho(t)\| \leq Ch^2 \|u(t)\|_{H^2} \leq Ch^2 (\|u^0\|_{H^2} + \int_0^t \|u_t\|_{H^2} d\tau), \quad t \geq 0.$$

In order to bound $\underline{\varrho}$, we note that by our definitions

$$(\underline{\varrho}_t, \chi) + a(\underline{\varrho}, \chi) = -(\varrho_t, \chi), \quad \forall \chi \in X_h.$$

Choosing $\chi = \underline{\varrho}$ and using the positivity of $a(\underline{\varrho}, \underline{\varrho})$, (1.11), and the fact that $\underline{\varrho}(0) = 0$, we easily find

$$\|\underline{\varrho}(t)\| \leq C \int_0^t \|\varrho_t\| d\tau \leq Ch^2 \int_0^t \|u_t\|_{H^2} d\tau, \quad t \geq 0.$$

Thus, together with (2.2) we have

$$\|\underline{u}_h(t) - u(t)\| \leq Ch^2 (\|u^0\|_{H^2} + \int_0^t \|u_t\|_{H^2} d\tau), \quad t \geq 0.$$

The theorem therefore follows from the inequality (1.13) which we now proceed to show. For this we use the elliptic regularity estimate (1.12) and differentiate (1.1) to obtain

$$(2.3) \quad \|u_t\|_{H^2} \leq C\|Lu_t\| \leq C(\|u_{tt}\| + \|f_t\|).$$

After integration it now remains to show

$$(2.4) \quad \int_0^t \|u_{tt}\| d\tau \leq C_T \varepsilon^{-1} (\|g_0\|_{H^\varepsilon} + \int_0^t \|f_t\|_{H^\varepsilon} d\tau), \quad t \leq T.$$

Recall that, by Duhamel's principle, we have

$$(2.5) \quad u(t) = E(t)u^0 + \int_0^t E(t-s)f(s) ds = E(t)u^0 + \int_0^t E(s)f(t-s) ds,$$

where $E(t)$ denotes the solution operator of the homogeneous case ($f = 0$) of (1.1). It is well-known that $E(t)$ is the analytic semigroup e^{-Lt} in L_2 generated by $-L$, and that

$$(2.6) \quad \|E(t)v\| \leq C\|v\| \quad \text{and} \quad \|E'(t)v\| \leq Ct^{-1}\|v\|,$$

where the inequalities easily follow, using Parseval's relation, by writing

$$E(t)v = \sum_{j=1}^{\infty} e^{-\lambda_j t} (v, \phi_j) \phi_j,$$

with $\{\lambda_j\}$ and $\{\phi_j\}$ the eigenvalues and eigenfunctions of L .

Differentiating (2.5) twice in time we find

$$(2.7) \quad u_t(t) = E(t)g_0 + \int_0^t E(t-s)f_t(s) ds,$$

and

$$(2.8) \quad u_{tt}(t) = E'(t)g_0 + f_t(t) + \int_0^t E'(t-s)f_t(s) ds.$$

Here we have that for $\varepsilon \in (0, 1/2)$

$$(2.9) \quad \|E'(t)v\| \leq Ct^{-1+\varepsilon/2}\|v\|_{H^\varepsilon}, \quad t > 0.$$

For this we first note that

$$(2.10) \quad \|E'(t)v\| \leq C\|E(t)Lv\| \leq C\|v\|_{H^2}, \quad \forall v \in H^2 \cap H_0^1.$$

By interpolation between this estimate and the second estimate in (2.6), (2.9) follows upon noting that $[L_2, H^2 \cap H_0^1]_{\varepsilon/2,2} = H^\varepsilon$, cf., e.g., [15, Corollary 1.4.4.5]. Here $[X, Y]_{\theta,q}$, $0 \leq \theta \leq 1$, $1 \leq q \leq \infty$, denote the Banach spaces intermediate between X and Y defined by the K -functional, which are used in interpolation theory, cf., e.g., [4, Chapter 5]. Finally, applying (2.9) in (2.8) we find

$$(2.11) \quad \begin{aligned} \int_0^t \|u_{tt}\| d\tau &\leq C \left(\int_0^t \tau^{-1+\varepsilon/2} \|g_0\|_{H^\varepsilon} d\tau + \int_0^t \|f_t\| d\tau \right. \\ &\quad \left. + \int_0^t \int_0^\tau (\tau-s)^{-1+\varepsilon/2} \|f_t(s)\|_{H^\varepsilon} ds d\tau \right) \\ &\leq Ct^{\varepsilon/2} \varepsilon^{-1} (\|g_0\|_{H^\varepsilon} + \int_0^t \|f_t\|_{H^\varepsilon} d\tau), \quad t \geq 0. \end{aligned}$$

which shows (2.4) and hence (1.13). This completes the proof. \square

Note that since (2.9) has been obtained by interpolation between L_2 and $H^2 \cap H_0^1$, the argument requires that $v = 0$ on $\partial\Omega$, if $\varepsilon \in (1/2, 2)$, but not for $\varepsilon \in (0, 1/2)$. Thus, in addition to reducing the regularity requirements on g_0 and f_t , compared to direct application of (2.10), using (2.9) makes it possible to avoid imposing unnatural boundary conditions on these functions.

3. THE FINITE VOLUME METHOD

In this section we begin, for completeness, by recalling some known preliminary material concerning the finite volume method, cf. [3], [7], [9], [11], [16], and [17], and then proceed with an error bound. The proof of the latter follows the lines of Theorem 2.1 for the finite element method, but uses the elliptic projection associated with the finite volume bilinear form.

For the analysis we introduce the interpolation operator $I_h : C(\Omega) \rightarrow Y_h$, defined by

$$I_h v = \sum_{z \in Z_h^0} v(z) \Psi_z,$$

where as before Ψ_z is the characteristic function of V_z . We note that $I_h : X_h \rightarrow Y_h$ is a bijection and bounded with respect to the L_2 -norm, and that $\Psi_z = I_h \Phi_z$ for $z \in Z_h^0$. Further, the bilinear form $(\chi, I_h \psi)$ is symmetric positive definite, thus an inner product on X_h , and the corresponding discrete norm is equivalent to the L_2 -norm, uniformly in h , i.e., with $c > 0$,

$$(3.1) \quad c \|\chi\| \leq \|\|\chi\|\| \leq C \|\chi\|, \quad \forall \chi \in X_h, \quad \text{where } \|\|\chi\|\| \equiv (\chi, I_h \chi)^{1/2}.$$

In fact, for z and w neighboring vertices in Z_h^0 , we have, with $W_z = \text{supp } \Phi_z$,

$$m_{zw} = (\Phi_z, \Psi_w) = \int_{W_z \cap V_w} \Phi_z dx = \frac{7}{36} |W_z \cap V_w| = \frac{7}{108} |W_z \cap W_w|$$

which is symmetric in z and w . From this we find

$$(3.2) \quad \sum_{w \neq z} m_{zw} \leq \frac{7}{18} |V_z|,$$

with equality when z is not a neighbor of a vertex of $\partial\Omega$. Since

$$(3.3) \quad m_{zz} = \int_{V_z} \Phi_z dx = \frac{11}{18} |V_z|,$$

we conclude that the mass matrix \mathcal{M} , cf. (1.8), is diagonally dominant which easily shows (3.1).

The bilinear form $a_h(\cdot, \cdot)$ of (1.6) may equivalently be written as

$$(3.4) \quad a_h(\chi, \eta) = \sum_K ((L\chi, \eta)_K + (A\nabla\chi \cdot n, \eta)_{\partial K}), \quad \forall \chi \in X_h, \eta \in Y_h.$$

Indeed, by integration by parts, we obtain, for $z \in Z_h^0$ and $K \in T_h$,

$$\int_{K_z} L\chi dx + \int_{\partial K_z \cap \partial K} A\nabla\chi \cdot n ds = - \int_{\partial K_z \cap \partial V_z} A\nabla\chi \cdot n ds.$$

and (3.4) hence follows by multiplication by $\eta(z)$ and by summation first over the triangles that have z as a vertex and then over the vertices $z \in Z_h^0$. Also, we easily see that, for any side e of a triangle in T_h ,

$$\int_e I_h \chi \, ds = \int_e \chi \, ds, \quad \forall \chi \in X_h.$$

Therefore, in the case of a constant coefficient matrix A we have

$$(3.5) \quad a_h(\chi, I_h \psi) = a(\chi, \psi), \quad \forall \chi, \psi \in X_h,$$

since $L\chi = 0$. For a smooth variable coefficient matrix A , one easily finds, cf., e.g., [7, Lemma 5.2], [17, Lemma 2],

$$|a(\chi, \psi) - a_h(\chi, I_h \psi)| \leq Ch \|\chi\|_{H^1} \|\psi\|_{H^1}, \quad \forall \chi, \psi \in X_h.$$

It follows that there exist positive constants c and h_0 such that

$$(3.6) \quad a_h(\chi, I_h \chi) \geq c \|\chi\|_{H^1}^2, \quad \forall \chi \in X_h, \quad h < h_0.$$

In particular, this shows that the stiffness matrix \mathcal{S} , cf. (1.8), is positive definite.

In view of the above, the Petrov-Galerkin equation in (1.5) may also be written in Galerkin form,

$$(u_{h,t}, I_h \chi) + a_h(u_h, I_h \chi) = (f, I_h \chi), \quad \forall \chi \in X_h, \quad t \geq 0.$$

We now turn to the error estimate for the finite volume method.

Theorem 3.1. *Let u_h and u be the solutions of (1.5) and (1.1), respectively, and assume that $u^0 = g_0 = 0$ on $\partial\Omega$ and $1 < p < p_0$, with p_0 as in (1.18). Then, if $u_h^0 = \tilde{R}_h u^0$, where \tilde{R}_h is the elliptic projection defined by (1.15),*

$$(3.7) \quad \|u_h(t) - u(t)\| \leq C_{T,p}(u^0, f) h^2, \quad t \leq T,$$

where

$$C_{T,p}(u^0, f) = \bar{C}_{T,p} (\|u^0\|_{W_p^3} + \|g_1\|_{L_p} + \int_0^t (\|f_t\|_{W_p^1} + \|f_{tt}\|_{L_p}) \, d\tau).$$

Proof. For the convenience of the reader we briefly show the error estimate (1.14) from [14, 16]. This time we write $u_h - u = (u_h - \tilde{R}_h u) + (\tilde{R}_h u - u) \equiv \tilde{\vartheta} + \tilde{\varrho}$. In view of (1.16) we have

$$(3.8) \quad \|\tilde{\varrho}(t)\| \leq Ch^2 \|u(t)\|_{W_p^3} \leq Ch^2 (\|u^0\|_{W_p^3} + \int_0^t \|u_t\|_{W_p^3} \, d\tau), \quad t \geq 0.$$

In order to bound $\tilde{\vartheta}$ we note that

$$(\tilde{\vartheta}_t, \eta) + a_h(\tilde{\vartheta}, \eta) = -(\tilde{\varrho}_t, \eta), \quad \forall \eta \in Y_h.$$

Choosing $\eta = I_h \tilde{\vartheta}$ and using the positivity of $a_h(\tilde{\vartheta}, I_h \tilde{\vartheta})$, (1.16), and the fact that $\tilde{\vartheta}(0) = 0$, we find easily

$$\|\tilde{\vartheta}(t)\| \leq C \int_0^t \|\tilde{\varrho}_t\| \, d\tau \leq Ch^2 \int_0^t \|u_t\|_{W_p^3} \, d\tau, \quad t \geq 0.$$

Together with (3.8) and (1.17), this shows at once

$$\|u_h(t) - u(t)\| \leq Ch^2(\|u^0\|_{W_p^3} + \int_0^t (\|u_{tt}\|_{W_p^1} + \|f_t\|_{W_p^1}) d\tau), \quad t \geq 0.$$

To prove the theorem it now remains to demonstrate the regularity estimate

$$\int_0^t \|u_{tt}\|_{W_p^1} d\tau \leq C(\|g_1\|_{L_p} + \int_0^t \|f_{tt}\|_{L_p} d\tau), \quad t \leq T.$$

If $g_0 \in W_p^2 \cap H_0^1$, $g_1 \in L_p$, and $f_{tt} \in L_1(0, T; L_p)$, differentiation in time of (2.7) shows

$$(3.9) \quad u_{tt}(t) = E(t)g_1 + \int_0^t E(t-s)f_{tt}(s) ds, \quad t \geq 0,$$

where we have used $f_t(0) - Lg_0 = f_t(0) - Lu_t(0) = u_{tt}(0) = g_1$. One can show that $E(t)$ is an analytic semigroup not only in L_2 but also in L_p , for $1 < p < \infty$, so that

$$(3.10) \quad \|E(t)v\|_{L_p} \leq C\|v\|_{L_p} \text{ and } \|E'(t)v\|_{L_p} \leq C_p t^{-1}\|v\|_{L_p}, \quad 1 < p < \infty.$$

This is proved in [19, Theorem 3.6, Chapter 7] for a domain with smooth boundary, but the arguments are easily seen to be valid also for a polygonal domain. In view of (3.10) and (1.17), we have for $p < p_0$, cf. (1.17),

$$\|E(t)v\|_{W_p^2} \leq C\|LE(t)v\|_{L_p} \leq C\|E'(t)v\|_{L_p} \leq Ct^{-1}\|v\|_{L_p}.$$

By interpolation between this and the first estimate in (3.10), and noting that $[L_p, W_p^2]_{1/2,1} \subset W_p^1$, cf. [4, Chapter 5], [5, Chapter 12], we get

$$(3.11) \quad \|E(t)v\|_{W_p^1} \leq C\|E(t)v\|_{[L_p, W_p^2]_{1/2,1}} \leq Ct^{-1/2}\|v\|_{L_p}.$$

Therefore

$$\begin{aligned} \int_0^t \|u_{tt}\|_{W_p^1} d\tau &\leq C\left(\int_0^t \tau^{-1/2}\|g_1\|_{L_p} d\tau + \int_0^t \int_0^\tau (\tau-s)^{-1/2}\|f_{tt}(s)\|_{L_p} ds d\tau\right) \\ &\leq Ct^{1/2}(\|g_1\|_{L_p} + \int_0^t \|f_{tt}\|_{L_p} d\tau), \quad t > 0. \quad \square \end{aligned}$$

We remark that instead the factor $t^{-1/2}$ in the right-hand side of (3.11), one could have an estimate with $t^{-1+\varepsilon}$ as in (2.9) in the proof of Theorem 2.1. However, since the estimate would then involve negative norms, we will refrain from elaborating on this.

4. ALTERNATIVE ANALYSIS OF THE FINITE VOLUME METHOD

In this section we give an alternative analysis of the finite volume method (1.5), in which the elliptic projection employed is the standard Ritz projection rather than the one based on the bilinear form $a_h(\cdot, \cdot)$. This time we shall show the following error estimate.

Theorem 4.1. *Let u_h and u be the solutions of (1.5) and (1.1), respectively, and assume $u^0 = g_0 = 0$ on $\partial\Omega$ and $\varepsilon \in (0, 1/2)$. Then, if $u_h^0 = R_h u^0$, with R_h defined by (1.10), we have for $t \leq T$,*

$$(4.1) \quad \|u_h(t) - u(t)\| \leq C_{T,\varepsilon}(u^0, f)h^2,$$

where

$$C_{T,\varepsilon}(u^0, f) = C_T \left(\|u^0\|_{H^2} + \|g_0\|_{H^1} + \varepsilon^{-1} \int_0^t \|f_t\|_{H^\varepsilon} d\tau \right. \\ \left. + \left(\int_0^t (\|f_t\|^2 + \|f\|_{H^1}^2) d\tau \right)^{1/2} \right).$$

Proof. This time we split the error as $u_h - u = (u_h - R_h u) + (R_h u - u) = \vartheta + \varrho$. We could use (2.2) and (1.13) to estimate ϱ as before, but since we will need more regularity from data in bounding θ than in the finite element method, we modify (2.11) by using $\varepsilon = 1$ in the term in g_0 and obtain

$$(4.2) \quad \|\varrho(t)\| \leq C_T h^2 \left(\|u^0\|_{H^2} + \|g_0\|_{H^1} + \varepsilon^{-1} \int_0^t \|f_t\|_{H^\varepsilon} d\tau \right), \quad t \leq T.$$

We now turn to the estimate for ϑ . Using (1.7), see that the error $u_h - u$ satisfies $((u_h - u)_t, \eta) + a_h(u_h - u, \eta) = 0$ for all $\eta \in Y_h$ and thus we have for ϑ

$$(4.3) \quad (\vartheta_t, \eta) + a_h(\vartheta, \eta) = -(\varrho_t, \eta) - a_h(\varrho, \eta), \quad \forall \eta \in Y_h.$$

For the analysis we introduce the error functionals

$$(4.4) \quad \begin{aligned} \varepsilon_h(f, \chi) &= (f, \chi) - (f, I_h \chi), \quad \forall f \in L_2, \chi \in X_h, \\ \varepsilon_a(\chi, \psi) &= a(\chi, \psi) - a_h(\chi, I_h \psi), \quad \forall \chi, \psi \in X_h. \end{aligned}$$

We note that

$$\begin{aligned} a_h(R_h u, I_h \vartheta) &= a(R_h u, \vartheta) - \varepsilon_a(R_h u, \vartheta) = a(u, \vartheta) - \varepsilon_a(R_h u, \vartheta) \\ &= (Lu, \vartheta) - \varepsilon_a(R_h u, \vartheta), \end{aligned}$$

and hence, since $a_h(u, I_h \vartheta) = (Lu, I_h \vartheta)$ by (1.7),

$$(4.5) \quad a_h(\varrho, I_h \vartheta) = \varepsilon_h(Lu, \vartheta) - \varepsilon_a(R_h u, \vartheta).$$

Choosing $\eta = I_h \vartheta$ in (4.3) we therefore have, since $Lu = f - u_t$,

$$(4.6) \quad (\vartheta_t, I_h \vartheta) + a_h(\vartheta, I_h \vartheta) = -(\varrho_t, I_h \vartheta) + \varepsilon_h(u_t - f, \vartheta) + \varepsilon_a(R_h u, \vartheta).$$

The following bounds for $\varepsilon_h(\cdot, \cdot)$ and $\varepsilon_a(\cdot, \cdot)$ are shown in [7, Lemmas 5.1 and 5.2].

Lemma 4.1. *Let $\chi \in X_h$, then*

$$\begin{aligned} |\varepsilon_h(f, \chi)| &\leq Ch^{i+j} \|f\|_{H^i} \|\chi\|_{H^j}, \quad f \in H^i, \quad i, j = 0, 1, \\ |\varepsilon_a(R_h v, \chi)| &\leq Ch^{i+j} \|v\|_{H^{1+i}} \|\chi\|_{H^j}, \quad v \in H^{1+i} \cap H_0^1, \quad i, j = 0, 1, \end{aligned}$$

Using the symmetry of $(\chi, I_h \psi)$ on X_h , together with (1.11) and (2.3) we therefore obtain from (4.6)

$$(4.7) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\vartheta\|^2 + a_h(\vartheta, I_h \vartheta) &\leq C \|\varrho_t\| \|\vartheta\| + Ch^2 (\|u_t\|_{H^1} + \|f\|_{H^1} + \|u\|_{H^2}) \|\vartheta\|_{H^1} \\ &\leq Ch^2 (\|u_t\|_{H^2} + \|f\|_{H^1}) \|\vartheta\|_{H^1} \\ &\leq Ch^2 (\|u_{tt}\| + \|f_t\| + \|f\|_{H^1}) \|\vartheta\|_{H^1}, \quad t > 0. \end{aligned}$$

Using (3.6) to kick back $\|\vartheta\|_{H^1}$, integrating, and using (3.1) we obtain, since $\vartheta(0) = 0$,

$$(4.8) \quad \|\vartheta(t)\|^2 \leq Ch^4 \int_0^t (\|u_{tt}\|^2 + \|f_t\|^2 + \|f\|_{H^1}^2) d\tau.$$

To complete the estimation of ϑ we now only need the regularity estimate

$$(4.9) \quad \int_0^t \|u_{tt}\|^2 d\tau \leq C(\|g_0\|_{H^1}^2 + \int_0^t \|f_t\|^2 d\tau), \quad t \leq T.$$

A more general such estimate is derived in Evans [12, Chapter 7], for $\partial\Omega$ smooth, using the method of Galerkin approximations. In the special case we need, the regularity of $\partial\Omega$ is not required. Finally, by combining (4.2), (4.8), and (4.9), we obtain the desired estimate (4.1). \square

5. SOME OTHER ESTIMATES

In this section we shall use our technique from Section 4, based on the standard Ritz projection, to derive error bounds also in H^1 and L_∞ norms. We begin with the bound in H^1 .

Theorem 5.1. *Let u_h and u be the solutions of (1.5) and (1.1), respectively, and assume that $u^0 = 0$ on $\partial\Omega$ and $\varepsilon \in (0, 1/2)$. Then, if $u_h^0 = R_h u^0$, with R_h defined in (1.10), we have*

$$(5.1) \quad \|u_h(t) - u(t)\|_{H^1} \leq C_{T,\varepsilon}(u^0, f)h, \quad t \leq T,$$

where

$$C_{T,\varepsilon}(u^0, f) = C_T \left(\|u^0\|_{H^2} + \|g_0\| + \left(\int_0^t \|f_t\|^2 d\tau \right)^{1/2} + \varepsilon^{-1} (\|g_0\|_{H^\varepsilon} + \int_0^t \|f_t\|_{H^\varepsilon} d\tau) \right).$$

Proof. In view of (1.11) we have

$$(5.2) \quad \|\varrho(t)\|_{H^1} \leq Ch \|u(t)\|_{H^2} \leq Ch (\|u^0\|_{H^2} + \int_0^t \|u_t\|_{H^2} d\tau), \quad t \geq 0.$$

Using (1.13) this shows that $\varrho(t)$ is bounded as desired.

We now turn to the estimation of ϑ , using the identity (4.3) with $\eta = I_h \vartheta_t$. Since, $a_h(\chi, I_h \psi) = a(\chi, \psi) - \varepsilon_a(\chi, \psi)$ then

$$a_h(\vartheta, I_h \vartheta_t) = \frac{1}{2} \frac{d}{dt} a(\vartheta, \vartheta) - \varepsilon_a(\vartheta, \vartheta_t).$$

Thus, from (4.3) and (4.5) we get

$$\|\vartheta_t\|^2 + \frac{1}{2} \frac{d}{dt} a(\vartheta, \vartheta) = -(\varrho_t, I_h \vartheta_t) + \varepsilon_a(\vartheta, \vartheta_t) - \varepsilon_h(Lu, \vartheta_t) + \varepsilon_a(R_h u, \vartheta_t).$$

Substituting $Lu = f - u_t$ this yields

$$(5.3) \quad \begin{aligned} \|\vartheta_t\|_{H^1}^2 + \frac{1}{2} \frac{d}{dt} a(\vartheta, \vartheta) &= -(\varrho_t, I_h \vartheta_t) + \varepsilon_a(\vartheta, \vartheta_t) + \varepsilon_h(u_t, \vartheta_t) \\ &\quad - \frac{d}{dt} \varepsilon_h(f, \vartheta) + \varepsilon_h(f_t, \vartheta) + \varepsilon_a(R_h u, \vartheta_t). \end{aligned}$$

Integrating this relation over $[0, t]$, $t \leq T$, using the fact that $\vartheta(0) = 0$, together with (1.11), Lemma 4.1, (3.1), and (2.3), we find

$$\|\vartheta(t)\|_{H^1}^2 \leq Ch^2(\|f(t)\|^2 + \int_0^t (\|u_t\|_{H^1}^2 + \|u\|_{H^2}^2 + \|f_t\|^2) d\tau) + C \int_0^t \|\vartheta\|_{H^1}^2 d\tau.$$

Using Gronwall's lemma and the simple estimate

$$\|u(t)\|_{H^2}^2 \leq C(\|u_t(t)\|^2 + \|f(t)\|^2) \leq C\left(\|u_t(t)\|^2 + \|f(0)\|^2 + \int_0^t \|f_t\|^2 d\tau\right),$$

now gives

$$(5.4) \quad \|\vartheta(t)\|_{H^1}^2 \leq Ch^2(\|f(0)\|^2 + \int_0^t (\|u_t\|_{H^1}^2 + \|f_t\|^2) d\tau), \quad t \leq T.$$

Similarly to (4.9), one can show

$$(5.5) \quad \int_0^t \|u_t\|_{H^1}^2 d\tau \leq C(\|g_0\|^2 + \int_0^t \|f_t\|^2 d\tau), \quad t \leq T,$$

by slightly modifying the proof of the corresponding result in [12]. Combined with (5.4), this shows the desired bound for $\vartheta(t)$. Together with (5.2) this completes the proof. \square

We now turn to some error estimates in the maximum-norm. Writing as usual $u_h - u = (u_h - R_h u) + (R_h u - u) = \vartheta + \varrho$ we will show $\|\vartheta\|_{L^\infty} = O(h^2(\log \frac{1}{h})^{1/2})$. At a corner the solution normally has a singularity, the strength of which depends on the angle at the corner. Below we show that $\|\varrho\|_{L^\infty} = O(h^\gamma)$ for any $\gamma < \beta = \pi/\omega$, cf. (1.18), and since $\beta > 1$ we can take $\gamma > 1$ as well. In our first result below we show a global maximum-norm error estimate of this order. This will be done using an analysis of Schatz [20] for the elliptic problem in a polygonal domain. We remark that when the domain has a smooth boundary it is known that $\|\varrho\|_{L^\infty} = O(h^2(\log \frac{1}{h})^2)$, see [23, Chapter 5, Lemma 5.6]. The loss of order of convergence only takes place near the corners, and in second result below we show a bound of the same order on domains $\Omega_0 \subset \Omega$, with $\bar{\Omega}_0$ not containing any vertex of Ω .

We now state and show the global error estimate.

Theorem 5.2. *Let u_h and u be the solutions of (1.5) and (1.1), respectively, and assume $u^0 = g_0 = 0$ on $\partial\Omega$, and that the family of triangulations $\{T_h\}_{0 < h < 1}$ is quasi-uniform. Then if $u_h^0 = R_h u^0$ we have, for any $\gamma < \beta$,*

$$(5.6) \quad \|u_h(t) - u(t)\|_{L^\infty} \leq C_{T,\gamma}(u^0, f)h^\gamma, \quad t \leq T,$$

where

$$C_{T,\gamma}(u^0, f) = \bar{C}_{T,\gamma} \left(\|u^0\|_{H^2} + \|g_0\|_{L^\infty} + \|g_0\|_{H^1} + \|g_1\| + \|f(0)\|_{L^\infty} \right. \\ \left. + \int_0^t \|f_t\|_{L^\infty} d\tau + \left(\int_0^t (\|f_t\|_{H^1}^2 + \|f_{tt}\|^2) d\tau \right)^{1/2} \right).$$

Proof. We begin by showing that for $\varrho = R_h u - u$ we have

$$(5.7) \quad \|\varrho(t)\|_{L^\infty} \leq C_\gamma h^\gamma \left(\|g_0\|_{L^\infty} + \|f(0)\|_{L^\infty} + \int_0^t \|f_t\|_{L^\infty} d\tau \right).$$

We shall assume that $1 < \gamma < \beta < 2$; the proof may easily be modified to cover the case $\beta \geq 2$. Letting $\delta = (\beta + \gamma)/2$ we have in view of [20, (0.8)] that

$$\|\varrho(t)\|_{L^\infty} \leq Ch^\delta \log \frac{1}{h} \|u(t)\|_{C^{1,\delta-1}} \leq Ch^\gamma \|u(t)\|_{C^{1,\delta-1}},$$

where $C^{1,\alpha}$ is the space of continuously differentiable functions whose first order derivative fulfill a uniform Hölder condition of order α . A standard imbedding argument, cf., e.g., [15, Theorem 1.4.5.2], shows

$$\|u(t)\|_{C^{1,\delta-1}} \leq C_\gamma \|u(t)\|_{W_p^2}, \quad \text{with } p = 2/(2 - \delta),$$

and since $p < 2/(2 - \beta)$, we have the elliptic regularity estimate, cf., e.g. [15, Theorem 5.2.7],

$$(5.8) \quad \|u\|_{W_p^2} \leq C \|Lu\|_{L^p}.$$

Combining (2.7) and $\|E(t)\|_{L^\infty} \leq 1$, we obtain

$$\|\varrho(t)\|_{L^\infty} \leq C_\gamma h^\gamma \|Lu\|_{L^\infty} \leq C_\gamma h^\gamma (\|u_t(t)\|_{L^\infty} + \|f(t)\|_{L^\infty}) \\ \leq C_\gamma h^\gamma (\|g_0\|_{L^\infty} + \|f(0)\|_{L^\infty} + \int_0^t \|f_t\|_{L^\infty} d\tau).$$

As indicated above, in the case that $\partial\Omega$ is smooth, we have $\|\varrho\|_{L^\infty} = O(h^2(\log \frac{1}{h})^2)$. This is seen by using the estimates

$$(5.9) \quad \|\varrho\|_{L^\infty} \leq C \log \frac{1}{h} \|I_h u - u\|_{L^\infty} \leq Ch^{2-2/p} \log \frac{1}{h} \|u\|_{W_p^2}$$

where the error bound for the interpolant is easily obtained by the Bramble Hilbert lemma, cf., e.g., [5, (4.4.29)], the elliptic regularity estimate

$$(5.10) \quad \|u\|_{W_p^2} \leq Cp \|Lu\|_{L^p}, \quad 2 \leq p < \infty,$$

and choosing $p = \log \frac{1}{h}$, cf. [23, Lemma 5.6]. The estimate (5.10) does not hold for polygonal domains for p large, cf. [15, Remark 4.3.2.6].

We now turn to the required estimate for $\vartheta = u_h - R_h u$. We will show

$$(5.11) \quad \|\vartheta(t)\|_{L^\infty} \leq Ch^2 \left(\log \frac{1}{h} \right)^{1/2} \left(\|u^0\|_{H^2} + \|g_0\|_{H^1} + \|g_1\| \right. \\ \left. + \left(\int_0^t (\|f_t\|_{H^1}^2 + \|f_{tt}\|^2) d\tau \right)^{1/2} \right).$$

In view of the well-known inequality, cf., e.g., [23, Chapter 5],

$$\|\chi\|_{L^\infty} \leq C \left(\log \frac{1}{h}\right)^{1/2} \|\nabla \chi\|, \quad \forall \chi \in X_h,$$

it suffices to demonstrate

$$\|\vartheta(t)\|_{H^1} \leq Ch^2 \left(\|u^0\|_{H^2} + \|g_0\|_{H^1} + \|g_1\| + \left(\int_0^t (\|f_t\|_{H^1}^2 + \|f_{tt}\|^2) d\tau \right)^{1/2} \right).$$

For this we rewrite (5.3) as

$$\begin{aligned} \|\vartheta_t\|^2 + \frac{1}{2} \frac{d}{dt} a(\vartheta, \vartheta) &= -(\varrho_t, I_h \vartheta_t) + \varepsilon_a(\vartheta, \vartheta_t) \\ &+ \frac{d}{dt} \varepsilon_h(u_t - f, \vartheta) - \varepsilon_h(u_{tt} - f_t, \vartheta) + \frac{d}{dt} \varepsilon_a(R_h u, \vartheta) - \varepsilon_a(R_h u_t, \vartheta). \end{aligned}$$

Since $\vartheta(0) = 0$, (1.11), Lemma 4.1, (3.1), and (4.8) give, for $t \geq 0$,

$$\begin{aligned} \|\vartheta(t)\|_{H^1}^2 &\leq Ch^4 \left(\|u_t(t)\|_{H^1}^2 + \|u(t)\|_{H^2}^2 + \|f(t)\|_{H^1}^2 \right. \\ &\quad \left. + \int_0^t (\|u_{tt}\|_{H^1}^2 + \|u_t\|_{H^2}^2 + \|f_t\|_{H^1}^2) d\tau \right) + C \int_0^t \|\vartheta\|_{H^1}^2 d\tau, \end{aligned}$$

and by Gronwall's lemma and obvious estimates we now have, for $t \leq T$,

$$\|\vartheta(t)\|_{H^1}^2 \leq Ch^4 \left(\|u^0\|_{H^2}^2 + \|g_0\|_{H^1}^2 + \int_0^t (\|u_{tt}\|_{H^1}^2 + \|f_t\|_{H^1}^2) d\tau \right).$$

To complete the proof we need the regularity estimate

$$\int_0^t \|u_{tt}\|_{H^1}^2 d\tau \leq C(\|g_1\|^2 + \int_0^t \|f_{tt}\|^2 d\tau), \quad t \leq T,$$

which can be found in [12] in the same way as (4.9).

We note that since we only require a $O(h^\beta)$ estimate for $\|\vartheta\|_{L^\infty}$, the regularity assumptions used in this part of the proof could be reduced. \square

We now turn to the error away from the corners, and demonstrate that there we can improve the convergence rate to essentially second order.

Theorem 5.3. *Let $\Omega_0 \subset \Omega$ be such that $\bar{\Omega}_0$ does not contain any vertex of Ω , and assume the family of triangulations $\{T_h\}_{0 < h < 1}$ is quasi-uniform. Let u_h and u be the solutions of (1.5) and (1.1), respectively, and assume $u^0 = g_0 = 0$ on $\partial\Omega$, $\varepsilon \in (0, 1/2)$ and $2 < p < 2/(2 - \beta)$. Then, if $u_h^0 = R_h u^0$, we have*

$$(5.12) \quad \|u_h(t) - u(t)\|_{L^\infty(\Omega_0)} \leq C_{T,p,\varepsilon}(u^0, f) h^2 \log \frac{1}{h}, \quad t \leq T,$$

where

$$\begin{aligned} C_{T,p,\varepsilon}(u^0, f) &= C_{T,p} \left(\|u^0\|_{H^2} + \|g_0\|_{H^1} + \|g_1\| + \|f(t)\|_{W_p^1} + \|f_t(0)\| \right. \\ &\quad \left. + \int_0^t \|f_{tt}\| d\tau + \left(\int_0^t (\|f_t\|_{H^1}^2 + \|f_{tt}\|^2) d\tau \right)^{1/2} + \varepsilon^{-1} \int_0^t \|f_t\|_{H^\varepsilon} d\tau \right). \end{aligned}$$

Proof. It follows from (5.11) in the proof of the previous theorem that $\|\vartheta\|_{L^\infty}$ is bounded as stated. It therefore suffices to consider ϱ , and for this we shall use the following lemma which is a special case of a result from a forthcoming paper of Schatz [21].

Lemma 5.1. *Let Ω be a convex polygonal domain in \mathbb{R}^2 and let $\Omega_0 \subset \Omega_1 \subset \Omega$, be such that Ω_1 does not contain any corners of Ω and the distance between $\partial\Omega_1 \cap \Omega$ and $\partial\Omega_0 \cap \Omega$ is positive. Then, if $v_h \in X_h$ satisfies*

$$a(v_h - v, \chi) = 0, \quad \forall \chi \in X_h,$$

we have, for any $\chi \in X_h$,

$$\|v_h - v\|_{L^\infty(\Omega_0)} \leq C \left(\log \frac{1}{h} \|v - \chi\|_{L^\infty(\Omega_1)} + \|v_h - v\|_{L_2} \right).$$

Let Ω_2 and Ω_3 be domains with $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \Omega$ and smooth boundaries and such that Ω_3 does not contain any corner of Ω . Assume further that the distances between $\partial\Omega_3 \cap \Omega$, $\partial\Omega_2 \cap \Omega$, and $\partial\Omega_1 \cap \Omega$ are positive. Let w be a smooth cutoff function such that $w|_{\Omega_2} = 1$ and $w|_{\partial\Omega_3 \cap \Omega} = 0$.

Let $\bar{I}_h : C(\Omega) \rightarrow X_h$, be the standard nodal interpolant. Then in view of the interpolation error estimate

$$\|\bar{I}_h u - u\|_{L^\infty(\Omega_1)} \leq Ch^2 \|u\|_{W_\infty^2(\Omega_2)},$$

Lemma 5.1 gives

$$\|\rho(t)\|_{L^\infty(\Omega_0)} \leq Ch^2 \log \frac{1}{h} \|u(t)\|_{W_\infty^2(\Omega_2)} + C \|\rho(t)\|.$$

Since the last term is bounded as desired by (4.2) it now remains to bound the first term on the right. Using first a Sobolev inequality and then an elliptic regularity estimate in Ω_3 (recalling that $\partial\Omega_3$ is smooth), and, finally, the elliptic regularity estimate (5.8) in Ω , we have, with $\tilde{u} = wu$,

$$\begin{aligned} \|u\|_{W_\infty^2(\Omega_2)} &\leq C \|u\|_{W_p^3(\Omega_2)} \leq C \|\tilde{u}\|_{W_p^3(\Omega_3)} \leq C \|L\tilde{u}\|_{W_p^1(\Omega_3)} \\ &\leq C (\|Lu\|_{W_p^1} + \|u\|_{W_p^2}) \leq C \|Lu\|_{W_p^1}. \end{aligned}$$

Using the differential equation, and then again a Sobolev inequality, followed by (2.3), we find

$$\begin{aligned} \|Lu\|_{W_p^1} &\leq C (\|f\|_{W_p^1} + \|u_t\|_{W_p^1}) \leq C (\|f\|_{W_p^1} + \|u_t\|_{H^2}) \\ &\leq C (\|f\|_{W_p^1} + \|u_{tt}\| + \|f_t\|). \end{aligned}$$

By (3.9), with $\|E(t)\| \leq 1$, we hence find

$$\|u(t)\|_{W_\infty^2(\Omega_2)} \leq C (\|f(t)\|_{W_p^1} + \|g_1\| + \|f_t(0)\| + \int_0^t \|f_{tt}\| d\tau),$$

which completes the proof of the desired estimate for ρ and hence of the theorem. \square

6. THE LUMPED MASS FINITE VOLUME METHOD

In addition to the finite volume method (1.5) studied above we consider also a lumped mass variant of this method, which we define by approximating the first integral in (1.4) by means of a simple quadrature formula. We thus seek $u_h(t) \in X_h$ satisfying

$$(6.1) \quad \begin{aligned} u_{h,t}(z, t)|V_z| - \int_{\partial V_z} (A\nabla u_h) \cdot n \, ds &= \int_{V_z} f \, dx, \quad \forall z \in Z_h^0, \, t \geq 0, \\ u_h(0) &= u_h^0. \end{aligned}$$

Note that this discrete scheme is no longer locally conservative.

In order to rewrite (6.1) in variational form we consider the quadrature formula

$$Q_{K,h}(f) = \frac{1}{3}|K| \sum_{z \in Z_h(K)} f(z) \approx \int_K f \, dx, \quad \text{for } K \in T_h,$$

and the associated bilinear form in $X_h \times Y_h$,

$$(\chi, \psi)_h = \sum_{K \in T_h} Q_{K,h}(\chi\psi) = \sum_{z \in Z_h^0} \chi(z)\psi(z)|V_z|, \quad \forall \chi \in X_h, \, \psi \in Y_h.$$

We note that $\|\chi\|_h := (\chi, I_h\chi)_h^{1/2}$ is a norm in X_h which is equivalent with the L_2 -norm, uniformly in h , i.e., with $c > 0$,

$$(6.2) \quad c\|\chi\| \leq \|\chi\|_h \leq C\|\chi\|, \quad \forall \chi \in X_h.$$

For an arbitrary $\eta \in Y_h$, by multiplying (6.1) by $\eta(z)$ and summing over all $z \in Z_h^0$, we obtain the Petrov-Galerkin formulation

$$(6.3) \quad (u_{h,t}, \eta)_h + a_h(u_h, \eta) = (f, \eta), \quad \forall \eta \in Y_h, \, t \geq 0.$$

For this analogue of (1.5) the mass matrix \mathcal{M} in (1.8) is replaced by the diagonal matrix $\overline{\mathcal{M}} = (\overline{m}_{zz})$ where $\overline{m}_{zz} = \|\Phi_z\|_h^2$. The name lumped mass method is inherited from its analogue for the finite element equation (1.2) and is motivated by the fact that with $W_z = \text{supp } \Phi_z$ and $V_z = \text{supp } \Psi_z$ we have, except when z is a neighbor of a boundary vertex, $W_z \subset \cup_w V_w$ and hence, cf. also (3.2) and (3.3),

$$\sum_{w \in Z_h^0} m_{zw} = (\Phi_z, \sum_w \Psi_w) = \int_{\cup_w V_w} \Phi_z \, dx = |V_z| = \overline{m}_{zz}.$$

When z has a neighbor on $\partial\Omega$ we have $\overline{m}_{zz} > \sum_{w \in Z_h^0} m_{zw}$.

We will now show that the L_2 -norm error bound of Theorem 4.1 remains valid for the lumped mass method (6.3).

Theorem 6.1. *Let u_h and u be the solutions of (6.3) and (1.1), respectively, and assume $u^0 = g_0 = 0$ on $\partial\Omega$, and $\varepsilon \in (0, 1/2)$. Then, if $u_h^0 = R_h u^0$, the error bound (4.1) holds.*

Proof. In view of our earlier estimate (4.2) for $\rho = R_h u - u$, it suffices to bound $\vartheta = u_h - R_h u$, which now satisfies

$$(6.4) \quad (\vartheta_t, \eta)_h + a_h(\vartheta, \eta) = -(\varrho_t, \eta) - a_h(\varrho, \eta) + \bar{\varepsilon}_h(R_h u_t, \eta), \quad \forall \eta \in Y_h,$$

where $\bar{\varepsilon}_h(\cdot, \cdot)$ is the quadrature error defined by

$$\bar{\varepsilon}_h(\chi, \eta) = (\chi, \eta) - (\chi, \eta)_h, \quad \forall \chi \in X_h, \eta \in Y_h.$$

Note that the equation (6.4) differs from (4.3) only in the first term on the left and in the addition of the last one on the right. We note that the above definition of $(\chi, \psi)_h$ may be used also for $\psi \in X_h$, and that $(\chi, \psi)_h = (\chi, I_h \psi)_h$ for $\chi, \psi \in X_h$. We therefore have, using the notation (4.4),

$$\bar{\varepsilon}_h(\chi, I_h \psi) = -\varepsilon_h(\chi, \psi) + ((\chi, \psi) - (\chi, \psi)_h).$$

It follows from, e.g., [23, Lemma 15.1], that

$$|(\chi, \psi)_h - (\chi, \psi)| \leq Ch^2 \|\chi\|_{H^1} \|\psi\|_{H^1}, \quad \forall \chi, \psi \in X_h.$$

and hence, using also Lemma 4.1, that

$$(6.5) \quad |\bar{\varepsilon}_h(R_h u_t, I_h \vartheta)| \leq Ch^2 \|u_t\|_{H^1} \|\vartheta\|_{H^1}.$$

Therefore, setting $\eta = I_h \vartheta$ in (6.4), and using (6.2) and (6.5), we obtain (4.8) as before, which shows the theorem. \square

We remark that the error bounds of Theorems 5.1, 5.2, and 5.3 also hold for the solution u_h of (6.3). In fact, the estimates for ϱ in the proofs of these theorems remain unchanged, and the bounds for ϑ in Section 5 may be shown from (6.4) similarly to the corresponding bounds in Theorem 6.1.

As in Theorem 3.1, instead of using R_h we could use \tilde{R}_h to obtain an L^2 -norm error estimate for the solution u_h of (6.3). In such a case $\tilde{\vartheta} = \tilde{R}_h u - u$ would satisfy

$$(\tilde{\vartheta}_t, \eta)_h + a_h(\tilde{\vartheta}, \eta) = -(\tilde{\varrho}_t, \eta) - \bar{\varepsilon}_h(\tilde{R}_h u_t, \eta), \quad \forall \eta \in Y_h.$$

and, in view of (6.2), (6.5), and (1.17) we would now obtain

$$\|\tilde{\vartheta}(t)\|^2 \leq Ch^4 \left(\int_0^t (\|u_{tt}\|_{W_p^1}^2 + \|f_t\|_{W_p^1}^2) d\tau \right), \quad \text{for } p < p_0.$$

However, since stronger assumptions on f_t than those in (4.8) are required to bound the integral, we shall not elaborate on this choice.

We recall that with the above notation the lumped mass finite element method for (1.1) is

$$(6.6) \quad (\underline{u}_{h,t}, \chi)_h + a(\underline{u}_h, \chi) = (f, \chi), \quad \forall \chi \in X_h, \quad t \geq 0.$$

We remark that in the case of a constant coefficient matrix A , the left sides of (6.6) and (6.3) with $\eta = I_h \chi$ are the same, cf. (3.5), so that the two lumped mass methods differ only in the treatment of the inhomogeneous term.

7. FULLY DISCRETE METHODS

In this section, for completeness, we will show by an example that our approach to the finite volume method applies also to fully discrete schemes. We consider the backward Euler method for the discretization in time of (1.5). Letting k be the time step, U^n the

approximation in X_h of $u(t)$ at $t = t_n = nk$, and $\bar{\partial}U^n = (U^n - U^{n-1})/k$, this method is defined by

$$(7.1) \quad \begin{aligned} (\bar{\partial}U^n, \eta) + a_h(U^n, \eta) &= (f(t_n), \eta), \quad \forall \eta \in Y_h, \quad n \geq 1, \\ U^0 &= u_h^0. \end{aligned}$$

We then have the following error estimate.

Theorem 7.1. *Let U^n and $u(t_n)$ be the solutions of (7.1) and (1.1), respectively, and assume $u^0 = g_0 = 0$ on $\partial\Omega$ and $\varepsilon \in (0, 1/2)$. Then, if $U^0 = R_h u^0$, we have*

$$(7.2) \quad \|U^n - u(t_n)\| \leq C_{T,\varepsilon}(u^0, f)h^2 + \bar{C}_T(u^0, f)k, \quad t_n \leq T,$$

where

$$C_{T,\varepsilon}(u^0, f) = C_T \left(\|u^0\|_{H^2} + \|g_0\|_{H^1} + \varepsilon^{-1} \int_0^{t_n} \|f_t\|_{H^\varepsilon} d\tau + \left(k \sum_{j=0}^n \|f^j\|_{H^1}^2 \right)^{\frac{1}{2}} \right)$$

and

$$\bar{C}_T(u^0, f) = \bar{C}_T \left(\|g_0\|_{H^1} + \left(\int_0^{t_n} \|f_t\|^2 d\tau \right)^{\frac{1}{2}} \right).$$

Proof. We write

$$U^n - u(t_n) = (U^n - R_h u(t_n)) + (R_h u(t_n) - u(t_n)) \equiv \vartheta^n + \varrho^n.$$

and since the estimate for $\rho^n = \rho(t_n)$ is the same as before, it suffices to bound ϑ^n . We have

$$(\bar{\partial}\vartheta^n, \eta) + a_h(\vartheta^n, \eta) = -(\bar{\partial}\varrho^n, \eta) - a_h(\varrho^n, \eta) + (u_t^n - \bar{\partial}u^n, \eta), \quad \forall \eta \in Y_h.$$

Choosing $\eta = I_h \vartheta^n$ we obtain, with $\|\cdot\|$ defined in (3.1), cf. (4.6),

$$\begin{aligned} & \frac{1}{2k} (\|\vartheta^n\|^2 - \|\vartheta^{n-1}\|^2) + \frac{1}{2k} \|\vartheta^n - \vartheta^{n-1}\|^2 + a_h(\vartheta^n, I_h \vartheta^n) \\ &= -(\bar{\partial}\varrho^n, I_h \vartheta^n) + \varepsilon_h (u_t^n - f^n, \vartheta^n) + \varepsilon_a (R_h u^n, \vartheta^n) + (u_t^n - \bar{\partial}u^n, I_h \vartheta^n) \\ &= -(\bar{\partial}\varrho^n, I_h \vartheta^n) + \varepsilon_h (\bar{\partial}u^n - f^n, \vartheta^n) + \varepsilon_a (R_h u^n, \vartheta^n) + (u_t^n - \bar{\partial}u^n, \vartheta^n) \end{aligned}$$

The three first terms on the right are bounded, as in (4.7), by

$$\begin{aligned} & Ch^2 (\|\bar{\partial}u^n\|_{H^2} + \|u^n\|_{H^2} + \|f^n\|_{H^1}) \|\vartheta^n\|_{H^1} \\ & \leq Ch^2 \left(k^{-1} \int_{t_{n-1}}^{t_n} \|u_t\|_{H^2} d\tau + \|u^n\|_{H^2} + \|f^n\|_{H^1} \right) \|\vartheta^n\|_{H^1} \end{aligned}$$

and the fourth by

$$\|u_t^n - \bar{\partial}u^n\| \|\vartheta^n\| \leq C \int_{t_{n-1}}^{t_n} \|u_{tt}\| d\tau \|\vartheta^n\|_{H^1} \leq C \left(k \int_{t_{n-1}}^{t_n} \|u_{tt}\|^2 d\tau \right)^{\frac{1}{2}} \|\vartheta^n\|_{H^1},$$

so that, after multiplication by $2k$ and kicking back $\|\vartheta^n\|_{H^1}$,

$$\begin{aligned} \|\vartheta^n\|^2 - \|\vartheta^{n-1}\|^2 &\leq Ch^4 \left(\int_{t_{n-1}}^{t_n} \|u_t\|_{H^2}^2 d\tau + k(\|u^n\|_{H^2}^2 + \|f^n\|_{H^1}^2) \right) \\ &\quad + Ck^2 \int_{t_{n-1}}^{t_n} \|u_{tt}\|^2 d\tau. \end{aligned}$$

In view of (3.1), using $\|u_t\|_{H^2} \leq C(\|u_{tt}\| + \|f_t\|)$ together with (4.9), and since $\vartheta^0 = 0$, summation completes the proof. \square

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