APPROXIMATION THEORY:
A volume dedicated to Blagovest Sendov
(B. Bojanov, Ed.), DARBA, Sofia, 2002, pp. 258-283.

Numerical Methods for Convection-Diffusion Problems on General Grids

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This paper summarizes a number of discretization and solution techniques that are commonly referred to as finite volume and mixed finite element methods. We describe some of them in more detail, that is, provide a common discretization strategy as well as address the issue of solving the resulting systems of linear algebraic equations. The paper focuses on locally conservative discretizations that include both unstructured and non–matching grids.

1. Introduction

The aim of this paper is to summarize the results of the authors and their collaborates, and other researchers in the area of construction and study of numerical methods for convection-diffusion-reaction equations. Such problems occur in mathematical modeling of wide range of scientific and technical phenomena such as heat and mass-transfer, magneto-statics and electro-statics, flow and transport in porous media (related to petroleum and ground-water applications) etc.

The advances in mathematical modeling have made it possible to set up complex models that describe the interaction of various physical processes. Further, the rapid developments in the computer technology, including computer graphics, visualization, and grid generation, allow a computational method to be set up on grids with millions of points and to handle systems of equations with tens of millions of unknowns. The increased complexity of the mathematical models in combination with grid generation, CAD/CAM, and other specialized systems force us to depart from the standard rectangular grids. Moreover, parallel and adaptive methods may often lead to grids that do not match along certain interfaces. This create a need for general, flexible, efficient,

*The work of R. Lazarov has been partially supported by the National Science Foundation under Grant DMS-9973328. The work of P. S. Vassilevski was performed under the auspices of the U. S. Department of Energy by the University of California Lawrence Livermore National Laboratory under contract W-7405-Eng-48.

and accurate approximation methods for differential equations in complex domains with solutions that may exhibit localized singular behavior due to various physical factors. In summary, the need of methods with properties that will guarantee fast, efficient, accurate computations has been a driving force in the numerical methods for differential equations in the last two decades.

In this paper we concentrate our attention on the following model second order elliptic boundary value problem:

$$\left\{\begin{array}{rcll} \mathcal{L}p \equiv \nabla\cdot\big(-a\nabla p + \underline{b}p\big) + c_0 p &=& f, & \text{in } \Omega,\\ p &=& g_D & \text{on } \Gamma_D,\\ -a\nabla p\cdot\mathbf{n} &=& 0 \ \text{ on } \ \Gamma_N^{out}, & \\ (-a\nabla p + \underline{b}p)\cdot\mathbf{n} &=& g_N \ \text{ on } \ \Gamma_N^{in}. & \end{array}\right. \tag{1.1}$$

Here Ω is a bounded polygonal domain in $\mathcal{R}^d$, $d = 2$ or 3 with a boundary Γ; $a = a(x)$ is $d \times d$ symmetric matrix, which is assumed uniformly in x positive definite and bounded in Ω, $\underline{b} = \underline{b}(x) = (b_1(x), \ldots, b_n(x))$ is a given bounded vector function, $c_0 = c_0(x)$ is the given bounded function, g_D and g_N are given boundary data, and $f = f(x) \in L_2(\Omega)$ is a given source function. We have also used the notation ∇p for the gradient of a scalar function p and $\nabla\cdot\underline{b}$ for the divergence of a vector function $\underline{b}$ in $\mathcal{R}^d$. The boundary Γ is split into two parts: $\Gamma \equiv \Gamma_D \cup \Gamma_N$ and further Γ_N is split into two parts, namely, an inflow boundary $\Gamma_N^{in} = \{x \in \Gamma_N : \mathbf{n}(x)\cdot\underline{b}(x) \le 0\}$ and an outflow part $\Gamma_N^{out} = \{x \in \Gamma_N : \mathbf{n}(x)\cdot\underline{b}(x) > 0\}$. Here $\mathbf{n}(x)$ is the unit outward vector normal to Γ at point x.

This problem is a prototype for flow and transport in porous media. For example, $p(x)$ can represent the pressure head in an aquifer or the concentration of a chemical dissolved and distributed in the ground-water due to processes of advection, diffusion, and absorption. In many cases $a = \epsilon I$, where I is the identity matrix in $\mathcal{R}^d$ and $\epsilon > 0$ is a small parameter. This corresponds to the important and difficult class of singularly perturbed convection–diffusion problem (see, e.g. the monograph of Ross, Stynes, and Tobiska [58]). This problem can be viewed also as the steady-state solution to a corresponding time dependent linear problem (see, e.g. the monograph of M. Feistauer [34]). Namely, $p = p(x,t)$ is a solution to the parabolic equation $\frac{\partial p}{\partial t} + \mathcal{L}p = f(x,t)$, $t > 0$, $x \in \Omega$ with appropriate initial and boundary. Finally, a nonlinear version of this problem with linear convective flux $\underline{b}p$ replaced by a non-linear flux $\underline{b}(p)$ appears in mathematical modeling of multi-phase flow in porous media and petroleum reservoirs.

Integrating the differential equation in (1.1) over an arbitrary volume $V \subset \Omega$ we get the so–called balance equation:

$$\int\limits_{\partial V} (-a\nabla p + \underline{b}p)\cdot\mathbf{n}\, d\varrho + \int\limits_V c_0 p\, dx = \int\limits_V f\, dx. \tag{1.2}$$

This equality gives the balance of the quantity expressed through the "flux" $\underline{\sigma} = -a\nabla p + \underline{b}p$ over any subset V. Discretization schemes that have this property over given finite set of volumes are called *locally conservative*. This means that on a discrete level the approximate solution satisfies the balance equation (1.2). For many practical problems local conservation property is very important and desired. For example, in ground-water modeling the variable $\underline{\sigma}$ can represent the mass flux. Without local conservation property the mass error can accumulate and the approximate solution may exhibit instability such as non-physical oscillations that may result also in loss of accuracy.

Further, the solution of the homogeneous equation (1.1) satisfies the *maximum principle*. This means that the solution of the homogeneous equation cannot achieve local extrema inside Ω. Monotonicity is another highly desired property of the approximate solution, oscillations are not physical and may lead to instabilities as well. In many particular cases such properties have the finite volume and mixed finite element methods.

Replacing the derivatives in (1.2) by finite differences has been a very successful approximation approach in the past. For comprehensive presentation of the main results and techniques we refer to the classical monograph of A.A. Samarskii [60]. Finite differences are still widely used for uniform rectangular or triangular grids. The main deficiency of the analysis in the classical theory of the finite differences is that it requires higher regularity of the solution, i.e. the error estimates are not optimal with respect of the regularity of the solution. To certain extend this deficiency has been overcome by Samarskii, Lazarov, and Makarov in [61] for diffusion-reaction equations. Further, various unconditionally stable schemes have been studied by Lazarov, Mishev, and Vassilevski in [44] where optimal with respect to the regularity of the solution error estimates have been established. Approximations on locally refined rectangular and triangular grids have been studied in [31, 32, 33, 66].

Extensions of the finite difference approximations to non-rectangular grids have been constructed and studied by B. Heinrich with his main accomplishments summarized in [37]. An interesting approach for construction of conservative finite difference schemes on arbitrary grids have been developed in the scientific school of A.A. Samarskii in the 80-es. The main results of this general approach, called *method of support operators*, are summarized in the monograph of M. Shashkov [62]. The method of support operators is based on the balance equation (1.2) and uses also relations between divergence and gradient operators in various discrete inner products. Theoretical results concerning stability, convergence, and error analysis are yet to be developed, for example, by relating this method to the well–understood mixed finite element method with Lagrange multipliers. The latter are needed in order to explicitly eliminate the vector unknown (flux) thus ending up with a problem for a single scalar unknown (pressure) as in the finite difference discretizations described in [62].

The main advantages of the direct finite differencing are related to: (1) simplicity, (2) local mass conservation, (3) use of harmonic averaging of the

coefficients and (4) weighted up-wind approximations, when necessary. However, they lack the flexibility of the method of finite elements when arbitrary meshes for general domains are involved and more general (than Dirichlet) boundary conditions are present. Some alternatives, based on overlapping grid discretizations are used to overcome this difficulty, but still theoretical results covering stability and error analysis are lacking. On the other hand, the theory of the finite element method is fully developed regarding optimal with respect to the order of approximation and regularity of the solution error estimates. Moreover, numerous implementations of the finite element method in production/CAD/CAM codes for stress analysis, heat and mass transfer and computational fluid dynamics have contributed to a full understanding of the merits and the deficiencies of the method.

This paper makes an attempt to put in a unified and consistent way the mathematical background of some widely used and popular as well as some recently proposed schemes, both based on the finite volume element or the mixed finite element methods. The structure of the remainder of the paper is as follows. We set up the notations and provide necessary background in Section 2. In Section 3 the main discretization strategies are formulated and whenever available main results concerning stability and error estimates are provided. Section 4 summarizes broadly the least-squares finite element method applied to the first order system (FOSLS) in our particular setting. Finally, in Section 5 we touch upon the important for the computational practice case of non-matching grid discretizations and outline some major solution strategies applied to resulting systems of linear algebraic equations.

2. Notations and Preliminaries

In what follows we use the following common notations. We shall use the Hilbert space $H^1_D(\Omega) = \{q \in H^1(\Omega) : \ q|_{\Gamma_D} = 0\}$ equipped with the standard L_2- and H^1-norms:

$$||q|| := (q,q)^{1/2}, \qquad ||q||_1 := ||q||_{H^1(\Omega)} := \{(q,q) + (\nabla q, \nabla q)\}^{1/2}.$$

Here $(\cdot,\cdot)$ is the inner product in $L_2(\Omega)$. Often we shall use L_2-inner product on Γ (or its parts Γ_N and Γ_D or other $(d-1)$-dimensional interfaces) denoted by $< p, q >_\Gamma = \int_\Gamma pq \, d\varrho$. Further, we use the Sobolev space $H^s(\Omega)$, s integer, as the set of functions with generalized derivatives up to order s belonging to the space $L_2(\Omega)$. This space is equipped with the usual norm denoted by $\|\cdot\|_{H^s(\Omega)}$.

For vector functions $\mathbf{v}$ defined on Ω we shall also use the Hilbert spaces $L_2(\Omega)$ and $H(div;\ \Omega)$:

$$H(div;\Omega) = \{\mathbf{v} : \quad \mathbf{v} \in L_2(\Omega) \ \text{ and } \ \nabla \cdot \mathbf{v} \in L_2(\Omega)\}.$$

The L_2- and $H(div)$-norms in this space are defined as

$$\|\mathbf{v}\|^2 = (\mathbf{v},\mathbf{v}), \qquad \|\mathbf{v}\|^2_{H(div)} = \|\mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2.$$

We shall use the subspace $H_N(div;\Omega) = \{\mathbf{v} \in H(div;\Omega) : \quad \mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_N\}$.

In this paper we shall use two different formulations of the problem (1.1), weak formulation in the space $H^1(\Omega)$ and weak formulation based on the mixed form of the problem, introduced by (2.3). For the first one the Dirichlet boundary conditions are essential and we shall assume that $g_D \equiv 0$ on Γ. For the mixed problem the original Neumann boundary conditions are essential and we shall assume in that case that $g_N \equiv 0$ on Γ_N. This is reflected in the definition of the spaces $H^1_D(\Omega)$ and $H_N(div;\Omega)$. The assumptions of the homogeneity of the boundary conditions are not essential for the methods. We make them just to simplify the exposition.

The weak formulation of the problem (1.1) is introduced by using the bilinear from $a(\cdot,\cdot)$ defined on $H^1_D(\Omega)\times H^1_D(\Omega)$ as:

$$a(p,q) \equiv (a\nabla p - \underline{b}p, \nabla q) + (c_0 p, q) + < \underline{b}\cdot\mathbf{n}\, p,\ q >_{\Gamma_N^{out}} . \tag{2.1}$$

Since the coefficients of the differential equation are bounded in Ω the bilinear form $a(\cdot,\cdot)$ is continuous in $H^1_D(\Omega)$. We shall assume that the form $a(\cdot,\cdot)$ is coercive in $H^1_D(\Omega)$, i.e. there is a constant $C_0 > 0$ such that

$$a(p,p) \geq C_0 ||p||_1^2, \quad \forall\ p \in H^1_D(\Omega).$$

A sufficient condition for the coercivity of the bilinear form is (see, e.g. [58]):

$$c_0(x) + \frac{1}{2}\,\nabla\cdot\underline{b}(x) \geq 0, \quad \forall\ x \in \Omega.$$

Then (1.1) has the following weak form: Find $p \in H^1_D(\Omega)$ such that

$$a(p,q) = (F,v) \equiv (f,q) - < g_N,\ q >_{\Gamma_N^{in}}, \quad \forall\ q \in H^1_D(\Omega). \tag{2.2}$$

This weak formulation is well suited for designing numerical methods using approximation in $H^1_D(\Omega)$. For example, this is the case when the approximate solution belongs to a conforming finite element space. However, this approach does not guarantee that the balance equation is satisfied over given set of volumes.

Often to get the local conservation property one has to work with discontinuous solution spaces or to use different spaces for the solution and for the test functions. These are the two main ways to achieve local conservation.

One of the most popular and widely used locally conservative method is based on the mixed form of the problem (1.1). In this formulation one introduces additional variable, called flux, and defines the operator $\mathcal{C}$

$$\mathbf{u} = -a\nabla p, \qquad \mathcal{C}p = \underline{b}\nabla p + c_0 p.$$

The original problem it then re-casted as a system of first order equations,

$$a^{-1}\mathbf{u} + \nabla p = 0, \qquad \nabla\cdot\mathbf{u} + \mathcal{C}p = f. \tag{2.3}$$

The boundary conditions play an essential role in the weak formulation of this mixed system. The simplest and most popular case is when $\underline{b} \equiv 0$, so that $\Gamma_N^{out} \equiv \Gamma_N$, and the equation (1.1) is a diffusion-reaction equation with boundary conditions $p = g_D$ on Γ_D, and $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ_N. Note, that we consider non-homogeneous Dirichlet boundary condition and homogeneous Neumann conditions on Γ_N. The weak mixed formulation then reads: find $p \in L_2(\Omega)$ and $\mathbf{u} \in H_N(div;\, \Omega)$ such that

$$\begin{aligned} (a^{-1}\mathbf{u},\, \mathbf{v}) - (p,\, \nabla \cdot \mathbf{v}) &= - < g_D,\, \mathbf{v} \cdot \mathbf{n} >_{\Gamma_D}, \quad && \forall\, \mathbf{v} \in H_N(div;\, \Omega), \\ -(\nabla \cdot \mathbf{u},\, q) - (c_0 p,\, q) &= -(f, q), && \forall\, q \in L_2(\Omega). \end{aligned} \tag{2.4}$$

This formulation is suitable for non–homogeneous Dirichlet boundary conditions, since the Dirichlet data g_D now is part of the variational formulation (2.4). In Subsection 3.3 we consider an approximation of the general case, namely, when $\underline{b} \not\equiv 0$.

3. Discretization Strategies

We assume that Ω is a polygonal domain (whenever needed convex) which is partitioned into triangles (in 2-D) or tetrahedra (in 3-D) called finite elements T. The elements are considered to be closed sets and the partitioning is denoted by $\mathcal{T}_h$. We assume that the partition $\mathcal{T}_h$ is locally quasi-uniform (or regular), that is $\text{meas}(T) \leq C\rho(T)^d$ with a constant C independent of the partition; here $\rho(T)$ is the radius of the largest ball contained in T. In the context of locally refined grids this means that neighboring finite elements are of approximately the same size while elements that are far away of each other may have very different sizes. Further, we denote by $\mathcal{E}_h$ the set of all edges (2-D) or faces (3-D) of finite elements in $\mathcal{T}_h$.

3.1. The finite element method

We define the finite element space S_h as set of piece-wise linear polynomials over elements $T \in \mathcal{T}_h$:

$$S_h = \{q \in C(\Omega) : \quad q|_T \quad \text{is linear for all} \quad T \in \mathcal{T}_h \quad \text{and} \; q|_{\Gamma_D} = 0\}.$$

The finite element approximation of the problem (2.2) is: find $p_h \in S_h$ such that

$$a(p_h, q) = (F, q) \equiv (f, q) - < g_N,\, q >_{\Gamma_N^{in}}, \qquad \forall \;\; q \in S_h. \tag{3.1}$$

This method is the most popular and widely used for solving a large variety of applied problems. Moreover, it has a straightforward generalization for transient problems and problems in solid mechanics. Furthermore, there is a number of efficient solution techniques for the resulting linear algebraic system.

However, this method has certain disadvantages which make necessary development of other methods especially for particular applications. For example, while this method is globally conservative, it may not satisfy a discrete version of the balance equation (1.2) locally. Moreover, special stabilization techniques are required in order to make the method well-posed in the case of convection-dominated problems (see, e.g. [38]). Further, monotone schemes are not produced by this approach and modifications are necessary (see, e.g. [64]). Below we present two ways of getting locally conservative approximations that are also unconditionally stable.

3.2. The finite (control) volume element method

The finite (control) volume method (also called box method) has been introduced as an alternative of the finite element method that will allow to obtain locally conservative schemes. It has been discussed in the early works of Angermann [1], Bank and Rose [6], Hackbusch [36] and Cai, Mandel and McCormick [14, 16]. Below we present one possible approach, namely the finite volume element method is viewed as a Petrov-Galerkin method for solving the problem (1.1). That is, the solution space is different from the test space. The solution space will be the finite element space S_h while the test space is described below.

To define the test space we introduce another partition (often called dual mesh) of the domain Ω into finite volumes. For a given finite element partition $\mathcal{T}_h$, we construct a dual mesh $\mathcal{T}_h^*$ (based upon $\mathcal{T}_h$), whose elements are called control volumes and denoted by V. In the finite volume methods there are various ways to introduce the control volumes. Almost all approaches can be described in the following general scheme, which we explain for tetrahedral elements T. In each element $T \in \mathcal{T}_h$ an internal point is selected. For the 3-D case, on each of the four faces $\overline{x_i x_j x_k}$ of T a point x_{ijk} is selected and on each of the six edges $\overline{x_i x_j}$ a point x_{ij} is selected. Then the internal point is connected to the points x_{ijk}, and the points x_{ijk} are connected to the points x_{ij} by straight lines (see Figure 1). The control volumes are associated to the vertices $x_i \in N_h$. Control volume associated with vertex x_i is denoted by V_i and defined as the union of the "quarter" elements $T \in \mathcal{T}_h$, which have x_i as a vertex (see Figure 1). The interface between two control volumes, V_i and V_j, is denoted by γ_{ij}.

For the 2-D case we will also use the construction of the control volumes in which the internal point is the circumcenter of the element T, i.e. the center of the circumscribed circle of T and x_{ij} are the midpoints of the edges of T. This type of control volume forms the so-called Voronoi meshes. Then obviously, γ_{ij} are the perpendicular bisectors of the three edges of T (see Figure 1). This construction requires that all finite elements are triangles of acute type, which we shall assume whenever such triangulation is used.

In addition to the finite element space S_h we define its dual volume element space S_h^* by

$$S_h^* = \{q \in L_2(\Omega) : \ q|_V \quad \text{is constant for all} \ \ V \in \mathcal{T}_h^* \ \ \text{and} \ q|_{\Gamma_D} = 0\}.$$

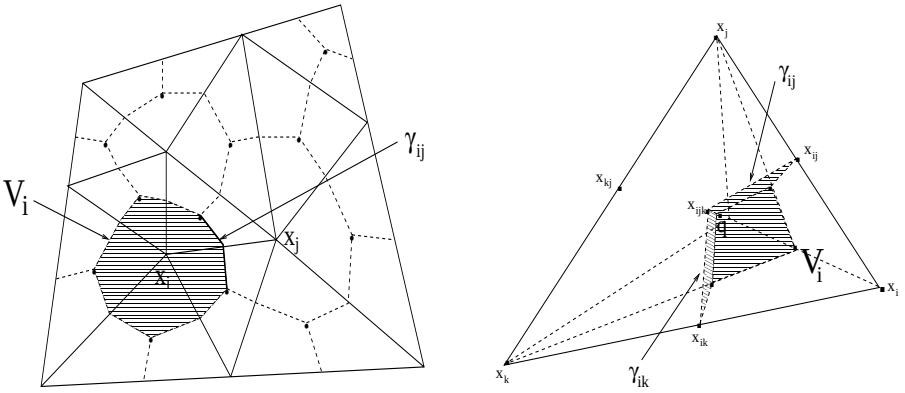

Figure 1: Left: Finite element and finite volume partitions in 2-D; Right: Contribution from one element to control volume V_i, γ_{ij} in 3-D; the internal point is the element medicenter and internal points for the faces are the medicenters of the faces.

Obviously, $S_h = \text{span}\{\phi_i(x) : \ x_i \in N_h^0\}$ and $S_h^* = \text{span}\{\chi_i(x) : \ x_i \in N_h^0\}$, where ϕ_i is the standard nodal linear basis function associated with the node x_i and χ_i is the characteristic function of the volume V_i. Further, we use the notations $N_h = \{p : p \ \text{ is a vertex of element } T \in \mathcal{T}_h\}$ and N_h^0 as the set of all vertices from N_h except those on Γ_D.

The finite volume element approximation of (1.1) reads as: Find $p_h \in S_h$ such that

$$a_h(p_h, q^*) \equiv D_h(p_h, q^*) + C_h(p_h, q^*) = F(q^*), \ \text{for all} \ q^* \in S_h^*. \tag{3.2}$$

Here the bilinear forms $D_h(p_h, q^*)$ and $C_h(p_h, q^*)$ are defined on $S_h \times S_h^*$, the linear form $F(q^*)$ on S_h^*, and are given, respectively, by

$$D_h(p_h, q^*) = \sum_{x_i \in N_h^0} q_i^* \left\{ - \int\limits_{\partial V_i \setminus \Gamma_N} a \nabla p_h \cdot \mathbf{n} d\varrho + \int\limits_{V_i} c_0 p_h dx \right\}, \tag{3.3}$$

$$F(q^*) = \sum_{x_i \in N_h^0} q_i^* \left\{ \int\limits_{V_i} f dx - \int\limits_{\partial V_i \cap \Gamma_N^{in}} g_N d\varrho \right\}. \tag{3.4}$$

Typically, in order to be able to handle different scales of a and $\underline{b}$ one discretizes the convective part using up-wind approximation. On rectangular grids this approximation is well understood and widely used in the context of finite difference schemes (see, [60]). In the case of arbitrary grids one can use the following approximation. Introduce the quantities $\alpha^+ = (\alpha + |\alpha|)/2$, $\alpha^- = (\alpha - |\alpha|)/2$. Next, define $\Pi(i)$ as the set of all indices j such that the

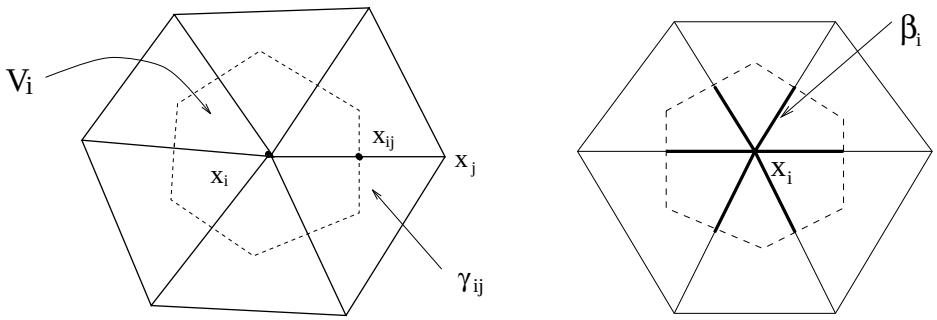

Figure 2: Control volumes with circumcenter as internal points (Voronoi meshes) and interface γ_{ij} of V_i and V_j. The rightmost picture shows the segments β_i.

interval (x_i, x_j) is an edge of an element in $\mathcal{T}_h$. Now the up-wind approximation of the convection part $\mathcal{C}$ of the operator $\mathcal{L}$ is:

$$C_h(p_h, q^*) = \sum_{x_i \in N_h^0} v_i^* \sum_{j \in \Pi(i)} \int_{\gamma_{ij}} \left[(\underline{b} \cdot \mathbf{n})^+ p_h(x_i) + (\underline{b} \cdot \mathbf{n})^- p_h(x_j) \right] d\varrho. \quad (3.5)$$

Theorem 1. *The following results are valid uniformly with respect to the mesh-size h:*

(1) the bilinear form $a_h(p_h, q^*)$ is bounded on $S_h \times S_h^*$ and satisfies the inequality

$$C\|p_h\|_1 \leq \sup_{q^* \in S_h^*} \frac{a_h(p_h, q^*)}{\|I_h q^*\|_1};$$

here I_h is the finite element nodal interpolation operator; consequently the solution u_h of the problem (3.2) is stable in H^1-norm;

(2) if p is H^2-regular, then the following error estimate holds:

$$\|p_h - p\|_1 \leq Ch\|p\|_{H^2(\Omega)}.$$

The proof of these results is a consequence of the construction of the method. For piece-wise constant coefficients the proof uses the equivalence of the finite volume approximation to the finite element approximation (see, e.g. [40]). For rectangular grids a detailed proof (including convergence in L_2-norm) is given in [44].

Similarly to the finite element method the discrete problem is unconditionally stable in H^1-norm. Moreover, the above approximation of the convection part of the operator produces an M-matrix. Therefore, if the diffusion part produces also an M-matrix (for example this will be the case if $a(x)$ is a diagonal matrix and the triangulation is of acute type) then the overall matrix

will be an M-matrix and the method will give monotone solution. Although the method is first order accurate combined with adaptive grid refinement it is very attractive for the applications. In [50] various a posteriori error estimators for finite volume element method have been studied, implemented, and tested on a large variety of 3-D problems (1.1). In particular, we have numerically solved problems with singular solutions due to concentrated source and sinks and boundary layers related to different scales of the diffusion matrix a and the convection field $\underline{b}$. The experiments in [50] demonstrate the flexibility of the method and its capability to capture the solution with a given accuracy on adaptive grids.

The above described up-wind approximation of the convection term is closely related to the discontinuous Galerkin approximation (see, e.g. the survey paper by Arnold, Brezzi, Cockburn, and Marini [4]) or to the Tabata scheme for Galerkin finite element method [64].

Remark 1. Voronoi meshes have some advantages in 2-D (see, e.g. [54]). A different type of weighted upwind approximation on Voronoi meshes in 2-D has been studied in [2]. However, these meshes are not well suited for adaptive grid refinement and their generalization to 3-D problems is not immediate or simple.

Remark 2. Further applications of this method to transient boundary-value problems have been reported in [23, 29]. The results in these works include optimal with respect to the regularity error estimates and estimates in L_2-norm.

3.3. Mixed finite element method

To achieve higher order of approximation one has to use the mixed finite element method. The lowest order mixed finite element method on rectangular meshes is a natural extension of the cell–centered finite difference methods. Although mixed methods are seldom used for approximation of convection-diffusion problems, it is still possible to formulate stable and convergent mixed approximations. This is essential for a computational environment of locally conservative methods.

The mixed finite element method is a discrete Galerkin form of (2.4) (with a necessary twist in order to cover the case $\underline{b} \not\equiv 0$). That is, one chooses a pair of finite element spaces $(\mathbf{V}_h,\ W_h) \subset (H(div;\Omega),\ L_2(\Omega))$, associated with a common triangulation $\mathcal{T}_h$ of Ω. The spaces are chosen such that the well–known LBB (Ladyzhenskaya–Babuška–Brezzi) stability condition is satisfied: namely, for a mesh-independent constant $\beta > 0$ the following estimate holds for any $q \in W_h$,

$$\beta \, \|q\| \leq \sup_{\mathbf{v} \in \mathbf{V}_h} \frac{(\nabla \cdot \mathbf{v},\ q)}{\|\mathbf{v}\|_{H(div)}}. \tag{3.6}$$

A simple example of stable pair of spaces is the lowest order Raviart–Thomas spaces (see, e.g.[13]); namely, for a partition $\mathcal{T}_h$ consisting of tetrahedra (triangles) or parallelepipeds (rectangles), W_h is the space of (discontinuous) piecewise constants, whereas $\mathbf{V}_h$ has continuous normal components $\mathbf{v}\cdot\mathbf{n}_E$ across the faces (edges) $E\in\mathcal{E}_h$. The vector function restricted to each element has the following polynomial form: for tetrahedra, $\mathbf{v}=(a+dx,\ b+dy,\ c+dz)$ and for parallelepipeds $\mathbf{v}=(a_1+b_1x,\ a_2+b_2y,\ a_3+b_3z)$. The coefficients (a,b,c,d) and (a_i,b_i), $i=1,\ldots,d$, are determined by the degrees of freedom, which are the values of $\mathbf{v}\cdot\mathbf{n}_E$ at the medicenters of the of the faces (edges) $E\in\mathcal{E}_h$ or the mean values of $\mathbf{v}\cdot\mathbf{n}_E$ over the faces (edges) $E\in\mathcal{E}_h$.

In order to describe the weak form of the second equation (2.3), $\nabla\cdot\mathbf{u}+\nabla\cdot(\underline{b}p)+c_0p=f$, we need to allow discontinuous functions p_h and q in the space W_h. Since the functions in W_h have traces from both sides of the faces (edges) of T, for a given function $q\in W_h$ we denote these traces bu q^o and q^i, where "o" stands for the outward (with respect to T) trace and respectively, "i" stands for the interior trace. The weak from of the second equation (2.3) is borrowed from the discontinuous Galerkin method (see, e.g. [41], pp. 189-196) by testing it by functions $q\in W_h$. Integrating over a particular $T\in\mathcal{T}_h$ we get the following contribution of the advection-reaction operator $\mathcal{C}p_h$ by introducing the bilinear form $C_T(p_h,q)$:

$$C_T(p_h,q)=(\mathcal{C}p_h,\ q)_T=<p_h^o-p_h^i,\ q^i(\underline{b}\cdot\mathbf{n})^->_{\partial T}+(c_0p_h,\ q)_T.$$

Here $\mathbf{n}$ is the outer unit normal vector to ∂T. Next, we sum over all $T\in\mathcal{T}_h$ so that, for $p_h,\ q\in W_h$ we get the following contribution of the operator $\mathcal{C}$ to the second equation:

$$C_h(p_h,\ q)=\sum_{T\in\mathcal{T}_h}<q^i,\ p_h^i(\underline{b}\cdot\mathbf{n})^++p_h^o(\underline{b}\cdot\mathbf{n})^->_{\partial T}+(c_0p_h,\ q). \tag{3.7}$$

Then the discrete mixed system for the problem (2.4) reads: Find $\mathbf{u}_h\in\mathbf{V}_h$ and $p_h\in W_h$ such that

$$\begin{aligned}(a^{-1}\mathbf{u}_h,\ \mathbf{v})\ -(p_h,\ \nabla\cdot\mathbf{v})&=-<g_D,\ \mathbf{v}\cdot\mathbf{n}>_{\Gamma_D}, &&\forall\ \mathbf{v}\in\mathbf{V}_h,\\ -(\nabla\cdot\mathbf{u}_h,\ q)\ -C_h(p_h,\ q)&=-(f,q), &&\forall\ q\in W_h.\end{aligned} \tag{3.8}$$

This discontinuous approximation (related also to up-wind approximation) ensures the non-negativity of C_h; namely, (see [41] or [46]),

$$\begin{aligned}C_h(q,\ q)\ =\ &\tfrac{1}{2}\sum_{E\in\mathcal{E}_h}<[q]^2,\ |\underline{b}\cdot\mathbf{n})|>_E\\ &+\tfrac{1}{2}<q^2,\ (\underline{b}\cdot\mathbf{n})^+>_{\Gamma+}-\tfrac{1}{2}<q^2,\ (\underline{b}\cdot\mathbf{n})^->_{\Gamma-}\\ &+\sum_{T\in\mathcal{T}_h}(c_0q,\ q)_T.\end{aligned} \tag{3.9}$$

The second equation of (3.8) expresses conservation of mass over each finite element $T\in\mathcal{T}_h$. Indeed, take in (3.8) a test function q to be the characteristic

function of a particular finite element T. Then the term $(\nabla\cdot\mathbf{u}_h,\ q)_T$ reduces to $(\nabla\cdot\mathbf{u}_h,\ 1)_T =< \mathbf{u}_h\cdot\mathbf{n}, 1 >_{\partial T}$ and expresses the mass influx in T through ∂T due to diffusion. Similarly, the first term of $C_h(p_h,\ q)$ reduces to $< p_h^i,\ \underline{b}\cdot\mathbf{n} >_{\partial T^+}$ $+ < p_h^o,\ \underline{b}\cdot\mathbf{n} >_{\partial T^-}$ and expresses the mass influx through ∂T due to convection (transport). Finally, the last term in $C_h(p_h,\ q)$ expresses mass change due to reaction/absorption. This approximation is closely related to the method studied by Jaffre [39] and Liu, Wang, and Yang [53]. In [53] error analysis, including interior estimates have been provided.

3.4. Mixed co–volume methods

The mixed co–volume methods have been developed as an alternative to the mixed finite element method to handle more general partitions and to work in the finite volume setting (see, e.g. [25, 27, 26]).

We present a co-volume (finite volume) method for approximation of the mixed system (2.3) in the general framework of Petrov-Galerkin method proposed and analyzed by Chou and Vassilevski in [24]. The idea is quite similar to the idea of departing form Galerkin method and using instead Petrov-Galerkin method for second order elliptic equations. Namely, we shall again use two different sets of spaces for the solution and for the test functions.

We take the solution space to be the lowest order Raviart-Thomas space (see, [13]) $(\mathbf{V}_h,\ W_h) \subset (H(div;\Omega),\ L_2(\Omega))$ already explained in Subsection 3.3 and used in the Galerkin approximation of the mixed system. To construct the test spaces we need a corresponding dual mesh. In fact, for the pressure p we use the same space W_h of piece-wise constant functions over the mesh $\mathcal{T}_h$. To construct the test space for the vector-function $\mathbf{v}$ we introduce a dual mesh $\mathcal{Q}_h$. For definiteness we consider the case of triangular (tetrahedral) mesh $\mathcal{T}_h$. The dual mesh is constructed in the following way: in each element $T \in \mathcal{T}_h$ the medicenter is connected with the vertices of the simplex (triangle in 2-D and tetrahedron in 3-D) so the simplex is split into $d+1$ simplexes. Two simplexes sharing a common face (edge) $E \in \mathcal{E}_h$ are added together (with the edge) so they form a co-volume $Q_E = T_E^- \cup E \cup T_E^+$ of the dual grid $\mathcal{Q}_h$ (see, Figure 3 for 2-D). On the boundary Γ_D the co-volume is either T_E^- or a T_E^+.

Then, the test space $\mathbf{V}_h^*$ will be a subspace of the space of piece-wise constant vector-functions that have continuous normal trace across the interior edges $E \in \mathcal{E}_h$; more precisely,

$$\begin{aligned}\mathbf{V}_h^* \subset \{\mathbf{v}:\ &\mathbf{v}|_K \text{ is constant},\ K = T^-,\ T^+,\\ &\mathbf{v}|_{T^-}\cdot\mathbf{n}_E = \mathbf{v}|_{T^+}\cdot\mathbf{n}_E,\ \text{for all } Q_E = T_E^-\cup E\cup T_E^+ \in \mathcal{Q}_h\}\,,\end{aligned} \tag{3.10}$$

where $\mathbf{n}_E$ is the unit normal vector to the face (edge) E with a fixed direction. Then by definition of a gradient of $p \in W_h$ (which is discontinuous function over volumes $Q_E \in \mathcal{Q}_h$) we have:

$$\sum_{Q\in\mathcal{Q}_h}(\nabla p,\mathbf{v})_{Q_E} = -\sum_{Q\in\mathcal{Q}_h}(p,\nabla\cdot\mathbf{v})_{Q_E} + \sum_{E\in\mathcal{E}_h} < p,\mathbf{v}\cdot\mathbf{n}_E >_{\partial Q_E},\quad \forall\ \ \mathbf{v}\in\mathbf{V}_h^*.$$

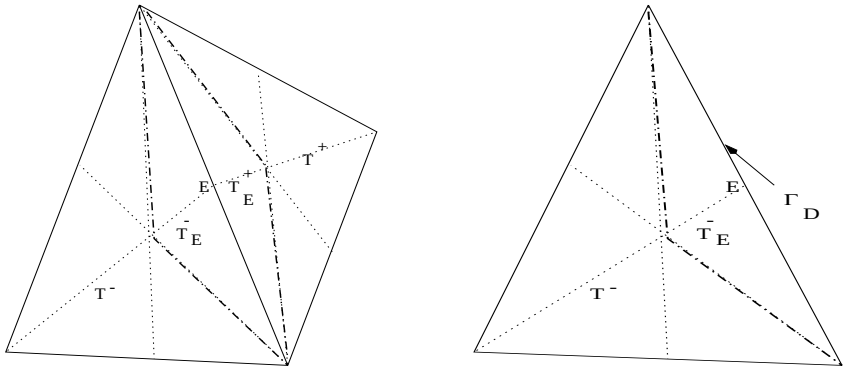

Figure 3: Left: a co-volumes sharing an internal edge $Q_E = T_E^- \cup E \cup T_E^+$; Right: a co-volume sharing an edge on Γ_D: $Q_E = T_E^-$.

Note that interior (to Ω) edge E has two neighboring simplexes T^+ and T^- from the neighboring volumes that share E. Further, $\mathbf{v}$ is constant over T_E^+ and T_E^+ and its normal component is continuous across E, that is, the jump $[\mathbf{v} \cdot \mathbf{n}_E] = 0$. Finally, $\nabla \cdot \mathbf{v} = 0$ over Q_E so that the following identity holds,

$$(\nabla p, \mathbf{v})_{Q_E} = < p, \mathbf{v} \cdot \mathbf{n} >_{\partial Q_E} = - < [p]_E, \ \mathbf{v} \cdot \mathbf{n}_E >_E \ .$$

Here, $[p]_E$ stands for the jump of p across the face (edge) E and the difference is taken of value from the element in the direction of $\mathbf{n}_E$ minus the value from the element in opposite direction of $\mathbf{n}_E$. Summing over all volumes $Q_E \in \mathcal{Q}_h$ we get the following approximation of the first equation of the mixed system (2.4) defined for $\mathbf{u}_h \in \mathbf{V}_h$, $\mathbf{v} \in \mathbf{V}_h^*$ and $p_h \in W_h$:

$$\sum_{Q \in \mathcal{Q}_h} (a^{-1}\mathbf{u}_h, \ \mathbf{v})_{Q_E} + \sum_{E \in \mathcal{E}_h} < [p_h]_E, \ \mathbf{v} \cdot \mathbf{n}_E >_E = 0.$$

Next, we need to discretize the second equation in (2.4). The discretization of the convective term is the same as in the mixed finite element approximation and is based on the idea of discontinuous Galerkin method. The evaluation of $(\nabla \cdot \mathbf{u}, \ q)$ can be done element-by-element and is straight forward since $\mathbf{u} \in \mathbf{V}_h$. Moreover, direct computations show that

$$(\nabla \mathbf{u}, \ q) = \sum_{T \in \mathcal{T}_h} (\nabla \mathbf{u}, \ q)_T = \sum_{E \in \mathcal{E}_h} < [q]_E, \ \mathbf{u} \cdot \mathbf{n}_E >_E \ .$$

To summarize, the mixed co–volume (Petrov–Galerkin) scheme reads: Find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in W_h$ such that

$$\begin{aligned} \sum_{Q \in \mathcal{Q}_h} (a^{-1}\mathbf{u}_h, \ \mathbf{v})_Q + \sum_{E \in \mathcal{E}_h} < [p_h]_E, \ \mathbf{v} \cdot \mathbf{n}_E >_E &= 0, \qquad \forall \ \mathbf{v} \in \mathbf{V}_h^*, \\ \sum_{E \in \mathcal{E}_h} < \mathbf{u}_h \cdot \mathbf{n}_E, \ [q]_E >_T - C_h(p_h, \ q) &= -(f, \ q), \ \forall \ q \in W_h. \end{aligned} \tag{3.11}$$

To complete the description of the mixed co-volume method we have to specify the trial space $\mathbf{V}_h^*$. Our construction will be based on $\mathbf{V}_h$, namely, we show that an image of a proper subspace of $\mathbf{V}_h$ will have the desired properties. It is clear that $\mathbf{V}_h^*$ should be isomorphic to the test space $\mathbf{V}_h$. Isomorphism here means that the dual space $\mathbf{V}_h^*$ must have as many degrees of freedom as $\mathbf{V}_h$, i.e. the dimensions of the space of test functions $\mathbf{v}$ and the space of solutions $\mathbf{u}_h$ are equal.

For each such function $\mathbf{v} \in \mathbf{V}_h$ one can define a function $\gamma_h \mathbf{v} \in \mathbf{V}_h^*$ as follows

$$\gamma_h \mathbf{v} = \begin{cases} \frac{1}{|E|} \int\limits_E \mathbf{v}^+ \, d\varrho, & \text{on } T_E^+, \\ \frac{1}{|E|} \int\limits_E \mathbf{v}^- \, d\varrho, & \text{on } T_E^-. \end{cases}$$

One notices that $[\mathbf{n}_E \cdot \gamma_h \mathbf{v}]_E = \frac{1}{|E|} \int\limits_E [\mathbf{n}_E \cdot \mathbf{v}] = 0$, that is, $\gamma_h \mathbf{v} \in \mathbf{V}_h^*$. It is also clear that $\gamma_h \mathbf{v} = 0$ implies that $\mathbf{v} \cdot \mathbf{n}_E = 0$ on the midpoint of E; hence, $\mathbf{v} = 0$ since these are the degrees of freedom which specify $\mathbf{v}$. That is, a natural trial space isomorphic to the test space $\mathbf{V}_h$ would be $\mathbf{V}_h^* \equiv \gamma_h \mathbf{V}_h$.

Problem (3.11) differs from the one obtained using the standard mixed system only by the transfer operator γ_h taking part in the first equation of (3.11). Here we used the fact that

$$\int\limits_E \gamma_h \mathbf{v} \cdot \mathbf{n}_E \, d\varrho = \int\limits_E \mathbf{v} \cdot \mathbf{n}_E \, d\varrho,$$

and that p_h is piece-wise constant over each finite element T.

The following theorem summarizes the results concerning the co-volume approximation (3.11). Its proof can be found in [24] (see, [25, 26, 27] for other approximations of this kind):

Theorem 2. *Let $\mathbf{u}_h$ and p_h be the solution of the discrete problem (3.11). Then*

$$\|\mathbf{u}_h\| + \|p_h\| \le C_0 \|f\|;$$

Let p be the solution of the problem (1.1), let $\mathbf{u} = -a\nabla p$, and assume that p is H^2-regular. Then the following error estimate is true:

$$\|\mathbf{u}_h - \mathbf{u}\| + \|p_h - p\| \le C_1 h (\|p\|_{H^1(\Omega)} + \|\mathbf{u}\|_{H^1(\Omega)}) + C_2 h^{1/2} \|p\|_{H^1(\Omega)}.$$

The constants C_0, C_1 and C_2 do not depend on the mesh size h. Furthermore, the constant C_2 can be taken as zero in the case of pure diffusion problem, i.e. $\underline{b} \equiv 0$.

4. First Order System Least-Squares (FOSLS)

Least-squares finite element approximations of second order problems have become a popular technique for deriving unconditionally stable approximations

including high order schemes. The idea of least-squares is quite old (see, e.g. the pioneering work of Neitaanmäki and J. Saranen [55]) but only recently a new development in the method has been accomplished (see, e.g. [10, 11, 12, 17, 18, 56]). For a comprehensive review of new recent results in least-squares method and their applications to a wide range of problems we refer to the paper by Bochev and Gunzburger, [9]. Attractive feature of this approach is that it leads to symmetric and positive definite discrete systems and allows approximations of high order for smooth solutions. The price one pays is an increased number of unknown functions. We explain the main idea of the least-squares method on the model problem (1.1) assuming that a homogeneous Dirichlet boundary condition is prescribed on the whole boundary, i.e. $\Gamma_D = \Gamma$.

As in the mixed method, we rewrite the original problem in the mixed form (2.3) and consequently form the least-squares functional:

$$J(\mathbf{v},\ q) \equiv \|a^{-1}\mathbf{v} + \nabla q\|_U^2 + \|f - \nabla\cdot\mathbf{v} - \mathcal{C}q\|_P^2. \tag{4.1}$$

Here, $\|\cdot\|_U$ and $\|\cdot\|_P$ are some norms in the sets of vector functions $\mathbf{v}$ and scalar functions q. Further, we discuss two possible norms that have been used in the least-squares method. Obviously, the solution $p \in H_0^1(\Omega)$ of the problem (1.1) and $\mathbf{u} = -a\nabla p$ gives this functional a value zero. It is also valid that for properly chosen norms the minimizer of the functional (4.1) in the corresponding spaces will be the solution of (1.1).

4.1. Least-squares based on L_2-inner product

The simplest and far the most popular least-squares method is based on the following choice of norms in (4.1): $\|\mathbf{u}\|_U^2 = (a^{-1}\mathbf{u},\ \mathbf{u})$ and $\|p\|_P^2 = (p,\ p)$, where $(\cdot,\cdot)$ is the standard L_2-inner product for scalar and vector functions defined on Ω. Then the L_2-inner product FOSLS which minimizes the quadratic functional (4.1) will lead to the following weak problem: find $p \in H_0^1(\Omega)$ and $\mathbf{u} \in H(div;\ \Omega)$ which satisfy the integral identity:

$$A(\mathbf{u},p;\ \mathbf{v},q) \quad = \quad F(\mathbf{v},q),\ \ \forall q \in H_0^1(\Omega),\ \ \text{and}\ \ \forall\ \ \mathbf{v} \in H(div;\ \Omega), \tag{4.2}$$

where

$$\begin{aligned} A(\mathbf{u},p;\ \mathbf{v},q) &:= (a^{-1}\mathbf{u} + \nabla p,\ \mathbf{v} + a\nabla q) + (\nabla\cdot\mathbf{u} + \mathcal{C}p,\ \nabla\cdot\mathbf{v} + \mathcal{C}q), \\ F(\mathbf{v},q) &:= (f,\ \mathcal{C}q). \end{aligned}$$

As proved in [17], the bilinear from $A(\mathbf{u},p;\ \mathbf{v},q)$ is bounded and coercive in the space $H(div;\ \Omega) \times H_0^1(\Omega)$ provided that the matrix $a(x)$ is uniformly positive definite and bounded in Ω.

This fundamental result leads to the following natural least-squares finite element method. Let S_h be the space of continuous piece-wise linear over the partition $\mathcal{T}_h$ scalar functions satisfying homogeneous Dirichlet boundary condition on Γ. Similarly, let $\mathbf{S}_h$ be the space of continuous piece-wise linear

over the partition $\mathcal{T}_h$ vector-functions. Then the least-squares finite element approximation to (4.2) is: find $p_h \in S_h$ and $\mathbf{u}_h \in \mathbf{S}_h$ such that

$$A(\mathbf{u}_h, p_h;\ \mathbf{v}, q) = F(\mathbf{v}, q), \quad \forall q \in S_h, \quad \text{and} \quad \forall \quad \mathbf{v} \in \mathbf{S}_h. \tag{4.3}$$

Theorem 3 (see, [17, 20, 56]). *The following results are valid:*

(1) the corresponding matrix of (4.3) *is symmetric, positive definite, and its condition number is of order* $O(h^{-2})$*;*

(2) the solution of the finite element method (4.3) *satisfies the a priori estimate*

$$\|p_h\|_{H^1(\Omega)} + \|\mathbf{u}_h\|_{L_2(\Omega)} \le C\|f\|_{L_2(\Omega)};$$

(3) if the solution of the problem (1.1) *is* H^3*-regular then the following error estimate is valid:*

$$\|p_h - p\|_{H^1(\Omega)} + \|\mathbf{u}_h - \mathbf{u}\|_{L_2(\Omega)} \le Ch\|p\|_{H^3(\Omega)}.$$

The constants C *in these inequalities are independent of the mesh step-size* h*.*

Here are some observation regarding this method. The discrete scheme (4.3) is not conservative (i.e., the discrete solution does not satisfy a balance equation). However, the method is very appealing since it leads to a symmetric positive definite problem without any conditions on the step-size h. Second, this methods adds another $3d$ unknowns per grid point, so it has increased memory requirement. On the other hand, the function $\mathbf{u}_h$ provides a direct approximation of the flux variable $\mathbf{u}$, which is continuous in the whole domain Ω. This property of the approximate solution might be very important in some applications, e.g. the case of flow in porous media. This method however, requires higher regularity of the solution, compared with the standard finite element method (3.1). Namely, for $O(h)$-convergence the least-squares finite element method requires H^3-regularity of the solution. This, in general, is not the case of polygonal domains Ω.

We should also note that the resulting discrete operator is not always close to a block diagonal operator of elliptic type; that is, the resulting operator, generally, couples strongly the different variables and this makes the construction of efficient preconditioners, including multigrid, a challenging task. This is an area of active research. Some progress has been made by the recently proposed spectral AMGe (algebraic multigrid finite element) method [22], which in essence, builds problem dependent coarse spaces in order to capture more closely the "algebraically smooth" components of the fine–grid functions. In other words, the spectral AMGe method builds, by local procedures, a coarse space that leads to a two–grid method which has a convergence factor bounded independently of the problem parameters (such as the PDE coefficients and the mesh size). A possible disadvantage of the method that it may lead to

high complexity (or equivalently, to too dense coarse matrices). More classical ILU-type methods have been used in [21] while multigrid utilizing matrix dependent coarse spaces and geometrically constructed coarse elements, have been demonstrated in [57].

Remark 3. We have discussed Dirichlet boundary conditions only. General boundary conditions can be made part of the least squares functional (with proper scaling) and this adds another dimension of flexibility to the method. For example, Dirichlet and Neumann boundary conditions in $H^{1/2}(\Gamma_D)$ and $H^{-1/2}(\Gamma_N)$ norms, respectively, can be added to the lest-squares functional (4.1). Finite element approximations based on multilevel method for such augmented functional the Poisson equation have been introduced and studied by Starke in [63].

Remark 4. For convection-dominated problems the constants in the inequalities in Theorem 3 depend on the ratio $\|a\|/|/\|\underline{b}\|$ and their dependence is not immediately available. To get schemes for which the dependence is weaker and explicit a hybrid discretization based on stream-line diffusion stabilization and least-squares has been proposed and studied by Lazarov, Tobiska, and Vassilevski in [48].

4.2. Least-squares based on H^{-1}-inner product

To overcome some of the deficiencies of the least-squares method based on L_2-inner product a more balanced set of norms in (4.1) has been proposed by Bramble, Lazarov, and Pasciak in [10]. In order to introduce the method we need to define a minus one inner product. First, we define the space $H^{-1}(\Omega)$ as the set of all functionals q for which the norm

$$\|q\|_{H^{-1}(\Omega)} = \sup_{\phi \in H_D^1(\Omega)} \frac{(q,\phi)}{\|\phi\|_{H^1(\Omega)}}$$

is finite. Here (q,ϕ) is the value of the functional q at ϕ.

Below we introduce this concept following [10]. We consider the following symmetric boundary value-problem: find $q \in H_D^1(\Omega)$ such that

$$q - \Delta q = f \text{ in } \Omega, \quad q = 0 \text{ on } \Gamma_D, \quad \nabla q \cdot \mathbf{n} = 0 \text{ on } \Gamma_N. \tag{4.4}$$

Let $\mathcal{D} : H^{-1}(\Omega) \mapsto H_D^1(\Omega)$ denote the solution operator for the above problem, i.e. for $f \in H^{-1}(\Omega)$, $\mathcal{D}f = q$ is the solution of (4.4). As proven in [10] we have

$$(q, \mathcal{D}q) = \sup_{\phi \in H_D^1(\Omega)} \frac{(q,\phi)^2}{\|\phi\|^2_{H^1(\Omega)}} = \|q\|^2_{H^{-1}(\Omega)}.$$

so that the inner product in $H^{-1}(\Omega)$ is given by $(q, \mathcal{D}q)$.

This suggests the following norms in (4.1):

$$\|\mathbf{u}\|_U^2 = (a^{-1}\mathbf{u},\ \mathbf{u}) \quad \text{and} \quad \|p\|_P^2 = (p,\ \mathcal{D}p).$$

An important property of these norms is that the corresponding bilinear from

$$A(\mathbf{u},p;\ \mathbf{v},q) := (a^{-1}\mathbf{u}+\nabla p,\ \mathbf{v}+a\nabla q) + (\nabla\cdot\mathbf{u}+\mathcal{C}p,\ \mathcal{D}(\nabla\cdot\mathbf{u}+\mathcal{C}q))$$

is bounded and coercive in $L_2(\Omega)^2 \times H_D^1(\Omega)$.

This functional provides a solid background for a construction of new type of least-squares method. The only problem is that the norm in H^{-1} is not readily computable. Next step is to replace this norm with an equivalent on the finite element space and yet computable norm. Let $\mathcal{D}_h : H^{-1}(\Omega) \mapsto S_h$ be defined as $\mathcal{D}_h f = q_h$, where q_h is the finite element solution of (4.4). Then we define $\mathcal{B}_h = h^2\mathcal{I} + \mathcal{D}_h$, where $\mathcal{I}$ is the identity operator, and form the least-squares functional

$$J(\mathbf{v},\ q) \equiv (a^{-1}\mathbf{v}+\nabla q,\ \mathbf{v}+a\nabla q) + (\mathcal{B}_h(f-\nabla\cdot\mathbf{v}-\mathcal{C}q),\ f-\nabla\cdot\mathbf{v}-\mathcal{C}q), \quad (4.5)$$

defined for $\mathbf{v} \in \mathbf{S}_h$ and $q \in S_h$.

Theorem 4 (see, [10]). *The following results are valid:*

(1) the quadratic functional (4.5) *has unique minimizer* $(\mathbf{u}_h,\ p_h) \in \mathbf{S}_h \times S_h$*;*

(2) if p *as a solution of* (1.1) *is* H^2*-regular, then the following error estimate holds with a constants* C *independent of the mesh-size* h*:*

$$\|p_h - p\|_{H^1(\Omega)} + \|\mathbf{u}_h - \mathbf{u}\|_{L_2(\Omega)} \le Ch\|p\|_{H^2(\Omega)}.$$

As seen from this theorem, this least-squares method has optimal convergence rate with respect to the regularity of the solution. In fact, in [10] it has been proven that the convergence of the minimizer of (4.5), with slight modification of the functional, is $O(h^r)$ if $p \in H^{1+r}(\Omega)$ for $0 < r < 1$. This approach has been applied in [11] directly to the equation (1.1) (not to the mixed system) with boundary conditions including oblique derivative. Further, extension to the equations of linear elasticity, including the case of incompressible materials, has been studied by Bramble, Lazarov, and Pasciak in [12].

On the negative side, this method is quite complex and computationally expensive due to the necessity to use the minus one inner product. This computationally expensive inner product can be replaced by a simpler one (see, the review paper [9]) based on the inverse inequality. Namely, $(q, \mathcal{D}q)$ is replaced by $h^2(q,q)$. In fact, this means that we skip the operator $\mathcal{D}_h$ in the definition of $\mathcal{B}_h$. This will lead to optimal error estimates, but the condition number of the resulting system is significantly larger, namely $O(h^{-4})$, and will result in higher solution costs.

5. Extensions and Conclusions

As mentioned at the beginning, the driving force in the theory and applications of numerical methods for differential equations has been the advances in the computer technology, mathematical modeling, and scientific computing.

In computer simulation of a complex process (or phenomenon) one needs to use various tools from natural sciences, mathematics, and informatics. Stable, accurate, and efficient approximations are only a part of the overall modeling process. To extract the necessary information about the process one needs to perform computations that might involve very large sets of equations. Efficient iterative methods based on preconditioning and parallel algorithms are the main tool to speed-up the solution process.

5.1. Non-conforming domain decomposition method

Domain decomposition (or substructuring) has become an important and standard tool for design of parallel numerical algorithms that utilize multiprocessor computer architectures. The idea of the method is conceptually very simple, namely, the domain (structure) is split up into subdomains (substructures). This makes it possible to assign one (or more) subdomains to a processor that may handle independently the meshing process and the approximation. Further, the processors communicate in order to solve the whole problem. In this area of active research we shall discuss two particular cases, namely, domain decomposition using non-matching grids and coupling different approximations in different subdomains.

In some situations (for example adaptive grid refinement) one is often left to deal with non–matching grids across subdomain interfaces. That is, one needs a technique to formulate an accurate discretization scheme by imposing certain matching conditions across these interface boundaries. Let us illustrate the main ideas on the example of two subdomains, i.e. $\Omega = \Omega_- \cup \Gamma \cup \Omega_+$, where Γ is the interface between Ω_- and Ω_+, i.e. $\Gamma = \partial\Omega_- \cap \partial\Omega_+$.

One approach is based on the so–called mortar method, proposed in the late 80-es in France (see, e.g. [8]). It imposes continuity in a weak sense, that is the jump of the discrete solution across all interfaces Γ is kept orthogonal to a multiplier space Λ,

$$\int_\Gamma [p]\ \mu = 0, \text{ for all } \mu \in \Lambda. \tag{5.1}$$

Then the subdomain problems, are coupled by simply finding computational basis in the constraint space satisfying the above orthogonality conditions. The mortar method provides a systematic way of constructing multiplier spaces and computational bases in the resulting constraint spaces.

Another approach, which better fits the finite volume methodology is to impose certain penalty on the generally discontinuous spaces, that is to add

the following penalty term in the bilinear form

$$\sum_{E\in\mathcal{E}_h\cap\Gamma}\frac{1}{|E|}\int_E [p][q]\; d\;\varrho.$$

Here, E is an edge of an element from one side of the triangulation on every interface Γ and the factor $\frac{1}{|E|}$ can be viewed as penalty. Typically, $|E|\simeq h\mapsto 0$, where h is the mesh size.

We shall not go here into more details for the above two approaches, rather we will refer to the existing literature (see, e.g. [7, 8, 30, 42, 51, 68]). A comprehensive study can be found in [69].

5.2. Coupling various methods

In some cases one may be required to couple different discretization methods applied to different parts of the domain. This approach is very natural for the so-called multi-physics mathematical models (widely used in the research group of M. Wheeler). For example, one may use a mixed discretization on one subdomain and a standard finite volume discretization on the other subdomain. The coupling is done through the interface boundary Γ. In this case penalty is not needed since the continuity is ensured by the flux variable $\mathbf{u}$ from the mixed side. Essentially, the discretization on the finite volume side uses $\mathbf{u}\cdot\mathbf{n}_-$ as a Neumann boundary condition and the discretization on the mixed side uses p^+ as a Dirichlet boundary condition coming from the finite volume side. More specifically assuming that $\mathbf{u}$ is the flux variable and p^- is the pressure variable on the left domain Ω_- and p_h^+ is the pressure unknown in the right domain Ω_+, one has on the mixed side (posed on Ω_-) the following system for $(\mathbf{u}_h, p_h^+)\in(\mathbf{V}_h,\ W_h)$ to solve,

$$\begin{array}{llll}(a^{-1}\mathbf{u}_h,\ \mathbf{v}) & +(p_h^-,\ \nabla\cdot\mathbf{v}) & = -<p_h^+,\ \mathbf{v}\cdot\mathbf{n}>_\Gamma, & \forall\ \mathbf{v}\in\mathbf{V}_h,\\ (\nabla\cdot\mathbf{u}_h,\ q) & +(\mathcal{C}p_h^-, q) & = (f,q)_{\Omega_-}, & \forall\ q\in W_h.\end{array}\tag{5.2}$$

Here, $(\mathbf{V}_h,\ W_h)$ is a stable mixed finite element pair defined on Ω_-. On the right domain one has a finite volume discretization with Neumann boundary conditions $-a\nabla p_h^+\cdot\mathbf{n}=-\mathbf{u}_h\cdot\mathbf{n}$ on the interface Γ. That is, one has, on Ω_+ the following discrete problem, find $p_h^+\in S_h$ such that

$$<\mathbf{u}_h\cdot\mathbf{n},\ q>_\Gamma - a_h^+(p_h^+,\ q) = -(f,\ q)_{\Omega_+},\quad \forall\ q\in S_h^*.\tag{5.3}$$

Here S_h and S_h^* are the spaces of finite elements and finite volumes functions defined on Ω_+, introduced in Subsection 3.2. Similarly, $a_h^+(p_h^+,\ q)$ is the corresponding bilinear form from Subsection 3.2 defined on Ω_+.

A simple iterative procedure for the coupled system would be: Given $p_h^{(n)+}$, an approximation to p_h^+, at step $n\geq 0$, one then solves a system like (5.2) with $p_+ := p_h^{(n)+}$ and determines (approximately) $\mathbf{u}^n\cdot\mathbf{n}$ on Γ. Then in order to

determine $p_h^{(n+1)+}$ one solves a finite volume problem like (5.3) with $\mathbf{u} \cdot \mathbf{n} := \mathbf{u}^n \cdot \mathbf{n}$ on Γ. This provides one step of the iterations.

In the case $\mathcal{C} = 0$ the convergence of the suggested method used as a preconditioner (and several other iteration methods) has been studied and numerically tested. More specifically, in [45] Lazarov, Pasciak, and Vassilevski give detailed analysis of the coupling of mixed and Galerkin methods for pure diffusion problem and study various optimal iteration methods. Similarly, coupling of mixed and finite volume methods for general convection-diffusion problems has been introduced and studied in [46].

5.3. Preconditioning

The matrices obtained by the finite volume method are typically non–symmetric even if the original elliptic operator $\mathcal{L}$ was self-adjoint. The up-wind approximation contributes in the global matrix an M–matrix and that makes the discrete convection operator invertible. There are no general strategies that lead to robust (i.e., problem or coefficient independent) solution methods.

In the case of diffusion-reaction equation, $\mathcal{C} = 0$, on a rectangular grid one gets a cell–centered discretization of the corresponding elliptic problem and there are efficient MG techniques to solve the discrete problem. In the mixed finite element setting, $\mathcal{C} = 0$, which is appropriate when a general coefficient matrix a is present, one approach is to use a preconditioned MINRES method where a number of block-diagonal preconditioners can be successfully applied.

One possibility is to use for the first block (corresponding to the flux variables) a mass-matrix preconditioner, whereas for the second block (corresponding to the pressure variable) one can use preconditioners coming form and equivalent interior penalty bilinear from as proposed by Rusten, Vassilevski, and Winther in [59]. In general, the second block gives rise to a non–conforming discretization of the pressure equation and any method, e.g. multigrid or based on domain decomposition is suitable in this case. Alternatively, one may wish to use for the first block preconditioners for the $H(div)$–bilinear form, for example, the hierarchical basis preconditioner as proposed and analyzed in Cai, Goldstein and Pasciak [15], which can be algebraically stabilized (see, e.g., [65]), or the domain decomposition type preconditioners as proposed by Arnold, Falk, and Winther in [5] and by Vassilevski and Wang in [67]. In the mixed co–volume setting the algebraically stabilized hierarchical basis preconditioners have been used in Chou, Kwak, and Vassilevski [27] and Chou and Vassilevski [24]. The second block (for the pressure variable), in the latter case is simply the mass matrix. Results concerning the eigenvalue estimates of the second type block diagonal preconditioners were derived by Lazarov and Vassilevski in [49].

The non-symmetric matrices arising in the co-volume mixed method can be treated as perturbations of the mixed finite element method. Therefore, any preconditioner for the mixed method can be used in GMRES iterations. The finite volume discretization in the case $\mathcal{C} \neq 0$ can be treated by exploiting suitable ordering of the unknowns along the vector field $\underline{b}$, which leads to very

efficient multigrid methods. For more details, see for example, the paper by Kim, Xu, and Zikatanov [43].

Acknowledgment

This paper is dedicated to the 70th birthday of Academician Blagovest Sendov - a pioneer in numerical analysis and computational mathematics in Bulgaria. Under his leadership the Department of Mathematical Modeling at the Bulgarian Academy of Sciences and the "St. Kliment Ohridski" University of Sofia had played leading role in establishing and developing *Mathematical Modeling* as a branch of contemporary mathematics. It is our pleasure and duty to acknowledge the vision, dedication, and contributions of Acad. Bl. Sendov to this important research area which has become a fundamental link between science and engineering on the basis of mathematics, informatics, and computer technology.

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