# ITERATIVE SOLUTION OF A COMBINED MIXED AND STANDARD GALERKIN DISCRETIZATION METHOD FOR ELLIPTIC PROBLEMS

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Abstract. In this paper, we consider approximation of a second order elliptic problem defined on a domain in two dimensional Euclidean space. Partitioning the domain into two subdomains, we consider a technique proposed by Wieners and Wohlmuth [23] for coupling mixed finite element approximation on one subdomain with a finite element approximation on other. We consider iterative solution of the resulting linear system of equations. This system is symmetric and indefinite (of saddle-point type). The stability estimates for the discretization imply that the algebraic system can be preconditioned by a block diagonal operator involving a preconditioner for $\mathbf{H}_{\mathrm{div}}$ (on the mixed side) and one for the discrete Laplacian (on the finite element side). Alternatively, we provide iterative techniques based on domain decomposition. Utilizing subdomain solves, the composite problem is reduced to a problem defined only on the interface between the two subdomains. We prove that the interface problem is symmetric, positive definite and well conditioned and hence can be effectively solved by a conjugate gradient iteration.

## 1. Introduction

One of the main problems in large scale scientific computation is the time required to set up a problem. In applications which involve partial differential equations on complicated domains, a great deal of effort is required to construct the mesh. Often, complex domains are built up from simpler ones. The mesh construction problem is greatly simplified if the simpler domains (i.e., the subdomains) can be meshed independently. This, however, results in meshes which do not align on the internal interfaces between subdomains. To get accurate approximation with such meshes, various techniques have been developed.

Since meshes do not align, the resulting spaces are necessarily nonconforming. Approximate continuity conditions are imposed by the use of a Lagrange multiplier [1], [2], [4], [5], [6], [7], [12], [14], [15], [16], [21]. There are two approaches for

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the analysis. The first treats the method as a nonconforming finite element approximation where the Lagrange multiplier constraints serve to define the nonconforming approximation subspace. The second approach is based on an appropriate Ladyzhenskaya-Babuška-Brezzi (LBB) condition. With the second approach, the discrete Lagrange multiplier is shown to approximate the continuous Lagrange multiplier, often a quantity of physical interest. In both cases, the Lagrange multiplier space needs to be strongly connected to the approximation in the subdomains. For the mortar finite element approximation, this connection comes from defining the Lagrange multiplier space from the mesh on one of the subdomains [6]. For the LBB condition, one often is required to use a multiplier space with a mesh size which is somewhat coarser than the mesh sizes on the subdomains [1], [3], [9], [14].

We consider an approximation technique proposed in [23] which utilizes a finite element discretization on one subdomain and a mixed finite element discretization on the other. This pair of approximations gives rise to a natural variational reformulation of the original problem into a saddle point problem involving the two variables (velocity/pressure) on the mixed side and the original variable (pressure) on the conforming finite element side. No additional multipliers need to be introduced.

The purpose of this paper is to develop iterative methods for the solution of the resulting system of algebraic equations. Because of the stability estimate, it is possible to precondition the full system if preconditioners for $H_{\text{div}}$ (on the mixed finite element subdomain) and $H^1$ (on the conforming finite subdomain) are available. Here, we consider domain decomposition approaches. The domain decomposition algorithms require solution of mixed and conforming finite element subproblems on the subdomain and reduce the problem to one on the interface between subdomains. We consider two algorithms of this type. The first iterates for the trace of the discrete solution on the interface while the second iterates for the trace of a discrete normal derivative on the interface. Both algorithms can be thought of as Neumann-Dirichlet in that the discrete subproblems correspond to problems with Neumann and Dirichlet boundary conditions on the respective subdomains.

The outline of the remainder of the paper is as follows. Section 2 gives the composite mixed/conforming variational formulation of the original problem. Section 3 describes the corresponding finite element discretization and its stability properties. The solution of the resulting system of algebraic equations is considered in Section 4. Finally, Section 5 gives the numerical results which illustrate the theory given in the earlier sections.

## 2. Variational formulation

Consider the model second order elliptic problem on a domain $\Omega$ contained in $R^2$,

$$\mathcal{L}p \equiv -\nabla \cdot a\nabla p = f(x), \quad x \in \Omega, \tag{2.1}$$

with, for example, homogeneous Dirichlet boundary conditions $p = 0$ on $\partial\Omega$. Here $a(x)$ is symmetric and uniformly in $\Omega$ positive definite $2 \times 2$ matrix with piece-wise smooth elements. With some abuse of the terminology we shall call the solutions of

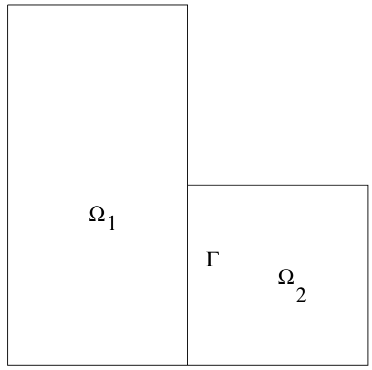


FIGURE 1. Domain partitioning of $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$.

the equation $\mathcal{L}p = 0$ harmonic functions. Of course, if $a$ is the identity matrix, then $p$ is harmonic in $\Omega$.

We partition $\Omega$ into two subdomains by a interface boundary $\Gamma$, i.e., let $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$ (see Figure 1). In $\Omega_1$ we will use a mixed setting of the problem (2.1). That is, we introduce the new (vector) variable $\mathbf{u} = -a\nabla p$. To distinguish between the problem settings we will write $p_1 = p|_{\Omega_1}$ and $p_2 = p|_{\Omega_2}$. The composite model will impose different smoothness requirements on the components $p_1$ and $p_2$. Indeed,

$$\begin{aligned} \mathbf{u} &\in H(\mathrm{div}, \Omega_1) \equiv \mathbf{V}, \\ p_1 &\in L_2(\Omega_1) \equiv Q_1, \\ p_2 &\in H_0^1(\Omega_2, \partial\Omega_2 \setminus \Gamma) = \{\phi \in H^1(\Omega_2)\,;\, \phi = 0 \text{ on } \partial\Omega_2 \setminus \Gamma\} \equiv Q_2. \end{aligned}$$

Note that $p_2$ is required to vanish on $\partial\Omega_2 \setminus \Gamma$. We will denote $\|\cdot\|_{\mathbf{V}}$ to be the $H_{\mathrm{div}}$ norm on $\mathbf{V}$.

We will use the following additional notation:

$$\begin{aligned} \langle p, q\rangle_\Gamma &= \int_\Gamma pq\, ds, \\ a(p,q) &= \int_{\Omega_2} a\nabla p \cdot \nabla q\, dx. \end{aligned} \tag{2.2}$$

Whenever there is no ambiguity will use $(.,.)$ to denote the $L_2$ inner product with respect to a domain (mostly $\Omega_1$ or $\Omega_2$). We will also use $H^s(\Omega)$ to denote the Sobolev space on $\Omega$ of order $s$ (see, for example, [18], [17]). The corresponding norm will be denoted $\|\cdot\|_{s,\Omega}$.

Testing the equation $a^{-1}\mathbf{u} + \nabla p = 0$ by a function $\underline{\chi} \in \mathbf{V}$, integrating by parts, using the zero boundary conditions for $p_1$ on $\partial\Omega_1 \setminus \Gamma$ and the fact the trace of $p_1$ on $\Gamma$ is the same as for $p_2$ on $\Gamma$ gives

$$(a^{-1}\mathbf{u},\ \underline{\chi}) - (p_1, \nabla\cdot\underline{\chi}) + \langle p_2, \underline{\chi}\cdot\mathbf{n}_1\rangle_\Gamma = 0, \quad \text{for all } \underline{\chi} \in \mathbf{V}. \tag{2.3}$$

The second equation is obtained by testing $\nabla \cdot \mathbf{u} = f$ on $\Omega_1$ by functions from $Q_1$. One gets

$$-(w_1, \nabla \cdot \mathbf{u}) = -(f, w_1), \quad \text{for all } w_1 \in Q_1. \tag{2.4}$$

Finally, testing the original equation (2.1) by a function $w_2 \in Q_2$ integrating by parts, using the zero boundary condition for $w_2$ on $\partial\Omega_2 \setminus \Gamma$ and the fact that $\mathbf{u} \cdot \mathbf{n}_1 = -a\nabla p_1 \cdot \mathbf{n}_1 = a\nabla p_2 \cdot \mathbf{n}_2$ on $\Gamma$ gives

$$\langle w_2, \mathbf{u} \cdot \mathbf{n}_1 \rangle_\Gamma - a(p_2, w_2) = -(f, w_2), \quad \text{for all } w_2 \in Q_2. \tag{2.5}$$

That is, the three unknowns $(\mathbf{u}, p_1, p_2) \in \mathbf{V} \times Q_1 \times Q_2$ satisfy the composite system

$$\begin{array}{lllll} (a^{-1}\mathbf{u},\ \underline{\chi}) & -(p_1, \nabla \cdot \underline{\chi}) & +\langle p_2, \underline{\chi} \cdot \mathbf{n}_1 \rangle_\Gamma & = 0, & \text{for } \underline{\chi} \in \mathbf{V}, \\ -(w_1, \nabla \cdot \mathbf{u}) & & & = -(f, w_1), & \text{for } w_1 \in Q_1, \\ \langle w_2, \mathbf{u} \cdot \mathbf{n}_1 \rangle_\Gamma & & -a(p_2, w_2) & = -(f, w_2), & \text{for } w_2 \in Q_2. \end{array} \tag{2.6}$$

2.1. **Well–posedness of the composite problem.** Following [23], we reorder the unknowns and consider the generalized system

$$\begin{array}{lllll} (a^{-1}\mathbf{u},\ \underline{\chi}) & +\langle p_2, \underline{\chi} \cdot \mathbf{n}_1 \rangle_\Gamma & -(p_1, \nabla \cdot \underline{\chi}) & = \langle F_1, \underline{\chi} \rangle, & \text{for } \underline{\chi} \in \mathbf{V}, \\ -\langle w_2, \mathbf{u} \cdot \mathbf{n}_1 \rangle_\Gamma & +a(p_2, w_2) & & = \langle F_2, w_2 \rangle, & \text{for } w_2 \in Q_2, \\ -(w_1, \nabla \cdot \mathbf{u}) & & & = \langle F_3, w_1 \rangle, & \text{for } w_1 \in Q_1. \end{array} \tag{2.7}$$

Here $F_1, F_2$, and $F_3$ are elements of the spaces $\mathbf{V}'$, $Q_2'$, and $Q_1'$ of bounded linear functionals in $\mathbf{V}$, $Q_1$, and $Q_2$, respectively. Finally, $\langle \cdot, \cdot \rangle$ denotes the pairing between a space and its dual.

The analysis of the above problem [23] is based on considering it as a block saddle-point problem of the form

$$\begin{array}{lll} \mathbf{A}\mathbf{u} + \mathbf{B}^T p & = & \tilde{F}_1, \\ \mathbf{B}\mathbf{u} & = & \tilde{F}_2. \end{array} \tag{2.8}$$

Here

$$\mathbf{A} : \mathbf{V} \times Q_2 \to (\mathbf{V} \times Q_2)', \quad \mathbf{B} : \mathbf{V} \times Q_2 \to Q_1', \quad \mathbf{B}^T \colon Q_1 \to (\mathbf{V} \times Q_2)',$$

are defined by

$$\begin{aligned} (\mathbf{A}(\mathbf{u}, q_2), (\mathbf{v}, r_2)) = (a^{-1}\mathbf{u}, \mathbf{v}) &+ \langle q_2, \mathbf{v} \cdot \mathbf{n}_1 \rangle_\Gamma \\ &- \langle r_2, \mathbf{u} \cdot \mathbf{n}_1 \rangle_\Gamma + a(q_2, r_2), \\ (\mathbf{B}(\mathbf{u}, q_2), r_1) = (\mathbf{B}^T r_1, (\mathbf{u}, q_2)) &= -(\nabla \cdot \mathbf{u}, r_1). \end{aligned} \tag{2.9}$$

Clearly, Ker $\mathbf{B} = \{(\mathbf{u}, q_2) :\ \nabla \cdot \mathbf{u} = 0\}$. It immediately follows that $\mathbf{A}$ is coercive on Ker $\mathbf{B}$. Moreover, the "inf-sup" condition corresponding to (2.9) is

$$\|p_1\|_{0,\Omega_1} \le C \sup_{\phi \in \mathbf{V}} \frac{(p_1, \nabla \phi)}{\|\phi\|_{\mathbf{V}}} \text{ for all } p_1 \in L^2(\Omega_1),$$

which is the standard condition for the mixed method on $\Omega_1$ alone. The following theorem is an immediate consequence [11].

**Theorem 2.1.** *There exists exactly one solution* $(\mathbf{u}, p_1, p_2)$ *of (2.7) in* $\mathbf{V} \times Q_1 \times Q_2$. *Moreover, there is a constant* $C$ *not depending on* $F_1 \in \mathbf{V}'$, $F_2 \in Q_1'$ *and* $F_3 \in Q_2'$ *such that*

$$\|\mathbf{u}\|_{\mathbf{V}} + \|p_1\|_{0,\Omega_1} + \|p_2\|_{1,\Omega_2} \le C \left[ \|F_1\|_{\mathbf{V}'} + \|F_2\|_{Q_1'} + \|F_3\|_{Q_2'} \right].$$

## 3. Finite element discretization

In this section, we present the finite element discretization of problem (2.7). Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be triangulations of $\Omega_1$ and $\Omega_2$. We assume that the triangles satisfy a minimal angle condition but need not align on the interface $\Gamma$. Let $(\mathbf{V}_h,\ W_1)$ be a stable pair of mixed finite element spaces associated with the triangulation $\mathcal{T}_1$, for example, BDM [10], BDFM [11], or RT [20]. Also, let $W_2$ be a conforming finite element space associated with $\mathcal{T}_2$. The functions in $W_2$ vanish on $\partial\Omega_2 \setminus \Gamma$. Then the discrete problem is as follows:

Find $\mathbf{u}_h \in \mathbf{V}_h$, $p_{1,h} \in W_1$ and $p_{2,h} \in W_2$ such that,

$$\begin{aligned} &(a^{-1}\mathbf{u}_h, \underline{\chi}) && +\langle p_{2,h}, \underline{\chi}\cdot\mathbf{n}_1\rangle_\Gamma && -(p_{1,h}, \nabla\cdot\underline{\chi}) && = \langle F_1, \underline{\chi}\rangle, \text{ for } \underline{\chi} \in \mathbf{V}_h,\\ &-\langle w_2, \mathbf{u}_h\cdot\mathbf{n}_1\rangle_\Gamma && +a(p_{2,h}, w_2) && && = \langle F_3, w_2\rangle, \text{ for } w_2 \in W_2,\\ &-(w_1, \nabla\cdot\mathbf{u}_h) && && && = \langle F_2, w_1\rangle, \text{ for } w_1 \in W_1. \end{aligned} \tag{3.1}$$

As in the continuous case [23], one groups together the spaces $\mathbf{V}_h$ and $W_2$. Then, one can rewrite the (3.1) in a matrix form similar to the (2.8) in which the corresponding block operators are denoted $\mathbf{A}_h$, $\mathbf{B}_h$ and $\mathbf{B}_h^T$. It is immediate that $\mathbf{A}_h$ is coercive on Ker $\mathbf{B}_h$ and the corresponding "inf-sup" condition is exactly that required for the mixed approximation pair $(\mathbf{V}_h, W_1)$, i.e. for all $p_1 \in W_1$,

$$\|p_1\|_{0,\Omega_1} \le C \sup_{\underline{\chi}\in\mathbf{V}_h} \frac{(p_1, \nabla\cdot\underline{\chi})}{\|\underline{\chi}\|_{\mathbf{V}}}. \tag{3.2}$$

The following result is an immediate consequence [11].

**Theorem 3.1.** *The discrete problem* (3.1) *is uniquely solvable and if the finite element spaces* $(\mathbf{V}_h, W_1)$ *satisfy the inf-sup (3.2) then the following a priori estimate holds for its solution:*

$$\|\mathbf{u}_h\|_{\mathbf{V}} + \|p_{1,h}\|_{0,\Omega_1} + \|p_{2,h}\|_{1,\Omega_2} \le C \left[ \|F_1\|_{\mathbf{V}'} + \|F_2\|_{Q_1'} + \|F_3\|_{Q_2'} \right]. \tag{3.3}$$

*The constant* $C$ *is independent of the mesh sizes* $h_1$ *of* $\mathcal{T}_1$ *and* $h_2$ *of* $\mathcal{T}_2$.

For the subsequent analysis, we shall need to use the $L^2$-projection operators $Q_{i,h} : Q_i \to W_i$, $i = 1, 2$ and the approximation operator $\Pi_h : \mathbf{V} \cap \mathbf{H}^\gamma(\Omega_1) \to \mathbf{V}_h$ associated with the mixed pair of subspaces. We assume that the operators $Q_{1,h}$, $Q_{2,h}$ and $\Pi_h$ satisfies the following properties:

**(A.1)** $\Pi_h$ is a stable operator from $\mathbf{V} \cap \mathbf{H}^1(\Omega_1)$ and:

(a) satisfies the commutativity property

$$\nabla \cdot \Pi_h \underline{\chi} = Q_{1,h} \nabla \cdot \underline{\chi} \text{ for all } \underline{\chi} \in \mathbf{V};$$

(b) if $\mathbf{w} \in \mathbf{V}$ satisfies $\mathbf{w} \cdot \mathbf{n}_1 = \mathbf{w}_h \cdot \mathbf{n}_1$ on $\Gamma$ for some $\mathbf{w}_h \in \mathbf{V}_h$, then $(\Pi_h \mathbf{w}) \cdot \mathbf{n}_1 = \mathbf{w}_h \cdot \mathbf{n}_1$ on $\Gamma$.

**(A.2)** There is an integer $k > 0$ such that for all $\gamma \in (0, k]$:

(a) $\|(I - \Pi_h)\mathbf{u}\|_{0,\Omega_1} \le C h_1^\gamma \|\mathbf{u}\|_{\gamma,\Omega_1}$ for all $\mathbf{u} \in \mathbf{V} \cap \mathbf{H}^\gamma(\Omega_1)$;

(b) $\|(I - Q_{1,h})p_1\|_{0,\Omega_1} \le C h_1^\gamma \|p_1\|_{\gamma,\Omega_1}$ for all $p_1 \in H^\gamma(\Omega_1)$;

(c) $\|(I - Q_{2,h})p_2\|_{0,\Omega_2} \le C h_2^{1+\gamma} \|p_2\|_{1+\gamma,\Omega_2}$ for all $p_2 \in H^{1+\gamma}(\Omega_2)$.

The above properties are standard for the well-known mixed finite element spaces (BDM [10], BDFM [11], and RT [20]) and their associated approximation operators. Similarly, the standard conforming Lagrangian finite element spaces will satisfy the last estimate. Moreover, since the $L^2$-projection is stable in $H^1(\Omega_2)$ as a consequence of (A.2.c) we have also the following estimate

$$\|(I - Q_{2,h})p_1\|_{1,\Omega_2} \le C h_2^\gamma \|p_2\|_{1+\gamma,\Omega_2}, \text{ for all } p_2 \in H^{1+\gamma}(\Omega_2).$$

The error analysis is quite straightforward, namely, we prove the following error estimate:

**Theorem 3.2.** *Let $(\mathbf{u}, p_1, p_2)$ and $(\mathbf{u}_h, p_{1,h}, p_{2,h})$ denote the solutions of (2.7) and (3.1), respectively. Let $0 < \gamma \le 1$ and assume that $\mathbf{u} \in \mathbf{H}^\gamma(\Omega_1)$, $\nabla \cdot \mathbf{u}, p_1 \in H^\gamma(\Omega_1)$, and $p_2 \in H^{1+\gamma}(\Omega_2)$. Then*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}} &+ \|p_1 - p_{1,h}\|_{0,\Omega_1} + \|p_2 - p_{2,h}\|_{1,\Omega_1} \\ &\le C\Big(h_1^\gamma \|\mathbf{u}\|_{\gamma,\Omega_1} + h_1^\gamma \|\nabla \cdot \mathbf{u}\|_{\gamma,\Omega_1} + h_1^\gamma \|p_1\|_{\gamma,\Omega_1} + h_2^\gamma \|p_2\|_{1+\gamma,\Omega_2}\Big) \end{aligned} \tag{3.4}$$

*with constant $C$ independent of $h_1$ and $h_2$.*

*Proof.* The approximation errors $\mathbf{e}_h = \Pi_h \mathbf{u} - \mathbf{u}_h$, $e_{1,h} = Q_{1,h} p_1 - p_{1,h}$, and $e_{2,h} = Q_{2,h} p_2 - p_{2,h}$ satisfy the discrete problem:

$$\begin{array}{llllll} (a^{-1}\mathbf{e}_h,\ \underline{\chi}) & -(e_{1,h}, \nabla \cdot \underline{\chi}) & +\langle e_{2,h}, \underline{\chi} \cdot \mathbf{n}_1 \rangle_\Gamma & = \langle \Phi_1, \underline{\chi} \rangle, & \text{for } \underline{\chi} \in \mathbf{V}_h, \\ -(w_1, \nabla \cdot \mathbf{e}_h) & & & = 0, & \text{for } w_1 \in W_1, \\ \langle w_2, \mathbf{e}_h \cdot \mathbf{n}_1 \rangle_\Gamma & & -a(e_{2,h}, w_2) & = \langle \Phi_3, w_2 \rangle, & \text{for } w_2 \in W_2, \end{array}$$

where,

$$\langle \Phi_1, \underline{\chi} \rangle = (a^{-1}(\Pi_h \mathbf{u} - \mathbf{u}), \underline{\chi}) + \langle Q_{2,h} p_2 - p_2, \underline{\chi} \cdot \mathbf{n}_1 \rangle_\Gamma$$

and

$$\langle \Phi_3, w_2 \rangle = -a(Q_{2,h} p_2 - p_2, w_2) + \langle w_2, (\Pi_h \mathbf{u} - \mathbf{u}) \cdot \mathbf{n}_1 \rangle_\Gamma.$$

By the approximation properties (A.2) of $\Pi_h$, $Q_{1,h}$, $Q_{2,h}$,

$$|\langle \Phi_1, \underline{\chi} \rangle| \le C\Big(h_1^\gamma \|\mathbf{u}\|_{\gamma,\Omega_1} + h_2^\gamma \|p_2\|_{1+\gamma,\Omega_2}\Big) \|\underline{\chi}\|_{\mathbf{V}}$$

and

$$\begin{aligned}|\langle \Phi_3, w_2\rangle| &\le C\Big(h_2^\gamma \|p_2\|_{1+\gamma,\Omega_2} + \|\Pi_h \mathbf{u} - \mathbf{u}\|_{\mathbf{V}}\Big)\|w_2\|_{1,\Omega_2}\\ &\le C\Big(h_2^\gamma \|p_2\|_{1+\gamma,\Omega_2} + h_1^\gamma \|\mathbf{u}\|_{\gamma,\Omega_1} + h_1^\gamma \|\nabla\cdot \mathbf{u}\|_{\gamma,\Omega_1}\Big)\|w_2\|_{1,\Omega_2}.\end{aligned}$$

The above two estimates and Theorem 3.1 show that $\|\mathbf{e}_h\|_{\mathbf{V}} + \|e_{1,h}\|_{0,\Omega_1} + \|e_{2,h}\|_{1,\Omega_2}$ is bounded by the right hand side of (3.4). The theorem immediately follows from this, the approximation properties of $\Pi_h$, $Q_{1,h}$ and $Q_{2,h}$ and the triangle inequality. □

## 4. Iterative Solution

We consider the problem of computing the solution of (3.1) in this section. We first consider preconditioning the composite system. This system is symmetric and indefinite. Preconditioners result from the *a priori* estimates for the discrete solution established in Theorem 3.1. The second approach is by domain decomposition. It uses the solution of subdomain problems to reduce to an iteration involving only unknowns on $\Gamma$.

4.1. **Preconditioning the composite saddle–point problem.** We first consider preconditioning the discrete algebraic system resulting from the composite problem. Let $\mathcal{X}$ denote the product space $\mathbf{V}_h \times W_2 \times W_1$ and consider the operator $\mathcal{A} : \mathcal{X} \to \mathcal{X}$ given by

$$\mathcal{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{T}^T & \mathbf{N}^T \\ -\mathbf{T} & A_2 & 0 \\ \mathbf{N} & 0 & 0 \end{bmatrix}. \tag{4.1}$$

Here

$$\begin{aligned}(\mathbf{A}_1 \underline{\chi}, \underline{\theta}) &= (a^{-1}\underline{\chi}, \underline{\theta}) \text{ for all } \underline{\chi}, \underline{\theta} \in \mathbf{V}_h,\\ (\mathbf{N}\underline{\chi}, w_1) &= (\mathbf{N}^T w_1, \underline{\chi}) = -(\nabla\cdot\underline{\chi}, w_1) \text{ for all } \underline{\chi} \in \mathbf{V}_h,\ w_1 \in W_1,\\ (\mathbf{T}\underline{\chi}, w_2) &= (\mathbf{T}^T w_2, \underline{\chi}) = \langle w_2, \underline{\chi}\cdot \mathbf{n}_1\rangle_\Gamma \text{ for all } \underline{\chi} \in \mathbf{V}_h,\ w_2 \in W_2,\\ (A_2 v_2, w_2) &= a(v_2, w_2) \text{ for all } v_2, w_2 \in W_2.\end{aligned}$$

We also consider the block diagonal operator

$$\mathcal{D} = \begin{bmatrix} \mathbf{\Lambda} & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & I \end{bmatrix}$$

where $(\mathbf{\Lambda}\underline{\chi},\ \underline{\theta}) = (a^{-1}\underline{\chi},\ \underline{\theta}) + (\nabla\cdot\underline{\chi},\ \nabla\cdot\underline{\theta})$ for all $\underline{\chi},\ \underline{\theta} \in \mathbf{V}_h$. By Theorem 3.1, for any $\mathbf{U} \in \mathcal{X}$,

$$|||\mathbf{U}|||_{\mathcal{D}}^2 \le C|||\mathcal{A}\mathbf{U}|||_{\mathcal{D}^{-1}}^2 = C \sup_{\theta\in\mathcal{X}} \frac{(\mathcal{A}\mathbf{U}, \theta)^2}{(\mathcal{D}\theta, \theta)} \le C|||\mathbf{U}|||_{\mathcal{D}}^2. \tag{4.2}$$

Here $|||\cdot|||_{\mathcal{D}}$ denotes the operator norm given by $|||\cdot|||_{\mathcal{D}} = (\mathcal{D}\cdot, \cdot)^{1/2}$ and the pairing $(\cdot,\cdot)$ denotes the inner-product in the product space $\mathcal{X}$.

In practice, one represents the above operators in terms of bases. Combining the bases for the three spaces which define $\mathcal{X}$ gives rise to a basis $\{\Psi_i\}_{i=1}^n$. Let $\bar{\mathcal{A}}$ be the matrix corresponding to the operator $\mathcal{A}$, i.e.,

$$\bar{\mathcal{A}}_{ij} = (\mathcal{A}\Psi_j, \Psi_i).$$

The matrix $\bar{\mathcal{D}}$ corresponding to $\mathcal{D}$ is defined analogously. The above inequality (4.2) can be rewritten in terms of matrices as

$$c_0(\bar{\mathcal{D}}x)\cdot x \le (\bar{\mathcal{A}}^T\bar{\mathcal{D}}^{-1}\bar{\mathcal{A}}x)\cdot x \le c_1(\bar{\mathcal{D}}x)\cdot x, \tag{4.3}$$

for all $x \in R^n$. Here $\bar{\mathcal{A}}^T$ denotes the transpose of the matrix $\bar{\mathcal{A}}$. The algebraic problem corresponding to (3.1) is to find the vector $x \in R^n$ satisfying

$$\bar{\mathcal{A}}x = b$$

for an appropriately defined $b$.

The inequality (4.3) implies that reformulated system

$$\bar{\mathcal{A}}^T\bar{\mathcal{D}}^{-1}\bar{\mathcal{A}}x = \bar{\mathcal{A}}^T\bar{\mathcal{D}}^{-1}b$$

can be preconditioned by $\bar{\mathcal{D}}^{-1}$. This can be solved by a rapidly convergent preconditioned iteration. In addition, the operators $\mathbf{\Lambda}$ and $A_2$ can be replaced by preconditioners. Instead of preconditioning the normal system one can alternatively precondition the original saddle–point system suing the same block–diagonal preconditioner $\overline{\mathcal{D}}^{-1}$ in the minimum residual method.

4.2. **Preconditioning reduced problems by interface domain decomposition.** We next consider strategies based on domain decomposition. Specifically, we consider the case when existing software is available for solving the mixed and finite element problems independently. The idea is to reduce the original problem (3.1) to a problem on $\Gamma$. We give two examples of such reductions. The reduced problems are symmetric, positive definite and well-conditioned with respect to appropriate inner-products. These are Dirichlet-Neumann domain decomposition algorithms.

To develop the reduced system, we first introduce $(\tilde{\mathbf{u}}_h, \tilde{p}_{1,h}, \tilde{p}_{2,h})$ in $\mathbf{V}_h \times W_1 \times W_2$ satisfying

$$\begin{array}{rcrcll} (a^{-1}\tilde{\mathbf{u}}_h, \underline{\chi}) & - & (\tilde{p}_{1,h}, \nabla\cdot\underline{\chi}) & = & \langle F_1, \underline{\chi}\rangle, & \text{for } \underline{\chi}\in\mathbf{V}_h,\\ -(w_1, \nabla\cdot\tilde{\mathbf{u}}_h) & & & = & \langle F_2, w_1\rangle, & \text{for } w_1\in W_1,\\ & - & a(\tilde{p}_{2,h}, w_2) & = & -\langle F_3, w_2\rangle, & \text{for } w_2\in W_2. \end{array} \tag{4.4}$$

The first two equations above correspond to a mixed finite element problem on $\Omega_1$ while the third is a finite element problem on $\Omega_2$. The remainder $(\mathbf{v}_h, q_{1,h}, q_{2,h}) = (\mathbf{u}_h - \tilde{\mathbf{u}}_h, p_{1,h} - \tilde{p}_{1,h}, p_{2,h} - \tilde{p}_{2,h})$ satisfies

$$\begin{array}{rcrcll} (a^{-1}\mathbf{v}_h, \underline{\chi}) & - & (q_{1,h}, \nabla\cdot\underline{\chi}) & = & -\langle \tilde{p}_{2,h} + q_{2,h}, \underline{\chi}\cdot\mathbf{n}_1\rangle_\Gamma, & \text{for } \underline{\chi}\in\mathbf{V}_h,\\ -(w_1, \nabla\cdot\mathbf{v}_h) & & & = & 0, & \text{for } w_1\in W_1,\\ & - & a(q_{2,h}, w_2) & = & -\langle w_2, (\tilde{\mathbf{u}}_h + \mathbf{v}_h)\cdot\mathbf{n}_1\rangle_\Gamma, & \text{for } w_2\in W_2. \end{array} \tag{4.5}$$

In our analysis we shall need the following trace inequalities:

$$\|w_2\|_{H^{1/2}_{0,0}(\Gamma)} \le C\|w_2\|_{1,\Omega_2} \quad \text{for all } w_2 \in Q_2, \tag{4.6}$$

and

$$\|\underline{\eta}\cdot\mathbf{n}\|_{H^{-1/2}(\Gamma)} \le \|\underline{\eta}\cdot\mathbf{n}\|_{H^{-1/2}(\partial\Omega_1)} \le C\|\underline{\eta}\|_{\mathbf{V}} \quad \text{for all } \underline{\eta}\in\mathbf{V}. \tag{4.7}$$

Here $H^{1/2}_{0,0}(\Gamma)$ is the interpolation space which is halfway between $H^1_0(\Gamma)$ and $L^2(\Gamma)$ and $H^{-1/2}(\Gamma)$ denotes its dual.

We will reformulate (4.5) in terms of operators on the spaces

$$W^{-1/2}(\Gamma) = \{\underline{\chi}\cdot\mathbf{n}_1 \text{ on } \Gamma \,:\, \underline{\chi}\in\mathbf{V}_h\}$$

and

$$W^{1/2}(\Gamma) = \{q|_\Gamma \,:\, q \in W_2\}.$$

We use the $H^{1/2}_{0,0}(\Gamma)$ norm (respectively, the $H^{-1/2}(\Gamma)$ norm) on $W^{1/2}(\Gamma)$ (respectively, $W^{-1/2}(\Gamma)$). Define $E:\ W^{1/2}(\Gamma)\mapsto W^{-1/2}(\Gamma)$ by $E\sigma = \mathbf{w}_h(\sigma)\cdot\mathbf{n}_1$ on $\Gamma$ where $(\mathbf{w}_h(\sigma), r)\in\mathbf{V}_h\times W_1$ is the solution of

$$\begin{aligned} &(a^{-1}\mathbf{w}_h(\sigma),\underline{\chi}) \quad -(r,\nabla\cdot\underline{\chi}) &&= \langle\sigma,\underline{\chi}\cdot\mathbf{n}_1\rangle_\Gamma,\ \text{for } \underline{\chi}\in\mathbf{V}_h,\\ &-(w_1,\nabla\cdot\mathbf{w}_h(\sigma)) &&= 0,\ \text{for } w_1\in W_1. \end{aligned} \tag{4.8}$$

The operator $E$ has a meaning of discrete Dirichlet-Neumann mapping on $\Gamma$. Indeed, by (4.8) a given data $\sigma$ is first projected by the operator $\Pi_h$ on the trace on $\Gamma$ of the normal component of $\mathbf{V}_h$ and then extended as discrete harmonic function $\mathbf{w}_h(\sigma)$ by the mixed finite element method (with homogeneous Neumann data on the rest of $\partial\Omega$); finally, the normal component $\mathbf{w}_h(\sigma)\cdot\mathbf{n}_1$ on $\Gamma$ is the desired Dirichlet-Neumann map.

In addition, define $S:\ W^{-1/2}(\Gamma)\to W^{1/2}(\Gamma)$ by $S\gamma = w_h(\gamma)$ on $\Gamma$ where $w_h(\gamma)\in W_2$ is the solution of

$$a(w_h(\gamma),q) = \langle\gamma,q\rangle_\Gamma \text{ for all } q\in W_2. \tag{4.9}$$

Clearly, $w_h(\gamma)$ is discrete harmonic. This is the Neumann-Dirichlet mapping generated by the discrete solution of the elliptic problem in $\Omega_2$.

It follows from (4.6) and (4.7) that both $S$ and $E$ are bounded operators on these spaces with bounds which do not depend on $h_1$ or $h_2$.

In terms of these operators, (4.5) becomes

$$\begin{aligned} \mathbf{v}_h\cdot\mathbf{n}_1 &= -E(\tilde{p}_{2,h} + q^\Gamma_{2,h})\\ q^\Gamma_{2,h} &= S((\tilde{\mathbf{u}}_h + \mathbf{v}_h)\cdot\mathbf{n}_1). \end{aligned} \tag{4.10}$$

Here $q^\Gamma_{2,h}$ is the trace of $q_{2,h}$ on $\Gamma$. Eliminating $\mathbf{v}_h\cdot\mathbf{n}_1$ gives

$$(I+SE)q^\Gamma_{2,h} = S(\tilde{\mathbf{u}}_h\cdot\mathbf{n}_1 - E\tilde{p}_{2,h}). \tag{4.11}$$

Note one can immediately recover the remainder $(\mathbf{v}_h, q_{1,h}, q_{2,h})$ from $q^\Gamma_{2,h}$ (or $\mathbf{v}_h\cdot\mathbf{n}_1$) by one additional solve on each subdomain. Thus, (4.11) reduces the problem of computing the remainder to a problem on the boundary $\Gamma$.

We consider the inner product on $W^{1/2}(\Gamma) \times W^{1/2}(\Gamma)$ defined by

$$\langle\langle v, w \rangle\rangle = a(\bar{v}, \bar{w})$$

where $\bar{v}$ and $\bar{w}$ respectively denote the discrete harmonic extensions of $v$ and $w$ in the space $W_2$. It is well-known that the corresponding norm on $W^{1/2}(\Gamma)$ is equivalent to the $H^{1/2}_{0,0}(\Gamma)$ norm. This equivalence holds uniformly independently of $h_2$. The following theorem shows that (4.11) can be effectively solved by conjugate gradient iteration.

**Theorem 4.1.** *The operator $SE$ is symmetric and positive semi-definite with respect to the inner product $\langle\langle \cdot, \cdot \rangle\rangle$. Moreover, $SE$ is bounded in the corresponding norm with bound $K$ independent of $h_1$ and $h_2$. Thus, $(I + SE)$ is symmetric and positive definite on $W^{1/2}(\Gamma)$ and has a condition number bounded by $K + 1$. The resulting conjugate gradient iteration converges with a rate bounded independently of $h_1$ and $h_2$.*

*Proof.* Let $\sigma$ be in $W^{1/2}(\Gamma)$. Note that $w_h(E\sigma)$ equals $SE\sigma$ on $\Gamma$ and is discrete harmonic. Thus, for any $\gamma \in W^{1/2}(\Gamma)$,

$$\langle\langle SE\sigma, \gamma \rangle\rangle = a(w_h(E\sigma), \bar{\gamma}) = \langle E\sigma, \gamma \rangle_\Gamma.$$

But $E\sigma = \mathbf{w}_h(\sigma) \cdot \mathbf{n}_1$. Using the fact that $(\nabla \cdot \mathbf{w}_h(\sigma), w_1) = 0$ for all $w_1 \in W_1$ gives

$$\langle\langle SE\sigma, \gamma \rangle\rangle = (a^{-1}\mathbf{w}_h(\sigma), \mathbf{w}_h(\gamma)).$$

This shows that $SE$ is symmetric and positive semi-definite.

Finally, it easily follows from the stability properties of the mixed finite element problem on $\Omega_1$ and (4.7) that

$$\langle SE\sigma, \sigma \rangle_\Gamma = (a^{-1}\mathbf{w}_h(\sigma), \mathbf{w}_h(\sigma)) \leq C\|\sigma\|^2_{H^{1/2}_{0,0}(\Gamma)}.$$

The theorem follows from the equivalence of the norm $\langle\langle \cdot, \cdot \rangle\rangle^{1/2}$ with the $H^{1/2}_{0,0}(\Gamma)$ norm on $W^{1/2}(\Gamma)$. □

**Remark 4.1.** *The operator $E$ is the discrete analogue of solving a problem on $\Omega_1$ with Dirichlet boundary conditions on $\Gamma$. The operator $S$ corresponds to solving a problem on $\Omega_2$ with Neumann boundary conditions on $\Gamma$.*

**Remark 4.2.** *The inner product which makes $I + SE$ into a symmetric and positive definite operator involves discrete harmonic extension with respect to the subspace $W_2$. This poses no additional computational problems. In fact, a carefully implemented conjugate gradient algorithm for (4.11) need only have one mixed solve on $\Omega_1$ and one finite element solve on $\Omega_2$ per iterative step (after startup).*

Alternatively, it is possible reduce to an equation for $\mathbf{v}_h \cdot \mathbf{n}_1$ on $\Gamma$ by eliminating $q^\Gamma_{2,h}$ in (4.10). One then gets

$$(I + ES)(\mathbf{v}_h \cdot \mathbf{n}_1) = -E(\tilde{p}_{2,h} + S(\tilde{\mathbf{u}}_h \cdot \mathbf{n}_1)). \tag{4.12}$$

Given $\mathbf{u}_h \cdot \mathbf{n}_1 \in W^{-1/2}(\Gamma)$, we define $(\bar{\mathbf{u}}_h, P) \in \mathbf{V}_h \times W_1$ by

$$\begin{aligned} \bar{\mathbf{u}}_h \cdot \mathbf{n}_1 &= \mathbf{u}_h \cdot \mathbf{n}_1 \text{ on } \Gamma, \\ (a^{-1}\bar{\mathbf{u}}_h, \underline{\chi}) - (\nabla \cdot \underline{\chi}, P) &= 0, \quad \text{for all } \underline{\chi} \text{ in } \mathbf{V}_h \text{ with } \underline{\chi} \cdot \mathbf{n}_1 = 0 \text{ on } \Gamma, \\ (\nabla \cdot \bar{\mathbf{u}}_h, q) &= 0, \text{ for all } q \text{ in } W_1. \end{aligned} \tag{4.13}$$

Let $\langle\langle \cdot, \cdot \rangle\rangle$ denote the inner-product

$$\langle\langle \mathbf{u}_h \cdot \mathbf{n}_1, \mathbf{u}_h \cdot \mathbf{n}_1 \rangle\rangle = (a^{-1}\bar{\mathbf{u}}_h, \bar{\mathbf{u}}_h) \text{ for all } \mathbf{u}_h \cdot \mathbf{n}_1 \in W^{-1/2}(\Gamma). \tag{4.14}$$

**Lemma 4.1.** *Assume that the $\Omega_1$ mesh restricted to $\Gamma$ consists of edge segments $\{\mathcal{E}_i\}$ which satisfy*

$$\textit{length}(\mathcal{E}_i) \geq Ch_1, \ \textit{for all } i. \tag{4.15}$$

*Then the inner-product defined by (4.14) gives rise to a norm which is equivalent (independently of $h_1$) to $\|\cdot\|_{H^{-1/2}(\Gamma)}$ on $W^{-1/2}(\Gamma)$.*

*Proof.* The bound

$$\|\mathbf{u}_h \cdot \mathbf{n}_1\|^2_{H^{-1/2}(\Gamma)} \leq C(a^{-1}\bar{\mathbf{u}}_h, \bar{\mathbf{u}}_h)$$

follows immediately from (4.7) and the fact that $\nabla \cdot \bar{\mathbf{u}}_h = 0$.

For the other direction, we reduce the problem to one of discrete (divergence–free) extension. Let $\mathbf{u}_h \cdot \mathbf{n}_1$ be in $W^{-1/2}(\Gamma)$. Suppose that we have defined $\tilde{\mathbf{u}}_h \in \mathbf{V}_h$ with $\tilde{\mathbf{u}}_h \cdot \mathbf{n}_1 = \mathbf{u}_h \cdot \mathbf{n}_1$ on $\Gamma$, $\nabla \cdot \tilde{\mathbf{u}}_h = 0$ in $\Omega_1$ and

$$\|\tilde{\mathbf{u}}_h\|_{\mathbf{V}} \leq C\|\mathbf{u}_h \cdot \mathbf{n}_1\|_{H^{-1/2}(\Gamma)}.$$

Let $\bar{\mathbf{u}}_h = \tilde{\mathbf{u}}_h + \mathbf{w}_h$. Then $\mathbf{w}_h \cdot \mathbf{n}_1 = 0$ on $\Gamma$ and

$$\begin{aligned} (a^{-1}\mathbf{w}_h, \underline{\chi}) - (\nabla \cdot \underline{\chi}, P) &= -(a^{-1}\tilde{\mathbf{u}}_h, \underline{\chi}), \text{ for all } \underline{\chi} \text{ in } \mathbf{V}_h \text{ with } \underline{\chi} \cdot \mathbf{n}_1 = 0 \text{ on } \Gamma, \\ (\nabla \cdot \mathbf{w}_h, q) &= 0, \text{ for all } q \text{ in } W_1. \end{aligned}$$

Here $P$ is as in (4.13). Since the mixed finite element pair is stable,

$$\|\mathbf{w}_h\|_{\mathbf{V}} \leq C\|\tilde{\mathbf{u}}_h\|_{\mathbf{V}} \leq C\|\mathbf{u}_h \cdot \mathbf{n}_1\|_{H^{-1/2}(\Gamma)},$$

it is immediate from the triangle inequality that

$$(a^{-1}\bar{\mathbf{u}}_h, \bar{\mathbf{u}}_h)^{1/2} \leq C\|\mathbf{u}_h \cdot \mathbf{n}_1\|_{H^{-1/2}(\Gamma)}.$$

This is the second inequality of the lemma. Thus, to complete the proof of the lemma, we need only construct $\tilde{\mathbf{u}}_h$. A regularity–free proof for two–dimensional domains is found in [19]. We below provide a simpler proof which relies on certain minimal regularity assumption.

We consider the function $\phi$ satisfying

$$\begin{aligned} \Delta\phi &= 0, \text{ in } \Omega_1, \\ \frac{\partial \phi}{\partial \mathbf{n}_1} &= \mathbf{u}_h \cdot \mathbf{n}_1, \text{ on } \Gamma, \\ \phi &= 0, \text{ on } \partial\Omega_1 \setminus \Gamma. \end{aligned}$$

Then

$$D(\phi, \theta) = F(\theta), \tag{4.16}$$

for all $\theta \in H^1(\Omega_1)$ with $\theta = 0$ on $\Omega_1 \setminus \Gamma$. Here $D(\cdot,\cdot)$ denotes the Dirichlet inner-product on $\Omega_1$ and $F$ denotes the functional $F(\theta) = \langle \mathbf{u}\cdot\mathbf{n}_1, \theta\rangle_\Gamma$. We define $\tilde{\mathbf{u}}_h = \Pi_h(\nabla\phi)$.

We clearly have that $(\nabla\phi)\cdot\mathbf{n}_1 = \mathbf{u}_h\cdot\mathbf{n}_1$ on $\Gamma$ and so, by (A.3), $(\Pi_h\nabla\phi)\cdot\mathbf{n}_1 = \mathbf{u}_h\cdot\mathbf{n}_1$ on $\Gamma$. Furthermore,

$$\|\nabla\phi\|_{\mathbf{V}} \le C\|\mathbf{u}_h\cdot\mathbf{n}_1\|_{H^{-1/2}(\Gamma)}.$$

In addition, for any $\gamma$ in $(0,1/2)$,

$$\|(I-\Pi_h)(\nabla\phi)\|_{\mathbf{V}} \le Ch_1^\gamma\|\phi\|_{1+\gamma,\Omega_1}.$$

For some $\gamma$ in $(0,1/2)$, we assume that the following regularity estimate holds for the mixed boundary value problem: Solutions of (4.16) satisfy

$$\|\phi\|_{1+\gamma,\Omega_1} \le C(\gamma)\|F\|_{-1+\gamma,\Omega_1}.$$

Now,

$$\|F\|_{-1+\gamma,\Omega_1} = \sup_\theta \frac{\langle \mathbf{u}_h\cdot\mathbf{n}_1,\theta\rangle_\Gamma}{\|\theta\|_{1-\gamma,\Omega_1}} \le C\sup_\theta \frac{\langle \mathbf{u}_h\cdot\mathbf{n}_1,\theta\rangle_\Gamma}{\|\theta\|_{1/2-\gamma,\Gamma}} = C\|\mathbf{u}_h\cdot\mathbf{n}_1\|_{H^{-1/2+\gamma}(\Gamma)}.$$

Here the supremum is over $\theta$ in $H^1(\Omega_1)$ with $\theta = 0$ on $\Omega_1\setminus\Gamma$. By the quasi-uniformity of the mesh on $\Gamma$,

$$\|\mathbf{u}_h\cdot\mathbf{n}_1\|_{H^{-1/2+\gamma}(\Gamma)} \le Ch_1^{-\gamma}\|\mathbf{u}_h\cdot\mathbf{n}_1\|_{H^{-1/2}(\Gamma)}.$$

Combining the above inequalities gives

$$\|\tilde{\mathbf{u}}_h\|_{\mathbf{V}} \le \|\nabla\phi\|_{\mathbf{V}} + \|(I-\Pi_h)\nabla\phi\|_{\mathbf{V}} \le C\|\mathbf{u}_h\cdot\mathbf{n}_1\|_{H^{-1/2}(\Gamma)}.$$

This completes the proof of the lemma. □

**Theorem 4.2.** *Assume that (4.15) holds. The operator $ES$ is symmetric and positive semi-definite with respect to the inner product $\langle\langle\cdot,\cdot\rangle\rangle$ defined by (4.14). Moreover, $ES$ is bounded in the corresponding norm with bound $K$ independent of $h_1$ and $h_2$. Thus, $(I+ES)$ is symmetric and positive definite on $W^{-1/2}(\Gamma)$ and has a condition number bounded by $K+1$. The resulting conjugate gradient iteration converges with a rate bounded independently of $h_1$ and $h_2$.*

*Proof.* A straightforward computation shows that for $\mathbf{u}_h\cdot\mathbf{n}_1, \mathbf{v}_h\cdot\mathbf{n}_1 \in W^{-1/2}(\Gamma)$,

$$\langle\langle ES(\mathbf{u}_h\cdot\mathbf{n}_1), \mathbf{v}_h\cdot\mathbf{n}_1\rangle\rangle = a(w_h(\mathbf{u}_h\cdot\mathbf{n}_1), w_h(\mathbf{v}_h\cdot\mathbf{n}_1))$$

where $w_h(\cdot)$ was defined by (4.9). This shows that $ES$ is symmetric and positive semi-definite. The theorem is a consequence of the *a priori* estimate

$$\|w_h(\mathbf{u}_h\cdot\mathbf{n}_1)\|_{1,\Omega_2} \le C\|\mathbf{u}_h\cdot\mathbf{n}_1\|_{H^{-1/2}(\Gamma)}$$

and Lemma 4.1. □

## 5. Numerical experiments

In this section we illustrate the method on the following two dimensional test example:

- the domain is $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$, where $\Omega_1 = (0,1)\times(0,1)$, $\Gamma = \{(1,y),\ 0 < y < b\}$, $b < 1$ is a given parameter, and $\Omega_2 = (1, 1+b)\times(0,b)$;
- the elliptic problem in $\Omega_1$ is $-\nabla \cdot a_1 \nabla p_1 = f_1$, where the coefficient matrix

$$a_1 = \left[ \begin{array}{cc} 1 + 10x^2 + y^2 & \frac{1}{2} + x^2 + y^2 \\ \frac{1}{2} + x^2 + y^2 & 1 + x^2 + 10y^2 \end{array} \right];$$

  the exact solution is $p_1(x,y) = (1-x)^2 x(1-y)y$, hence $\mathbf{u} = -a_1 \nabla p_1$.
- the elliptic problem in $\Omega_2$ is $-\nabla \cdot a_2 \nabla p_2 = f_2$, where the coefficient matrix is just the identity, i.e. $a_2 = I$, and the exact solution is $p_2(x,y) = 10^5(1+b-x)(x-1)^2 y(b-y)$.

Note that,

$$p(x,y) = \left\{ \begin{array}{ll} p_1(x,y), & \text{in } \Omega_1, \\ p_2(x,y), & \text{in } \Omega_2 \end{array} \right.$$

is an $H^1(\Omega)$–function since $[p]|_\Gamma = 0$ and $(a_1 \nabla p_1)\cdot \mathbf{n}_1 = (a_2 \nabla p_1)\cdot \mathbf{n}_1$ on $\Gamma$. Also, $p$ vanishes on $\partial\Omega$.

To discretize the problem we used lowest order Raviart–Thomas spaces on uniform triangular mesh of size $h_1$ in $\Omega_1$ and conforming piecewise linear functions over uniform triangles in $\Omega_2$ with mesh-size $h_2$. We write the resulting linear system in the form

$$\left[ \begin{array}{ccc} \mathbf{A}_1 & \mathbf{N}^T & \mathbf{T}^T \\ \mathbf{N} & 0 & 0 \\ \mathbf{T} & 0 & -A_2 \end{array} \right] \left[ \begin{array}{c} \mathbf{u}_h \\ p_{1,\,h} \\ p_{2,\,h} \end{array} \right] = \left[ \begin{array}{c} 0 \\ \mathbf{f}_1 \\ \mathbf{f}_2 \end{array} \right].$$

The corresponding Neumann problems on $\Omega_2$ we solve exactly by $LU$ factorization. Similarly, to compute the actions of the Schur complement $\tilde{S}$, that corresponds to discrete harmonic extension in $\Omega_2$, i.e., $< \tilde{S} w_2, v_2 >_\Gamma = (a_2 \nabla \widetilde{w}_2,\ \nabla \widetilde{v}_2)$, where $\widetilde{v}_2$ and $\widetilde{w}_2$ are the discrete harmonic extensions of $v_2$ and $w_2$ – piecewise linear functions on the interface $\Gamma$ (vanishing at the end points of $\Gamma$), we compute the matrix representation of $\tilde{S}$ explicitly, by appropriate exact factorization of the subdomain (Neumann) stiffness matrix $A_2$. Thus, the above discrete problem is reduced to a problem for the unknown $\mathbf{u}_h$, $p_{1,\,h}$, and $p^\Gamma_{2,\,h} := p_{2,\,h}|_\Gamma$ by solving a Neumann problem on $\Omega_2$, i.e., by solving $A_2 \widetilde{p}_{2,\,h} = \mathbf{f}_2$. The latter leads to,

$$\left[ \begin{array}{ccc} \mathbf{A}_1 & \mathbf{N}^T & \mathbf{T}^T_\Gamma \\ \mathbf{N} & 0 & 0 \\ \mathbf{T}_\Gamma & 0 & -\tilde{S} \end{array} \right] \left[ \begin{array}{c} \mathbf{u}_h \\ p_{1,h} \\ p^\Gamma_{2,h} \end{array} \right] = \left[ \begin{array}{c} -\mathbf{T}^T_\Gamma \widetilde{p}_{2,\,h} \\ \mathbf{f}_1 \\ 0 \end{array} \right].$$

We used the following solution methods:

- the MINRES (minimum residual method) for the above reduced system with the following preconditioner:

$$\begin{bmatrix} \mathbf{B}_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \tilde{S} \end{bmatrix};$$

  Here, $\mathbf{B}_1$ stands for an algebraically stabilized version of the hierarchical basis method (HB) from [13]. Details on the algebraic stabilization of the HB methods are found in [22]. The $I$ is the (diagonal) mass matrix and $\tilde{S}$ is a discrete Neumann–Dirichlet mapping, which we invert exactly. (I.e., it corresponds to solving a discrete Neumann problem in $\Omega_2$ and restrict the result to $\Gamma$.) Note that the thus described preconditioned MINRES method is exactly equivalent to the preconditioned MINRES method applied to the original (unreduced) problem with the block-diagonal preconditioner

$$\begin{bmatrix} \mathbf{B}_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & A_2 \end{bmatrix},$$

  since the successive iterates and search directions are discrete harmonic (with respect to $A_2$) functions in $\Omega_2$.
- the CG method applied to the reduced problem (4.11); the stopping criterion here was until relative residual reduction of $10^{-6}$ has been reached.

Let the grid in $\Omega_1$ have mesh-nodes denoted by $(x_i, y_j)$, $0 \le i \le n_x$, $0 \le j \le n_y$, $n_x = n_y = 1/h_1$, $h_x = h_y := h_1$. Most of the errors are computed at the shifted by half step-size points, namely we use also the points $x_{i-1/2} = x_i - 0.5h_x$ and $y_{j-1/2} = y_j - 0.5h_y$. Finally, $I_h$ stands for the finite element interpolation operator.

In the Table 1 we show:

(i) $$\delta_p = \|I_h p_1 - p_{1,h}\|_h \equiv \left[\sum_{i=1}^{n_x}\sum_{j=1}^{n_y} h_x h_y (p_1(x_{i-1/2}, y_{j-1/2}) - p_{1,h}(x_{i-1/2}, y_{j-1/2}))^2\right]^{\frac{1}{2}},$$

i.e., a discrete $L^2$–norm of the error $p_1 - p_h$;

(ii) $$\delta_{u_1} = \|I_h u_1 - u_{h,\,1}\|_h \equiv \left[\sum_{i=0}^{n_x}\sum_{j=1}^{n_y} h_x h_y (u_1(x_i, y_{j-1/2}) - u_{h,\,1}(x_i, y_{j-1/2}))^2\right]^{\frac{1}{2}},$$

i.e., a discrete $L^2$–norm of the error $u_1 - u_{h,1}$;

(iii) $$\delta_{u_2} = \|I_h u_2 - u_{h,\,2}\|_h \equiv \left[\sum_{i=1}^{n_x}\sum_{j=0}^{n_y} h_x h_y (u_2(x_{i-1/2}, y_j) - u_{h,\,2}(x_{i-1/2}, y_j))^2\right]^{\frac{1}{2}},$$

i.e., a discrete $L^2$–norm of the error $u_2 - u_{h,\,2}$;

(iv)

$$\delta_{u_{\mathrm{int}}} = \|I_h(\mathbf{u} - \mathbf{u}_h)\|_h \equiv$$
$$\left[\sum_{i=1}^{n_x}\sum_{j=1}^{n_y} h_x h_y \left((\mathbf{u}\cdot\mathbf{n})(x_{i-1/2}, y_{j-1/2}) - (\mathbf{u}_h\cdot\mathbf{n})(x_{i-1/2}, y_{j-1/2})\right)^2\right]^{\frac{1}{2}},$$

| | $h_1 = 1/16$ $h_2 = b/16$ | $h_1 = 1/32$ $h_2 = b/32$ | $h_1 = 1/64$ $h_2 = b/64$ | $h_1 = 1/128$ $h_2 = b/128$ | $\approx$ order |
|---|---|---|---|---|---|
| $\delta_p$ | 3.18e-2 | 7.57e-3 | 1.83e-3 | 4.57e-4 | 2 |
| $\delta_{u_1}$ | 0.5749 | 0.1343 | 3.27e-2 | 7.87e-3 | 2 |
| $\delta_{u_2}$ | 0.3617 | 8.87e-2 | 2.21e-2 | 5.51e-3 | 2 |
| $\delta_{u_{\rm int}}$ | 0.3792 | 9.42e-2 | 2.37e-2 | 5.93e-3 | 2 |
| $\delta_{p_2}$ | 0.1519 | 3.44e-2 | 7.71e-3 | 1.91e-3 | 2 |
| # iterations | 57 | 71 | 86 | 92 | |
| $\varrho$ | 0.69 | 0.74 | 0.78 | 0.79 | |

Table 1. Error behavior and iteration counts for the composite problem; $b = 0.55$.

| $h_1$ | $h_2$: $b/16$ | $b/32$ | $b/64$ | $b/128$ |
|---|---|---|---|---|
| 1/16 | 11, 0.21 | 12, 0.26 | 13, 0.30 | 13, 0.30 |
| 1/32 | 12, 0.30 | 15, 0.39 | 15, 0.39 | 15, 0.39 |
| 1/64 | 10, 0.22 | 14, 0.36 | 16, 0.39 | 15, 0.39 |
| 1/128 | 9, 0.21 | 11, 0.27 | 15, 0.38 | 16, 0.40 |

Table 2. Number of CG iterations and average reduction factors for solving the system $(I + SE)q_{2,h} = rhs_{2,h}$; $b = 0.55$.

i.e., a discrete $L^2$–norm of the error $\mathbf{u} \cdot \mathbf{n} - \mathbf{u}_h \cdot \mathbf{n}$, where $\mathbf{n}$ is the unit normal vector to the edge with end-points $(x_{i-1}, y_{j-1})$ and $(x_i, y_j)$;

(v) a discrete $H^{1/2}_{0,0}(\Gamma)$–norm of the error $(I_h p_2 - p_{h_2})|_\Gamma$;

(vi) the number of iterations of the preconditioned MINRES method;

(vii) an average reduction factors $\varrho$.

The second test demonstrates the convergence of the CG method applied to the matrix of the reduced problem (4.11). We have chosen a random $r.h.s._{2,\, h_2}$, and the iterations were stopped after the norm of the residual has been relatively reduced by $10^{-6}$. Here we varied the meshes $h_1$ and $h_2$ to see the sensitivity of the method with respect to the discrepancy of the grids. As it is seen, the convergence appears to be fairly insensitive to the mesh sizes, all in good agreement with the theory (see, e.g., Theorem 4.1).

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